APPLICATION OF THE WIGNER FORMALISM TO A
SLIGHTLY RELATIVISTIC QUANTUM PLASMA

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CHAPTER I

INTRODUCTION

The general description of a system of many interacting particles and determination of its properties is a very difficult subject. As expected, only a few idealized systems have been solved exactly. There are many approaches to the problem involving approximations or insoluble symbolic equations with various initial assumptions and a maze of pathways leading from them. Perhaps the most useful goal is the development of a relativistic quantum statistical mechanics. A brief discussion of these terms, taken in reverse order, follows.

Statistical mechanics attempts to describe a system of particles without relying on its detailed microscopic behavior, but rather predicting probable behavior based upon statistical information or some averaging process. Although the detailed coupled equations can be written down, it is hopeless to grind out solutions involving the initial conditions of $6N+1$ variables for an $N$-particle system where $N$ may be on the order of $10^{23}$, give or take a few orders of magnitude. In spite of simplifying assumptions, statistical mechanics can produce quite accurate results. Therefore some form of statistical mechanics appears to be a logical
foundation for a many-particle system. In this study the properties of a system are described by a distribution function \( f(\{q_i, \{p_i, t\} \) depending on the spatial coordinates \( \{q_i\} = (\bar{q}, \bar{q}_1, ..., \bar{q}_N) \), the momenta \( \{p_i\} = (\bar{p}, \bar{p}_1, ..., \bar{p}_N) \), and time. The physical meaning of this function is that the quantity

\[
J_p, c)p = \int (\bar{q}, \bar{q}_1, ..., \bar{q}_N) \int (\bar{p}, \bar{p}_1, ..., \bar{p}_N) \]

represents the joint probability that at time \( t \) particle one is within a range \( \delta \bar{q} \) of \( \bar{q} \), and \( \delta \bar{p} \) of \( \bar{p} \), particle two is within a range \( \delta \bar{q}_1 \) of \( \bar{q}_1 \) and \( \delta \bar{p}_1 \) of \( \bar{p}_1 \), and so forth for each particle. The integral of this expression is normalized to unity, indicating that each particle is located in a coordinate and momentum range in phase space. The function is useful in the sense that the average of a macroscopic quantity is obtained by integrating the product of \( f \) with a corresponding weight factor in an appropriate manner (1, pp. 5-7). This idea will be developed more fully later.

Since the particles are microscopic entities, quantum effects may become important. These arise mainly from the indistinguishability of identical particles and Heisenberg's uncertainty principle. The former concept leads to the conclusion that a system of identical particles can be described by one of two statistical descriptions, depending on the spin associated with them (2, pp. 91-137). This study will be limited to Fermi-Dirac statistics, which includes electron systems.
The density matrix (3, pp. 90-109) provides a statistical framework for quantum-mechanical systems. This is not, however, the quantum analog of classical distribution functions. In order to develop such an analog so that classical limits may be easily taken, a transformation is applied to the density matrix. The resulting quantity is called the Wigner distribution function. There are many forms of the Wigner distribution function and methods of its application. Besides the usual quantum formulation (4, p. 221; 5, p. 361), many researchers go over to the formalism of second quantization (1, 6), where the number of particles can vary. The latter approach introduces fields and their infinite number of degrees of freedom. This study will pursue the former course, dealing throughout only with particle coordinates and without using special evaluation methods such as diagram techniques.

If the particles under consideration have velocities on the order of magnitude of the speed of light, then relativistic effects become important. These conditions are found at high temperatures in the study of the interior of white dwarf stars (7, p. 20) or at high densities on the order of $6 \times 10^{29}$ electrons per cubic centimeter (8, p. 428). Relativistic particles may also be emitted in beta decay. Since the purpose of this study is to investigate the properties of a randomized gas, such relativistic systems as particle accelerators where there is a high degree of
particle correlation will not be considered. It should be kept in mind that relativistic effects for a classical system can be obtained in the limit of $\hbar \to 0$.

