LINEAR FIRST-ORDER DIFFERENTIAL-DIFFERENCE EQUATIONS OF RETARDED TYPE WITH CONSTANT COEFFICIENTS

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Differential-difference equations are essentially like pure differential equations, except that a differential equation contains one or more derivatives of an unknown function, and a differential-difference equation can contain one or more derivatives of an unknown function, for example $u'(t)$, $u''(t)$, together with any number of functions of the form $u(t-\omega_1)$, $u(t-\omega_2)$, $\cdots$, $u(t-\omega_n)$, and possibly some of their derivatives. The numbers ω_1, ω_2 are given and are called retardations or spans. Some examples of differential-difference equations are

(1-1), $u''(t) - u'(t-1) + 2u(t) = 0$,

(1-2), $u'(t) + u(t-1) - u(t-\sqrt{2}) = 0$,

(1-3), $u'(t) + 2u(t) - 2u(t-1) = e^t$.

This paper is concerned with equations in which u is regarded as a function of a single independent variable which will be called t. Therefore, all derivatives are ordinary rather than partial derivatives. The customary meanings of differential order and difference order of an equation are observed. That is, differential order of an equation is the order of the highest derivative appearing, and difference order is one less than the distinct number of arguments ω_1,

$\omega_2, \cdots$. For example, equation (1-1) is of order 2 in derivatives and of order 1 in differences, but equation (1-2) is of order 1 in derivatives and of order 2 in differences.

The general form of a differential-difference equation of differential order n and difference order m is

$$F\Big[t,\ u(t),\ u(t-\omega_1),\ \cdots,\ u(t-\omega_m),\ u'(t),\ u'(t-\omega_1),\ \cdots,\ \cdots,\ u^{(n)}(t),\ u^{(n)}(t-\omega_1),\ \cdots,\ u^{(n)}(t-\omega_m)\Big] = 0,$$

where F is a given function of $1+(m+1)(n+1)$ variables and the numbers $\omega_1, \omega_2, \cdots, \omega_m$ are also given. This paper, however, is concerned with equations of order 1 in both derivatives and differences. Equation (1-3) is such an equation. Also, F and u are here required to be real functions of real variables, and the numbers $\omega_1, \omega_2, \cdots, \omega_m$ must be real. The coefficients are assumed to be constants. Complex solutions may result, but real solutions are of primary interest here.

With the foregoing restrictions, the general form for a differential-difference equation is reduced to

(1-4), $a_0u'(t) + a_1u'(t-\omega) + b_0u(t) + b_1u(t-\omega) = f(t).$

The theory of the equations of this form exhibits most of the features of the more general theory, while avoiding a lot of the detail involved.

It is well known that differential equations have great value in application to physical situations. A complete study of the rate of change of a physical system, though, involves not only the present state of the system but also its past history.

In particular, equation (1-4) represents several types of applications. If $a_o = a_1 = 0$ or $b_o = b_1 = 0$, the equation is a pure difference equation. If $a_o = b_o = 0$ or $a_1 = b_1 = 0$, it is an ordinary differential equation. These are not discussed here, because they have been treated extensively elsewhere. There are, however, three distinct types of differential-difference equations represented by equation (1-4). An equation of that form is of retarded type if $a_o \neq 0$ and $a_1 = 0$. (Such equations are also called delay differential equations and hystero-differential equations.) It is of neutral type if $a_o \neq 0$ and $a_1 \neq 0$. It is of advanced type if $a_o = 0$ and $a_1 \neq 0$.

In applications, usually t represents time. Thus an equation of retarded type can be used to represent the behavior of a system where the rate of change of the quantity involved depends on past and present values of the quantity. An equation of neutral type can represent a system in which the present rate of change of the quantity depends on the past rate of change as well as the past and present values of the quantity. An equation of advanced type can represent a system in which the rate of change of the quantity depends either on the past value and the past rate of change or on the present and future values of the quantity.

The discussion here is limited to equations of retarded type since they are in some ways simpler to work with than equations of neutral or advanced type, and yet the theory of equations of retarded type is generally representative of all three.

As is the case with ordinary differential equations, the main objective in studying differential-difference equations is to solve them: that is, to find a particular function (or functions) $u(t)$ which satisfies the equation.

There is actually a clearly defined process which can be used to find solutions over finite intervals. Called the continuation process, it is a means of determining a unique solution $u(t)$ by first specifying that $u(t)$ have a particular value over an initial interval, $0 < t < \omega$. The value of the solution is then determined for the next interval, $\omega < t < 2\omega$. The process can be continued to obtain $u(t)$ for any desired finite interval. If, after three or four such steps, a pattern is evident, then a formula is written for $u(t)$, $n\omega < t < (n+1)\omega$.

A method of solution which corresponds to a method used for ordinary differential equations is that of obtaining a solution which is the sum of simple exponential solutions. When some simple exponential solutions are determined, then linear combinations of these are also solutions. Operator symbolism is sometimes useful in obtaining exponential solutions. The operators are manipulated and then applied to the differential-difference equation, thereby obviating burdensome manipulation of the equation itself.

For some uses, it is expedient to have an estimate of the magnitude of solutions. By using certain theorems, an estimate can be obtained of the size of a solution, of the difference between the values of two solutions of the same equation, or of the difference between the values of the solutions of two different equations. Previously, the key theorem for obtaining these estimates was proved using Laplace transforms. Here it is proved using integration by parts.

Also, there is a method of solving differential-difference equations which consists of obtaining a solution in the form of a definite integral. Theorems are proved to show how this can be done and to demonstrate the form which such solutions take.

To obtain exponential solutions $u(t) = e^{st}$, the characteristic roots s must be found. A particular group of equations $u'(t) - u(t-\omega) = 0$ is studied to find the characteristic roots which lead to solutions e^{st}. It is determined that there exist an infinite number of such solutions, and therefore there is a series expansion for

$$u(t) = \sum_{r=1}^{\infty} p_R(t)e^{s_R t}.$$

CHAPTER II

CONTINUATION PROCESS

One of the fundamental methods for the evaluation of differential-difference equations is the continuation process in which the solution is extended forward by increasing t interval by interval. The process provides a method of proving that a solution exists and for calculating the solution over any desired finite interval.

Suppose $\quad u'(t) = 1 + u(t-1), \quad t > 1,$

with the initial condition $u(t) = 1, \quad 0 \le t \le 1.$

Then, for $1 \le t \le 2$, $u'(t) = 1 + u(t-1) = 2.$

By integrating $u'(t)$ with respect to $t-1$,

$$u(t) = 2(t-1) + c.$$

Since $u(1) = 1$, $c = 1$.

Therefore, $\quad u(t) = 2(t-1) + 1, \quad 1 \le t \le 2.$

Similarly, for $2 \le t \le 3$, $\quad u'(t) = 1 + u(t-1)$

$$= 2(t-2) + 2.$$

$$u(t) = (t-2)^2 + 2(t-2) + c.$$

Since $u(2) = 3$, $c = 3$.

Therefore, $\quad u(t) = (t-2)^2 + 2(t-2) + 3, \quad 2 \le t \le 3.$

For $3 \le t \le 4$, $\quad u'(t) = 1 + u(t-1)$

$$= (t-3)^2 + 2(t-3) + 4.$$

$$u(t) = 1/3(t-3)^3 + (t-3)^2 + 4(t-3) + c.$$

Since $u(3) = 6$, $c = 6$.

Therefore, $u(t) = 1/3(t-3)^3 + (t-3)^2 + 4(t-3) + 6, \; 3 \leq t \leq 4$.

For $4 \leq t \leq 5$, $u'(t) = 1/3(t-4)^3 + (t-4)^2 + 4(t-4) + 7$.

Since $u(4) = 34/3$,

$u(t) = 1/12(t-4)^4 + 1/3(t-4)^3 + 2(t-4)^2 + 7(t-4) + 34/3$,

$4 \leq t \leq 5$. A pattern for a general solution is suggested, but it is not a compact formula, and is not presented here.

Consider also the following example. (Note: For each integration to find $u(t)$ for $n \leq t \leq n+1$, the appropriate constant of integration is found by computing $u(n)$ using $u(t)$ for $n-1 \leq t \leq n$.)

Suppose $u'(t) = 2u(t-1)$, $t > 1$, with the initial condition

$$u(t) = t, \quad 0 \leq t \leq 1.$$

For $1 \leq t \leq 2$, $u'(t) = 2(t-1)$, and

$$u(t) = (t-1)^2 + 1.$$

For $2 \leq t \leq 3$, $u'(t) = 2(t-2)^2 + 2$, and

$$u(t) = 2/3(t-2)^3 + 2(t-2) + 2.$$

For $3 \leq t \leq 4$, $u'(t) = 4/3(t-3)^3 + 4(t-3) + 4$, and

$$u(t) = 1/3(t-3)^4 + 2(t-3)^2 + 4(t-3) + 14/3.$$

For $4 \leq t \leq 5$, $u'(t) = 2/3(t-4)^4 + 4(t-4)^2 + 8(t-4) + 28/3$,

and thus $$u(t) = 2/15(t-4)^5 + 4/3(t-4)^3 + 4(t-4)^2 + 28/3(t-4) + 11.$$

As in the previous example a general solution is suggested, but it is not simple and is not shown here.

An equation of the type $u'(t) = a_0u(t) + b_1u(t-\omega)$ containing the additional $u(t)$ term presents a somewhat different problem, which is illustrated by the following example.