There are various problems involved in relativistic theories. The concept of action at a distance is damaged due to retarded potentials arising from the finite times of propagation. A many-particle theory will not necessarily have to be cast into covariant form since there is no Lorentz frame of observation in which relativistic effects vanish. In addition, only low-order approximations will be carried out here. There are many starting points for developing a relativistic statistical theory (7, p. 2-8), most beginning with some assumptions about the system's Hamiltonian. It is not obvious, however, that a well-defined Hamiltonian exists in the relativistic case. The method used in this study involves the two-body Darwin Hamiltonian (9) correct to order $(\frac{v}{c})^2$. Leaf (10) shows that while this provides an adequate description for the case of a particle pair, a Hamiltonian does not exist when higher-order terms are included: momentum is no longer conserved. In spite of the fact that many-body Darwin Hamiltonians have been derived (11), no published calculations have been made using them. Some work has been done with the two-body form (12; 13, p. 84-90), but not in the form of a kinetic equation.
The following development includes some of the devices mentioned above. A slightly relativistic fermion gas is described by the dynamical theory obtained from the Wigner distribution function. The problem is approached in a self-consistent manner including the two-body Darwin Hamiltonian. The goal is to find the departures from equilibrium and dispersion relations for wave propagation in the gas.
CHAPTER BIBLIOGRAPHY


CHAPTER II
THE WIGNER FORMALISM

The Density Matrix

Consider an isolated system on $N$ identical, interacting, structureless particles described by the coordinates $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_N$ and momenta $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N$. In general the system can be described quantum-mechanically as a mixture of pure states $\Psi^{(i)}$, each of which has a statistical weight $\omega^{(i)}$, by

$$\Psi = \sum_{i=1}^{N} \omega^{(i)} \Psi^{(i)}$$

where $\omega^{(i)} \geq 0$ and $\sum_{i=1}^{N} \omega^{(i)} = 1$

Each of the pure states can be expanded in terms of a complete set of orthonormal eigenvectors $\phi_m$ of an operator.

$$\Psi^{(i)} = \sum_{m} C_m^{(i)} \phi_m$$

Then the expectation value of some operator $\Omega$ in the pure state $\Psi^{(i)}$ is

$$\langle \Omega \rangle \approx \langle \Psi^{(i)} | \Omega | \Psi^{(i)} \rangle = \sum_{m,n} C_m^{(i)*} C_n^{(i)} \langle \phi_m | \Omega | \phi_n \rangle = \sum_{m,n} C_m^{(i)*} C_n^{(i)} \Omega_{mn}$$

The system average of $\Omega$ is formed by the linear combination

$$\langle \Omega \rangle = \sum_{i=1}^{N} \omega^{(i)} \langle \Omega \rangle_i = \sum_{m,n} \Omega_{mn} \sum_{i=1}^{N} \omega^{(i)} C_m^{(i)*} C_n^{(i)}$$

Now define a quantity called the density matrix of the system for which the matrix elements of this quantity $\rho$ are
\[
\rho_{nm} = \sum_i W(i) |C_m(i)\rangle \langle C_n(i)|
\]

Then \[ \langle \Omega \rangle = \sum_{m,n} \Omega_{mn} \rho_{nm} = \sum_n (\Omega \rho)_{nm} = \text{Tr} (\Omega \rho) . \]

Knowledge of the density matrix therefore provides a method of calculating the expectation value of a given operator. The transition to the continuous case is not difficult. If the system is in a pure state the density matrix becomes, in the \( \phi \)-representation,

\[
\rho_{nm} = C_m^* C_n
\]

\[
\rho_{nn} = |C_n|^2 \quad \text{so} \quad 0 \leq \rho_{nn} \leq 1 .
\]

In the Schrödinger picture operators are time-independent and state vectors follow the Schrödinger equation.

\[
\mathcal{H}\psi = i\hbar \frac{\partial \psi}{\partial t}
\]

where \( \mathcal{H} \) is the system Hamiltonian. Then

\[
(\varphi_m, \mathcal{H}\psi) = \sum_n (\varphi_m, \mathcal{H} C_n \varphi_n) = \sum_n C_n (\varphi_m, \mathcal{H} \varphi_n)
\]

\[
= \sum_n \mathcal{H}_{mn} C_n
\]

\[
= i\hbar (\varphi_m, \frac{\partial \psi}{\partial t}) = \sum_n i\hbar \frac{\partial C_n}{\partial t} (\varphi_m, \varphi_n) = i\hbar \frac{\partial C_n}{\partial t} .
\]

Similarly,

\[
\sum_n \mathcal{H}_{mn}^* C_n^* \quad -i\hbar \frac{\partial C_n^*}{\partial t} .
\]

Therefore

\[
\frac{\partial}{\partial t} \rho_{nm} = \frac{\partial C_n^*}{\partial t} C_n + a_m^* \frac{\partial C_n}{\partial t}
\]

\[
= i\hbar \sum_k \{ \mathcal{H}_{nk} C_k C_n + C_m^* (\mathcal{H}_{nk} C_k) \} .
\]
In terms of commutator brackets this can be written

\[ \frac{\partial \rho}{\partial t} = \frac{i}{\hbar} \{ \rho, H \}. \]  

(II-1)

The Wigner Distribution Function

The elements of the density matrix in coordinate representation and Dirac notation, \( \langle q | \rho | q' \rangle \), are defined over a 6N-dimensional configuration space. In order to develop a theory for quantum-mechanical systems analogous to the classical method of distribution functions, it is advantageous to define a quantum distribution function \( F(\vec{q}, \vec{p} : t) \) closely related to the density matrix, but defined over a 6N-dimensional phase space. A function which provides such a description will not be unique, in the sense that some other function obtained by another method may serve just as well. The method below is a straightforward development of such a function assuming only the definition of a phase-space distribution function and the correspondence between classical variables and their corresponding quantum operators. The result is identical to the function
proposed in 1932 by Wigner (1), so it will be referred to as the Wigner distribution function.

Let $G(\hat{q}, \hat{p})$ be a classical function defined over phase space with the expectation value $\langle G \rangle$. According to Weyl (2), if this function is the Fourier transform of a function $N(\{q\}, \{p\})$, then the corresponding quantum operator $\hat{G}(\{\hat{q}\}, \{\hat{p}\})$ is defined by the following relations.

$$G(\hat{q}, \hat{p}) = \int N(\{\hat{q}\}, \{\hat{p}\}) e^{\frac{i}{\hbar} \sum_{k=1}^{n} (\hat{q}_k \hat{p}_k \mp \hat{p}_k \hat{q}_k)} d\hat{q}_1 d\hat{p}_1 \cdots d\hat{q}_n d\hat{p}_n.$$ 

The quantities $\hat{q}$ and $\hat{p}$ are the operators corresponding to the classical variables $\hat{q}$ and $\hat{p}$. The first equation is solved for

$$N(\{\hat{q}\}, \{\hat{p}\}) = \frac{1}{(2\pi \hbar)^n} \int G(\hat{q}, \hat{p}) e^{-\frac{i}{\hbar} \sum_{k=1}^{n} (\hat{q}_k \hat{p}_k \mp \hat{p}_k \hat{q}_k)} d\hat{q}_1 d\hat{p}_1 \cdots d\hat{q}_n d\hat{p}_n,$$

and then substituted into the second equation, giving

$$\hat{G}(\{\hat{q}\}, \{\hat{p}\}) = \frac{1}{(2\pi \hbar)^n} \int G(\hat{q}, \hat{p}) e^{\frac{i}{\hbar} \sum_{k=1}^{n} (\hat{q}_k \hat{p}_k \mp \hat{p}_k \hat{q}_k)} e^{\frac{i}{\hbar} \sum_{k=1}^{n} (\hat{q}_k \hat{p}_k \mp \hat{p}_k \hat{q}_k)} d\hat{q}_1 d\hat{p}_1 \cdots d\hat{q}_n d\hat{p}_n.$$ 

The matrix elements of $\hat{G}$ now depend on the positive exponential. This can be simplified somewhat by the use of McCoy's theorem: if the commutator of two operators $\Omega_1$ and $\Omega_2$ is a complex number $c$, then

$$e^{\Omega_1 + \Omega_2} = e^{\Omega_1} e^{\Omega_2} e^{-\frac{c}{2}}.$$ 

Let $\Omega_1 = \hat{p}$ and $\Omega_2 = \hat{q}$. Since $[\hat{p}, \hat{q}] = \hbar \Omega_{ik}$,
The positive exponential then becomes

\[ e^{\hat{A} \hat{p}_3 + \hat{B} \hat{q}_3} = e^{\hat{A} \hat{p}_3} e^{\hat{B} \hat{q}_3} e^{\hat{A} \hat{B}}. \]

The matrix elements are then

\[ \langle q'_3 | e^{\hat{A} \sum_j (\hat{p}_j + \hat{q}_j)} | q''_3 \rangle = e^{-\xi_{13} \hat{q}_3} e^{\xi_{13} \hat{p}_3} \prod_j \langle q'_3 | e^{\xi_{13} \hat{p}_3} | q''_3 \rangle \]

since

\[ e^{\xi_{13} \hat{q}_3} | q''_3 \rangle \cdot e^{\xi_{13} \hat{p}_3} | q''_3 \rangle. \]