Suppose $u'(t) = 1 - u(t) - \left[1 - u(t-\omega)\right] e^{-\omega}, \quad t \geq \omega$, and

$$u(t) = 1 - e^{-t}(1+t), \quad 0 \leq t \leq \omega.$$

Or, for $0 \leq t \leq \omega$, $\quad u(t)e^{t} = e^{t} - t - 1.$

$\omega \leq t \leq 2\omega$, $\quad u(t-\omega) = 1 - e^{-t+\omega}(1+t-\omega).$

$$u'(t) = 1 - u(t) - \left[1 - u(t-\omega)\right] e^{-\omega}$$
$$= 1 - u(t) - e^{-\omega}\left[1 - 1 + e^{-t+\omega}(1+t-\omega)\right].$$

Thus $\quad u'(t) + u(t) = 1 - e^{-t}(1+t-\omega).$

Solving the differential equation gives

$$u(t)e^{t} = \int e^{t} - (1+t-\omega)\, dt + c$$
$$= e^{t} - 1/2(t-\omega)^{2} - (t-\omega) + c.$$

Since $u(\omega)e^{\omega} = e^{\omega} - \omega - 1$, $\quad c = -\omega - 1.$

Therefore $\quad u(t) = 1 - 1/2(t-\omega)^{2}e^{-t} - (t-\omega)e^{-t} - (\omega+1)e^{-t},$

$$\omega \leq t \leq 2\omega.$$

For $2\omega \leq t \leq 3\omega$,

$$u(t-\omega) = 1 - 1/2(t-2\omega)^{2}e^{-t+\omega} - (t-2\omega)e^{-t+\omega} - (\omega+1)e^{-t+\omega}.$$
$$u'(t) = 1 - u(t) - e^{-\omega}\left[1 - 1 + 1/2(t-2\omega)^{2}e^{-t+\omega} + (t-2\omega)e^{-t+\omega} + (\omega+1)e^{-t+\omega}\right].$$

Thus $u'(t) + u(t) = 1 - 1/2(t-2\omega)^{2}e^{-t} - (t-2\omega)e^{-t} - (\omega+1)e^{-t}.$

$$u(t)e^{t} = \int e^{t} - 1/2(t-2\omega)^{2} - (t-2\omega) - (\omega+1)\, dt + c$$
$$= e^{t} - 1/6(t-2\omega)^{3} - 1/2(t-2\omega)^{2} - (\omega+1)(t-2\omega) + c.$$

Since $u(2\omega)e^{2\omega} = e^{2\omega} - 1/2\omega^{2} - 2\omega - 1$, $\quad c + -1/2\omega^{2} - 2\omega - 1.$

Therefore $u(t) = 1 - 1/6(t-2\omega)^{3}e^{-t} - 1/2(t-2\omega)^{2}e^{-t} -$

$$(\omega+1)(t+2\omega)e^{-t} - (1/2\omega^{2}+2\omega+1), \quad 2\omega \leq t \leq 3\omega$$

By the same method, for $3\omega \le t \le 4\omega$,

$$u(t)e^t = e^t - 1/24(t-3\omega)^4 - 1/6(t-3\omega)^3 + (\omega+1)(t-3\omega)^2 - (1/2\omega^2+2\omega+1)(t-3\omega) - (1/6\omega^3+2\omega^2+3\omega+1).$$

In general, for $n\omega \le t \le (n+1)\omega$,

$$u(t)e^t = e^t - 1/(n+1)!\ (t-n\omega)^{n+1} - 1/n!\ (t-n\omega)^n - C_1(t-n\omega)^{n-1} - C_2(t-n\omega)^{n-2} - \cdots - C_{n-1}(t-n\omega) - C_n,$$

where $C_n = 1/n!\omega^n + 1/(n+1)!\omega^{n-1} + C_1\omega^{n-2} + C_2\omega^{n-3} + \cdots + C_{n-1}\omega$.

In the following problem, the initial interval is $0 \le t \le 2$. The first step in the continuation process for the interval $2 \le t \le 3$ is not complex. The next step, $(3 \le t \le 4)$, involves shifting the limits of the integral involved and dividing the integral into two parts. The process can be continued further but is not continued here because a general solution is not readily identifiable.

Suppose $u'(t) = u(t-1) + \int_1^2 u(t-s)\, ds, \quad t > 2,$

with $\quad u(t) = 1, \quad 0 \le t \le 2.$

Then for $2 \le t \le 3$, $u'(t) = 1 + \int_1^2 1\, ds = 2$, and

$$u(t) = 2t-3.$$

For $3 \le t \le 4$, $u'(t) = 2(t-1) - 3 + \int_1^2 u(t-s)\, ds.$

$$u(t-s) = \begin{cases} 1, & \text{if } 1 \le t-s \le 2, \text{ or} \\ 2(t-s)-3, & \text{if } 2 \le t-s \le 3. \end{cases}$$

It follows that $u(t-s) = \begin{cases} 1, & \text{if } s \ge t-2, \text{ or} \\ 2t-3-2s, & \text{if } s \le t-2. \end{cases}$

Therefore, $u'(t) = 2t - 5 + \int_1^{t-2} 2t-s-2s\ ds + \int_{t-2}^{2} 1\ ds$

$$= 2t - 5 + \Big[(2t-3)s - s^2\Big]_{s=1}^{t-2} + \Big[s\Big]_{s=t-2}^{2},$$

which reduces to $t^2 - 4t - 5$.

Hence, $u(t) = 1/2t^3 - 2t^2 - 5t - 3, \quad 3 \le t \le 4.$

In the following example, the continuation process must be performed in half-unit steps.

Suppose $u'(t) = u'(t-1) + \int_{\frac{1}{2}}^{1} u(t-s)\ ds, \quad t > 1$, with

$u(t) = 1, \quad 0 \le t \le 1.$

Then, for $1 \le t \le 3/2$, $u'(t) = 0 + \int_{\frac{1}{2}}^{1} u(t-s)\ ds.$

$1 \le t \le 3/2$ and $-1 \le -s \le -1/2$; so $0 \le t-s \le 1$.

Hence $$u'(t) = \int_{\frac{1}{2}}^{1} 1\ ds = s\Big]_{s=\frac{1}{2}}^{1} = 1/2.$$

Therefore $u(t) = 1/2t + 1/2, \quad 1 \le t \le 3/2.$

For $3/2 \le t \le 2$, $u'(t) = 0 + \int_{\frac{1}{2}}^{1} u(t-s)\ ds.$

$$u(t-s) = \begin{cases} 1, & \text{if } 1/2 \le t-s \le 1, \text{ or} \\ 1/2(t-s)+1/2, & \text{if } 1 \le t-s \le 3/2. \end{cases}$$

$$u(t-s) = \begin{cases} 1, & \text{if } s \ge t-1, \text{ or} \\ 1/2(t-s+1), & \text{if } s \le t-1. \end{cases}$$

Hence $u'(t) = \int_{\frac{1}{2}}^{t-1} 1/2(t+1) - 1/2s\ ds + \int_{t-1}^{1} 1\ ds$

$$= \Big[1/2(t+1)s - 1/4s^2\Big]_{s=\frac{1}{2}}^{t-1} + \Big[s\Big]_{s=t-1}^{1},$$

which reduces to $1/4t^2 - 3/4t - 7/16$.

Therefore $u(t) = 1/12t^3 - 3/8t^2 - 7/16t + 79/32, \quad 3/2 \le t \le 2.$

For $2 \leq t \leq 5/2$, $u'(t) = 1/2 + \int_{\frac{1}{2}}^{1} u(t-s)\, ds$.

$$u(t-s) = \begin{cases} 1/2(t+1-s), \text{ if } 1 \leq t-s \leq 3/2, \text{ or} \\ 1/12(t-s)^3 - 3/8(t-s)^2 - 7/16(t-s) + 79/32, \\ \qquad \text{if } 3/2 \leq t-s \leq 2. \end{cases}$$

$$\text{Therefore } u'(t) = 1/2 + \int_{\frac{1}{2}}^{t-\frac{3}{2}} \left[1/12(t-s)^3 - 3/8(t-s)^2 - 7/16(t-s) + 79/32\right] ds + \int_{t-\frac{3}{2}}^{1} 1/2(t-1-s)\, ds, \; 2 \leq 5 \leq 5/2.$$

$u'(t)$ is determined by solving the two above definite integrals. A subsequent integration and computation of the constant of integration gives $u(t)$ for $2 \leq t \leq 5/2$.

CHAPTER III

EXPONENTIAL SOLUTIONS AND LINEAR OPERATORS

Often, use of the continuation process does not lead to a formula which gives the value of the solution over all finite intervals, or even if such a formula is apparent, it may not be useful in isolating certain properties of the solution. Other methods of solution are needed. One method, which is also used to solve pure differential equations, is the constructing of sums of simple exponential solutions. Operator symbolism is also of some use in solving differential-difference equations.

For convenience, the linear operator $L(u)$ is here defined by the following equation:

(3-1), $L(u) = a_0u'(t) + b_0u(t) + b_1u(t-\omega)$.

An equation of the form $L(u) = 0$ is by definition homogeneous, and an equation of the form $L(u) = f$ is inhomogeneous. The linearity of $L(u)$ leads to the observation that if $u_1(t)$ and $u_2(t)$ are any two solutions of the equation $L(u) = 0$, and if c_1 and c_2 are any two constants, $c_1u_1(t) + c_2u_2(t)$ is also a solution of $L(u) = 0$ because $L(c_1u_1+c_2u_2) = c_1L(u_1) + c_2L(u_2) = 0$.