Now expand the ket \( | q''_3 \rangle \) in momentum eigenfunctions.

\[ | q''_3 \rangle \cdot \int dp'_3 \langle p'_3 | q''_3 \rangle | p'_3 \rangle \]

where \( \langle p'_3 | q''_3 \rangle \cdot (\frac{1}{2\pi \hbar})^3 e^{i\hat{p}_3 \cdot \hat{q}_3''} \cdot \langle q''_3 | p'_3 \rangle \) is the inner product of \( p'_3 \) with \( q''_3 \).

The remaining matrix element now becomes

\[ \prod_j \langle q'_3 | e^{\xi_{13} \hat{p}_3} | q''_3 \rangle = \prod_j \left( \int dp'_3 \langle p'_3 | q''_3 \rangle \langle q'_3 | e^{\xi_{13} \hat{p}_3} | p'_3 \rangle \right) \]

\[ = \prod_j \left( \frac{1}{(2\pi \hbar)^3} \int dp'_3 e^{\xi_{13} (p'_3 - q''_3 + z'_3)} \right) \]

\[ = \prod_j \sigma (q''_3 - q''_3 + z'_3). \]

Now the matrix elements of \( \hat{G} \) can be written

\[ \langle q''_3 | \hat{G} | q''_3 \rangle = \frac{1}{(2\pi \hbar)^3} \int G(i \hat{p}_3, i \hat{q}_3) e^{\xi_{13} (\hat{p}_3 + \hat{q}_3)} e^{\xi_{13} \hat{p}_3 \cdot (\hat{q}_3'' - \hat{q}_3'')} \prod_i \sigma (q''_3 - q''_3 + z'_3) \prod d\hat{q}_3.d\hat{p}_3.dz_3.d\hat{\gamma}_3: \]

\[ \cdot \frac{1}{(2\pi \hbar)^3} \int G(i \hat{p}_3, i \hat{q}_3) e^{\xi_{13} \hat{p}_3 \cdot (\hat{q}_3'' - \hat{q}_3')} e^{\xi_{13} \hat{p}_3 \cdot (\hat{q}_3'' + \hat{z}_3')} \prod_i \sigma (q''_3 - q''_3 + z'_3) \prod d\hat{q}_3.d\hat{p}_3.dz_3.d\hat{\gamma}_3: \]
\[ \frac{1}{(2\pi \hbar)^3N} \int G(\{\vec{q}_i, \vec{\pi}_i\}) e^{-\frac{i}{\hbar} \sum_j \vec{q}_j \cdot \vec{p}_j} \prod_i d\vec{q}_i d\vec{\pi}_i \]

\[ = \frac{1}{(2\pi \hbar)^3N} \int G\left(\{\frac{\vec{q}'' + \vec{q}'}{2}, \{\vec{\pi}\}\right) e^{-\frac{i}{\hbar} \sum_j \vec{q}_j \cdot \vec{p}_j} \prod_i d\vec{q}_i d\vec{\pi}_i \]

This expression is inserted into the coordinate representation of the density matrix for the expectation value of \( G \).

\[ \langle G \rangle = \text{Tr}(\hat{G} \hat{\rho}) \]

\[ = \int \langle q' | \hat{G} \hat{\rho} | q' \rangle dq' \]

\[ = \int \langle q'' | \hat{G} | q'' \rangle \langle q' | \hat{\rho} | q' \rangle dq'' dq' \]

\[ = \frac{1}{(2\pi \hbar)^2} \int G\left(\{\frac{\vec{q}'' + \vec{q}'}{2}, \{\vec{\pi}\}\right) e^{-\frac{i}{\hbar} \sum_j \vec{q}_j \cdot \vec{p}_j} \langle q'' | \hat{\rho} | q'' \rangle \prod_i d\vec{q}_i d\vec{\pi}_i \]

In order to simplify the argument of \( G \), use the substitution

\[ \vec{q}'' = \frac{\vec{q} + \frac{\vec{q}'}{2}}{2} \]

The old variables can be written

\[ \vec{q}' = \vec{q} - \frac{\vec{q}'}{2} \]

Since the Jacobian of the transformation is \( \hbar \), then

\[ \langle G \rangle = \frac{1}{(2\pi \hbar)^{2N}} \int G(\{\vec{q}, \{\vec{\pi}\}\}) e^{-\frac{i}{\hbar} \sum_j \vec{q}_j \cdot \vec{p}_j} \langle \vec{q} + \frac{\vec{q}'}{2} | \hat{\rho} | \vec{q} - \frac{\vec{q}'}{2} \rangle \prod_i d\vec{q}_i d\vec{\pi}_i d\vec{q}' d\vec{\pi}' \]

If \( F(\{\vec{q}, \{\vec{\pi}\}, t) \) is a distribution function it satisfies

\[ \langle G \rangle = \int G(\{\vec{q}, \{\vec{\pi}\}\}) F(\{\vec{q}, \{\vec{\pi}\}, t) \prod_i d\vec{q}_i d\vec{\pi}_i \]

Subtracting these last two equations and requiring the integral to vanish independent of \( G \), the quantum distribution function is found to be
\[ F(\{q_i, p_i, t\}) = \frac{1}{(2\pi)^n} \int e^{-i \sum p_i \cdot \hat{q}_i} \left\langle \hat{q}_0^{1/2} \hat{p}_0^{1/2} \mid \rho \mid \hat{q}_0^{1/2} \hat{p}_0^{1/2} \right\rangle \prod_i d\hat{q}_i. \]  

This is the Wigner distribution function. Pool (3) shows that the Weyl correspondence is the only connection between classical and quantum mechanics that allows the calculation of expectation values by integrating the corresponding classical function with the Wigner distribution function over phase space. Hence, any other quantum distribution function requires a different correspondence between classical and quantum mechanics. He also claims to have the only rigorous proof that the Wigner function is square-integrable over phase space.

One of the drawbacks of the Wigner distribution function is that it is not positive definite, as classical distribution functions are required to be. In fact, Sudarshan shows that a positive definite function may be given a negative expectation value (4, pp. 178-188). This is not unexpected since the concept of a quantum-mechanical phase space is not well defined, due to the uncertainty principle. But generally these difficulties refer to situations which are immeasurable in the sense that the irregularities occur below the uncertainty limit \( \Delta p \Delta q = \frac{\hbar}{2} \). Also, the distribution function can be made positive definite by integration over a region of phase space of order \( \hbar^{3N} \) (5, p. 238). The ultimate evaluation of a theory is based on its ability to describe and predict, and on this point
the Wigner distribution function is quite satisfactory. Some of its properties are listed below. It should be noted that the first two integrals are positive definite since the result does not involve the uncertainty principle.

\[
\int F(\{q, p\}_t) \, dq \, dp = |\psi(\{q, t\})|^2
\]

= configuration-space probability density

\[
\int F(\{q, p\}_t) \, dq = |\phi(\{p, t\})|^2
\]

= momentum-space probability density

\[
\int F(\{q, p\}_t) \, dq \, dp = 1
\]

\[
F^*(\{q, p\}_t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-i \frac{q \cdot \tau}{\hbar}} \langle \frac{q}{\hbar} - \frac{p}{\hbar} | \rho | \frac{q}{\hbar} + \frac{p}{\hbar} \rangle \, dp \, dq
\]

The last relation shows that \( F(\{q, p\}_t) \) is real.

The Kinetic Equation

The time development of \( F(\{q, p\}_t) \) is given by the partial time derivative using equation II-1.

\[
\frac{\partial F}{\partial t} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-i \frac{q \cdot \tau}{\hbar}} \frac{\partial}{\partial \tau} \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \rho | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle \, dp \, dq
\]

where

\[
\frac{\partial}{\partial \tau} \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \rho | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle = \frac{i}{\hbar} \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \rho \dot{\chi} - \dot{\rho} \chi | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle
\]

\[
= \frac{i}{\hbar} \left[ \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \rho | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \chi \dot{\rho} | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle - \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \rho | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle \langle \frac{q}{\hbar} + \frac{p}{\hbar} | \dot{\chi} \rho | \frac{q}{\hbar} - \frac{p}{\hbar} \rangle \right] \]
Each of the density matrix elements above can be cast into the form appearing in the Wigner function by transformations. These are

\[ \tilde{q}', \tilde{p}' \rightarrow \tilde{q}, \tilde{p} \]