Thus new solutions are generated by forming linear combinations of known solutions.

Similarly, if $v(t)$ is a solution of $L(u) = f$, and if $y(t)$ is a solution of $L(u) = 0$, then $v + y$ is a solution of $L(u) = f$, because $L(v+y) = L(v) + L(y) = f$. Thus, the solution of the inhomogeneous equation $L(u) = f$, with the initial condition $u = g$ for $t_o \leq t \leq t_o + \omega$, can be found by adding the solutions w and v of the simpler problems

$L(w) = f$, $w = 0$ for $t_o \leq t \leq t_o + \omega$, and

(3-2), $L(v) = 0$, $v = g$ for $t_o \leq t \leq t_o + \omega$. Therefore, a study of the homogeneous equation (3-2) is in order.

Since solutions for homogeneous equations can be generated as linear combinations of simple solutions, some simple solutions must be found for (3-2). As with pure differential equations, the simple solutions are exponentials. Suppose

$$L(e^{st}) = a_o s e^{st} + b_o e^{st} + b_1 e^{s(t-\omega)}$$
$$= (a_o s + b_o + b_1 e^{-\omega s}) e^{st}.$$

Then, e^{st} is a solution of $L(u) = 0$, for all t, if and only if s is a zero of the function

$$h(s) = a_o s + b_o + b_1 e^{-\omega s}.$$

The function $h(s)$ above, which is associated with the equation $L(u) = 0$, is by definition the characteristic function of L. The equation $h(s) = 0$ is by definition the characteristic equation of L. The roots of $h(s) = 0$ are by definition the characteristic roots of L. A root of $h(s)$ is a root of multiplicity m if $h(s)$ and its first $m-1$ derivatives are zero.

There is a solution, which is possibly complex, of $L(u) = 0$ for each characteristic root; and multiple characteristic roots generate a number of independent solutions, a fact which leads to the following theorem.

Theorem 3-1.

The equation $\quad L(u) = a_0u'(t) + b_0u(t) + b_1u(t-\omega) = 0$ is satisfied by $\sum p_k(t)e^{s_k t}$ where $\{s_k\}$ is any sequence of characteristic roots and $p_k(t)$ is some polynomial with degree less than the multiplicity of s_k. The sum may be either finite or infinite with conditions which are suitable to insure convergence.

Proof of Theorem 3-1 consists of several observations.

First, $\quad h'(s) = a_0 - b_1\omega e^{-\omega s}.$

Thus, $\quad h^{(k)}(s) = (-1)^k b_1\omega^k e^{-\omega s}, \ k = 2,3, \cdots.$

Then, for any $n \geq 1$,

$$L(t^n e^{st}) = a_0(t^n se^{st} + nt^{n-1}e^{st}) + b_0 t^n e^{st} + b_1(t-\omega)^n e^{s(t-\omega)}.$$

Consider the last term, $b_1(t-\omega)^n e^{s(t-\omega)}$, of the equation above. If $(t-\omega)^n$ is expanded by the binomial theorem, the kth term, $0 \leq k \leq n$, of $b_1(t-\omega)^n e^{s(t-\omega)}$ is

$$\binom{n}{k}(-1)^k b_1 t^{n-k}\omega^k e^{st-\omega s}$$

$$= \binom{n}{k}h^{(k)}(s)t^{n-k}e^{st}.$$

Therefore, $\quad L(t^n e^{st}) = e^{st}\sum_{k=0}^{n}\binom{n}{k}t^{n-k}h^{(k)}(s).$

Thus, $L(t^n e^{st}) = 0$ for any integer n, $0 \leq n \leq m-1$, where s is a characteristic root of multiplicity m, since $h(s)$ and its first $m-1$ derivatives are all zero. Given a root s of multiplicity m, there are therefore m functions,

$$e^{st}, te^{st}, \ldots, t^{m-1}e^{st},$$

which are all linearly independent solutions of $L(u) = 0$, for all real numbers t. Hence, if $p(t)$ is any polynomial of degree not greater than $m-1$, $p(t)e^{st}$ is a solution of the equation $L(u) = 0$ since that equation is linear and homogeneous. Thus, Theorem 3-1 is justified.

Suppose $u'(t) = u(t-1)$. $u = e^{st}$ is a solution if and only if s is a zero of $h(s)$. Here, $h(s) = s - e^{-s}$. Hence, for all s such that $s = e^{-s}$, $u = e^{st}$ is a solution of $u'(t) = u(t-1)$.

Suppose $u'(t) = u(t-1) + \int_1^2 u(t-r)dr$. Then, it follows that $u(t) = e^{st}$ is a solution if s is a solution of $s^2 = (s+1)e^{-s} - e^{-2s}$.

$$\begin{aligned} u(t-1) + \int_1^2 u(t-r)dr &= e^{s(t-1)} + \int_1^2 e^{s(t-r)}dr \\ &= e^{s(t-1)} + \frac{e^{s(t-2)}}{-s} - \frac{e^{s(t-1)}}{-s} \\ &= \frac{se^{st}(se^{-s} - e^{-2s} + e^{-s})}{s^2} \\ &= \frac{se^{st}\left[(s+1)e^{-s} - e^{-2s}\right]}{s^2} \\ &= se^{st} = u'(t). \end{aligned}$$

The problem of finding the characteristic roots s_k is discussed at greater length in Chapter VI, where a specific equation, $u'(t) = -u(t-\omega)$ is examined.

Linear operators are sometimes helpful in solving differential-difference equations. The operators D, Δ, and E are here defined by the relations

$$Du(t) = u'(t), \ \Delta u(t) = u(t+1) - u(t), \ Eu(t) = u(t+1).$$

For any real numbers k and ω define the following:

$$(kE)^{\omega}u(t) = k^{\omega}u(t+\omega),$$

$$(kD)u(t) = ku'(t),$$

$$(k\Delta)u(t) = k\left[u(t+1)-u(t)\right].$$

If D and E are any two operators, define sums and products as is usual:

$$(D+E)u = Du + Eu,$$

$$(DE)u = D(Eu).$$

Two operators are by definition equal if they produce the same results when applied to any function. Consequently, the commutative law holds. For example,

$$\begin{aligned}(D+\Delta)u &= Du + \Delta u \\ &= u'(t) + \left[u(t+1) - u(t)\right] \\ &= \left[u(t+1)-u(t)\right] + u'(t) \\ &= \Delta u + Du = (\Delta + D)u.\end{aligned}$$

And, $(\Delta E)u = \Delta(Eu) = \Delta\left[u(t+1)\right]$

$$= u(t+2)-u(t+1)$$

$$= E\left[u(t+1)-u(t)\right]$$

$$= E(\Delta u)$$

$$= (E\Delta)u.$$

Corresponding proofs show the commutative law of addition for $(D+E)u$ and $(\Delta+E)u$ and the commutative law of multiplication for $(DE)u$ and $(D\Delta)u$.

The associative laws also hold. For example,

$$\left[(D\Delta)E\right]u = (D\Delta)(Eu) = (D\Delta)\left[u(t+1)\right]$$

$$= D\left[u(t+2)-u(t+1)\right]$$

$$= D\left\{\Delta\left[u(t+1)\right]\right\}$$

$$= D\left[\Delta(Eu)\right]$$

$$= D(\Delta E)u$$

$$= \left[D(\Delta E)\right]u.$$

The distributive law likewise holds. For example,

$$\Delta(E+D)u = \Delta\left[u(t+1) + u'(t)\right]$$

$$= u(t+2)-u(t+1) + u'(t+1)-u'(t)$$

$$= \left[u(t+2)-u(t+1)\right] + \left[u'(t+1)-u'(t)\right]$$

$$= \Delta\left[u(t+1)\right] + \Delta\left[u'(t)\right]$$

$$= \Delta(Eu) + \Delta(Du)$$

$$= (\Delta E)u + (\Delta D)u$$

$$= (\Delta E+\Delta D)u.$$

The operator $e^{\omega D}$ is here defined by the series

$$e^{\omega D} = 1 + \omega D + \frac{\omega^2 D^2}{2!} + \cdots.$$

Consequently, $e^{\omega D}u = 1 + \omega u'(t) + \frac{\omega^2 u''(t)}{2!} + \cdots$,

which is the Taylor's expansion for $u(t+\omega)$, where $u(t) = 1$.

However,

$$\begin{aligned} u(t+\omega) &= 1^{\omega}u(t+\omega) \\ &= E^{\omega}u \end{aligned}$$

Therefore, $e^{\omega D} = E^{\omega}$.

Suppose an operator q is defined in such a manner that

$$q(D) = a_0De^{\omega D} + b_0e^{\omega D} + b_1.$$

Then

$$\begin{aligned} q(D)u(t) &= (a_0De^{\omega D}+b_0e^{\omega D}+b_1)u(t) \\ &= (a_0DE^{\omega}+b_0E^{\omega}+b_1)u(t) \\ &= \left[a_0Du(t+\omega) + b_0u(t+\omega) + b_1\right]u(t) \\ &= a_0u'(t+\omega) + b_0u(t+\omega) + b_1u(t). \end{aligned}$$

Therefore the equation

$$a_0u'(t+\omega) + b_0u(t+\omega) + b_1u(t) = f(t)$$

can be written as

(3-3), $q(D)u(t) = f(t)$, where

$$q(D) = a_0De^{\omega D} + b_0e^{\omega D} + b_1.$$

Operators can be used to an advantage because of the relative ease with which they can be manipulated and then applied to differential-difference equations. For any constants k and c and any function f which is sufficiently differentiable,

(3-4), $q(D)(ke^{ct}) = q(c)ke^{ct}$.

For a proof, it is sufficient to examine the effects of the operators involved.

$$q(D)(ke^{ct}) = (a_o De^{\omega D} + b_o e^{\omega D} + b_1)(ke^{ct})$$

$$= a_o Dke^{c(t+\omega)} + b_o ke^{c(t+\omega)} + b_1 ke^{ct}$$

$$= a_o kce^{c(t+\omega)} + b_o ke^{c(t+\omega)} + b_1 ke^{ct}.$$

$$q(c)ke^{ct} = (a_o ce^{\omega c} + b_o e^{\omega c} + b_1)ke^{ct}$$

$$= a_o cke(\omega c+ct) + b_o ke^{\omega c+ct} + b_1 ke^{ct}$$

$$= a_o kce^{c(t+\omega)} + b_o ke^{c(t+\omega)} + b_1 ke^{ct}.$$

Let the expression

(3-5), $$q^{-1}(D)f(t) = (a_o De^{\omega D} + b_o e^{\omega D} + b_1)^{-1} f(t)$$

denote any particular solution of the equation

$$au'(t+\omega) + b_c u(t+\omega) + b_1 u(t) = f(t).$$ Then,

(3-6), $$q(D)\left[q^{-1}(D)f(t)\right] = f(t),$$ can be proved.

Proof: $q(D)\left[q^{-1}(D)f(t)\right] = q(D)u(t)$, by definition,

$= f(t)$, equation (3-3).

Similarly,

(3-7), $$q^{-1}(D)(ke^{ct}) = \frac{ke^{ct}}{q(c)},$$ provided $q(c) \neq 0$.

Proof: $q(D)\left[q^{-1}(D)(ke^{ct})\right] = ke^{ct}$, equation (3-6).

$$q(D)\frac{ke^{c}}{q(c)} = \frac{q(D)(ke^{ct})}{q(c)}$$

$$= \frac{q(c)(ke^{ct})}{q(c)},$$ equation (3-4),

$= ke^{ct}$; and likewise,

(3-8), $$q^{-1}(D)(e^{ct}\sin lt) = \mathrm{Im}\left[q^{-1}(D)e^{(c+il)t}\right].$$

Proof: $e^{ct}\sin lt = \mathrm{Im}e^{(c+il)t}$

$$q^{-1}(D)(e^{ct}\sin lt) = q^{-1}(D)\left[\mathrm{Im}e^{(c+il)t}\right]$$
$$= \mathrm{Im}\left[q^{-1}(D)e^{(c+il)t}\right]$$

The operators and methods above can be used in finding a particular solution $q^{-1}(D)f(t)$ of each of the following equations.

Suppose $u'(t+1) - u(t) = 1$.

From equation (3-5),

$$\begin{aligned} q^{-1}(D)f(t) &= (De^{D} - 1)^{-1}(1), \quad f(t) = 1, \\ &= -1(1-De^{D})^{-1}(1) \\ &= -1(1-De^{D} + D^2e^{2D} - D^3e^{3D} + \cdots)(1) \\ &= -1 + 0 - 0 + 0 - \cdots. \end{aligned}$$

Therefore a particular solution is -1, because

$$De^{D}(1) = D(2) = 0,$$
$$D^2e^{2D}(1) = D^2(3) = D(0) = 0, \text{ etc.}$$

Suppose $u'(t+\pi) - u(t) = \sin t$.

From equation (3-8),

$$\begin{aligned} q^{-1}(D)\sin t &= \mathrm{Im}\left[q^{-1}(D)e^{it}\right] \\ &= \mathrm{Im}\left[\frac{e^{it}}{q(i)}\right], && \text{equation (3-7)}, \\ &= \mathrm{Im}\left[\frac{e^{it}}{ie^{\pi i}-1}\right], && \text{equation (3-3)}. \end{aligned}$$

Therefore a particular solution is $\mathrm{Im}\left[\dfrac{e^{it}}{ie^{\pi i}-1}\right]$.

Suppose $u(t+1) - u(t) = t^k$.

From equation (3-5),

$$
\begin{aligned}
q^{-1}(D)f(t) &= (De^D-1)^{-1}t^k \\
&= -1(1-De^D)^{-1}t^k \\
&= -1(1-De^D + D^2e^{2D} - D^3e^{3D} + \cdots)t^k \\
&= -1(1-DE + D^2E^2 - D^3E^3 + \cdots)t^k \\
&= -t^k + k(t+1)^{k-1} - k(k-1)(t+2)^{k-2} + \cdots,
\end{aligned}
$$

which is therefore a particular solution $q^{-1}(D)f(t)$.

CHAPTER IV

ESTIMATES OF THE MAGNITUDE OF SOLUTIONS

For some applications an estimate of the magnitude of solutions of differential-difference equations is an invaluable tool. It is possible to estimate the size of the difference between two solutions of the same equation and between solutions of two different equations. An estimate can also be made of the size of the solution of a single equation. Theorem 4-1 will be used to prove a lemma, which will then be used in estimating the magnitude of solutions.

Theorem 4-1.

> For any real numbers a and b, $a < b$, let K_{ab} denote the class of real-valued continuous functions defined on $[a,b]$. Let $T = T_{\alpha\beta\gamma}$ be an operator from $K_{\alpha\gamma}$ to $K_{\beta\gamma}$, where α, β, γ are fixed numbers, $\alpha \leq \beta \leq \gamma$. Assume that
>
> (a) T is monotone in the sense that if $u \in K_{\alpha\gamma}$, $v \in K_{\alpha\gamma}$, and $u(t) \leq v(t)$ for $\alpha \leq t \leq \theta$, where $\beta \leq \theta \leq \gamma$, then $(Tu)(t) \leq (Tv)(t)$ for $\beta \leq t \leq \theta$,
>
> (b) T is contracting at a point in the sense that there is a function $g \in K_{\alpha\gamma}$ such that $(Tg)(t) < g(t)$ for $\beta \leq t \leq \gamma$. Let u be a function of class $K_{\alpha\gamma}$ such that
> $$u(t) \leq (Tu)(t), \quad \beta \leq t \leq \gamma,$$
> $$u(t) < g(t), \quad \alpha \leq t \leq \beta,$$
> Then $u(t) < g(t)$ for $\alpha \leq t \leq \gamma$.

Proof: This theorem can be proved by contradiction. If the conclusion is false, there is a smallest number t^* such that $u(t) < g(t)$, $\alpha \leq t < t^*$, but $u(t^*) = g(t^*)$. Here $\beta < t^* < \gamma$. Using properties

(a) and (b), we deduce that

$$(Tu)(t) \le (Tg)(t) < g(t), \quad \beta \le t \le t^*.$$

Therefore $u(t) < g(t)$, $\beta < t \le t^*$, contradicting $u(t^*) = g(t^*)$.[1]

This theorem is used now to prove the following important lemma.

Lemma 4-1.

If $w(t)$ is positive and monotone nondecreasing, $u(t) \ge 0$, $v(t) \ge 0$, all three functions are continuous, and if

$$u(t) \le w(t) + \int_a^t u(t_1)v(t_1)\,dt_1, \quad a \le t \le b,$$

then $u(t) \quad w(t)\exp\left[\int_a^t v(t_1)\,dt_1\right], \quad a \le t \le b.$

Proof: Theorem 4-1 can be applied by letting $\alpha = \beta = a$ and $\gamma = b$. By assuming the hypothesis, the theorem is now of the form:

If $u(t) \le (Tu)(t), \quad a < t < b,$

and $u(t) < g(t), \quad t = a,$

then $u(t) < g(t), \quad a \le t \le b.$

Let T be defined by

$$(Tu)(t) = w(t) + \int_a^t u(t_1)v(t_1)\,dt_1, \quad a \le t \le b,$$

and g by

$$g(t) = (1+\epsilon)w(t)\exp\left[\int_a^t v(t_1)\,dt_1\right], \quad a \le t \le b, \quad \epsilon > 0.$$

---

[1] Richard Bellman and Kenneth L. Cooke, Differential-Difference Equations (New York, 1963), pp. 61-62.

Then, by Theorem 4-1,

if $\quad u(t) \le w(t) + \int_a^t u(t_1)v(t_1)dt_1, \quad a \le t \le b,$

and $\quad u(a) < g(a),$

then $\quad u(t) < (1+\epsilon)w(t)\exp\left[\int_a^t v(t_1)dt_1\right].$

thus $\quad u(t) \le w(t)\exp\left[\int_a^t v(t_1)dt_1\right], \quad a \le t \le b.$

But it is in this case known to be true that $u(a) < g(a)$:

$$g(a) = (1+\epsilon)w(a)\exp\left[\int_a^a v(t_1)dt_1\right], \quad \epsilon > 0,$$

$$= (1+\epsilon)w(a)e^0;$$

$$u(a) \le w(a) + \int_a^a u(t_1)v(t_1)dt_1$$

$$\le w(a);$$

and since $u(a) \le w(a)$,

then $\quad u(a) < (1+\epsilon)w(a) = g(a), \quad \epsilon > 0,$

and thus $\quad u(a) < g(a)$.