In both cases the Jacobian is \(2^{3N}\). Using these substitutions the kinetic equation can be written

\[
\frac{\partial F}{\partial t} = \frac{i}{\hbar^2} \int \left[ e^{\frac{\gamma}{h} (\tilde{q} - \tilde{q}')} \left( \tilde{q}, \tilde{p}, \tilde{q}', \tilde{p}' \right) \frac{1}{2} \left( \langle \tilde{q}|\tilde{p} \rangle \langle \tilde{q}|\tilde{p} \rangle - \langle \tilde{q}'|\tilde{p}' \rangle \langle \tilde{q}'|\tilde{p}' \rangle \right) \right] \prod d\tilde{q} d\tilde{p}
\]

Inversion of equation II-2 gives an expression for the matrix

\[
\langle \tilde{q} + \frac{h}{2} \tilde{p} | \tilde{p} | \tilde{q} - \frac{h}{2} \tilde{p} \rangle = \int e^{\frac{\gamma}{h} (\tilde{q} - \tilde{q}')} F(\tilde{q}, \tilde{p}); \tilde{q}', \tilde{p}' \rangle \prod d\tilde{p}'.
\]

Now the kinetic equation becomes

\[
\frac{\partial F}{\partial t} = \frac{i}{\hbar^2} \int \left[ e^{\frac{\gamma}{h} (\tilde{q} - \tilde{q}')} \left( \tilde{q}, \tilde{p}, \tilde{q}', \tilde{p}' \right) \frac{1}{2} \left( \langle \tilde{q}|\tilde{p} \rangle \langle \tilde{q}|\tilde{p} \rangle - \langle \tilde{q}'|\tilde{p}' \rangle \langle \tilde{q}'|\tilde{p}' \rangle \right) \right] \prod d\tilde{q} d\tilde{p} d\tilde{q}' d\tilde{p}'.
\]

This equation is simplified somewhat if the \(\tilde{q}\)-dependence appears only in the exponentials. A convenient transformation is

\[
\tilde{q} = \tilde{q} + \frac{h}{2} \tilde{p}, \quad \tilde{q}' = \tilde{q}' + \frac{h}{2} \tilde{p}'.
\]

The Jacobian of this transformation is \((\frac{h}{2})^3\), so
Reduced Distribution Functions

Most of the macroscopic quantities of interest whose averages are calculated with the Wigner function are functions of the positions and momenta of a small number of particles, usually one or two. Then most of the coordinates and momenta can be integrated over immediately. This leads to the definition of distribution functions which depend on the positions and momenta of a few particles, called reduced distribution functions.

If \( G = G(\vec{\eta}, \vec{p}) \), then

\[
\langle G \rangle = \int G(\vec{\eta}, \vec{p}) F(\{\vec{q}, \vec{p}\}, t) \prod_{\alpha} d\vec{q}_\alpha d\vec{p}_\alpha
\]

\[
= \int G(\vec{p}, \vec{\eta}) \left[ \int F(\{\vec{q}, \vec{p}\}, t) \prod_{\alpha} d\vec{q}_\alpha d\vec{p}_\alpha \right] d\vec{q}_\alpha d\vec{p}_\alpha
\]

The quantity in brackets is a function of \( \vec{\eta} \) and \( \vec{p} \) only, providing the suggested reduction. Since the \( N \) particles are assumed to be indistinguishable, there are \( N \)
such integrals which will give the same result since the subscript one refers to a single particle, and not particle number one.

\[ \langle G \rangle = \frac{1}{N} \int G(\vec{q}, \vec{p}) \ F_i(\vec{q}, \vec{p}) \ d\vec{q} \ d\vec{p}, \]

where

\[ F_i(\vec{q}, \vec{p}) = N \int F(\{\vec{q}_i, \vec{p}_i\}, t) \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i = \sum_{i=1}^{N} \int F(\{\vec{q}_i, \vec{p}_i\}, t) \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i. \]

In general, the s-body distribution function is defined

\[ F_s(\vec{q}_1, \vec{q}_2, ..., \vec{q}_s, \vec{p}_1, ..., \vec{p}_s, t) = \frac{N!}{(N-s)!} \int F_s(\{\vec{q}_1, \vec{p}_1, t\}) \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i. \]