Therefore, the condition which states "if $u(a) < g(a)$" is removed, and the result is Lemma 4-1.

Lemma 4-1 is used to prove some theorems concerning the size of differential-difference equation solutions. It is first necessary to define a function of class C^k on an interval.

The set of all real functions having k continuous derivatives on an open interval $t_1 < t < t_2$ is denoted by

$C^k(t_1,t_2)$. If f is a member of this set, then it is written $f\in C^k(t_1,t_2)$, or $f\in C^k$ on (t_1,t_2). If $f\in C^k(t_1,t_2)$ for every $t_2>t_1$, it is written $f\in C^k(t_1,\infty)$.

A function f is said to be of class C^k on $[t_1,t_2)$ if it is of class C^k on (t_1,t_2), if it has a right-hand kth derivative at t_1, and if the function $f^{(k)}(t)$ defined over $t_1\le t\le t_2$ by these values is continuous from the right at t_1.

Theorem: Let $u_1(t)$ and $u_2(t)$ be solutions of the following equation.

(4-1) $\qquad L(u) = a_o u'(t) + b_o u(t) + b_1 u(t-\omega)=f(t)$

which are of class C^2 on $[0,\infty)$. Suppose that f is of class C^1 on $[0,\infty)$, and let $m = \max\limits_{0\le t\le\omega}|u_1(t)-u_2(t)|$.

Then there is a positive constant c, depending only on the coefficients a_o, b_o, and b_1, such that

$$|u_1(t)-u_2(t)| \le me^{ct}, \quad t\ge 0.$$

Proof: Integrating $L(u_1)$ and $L(u_2)$ from ω to t and taking the difference gives

$$a_o[u_1(t)-u_2(t)] = a_o[u_1(\omega)-u_2(\omega)]$$
$$+ b_o\int_\omega^t u_2(t_1)-u_1(t_1)dt_1 + b_1\int_\omega^t u_2(t_1-\omega)-u_1(t_1-\omega)dt_1.$$

Then,

$$\left|u_1(t)-u_2(t)\right| \le \frac{|a_o|m}{|a_o|} + \frac{|b_o|}{|a_o|}\int_0^t \left|u_2(t_1)-u_1(t_1)\right| dt_1$$

$$+ \frac{|b_1|}{|a_o|}\int_0^{t-\omega} \left|u_2(t_1)-u_1(t_1)\right| dt_1$$

$$\le m + \frac{|b_o| + |b_1|}{|a_o|}\int_0^t \left|u_2(t_1)-u_1(t_1)\right| dt_1.$$

Let $c = \dfrac{|b_o| + |b_1|}{|a_o|}$.

Then $\left|u_1(t)-u_2(t)\right| \le m + c\int_0^t \left|u_2(t_1)-u_1(t_1)\right| dt_1$,

and by Lemma 4-1,

$$\left|u_1(t)-u_2(t)\right| \le me^{ct}.$$

Theorem: Let $u_1(t)$ and $u_2(t)$, of class C^2 on $[0, \infty)$ be solutions of $L(u) = f_1$ and $L(u) = f_2$, respectively, when f_1 and f_2 are of class C^1 on $[0, \infty)$.

Let $m = \max\limits_{0 \le t \le \omega} \left|u_1(t)-u_2(t)\right|$.

Then

$$\left|u_1(t)-u_2(t)\right| \le \left[m + \int_0^t \frac{\left|f_1(t_1)-f_2(t_1)\right| dt_1}{|a_o|}\right] e^{ct}, \quad t \ge 0.$$

Proof: Integrating $L(u) = f_1$ and $L(u) = f_2$ from ω to t and taking the difference gives

$$a_o u_1(t) - a_o u_2(t) = a_o u_1(\omega) - a_o u_2(\omega) + \int_\omega^t \left|f_1(t_1)-f_2(t_1)\right| dt_1$$

$$-b_o \int_\omega^t \left|u_1(t_1)-u_2(t_1)\right| dt_1 - b_1 \int_\omega^t \left|u_1(t-\omega) - u_2(t-\omega)\right| dt_1.$$

Then,

$$|u_1(t)-u_2(t)| \leq |u_1(\omega)-u_2(\omega)| + |a_o|^{-1}\int_0^t |f_1(t_1)-f_2(t_1)|\,dt_1$$

$$+ \frac{|b_o|}{|a_o|}\int_0^t |u_1(t_1)-u_2(t_1)|\,dt_1 + \frac{|b_1|}{|a_o|}\int_0^t |u_1(t_1)-u_2(t_1)|\,dt_1$$

$$\leq m + |a_o|^{-1}\int_0^t |f_1(t_1)-f_2(t_1)|\,dt_1$$

$$+ \frac{|b_o|+|b_1|}{|a_o|}\int_0^t |u_1(t_1)-u_2(t_1)|\,dt_1.$$

Let $c = \dfrac{|b_o|+|b_1|}{|a_o|}$.

Then by Lemma 4-1,

$$|u_1(t)-u_2(t)| \leq \left[m + |a_o|^{-1}\int_0^t |f_1(t_1)-f_2(t_1)|\,dt_1\right]e^{ct}, \quad t > 0.$$

The following theorem is not proved here, but it can be proved in the same manner as in the two immediately preceding theorems. It is stated here, however, and used in a subsequent theorem.

Theorem 4-2.

Let $u(t)$ be a solution of $L(u) = f(t)$, which is of class C^1 on $[0, \infty)$. Suppose that f is of class C^0 on $[0, \infty)$ and that $|f(t)| \leq c_1 e^{c_2 t}$, $t \geq 0$, where c_1 and c_2 are positive constants. Let $m = \max\limits_{0 \leq t \leq \omega} |u(t)|$.

Then there are positive constants c_3 and c_4 which depend only on the coefficients c_2, a_o, b_o, and b_1 such that

$$|u(t)| \leq c_3(c_1+m)e^{c_4 t}, \quad t \geq 0.$$

Theorem: Let $u_1(t)$ and $u_2(t)$, of class C^2 on $[0, \infty)$, be solutions of $L_1(u) = f_1$ and $L_2(u) = f_2$, respectively, where f_1 and f_2 are of class C^1 on $[0, \infty)$, and where

$$L_1(u) = a_o u'(t) + b_o u(t) + b_1 u(t-\omega),$$

$$L_2(u) = a_o u'(t) + (b_o+\epsilon_o)u(t) + (b_1+\epsilon_1)u(t-\omega).$$

Let $\left|f_i(t)\right| \leq c_1 e^{c_2 t}$, $t \geq 0$, $i = 1,2$, $c_1 > 0$, $c_2 > 0$.

Let $m = \max\limits_{0 \leq t \leq \omega} \left|u_1(t)-u_2(t)\right|$, $t \geq 0$.

Let $\epsilon = \max\left(\left|\epsilon_o\right|, \left|\epsilon_1\right|\right)$, $m_2 = \max\limits_{0 \leq t \leq \omega}\left|u_2(t)\right|$.

Then, there are positive constants c, c_4, and c_5 which depend only on c_1, c_2, and a_o, b_1, b_o, such that

$$\left|u_1(t)-u_2(t)\right| \leq \Big[m + \left|a_o\right|^{-1}\int_0^t \left|f_1(t_1)-f_2(t_1)\right| dt_1$$
$$+ \epsilon c_5(c_1+m_2)e^{c_4 t}\Big]e^{ct}, \quad t \geq 0.$$

Proof: Integrating $L_1(u) = f$ from ω to t gives

$$a_o u_1(t) = a_o u_1(\omega) + \int_\omega^t f_1(t_1)dt_1 - b_o \int_\omega^t u_1(t_1)dt_1$$
$$- b_1 \int_\omega^t u_1(t_1-\omega)dt_1.$$

$$a_o u_2(t) = a_o u_2(\omega) + \int_\omega^t f_2(t_1)dt_1 - (b_o+\epsilon_o)\int_\omega^t u_2(t_1)dt_1$$
$$-(b_1+\epsilon_1)\int_\omega^t u_2(t_1-\omega)dt_1.$$

Then,

$$\left|u_1(t)-u_2(t)\right| \le \left|u_1(\omega)-u_2(\omega)\right| + \left|a_o\right|^{-1}\int_0^t \left|f_1(t_1)-f_2(t_1)\right| dt_1$$

$$+ \frac{|b_o|}{|a_o|}\int_\omega^t \left|u_1(t_1)-u_2(t_1)\right| dt_1$$

$$+ \frac{|b_1|}{|a_o|}\int_\omega^{t-\omega} \left|u_1(t_1)-u_2(t_1)\right| dt_1$$

$$+ \frac{|\epsilon_o|}{|a_o|}\int_\omega^t \left|u_2(t_1)\right| dt_1$$

$$+ \frac{|\epsilon_1|}{|a_o|}\int_\omega^{t-\omega} \left|u_2(t_1)\right| dt_1$$

$$\le m + \left|a_o\right|^{-1}\int_0^t \left|f(t_1)-f_2(t_1)\right| dt_1$$

$$+ \frac{|b_o| + |b_1|}{|a_o|}\int_0^t \left|u_1(t_1) - u_2(t_1)\right| dt_1$$

$$+ \frac{|\epsilon_o| + |\epsilon_1|}{|a_o|}\int_0^t \left|u_2(t_1)\right| dt_1.$$

Let $c = \dfrac{|b_o| + |b_1|}{|a_o|}$. By Theorem 4-2, $\left|u_2(t)\right| \le c_3(c_1+m_2)e^{c_4 t}$.

Thus, $$\left|u_1(t)-u_2(t)\right| \le m + \left|a_o\right|^{-1}\int_0^t \left|f_1(t_1)-f_2(t_1)\right| dt_1$$

$$+c\int_0^t \left|u_1(t_1)-u_2(t_1)\right| dt_1 + \frac{2\epsilon}{|a_o|}\int_0^t c_3(c_1+m_2)e^{c_4 t}.$$

Let $c_5 = \dfrac{2c_3}{a_o}$.

Then $$\left|u_1(t)-u_2(t)\right| \le m + \left|a_o\right|^{-1}\int_0^t \left|f_1(t)-f_2(t)\right| dt_1$$

$$+ \epsilon c_5(c_1+m_2)e^{c_4 t} e^{ct}.$$

CHAPTER V

SOLUTIONS IN THE FORM OF A DEFINITE INTEGRAL

Another technique used to solve differential-difference equations is discussed in this chapter. This method involves obtaining a solution in the form of a definite integral by the use of a function $k(t)$ which has certain peculiar qualities and which satisfies the equation

$$a_0 k'(t) + b_0 k(t) + b_1 k(t-\omega) = 0.$$

The solution of the differential-difference equation $L(u) = a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t)$, $t > \omega$, $a_0 \neq 0$, which satisfies the initial condition $u(t) = g(t)$, $0 \leq t \leq \omega$, if g is $C^0 [0, \omega]$, f is $C^0 [0, \infty)$, and

$$\left| f(t) \right| \leq c_1 e^{c_2 t}, \ t \geq 0, \ c_1 > 0, \ c_2 > 0,$$

can be represented by a definite integral.

Let $k(t)$ be the unique function with the following properties:

(1) $k(t) = 0$, $t < 0$;

(2) $k(0) = (a_0)^{-1}$;

(3) $k(t)$ is of class C^0 on $[0, \infty)$;

(4) $k(t)$ satisfies the equation $a_0 k'(t) + b_0 k(t) + b_1 k(t-\omega) = 0$, $t > 0$.

Theorem: Suppose $u(t)$ is a continuous solution of

(5-1), $a_o u'(t) + b_o u(t) + b_1 u(t-\omega) = f(t)$, $t>\omega$, $a_o \neq 0$,

with the initial condition $u(t) = g(t)$, $0 \leq t \leq \omega$.

If g is C^0 on $[0, \infty)$ and f is C^0 on $[0, \infty)$, then for $t>\omega$,

$$u(t) = a_o g(\omega)\ k(t-\omega) - b_1 \int_0^{\omega} g(t_1)\ k(t-t_1-\omega)\ dt_1 + \int_{\omega}^{t} f(t_1)\ k(t-t_1) dt_1.$$

Proof: Let $t>\omega$. Integrating equation (5-1) from ω to t gives

$$\int_{\omega}^{t} f(t_1)k(t-t_1)dt_1 = a_o \int_{\omega}^{t} u'(t_1)k(t-t_1)dt_1 + b_o \int_{\omega}^{t} u(t_1)k(t-t_1)dt_1 + b_1 \int_{\omega}^{t} u(t_1-\omega)k(t-t_1)dt_1.$$

Integrating by parts,

$$\begin{aligned}\int_{\omega}^{t} f(t_1)k(t-t_1)dt_1 &= \Big[a_o u(t_1)k(t-t_1)\Big]_{\omega}^{t} \\ &+ a_o \int_{\omega}^{t} u(t_1)k'(t-t_1)dt_1 + b_o \int_{\omega}^{t} u(t_1)k(t-t_1)dt_1 \\ &+ b_1 \int_{\omega}^{t} u(t_1-\omega)k(t-t_1)dt_1 = a_o u(t)k(0) - a_o u(\omega)k(t-\omega) \\ &- b_o \int_{\omega}^{t} u(t_1)k(t-t_1)dt_1 - b_1 \int_{\omega}^{t-\omega} u(t_1)k(t-t_1-\omega)dt_1 \\ &+ b_o \int_{\omega}^{t} u(t_1)k(t-t_1)dt_1 + b_1 \int_{\omega}^{t} u(t_1-\omega)k(t-t_1)dt_1.\end{aligned}$$

Here the expression $-b_o \int_\omega^t u(t_1)k(t-t_1)dt_1$

$$-b_1 \int_\omega^t u(t_1)k(t-t_1-\omega)dt_1$$

is substituted for $a_o \int_\omega^t u(t_1)k'(t-t_1)dt_1$, except the limits on the integral with the coefficient b_1 are ω to $t-\omega$ because with limits from $t-\omega$ to ω the value of the integral is zero. Hence,

$$\int_\omega^t f(t_1)k(t-t_1)dt_1 = u(t) - a_o u(\omega)k(t-\omega)$$
$$- b_1 \int_\omega^{t-\omega} u(t_1)k(t-t_1-\omega)dt_1 + b_1 \int_\omega^t u(s)k(t-s-\omega)ds$$
$$= u(t) - a_o g(\omega)k(t-\omega) + b_1 \int_0^\omega g(t_1)k(t-t_1-\omega)dt_1.$$

Therefore,

$$(5\text{-}2), \quad u(t) = a_o g(\omega)k(t-\omega) - b_1 \int_0^\omega g(t_1)k(t-t_1-\omega)dt_1 + \int_\omega^t f(t_1)k(t-t_1)dt_1, \quad t>\omega .$$

Incidentally, the right-hand member of (5-2) is zero, not $g(t)$, when $0 < t < \omega$.

This is true because

(1) $(t-\omega) < 0$, and thus $k(t-\omega) = 0$,

(2) $(t-t_1-\omega) < 0$, $0 \le t_1 \le \omega$, and thus $k(t-t_1-\omega) = 0$,

(3) $(t-t_1) < 0$, $\omega \le t_1 \le t$, and thus $k(t-t_1) = 0$.

Note: Since $a_o k'(t) + b_o k(t) + b_1 k(t-\omega) = 0$, $t-\omega<0$ when $t<\omega$, and thus $k(t-\omega) = 0$, $t<\omega$. Consequently, $a_o k'(t) + b_o k(t) = 0$, $t<\omega$.

Let $t > 2\omega$. Then, by differentiating equation (5-2),

$$u'(t) = \frac{d}{dt}\Big[a_0 g(\omega)k(t-\omega) - b_1\int_0^{\omega} g(t_1)k(t-t_1-\omega)dt_1$$
$$+ \int_{\omega}^{t} f(t_1)k(t-t_1)dt_1\Big]$$
$$= a_0 g(\omega)k'(t-\omega) - b_1\int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 + f(t)k(0)$$
$$+\int_{\omega}^{t} f(t_1)k'(t-t_1)dt_1$$
$$= a_0 g(\omega)k'(t-\omega) - b_1\int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 + (a_0)^{-1}f(t)$$
$$+\int_{\omega}^{t} f(t_1)k'(t-t_1)dt_1, \quad t > 2\omega.$$

Hence,

$$a_0u'(t) + b_0u(t) + b_1u(t-\omega) = a_0\Big[a_0 g(\omega)k'(t-\omega)$$
$$-b_1\int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 + (a_0)^{-1}f(t)$$
$$+\int_{\omega}^{t} f(t_1)k'(t-t_1)dt_1\Big] + b_0\Big[a_0 g(\omega)k(t-\omega)$$
$$-b_1\int_0^{\omega} g(t_1)k(t-t_1-\omega)dt_1 + \int_{\omega}^{t} f(t_1)k(t-t_1)dt_1\Big]$$
$$+ b_1\Big[a_0 g(\omega)k(t-2\omega) - b_1\int_0^{\omega} g(t_1)k(t-t_1-2\omega)dt_1$$
$$+\int_{\omega}^{t} f(t_1)k(t-t_1-\omega)dt_1\Big]$$

$$= a_0^2 g(\omega)k'(t-\omega) - a_0 b_1 \int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 + f(t)$$
$$+ a_0 \int_{\omega}^{t} f(t_1)k(t-t_1)dt_1 + b_0 a_0 g(\omega)k(t-\omega)$$
$$- b_0 b_1 \int_0^{\omega} g(t_1)k(t-t_1-\omega)dt_1 + b_0 \int_{\omega}^{t} f(t_1)k(t-t_1)dt_1$$
$$+ b_1 a_0 g(\omega)k(t-2\omega) - b_1^2 \int_0^{\omega} g(t_1)k(t-t_1-2\omega)dt_1$$
$$+ b_1 \int_{\omega}^{t} f(t_1)k(t-t_1-\omega)dt_1$$
$$= f(t) + a_0 g(\omega)\Big[a_0 k'(t-\omega) + b_0 k(t-\omega) + b_1 k(t-2\omega)\Big]$$
$$- b_1 \int_0^{\omega} g(t_1)\Big[a_0 k'(t-t_1-\omega) + b_0 k(t-t_1-\omega)$$
$$+ b_1 k(t-t_1-2\omega)\Big]dt_1 + \int_{\omega}^{t} f(t_1)\Big[a_0 k'(t-t_1) + b_0 k(t-t_1)$$
$$+ b_1 k(t-t_1-\omega)\Big]dt_1$$
$$= f(t), \ t > 2\omega.$$

Therefore $a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t)$ is satisfied for $t > 2\omega$.