In terms of the density matrix this becomes

\[ F_s(\vec{q}_1, \vec{q}_2, ..., \vec{q}_s, \vec{p}_1, ..., \vec{p}_s, t) = \frac{N!}{(N-s)!} \left\{ \prod_{i=2}^{N} \int e^{i \vec{q}_i \cdot \vec{p}_i} \frac{1}{(2\pi)^3} \right\} \langle \{ \vec{q}_{k+s}, \vec{q}_{k+2}, ..., \vec{q}_s, \vec{p}_{k+s}, \vec{p}_{k+2}, ..., \vec{p}_s \} | \{ \vec{q}_1, \vec{p}_1, t\} \rangle \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i \]

where \( k = 1, 2, ..., s \) and

\[ \langle \{ \vec{q}_{k+s}, \vec{q}_{k+2}, ..., \vec{q}_s, \vec{p}_{k+s}, \vec{p}_{k+2}, ..., \vec{p}_s \} | \{ \vec{q}_1, \vec{p}_1, t\} \rangle = \frac{N^!}{(N-s)!} \prod_{i=2}^{N} \langle \vec{q}_i, \vec{p}_i | \vec{q}_{k+s}, \vec{q}_{k+2}, ..., \vec{q}_s, \vec{p}_{k+s}, \vec{p}_{k+2}, ..., \vec{p}_s \rangle \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i. \]

Since

\[ \frac{\partial F}{\partial t} = N \int \frac{\partial F}{\partial t} \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i, \]

the one-body analogy of equation II-3 is

\[ \frac{1}{N} \frac{\partial F}{\partial t} - \frac{1}{\hbar(2\pi)^3} \int [H(\vec{q}, \vec{p}) - H(\vec{q}_1, \vec{p}_1)] F_s(\vec{q}_1, \vec{p}_1, t) \prod_{i=2}^{N} e^{i \vec{q}_i \cdot \vec{p}_i} \prod_{i=2}^{N} d\vec{q}_i \ d\vec{p}_i. \]
Inclusion of Relativistic Effects

Since electromagnetic interactions are transmitted with velocity \( c \), the speed of light, the force on a particle at some instant depends on the positions of the particles at some earlier time. An appropriate mechanical description of this retardation effect can be formulated by relating the field and potential of a particle to its position and velocity at an earlier time.

The equation for the electric potential in terms of charge density \( \rho \),

\[
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho,
\]

can be solved for \( \phi \) as

\[
\phi(\vec{q},t) = \int \frac{\rho(\vec{q}',t-\frac{|\vec{q} - \vec{q}'|}{c})}{|\vec{q} - \vec{q}'|} d\vec{q}'
\]

Expanding the denominator in a binomial fashion for small velocities with use of conservation of charge, it becomes

\[
\phi(\vec{q},t) = \frac{e}{|\vec{q} - \vec{q}'|} \cdot \frac{e}{2c^2} \frac{\partial}{\partial t} \frac{\vec{v}}{|\vec{q} - \vec{q}'|}
\]

\[
= \frac{e}{|\vec{q} - \vec{q}'|} - \frac{e}{2c^2} \frac{\partial}{\partial t} \left\{ \frac{\vec{v}}{|\vec{q} - \vec{q}'|} \right\}
\]
where the first term is the expected Coulomb potential and the second term is a relativistic correction which can be neglected for small velocities.

The solution of the equation for the vector magnetic potential requires only the usual first term

\[ \vec{A}(\vec{q}) = \frac{1}{c} \int \frac{\rho(\vec{v}')}{|\vec{q} - \vec{q}'|} d\vec{q}' = \frac{e\vec{v}'}{|\vec{q} - \vec{q}'|} \]

since it is of the same order as the preceding correction term. The electric term can be simplified by taking advantage of the gauge invariance of the potentials (6, p. 181)

\[ \phi \rightarrow \phi + \frac{1}{c} \frac{\partial \phi}{\partial t}, \quad A_i \rightarrow A_i - \frac{\partial \phi}{\partial x_i} \]

by choosing

\[ \chi = \frac{e}{2c} \frac{\vec{v}' \cdot (\vec{q} - \vec{q}')}{|\vec{q} - \vec{q}'|} \]

The electric and magnetic potentials now have the form

\[ \Phi(\vec{q}) = \frac{e}{|\vec{q} - \vec{q}'|} \]

\[ A_i = \frac{e}{2c} \left( \frac{\vec{v}' \cdot (\vec{q} - \vec{q}')}{|\vec{q} - \vec{q}'|} + \sum_{k \neq i} \frac{\nu_k (\vec{q}_k - \vec{q}') \cdot (\vec{q}_k - \vec{q}')}{|\vec{q}_k - \vec{q}'|^3} \right) \]

It is easy to show that the divergence of \( \vec{A} \) is zero and that its curl is the magnetic field, i.e., the cross product of \( \vec{v}' \) and the electric field.