The following discussion proves that $a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t)$ is also satisfied for $\omega < t < 2\omega$.

Theorem: Suppose $u(t)$ is a solution of (5-1) for $\omega < t < 2\omega$. Then
$$u(t) = a_0 g(\omega)k(t-\omega) - b_1 \int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1 + \int_{\omega}^{t} f(t_1)k(t-t_1)dt_1, \ \omega < t < 2\omega.$$

Proof: Assume $\omega < t < 2\omega$.

Integrating (5-1) from ω to t gives

$$\int_\omega^t f(t_1)k(t-t_1)dt_1 = a_o\int_\omega^t u'(t_1)k(t-t_1)dt_1$$
$$+ b_o\int_\omega^t u(t_1)k(t-t_1)dt_1 + b_1\int_\omega^t u(t-\omega)k(t-t_1)dt_1.$$

Integrating by parts,

$$\int_\omega^t f(t_1)k(t-t_1)dt_1 = \Big[a_o u(t_1)k(t-t_1)\Big]_\omega^t + a_o\int_\omega^t u(t_1)k'(t-t_1)dt_1$$
$$+ b_o\int_\omega^t u(t_1)k(t-t_1)dt_1 + b_1\int_\omega^t u(t_1-\omega)k(t-t_1)dt_1$$
$$= a_o u(t)k(0) - a_o u(\omega)k(t-\omega)$$
$$- b_o\int_\omega^t u(t_1)k(t-t_1)dt_1 - b_1\int_\omega^t u(t_1)k(t-t_1-\omega)dt_1$$
$$+ b_o\int_\omega^t u(t_1)k(t-t_1)dt_1 + b_1\int_\omega^t u(t_1-\omega)k(t-t_1)dt_1.$$

The term $-b_1\int_\omega^t u(t_1)k(t-t_1-\omega)dt_1 = 0$, because $(t-t_1-\omega) < 0$.

Thus $\int_\omega^t f(t_1)k(t-t_1)dt_1 = u(t) - a_o u(\omega)k(t-\omega)$

$$+ b_1\int_0^{t-\omega} u(s)k(t-s-\omega)ds = u(t) - a_o g(\omega)k(t-\omega)$$
$$+ b_1\int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1.$$

Therefore,

$$(5\text{-}3),\quad u(t) = a_o g(\omega)k(t-\omega) - b_1\int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1$$
$$+ \int_\omega^t f(t_1)k(t-t_1)dt_1, \quad \omega < t < 2\omega.$$

Let $\omega < t < 2\omega$. Then, by differentiating equation (5-3),

$$\begin{aligned}
u'(t) &= a_0 g(\omega)k'(t-\omega) - b_1 g(t-\omega)k(t-t+\omega-\omega) \\
&\quad - b_1 \int_0^{t-\omega} g(t_1)k'(t-t_1-\omega)dt_1 + f(t)k(t-t) \\
&\quad + \int_\omega^t f(t_1)k'(t-t_1)dt_1 = a_0 g(\omega)k'(t-\omega) \; a_0^{-1}b_1 g(t-\omega) \\
&\quad - b_1 \int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 + a_0^{-1}f(t) \\
&\quad + \int_\omega^t f(t_1)k'(t-t_1)dt_1, \quad \omega < t < 2\omega.
\end{aligned}$$

The term $-b_1\int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1$ replaces the term

$$- b_1 \int_0^{t-\omega} g(t_1)k'(t-t_1-\omega)dt_1.$$

This occurs because the value of the integral from $t-\omega$ to t is zero, caused by $t-t_1-\omega < 0$, which results in $k'(t-t_1-\omega) = 0$.

Hence, for $\omega < t < 2\omega$,

$$\begin{aligned}
&a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = \\
&\quad a_0\Big[a_0 g(\omega)k'(t-\omega) - a_0^{-1}b_1 g(t-\omega) - b_1\int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 \\
&\qquad + a_0^{-1}f(t) + \int_\omega^t f(t_1)k'(t-t_1)dt_1\Big] \\
&\quad + b_0\Big[a_0 g(\omega)k(t-\omega) - b_1\int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1 \\
&\qquad + \int_\omega^t f(t_1)k(t-t_1)dt_1 + b_1 g(t-\omega)\Big]
\end{aligned}$$

$$= f(t) + a_o g(\omega)\left[a_o k'(t-\omega) + b_o k(t-\omega)\right] - b_1 g(t-\omega)$$

$$+ b_1 g(t-\omega) - a_o b_1 \int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1$$

$$+ \int_{\omega}^{t} f(t_1)\ a_o k'(t-t_1) + b_o k(t-t_1)\ dt_1$$

$$- b_o b_1 \int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1$$

$$= f(t) + a_o g(\omega)\left[a_o k'(t-\omega) + b_o k(t-\omega)\right]$$

$$- a_o b_1 \int_0^{\omega} g(t_1)k'(t-t_1-\omega)dt_1 - b_o b_1 \int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1$$

$$+ \int_{\omega}^{t} f(t_1)\left[a_o k'(t-t_1) + b_o k(t-t_1)\right]dt_1.$$

But $a_o g(\omega)\left[a_o k'(t-\omega) + b_o k(t-\omega)\right] = 0$, because $0 < (t-\omega) < \omega$,

resulting in $b_1 k(t-2\omega) = 0$.

Also, $\int_{\omega}^{t} f(t_1)\left[a_o k'(t-t_1) + b_o k(t-t_1)\right]dt_1 = 0$, because

$(t-t_1) < \omega$, resulting in $b_1 k(t-t_1-\omega) = 0$.

The two terms $-a_o b_1 \int_0^{\omega} g(t_1)k'(t-t-\omega)dt_1$ and

$-b_o b_1 \int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1$ can be combined because

$$\int_0^{t-\omega} g(t_1)k(t-t_1-\omega)dt_1 = \int_0^{\omega} g(t_1)k(t-t_1-\omega)dt_1$$

$$- \int_{t-\omega}^{\omega} g(t_1)k(t-t_1-\omega)dt_1.$$

But $\int_{t-\omega}^{\omega} g(t_1)k(t-t_1-\omega)dt_1 = 0$, because $(t-t_1-\omega) < 0$.

Thus,

$$a_0u'(t) + b_0u(t) + b_1u(t-\omega) = f(t)$$

$$-b_1 \int_0^{\omega} g(t_1) \Big[a_0k'(t-t_1-\omega)$$

$$+ b_0k(t-t_1-\omega)\ dt_1\Big]$$

$$= f(t),$$

since the integral involved is zero because $(t-t_1-\omega) < \omega$, resulting in $b_1k(t-t_1-2\omega) = 0$.

Therefore $a_0u'(t) + b_0u(t) + b_1u(t-\omega) = f(t)$ is satisfied for $\omega < t < 2\omega$.

CHAPTER VI

CHARACTERISTIC ROOTS AND SERIES EXPANSION OF $u'(t) + u(t-\omega) = 0$

In Chapter III it was proved that the equation $L(u) = a_0u'(t) + b_0u(t) + b_1u(t-\omega) = 0$ is satisfied by $\sum p_k(t)e^{s_k(t)}$, where $\{s_k\}$ is any sequence of characteristic roots of L, $p_k(t)$ is a polynomial of degree less than the multiplicity of s_k, and the sum is either finite or is infinite with suitable conditions to ensure convergence. That is, $L(u) = 0$ is satisfied by a sum of simple exponential solutions of the form $e^{s_k t}$ with coefficients $p_k(t)$. Reiterating, since $L(e^{st}) = (a_0s+b_0+b_1e^{-\omega s})e^{st}$, $h(s) = a_0s+b_0+b_1e^{-\omega s}$ is called the characteristic function of L; $h(s) = 0$ is called the characteristic equation of L; and a solution of $h(s) = 0$ is called a characteristic root of L. Thus a characteristic root s results in a solution e^{st}.

The problem discussed here is how to determine these characteristic roots. To facilitate the discussion, an analysis will be made of a particular group of differential-difference equations which have the form $u'(t) + u(t-\omega) = 0$ or $u'(t) = -u(t-\omega)$, where ω is a real number >0.

For $u(t) = e^{st}$, the equation is $se^{st} + e^{s(t-\omega)} = 0$. Therefore the characteristic function is $h(s) = s + e^{-\omega s}$, and the characteristic equation $s + e^{-\omega s} = 0$ must be solved for s. To simplify the problem somewhat, a substitution is made. Let $z = -s$. Thus the equation $e^{\omega z} = z$ must be solved for z.

Separating z into real and imaginary parts and changing to polar coordinates results in: $-s = z = x+iy$ $= r(\cos\theta + i \sin\theta)$, $x = r\cos\theta$, $y = r\sin\theta$; $\exp \omega z$ $= e^{\omega x}(\cos \omega y + i \sin \omega y)$. Then the form of $e^{\omega z} = z$ is changed to $e^{\omega x}(\cos \omega y + i \sin \omega y) = r(\cos\theta + i \sin\theta)$. Thus z is a zero of $e^{\omega z} = z$ if $e^{\omega x} = r$ and $\omega y = \theta$. But $r^2 = x^2 + y^2 = e^{2\omega x}$. Therefore z is a solution of $e^{\omega z} = z$ if $y = \pm\sqrt{e^{2\omega x}-x^2}$ and $y = \frac{\theta}{\omega}$.

First consider the case of $\omega = 1$. To obtain exponential solutions of $u'(t) + u(t-1) = 0$, it is necessary to solve the characteristic equation $e^z = z$. And z is a solution of $e^z = z$ if $y = \pm\sqrt{e^{2x}-x^2}$ and $y = \theta$.

For $\pi < \theta < 2\pi$, z must be in the third or fourth Cartesian quadrant (because of the relationship between Cartesian and polar coordinates). But $y = \theta$. Therefore, for z to be in the third or fourth quadrant, y must be negative. However y cannot be negative because it is equal to θ, which is a positive number between π and 2π. It is thus evident that there is no solution z for y between π and 2π. The same argument can also be used to show that there is no solution z for $3\pi < y < 4\pi$, $5\pi < y < 6\pi$, $\cdots$, $(2n-1)\pi < y < 2n\pi$, $\cdots$. Also, if $0 < y < \pi$, then $0 < \sqrt{e^{2x}-x^2} < \pi$. However, if $x \le 0$, $\sqrt{e^{2x}-x^2} \le 1$ and thus $y = \theta \le 1 < \pi/2$. Therefore z must be in the first quadrant, a contradiction of $x \le 0$. It is evident that there is no solution z for y between $\pi/2$ and π. The same argument can be used to show that there is no solution z for $5\,\pi/2 < y < 3\pi$, $9\,\pi/2 < y < 5\pi$, $\cdots$, $(4n-3)\,\pi/2 < y < (2n-1)\pi$, $\cdots$.

For $y > 0$, since $y = \theta$ and $y = +\sqrt{e^{2x}-x^2}$ are single valued functions, there exists one solution z for each interval $y_n = (2n-2)\pi$ to $(4n-3)\,\pi/2$, $n = 1,2, \cdots$. Likewise for $y < 0$, there is a solution $z = x + -iy$ corresponding to each solution for $y > 0$. That is, each solution z has a complex conjugate which is also a solution. It is necessary here to recall that the desired solutions are the characteristic roots $s = -z$. Let the solution z be called $\overline{z_1} = x_1 + iy_1$, $0 < y_1 < \pi/2$ and $x_1 \ge 0$, $\overline{z_2} = x_2+iy_2$, $2\pi < y_2 < 5\,\pi/2$ and $x_2 > 0$, etc., and the complex conjugate solutions be called $z_1 = x_1 - iy_1$, $0 < y_1 < \pi/2$ and $x_1 > 0$, etc. Then the desired solutions s are $s_1 = -x_1 + iy_1$, $0<y_1< \pi/2$ and $x_1 \ge 0$, $s_2 = -x_2 + iy_2$, $2\pi<y_2<5\,\pi/2$ and $x_2 > 0$, etc., and the complex conjugate solutions are $\overline{s_1} = -x_1-iy_1$, $0 < y_1 < \pi/2$ and $x_1 > 0$, etc. The graph of these solutions for $y > 0$ shows $s_1 = (x_1,y_1)$, where $x_1 < 0$ and $0 < y_1 < \pi/2$, and $s_2 = (x_1,x_2)$, where $x_2 < x_1$ and $2\pi < y_2 < 5\,\pi/2$.

The coordinates of the solutions are such that $x_n > x_{n+1}$ and $y_n < y_{n+1}$, for $y > 0$. For large values of x, the equation $y = \sqrt{e^{2x} - x^2}$ is very much like the graph of the familiar equation $y = e^x$.

There is a corresponding set of complex conjugate solutions in the third quadrant. It is evident that for $\omega = 1$, any solusion $s_n = x_n + iy_n$, $x_n < 0$ and the value of y_n is known to be within a certain range, and $|x_n| < |y_n|$, $|x_{n+1}| < |y_{n+1}|$.

For values of $\omega \neq 1$, the solutions form a pattern which for large values of x is much like the graph of $e^{\omega x}$. Thus, for values of $\omega > 1$, the curve rises more steeply than the curve for $\omega = 1$. That is, the ratio of y to x is larger than for $\omega = 1$. Conversely, for values of $0 < \omega < 1$, the ratio of y to x is smaller than for $\omega = 1$. However, for all $\omega > 0$, as $|s| \to \infty$ along the curve, the curve becomes nearly parallel to the imaginary axis.

In the case of $\omega = 1$, all solutions are such that $x_n < 0$, that is, the real part of each complex solution is negative. If $\omega = \pi/2$, the first solution, s_1 (and its complex conjugate s_1), has positive real part $x_1 = 0$, and all other solutions have negative real parts. For each $\omega > \pi/2$, there exists some integer $i = 1, 2, \cdots$, such that the solutions in the first i intervals of $2(i-1)\pi/\omega < y_i < (4i-3)\pi/2\omega$ have positive real parts. All subsequent solutions have negative real parts. As ω is increased, the integer i increases, but in all cases i will be a finite number.

The value of ω also effects the range of y. For all ω, $2(n-1)\pi/\omega < y_n < (4n-3)\pi/2\omega$.

For many purposes, it is expedient to have an expansion of u(t) in the form of an infinite series, $\sum p_R(t)e^{s_R t}$, where the sum is over all characteristic roots s_R, and where $p_R(t)$ is a polynomial in t if s_R is a multiple root. It is difficult to prove such an expansion theorem rigorously. One reason is that for different values of ω, the distribution of the solutions s_R is different. Therefore, the theorem is proved here for $k'(t) + k(t-\omega) = 0$, where k is as defined in Chapter V.

Theorem: Suppose $k'(t) = -k(t-\omega)$. Let $a_j = \frac{3\pi}{2\omega} + \frac{2\pi j}{\omega}$, $j = 0, 1, 2, \cdots$. Let s_j denote the rectangle formed by segments of $y = -a_j$, $x = 0$, $y = a_j$, $x = -a_j$. Let $k_j(t)$ be the sum of the residues of $\frac{e^{ts}}{s+e^{-\omega s}}$ in the band $|y| < a_j$. Then

$k(t) = \lim_{j\to\infty} k_j(t)$.

Proof: The first rectangle s_0 contains two poles of $\frac{e^{ts}}{s+e^{-\omega s}}$. Each additional rectangle encompasses two subsequent poles (a characteristic root and its complex conjugate). Then

(6-1), $k_j(t) = \int_{s_j} \frac{e^{ts}}{s+e^{-\omega s}}\, ds$, by the Residue Theorem. The integral (6-1) around s_j is the sum of four integrals, one for each side of the rectangle s_j. Thus, let $k_j(t) = I_1 + I_2 + I_3 + I_4$. I_1 corresponds to the integral over the vertical line segment which forms the right side of the rectangle. I_2 corresponds to the integral over the horizontal line segment which forms the top side of the rectangle. Likewise I_3 and I_4 correspond, respectively, to the integrals over the lines which are the left and bottom sides of the rectangle. Since the Laplace Transform for $k(t)$ is $\int_0^\infty e^{-ts}k(t)\, dt = \frac{1}{s+e^{-\omega s}}$, therefore $k(t) = \lim_{j\to\infty} I_1$, by the Laplace Inversion Theorem. It must therefore be proved that $\lim_{j\to\infty} (I_2 + I_3 + I_4) = 0$. Since $I_4 = \overline{I_2}$, it suffices to show that $\lim_{j\to\infty} I_2 = 0$ and $\lim_{j\to\infty} I_3 = 0$.

Thus, $I_3 = i\int_{a_j}^{-a_j} \frac{e^{t(-a_j+iy)}}{-a_j+iy+e^{-\omega(-a_j+iy)}}\, dy$.

There exists a J such that if $j > J$, then $e^{\omega a_j} \geq 4a_j$. For such j, $|e^{-\omega(-a_j+iy)}| \geq 2|-a_j+iy|$, so that

$|-a_j + iy + e^{-\omega(-a_j+iy)}| \geq 1/2e^{\omega a_j}$. This implies that

$|I_3| \leq \int_{-a_j}^{a_j} \frac{e^{-ta_j}}{1/2e^{\omega a_j}}\, dy = 4a_je^{-(t+\omega)a_j}$, which implies

$\lim_{j\to\infty} I_3 = 0$, since t is positive.

Also, $I_2 = \int_0^{-a_j} \frac{e^{t(x+ia_j)}}{x+ia_j+e^{-\omega(x+ia_j)}}\,dx$. Since $e^{-\omega i a_j} = 1$,

$$|I_2| \le \int_{-a_j}^{0} \frac{e^{tx}}{|x+i(a_j+e^{-\omega x}|}\,dx \le \frac{1}{a_j} \int_{-a_j}^{0} e^{tx} = \frac{1-e^{-ta_j}}{a_j t} < \frac{1}{ta_j}.$$

Therefore, $\lim_{j\to\infty} I_2 = 0$.

Each of the residues is of the form $C_n e^{s_n t}$, $(\omega \neq 1/e)$, and therefore $k(t) = \sum_{R=1}^{\infty} \left[C_R e^{s_R t} + \bar{C}_R e^{\bar{s}_R t}\right]$. In Chapter V, it was proved that $u(t) = a_0 g(\omega)k(t-\omega)$

$$-b_1 \int_0^{\omega} g(t_1)k(t-t_1-\omega)dt_1 + \int_{\omega}^{t} f(t_1)k(t-t_1)dt_1.$$

Therefore, by substituting the exponential sum for $k(t)$ in the integral representation of $u(t)$ and integrating term by term, the result is a series solution for $u(t)$.

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