The Lagrangian for a system of charged particles with relativistic corrections is, approximately,

\[ L = \sum_a \left\{ -\frac{m_a c^2}{2} \left[ 1 - \left( \frac{v_a}{c} \right)^2 \right]^{1/2} + \frac{e}{c} \vec{v}_a \cdot \vec{A} - e\phi \right\} \]

\[ -\sum_a m_a c^2 \left( -\frac{v_a}{c} \right)^2 + \frac{e^2}{4c} \sum_a \sum_{a' \neq a} \left\{ \frac{\vec{v}_a \cdot \vec{v}_a}{|\vec{q} - \vec{q}'|^3} + \frac{\vec{v}_a \cdot (\vec{q}_a - \vec{q}') \vec{v}_a \cdot (\vec{q}_a - \vec{q}')}{|\vec{q}_a - \vec{q}'|^3} \right\} + \frac{e}{c} \sum_a \vec{v}_a \cdot \vec{A}_{\text{ext}} \]

\[ -\frac{1}{2c} \sum_{a \neq a'} \frac{e^2}{|\vec{q}_a - \vec{q}_a'|} \]
where $\tilde{A}_a$ is an external magnetic vector potential. Retaining terms through order ($\gamma$) and omitting the rest-mass energy, the system Hamiltonian is found to be

$$\mathcal{H} = \sum_a \tilde{v}_a \cdot \frac{\partial}{\partial \tilde{v}_a} - L$$

$$= \sum_a \frac{1}{2} m \tilde{v}_a^2 + \sum_a \frac{3}{8} m \tilde{v}_a^4 + \frac{1}{2} \sum_{a(\neq b)} \frac{e}{4c^2} \frac{\tilde{v}_a \tilde{v}_b}{|\tilde{q}_a - \tilde{q}_b|^3} + \frac{e}{4c^2} \frac{\tilde{v}_a \tilde{v}_b (\tilde{q}_a \cdot \tilde{q}_b) \tilde{v}_c (\tilde{q}_a \cdot \tilde{q}_b)}{|\tilde{q}_a - \tilde{q}_b|^3}.$$  

The velocities $\tilde{v}_a$ can be found in terms of the canonical momenta by

$$\tilde{p}_a = m \tilde{v}_a (1 + \frac{e}{c^4} \tilde{A}_a) + \frac{e}{2c^2} \left\{ \frac{\tilde{v}_a}{|\tilde{q}_a|^3} \right\} + \frac{e}{c} \tilde{A}_a.$$

A first approximation is found by neglecting terms with a coefficient $\frac{1}{\alpha}$,

$$\tilde{p}_a \approx m \tilde{v}_a + \frac{e}{c} \tilde{A}_a$$

and substituting this into the previous equation to obtain

$$\tilde{p}_a = \frac{m \tilde{v}_a (1 + \frac{e}{c^4} \tilde{A}_a)}{1 + \frac{e}{c^4} \tilde{A}_a} = \sum_a \frac{e}{2c^2} \left\{ \frac{\tilde{v}_a}{|\tilde{q}_a|^3} \right\} + \frac{e}{c} \tilde{A}_a.$$

The velocity is, through order ($\gamma$),

$$\tilde{v}_a = \frac{\tilde{p}_a - \frac{e}{c} \tilde{A}_a}{m} (1 + \frac{e}{c^4} \tilde{A}_a) - \sum_a \frac{e}{2c^2} \left\{ \frac{\tilde{v}_a}{|\tilde{q}_a|^3} \right\} + \frac{e}{c} \tilde{A}_a.$$

The Hamiltonian can now be expressed in terms of the canonical momenta by

$$\mathcal{H} = \sum_a \left\{ \frac{(\tilde{p}_a - \frac{e}{c} \tilde{A}_a)^2}{2m} \left[ 1 - \left( \frac{(\tilde{p}_a - \frac{e}{c} \tilde{A}_a)}{m \tilde{v}_a} \right)^2 \right] - \frac{e}{c} \frac{\tilde{v}_a (\tilde{p}_a - \frac{e}{c} \tilde{A}_a) \cdot (\tilde{p}_a - \frac{e}{c} \tilde{A}_a)}{1 + \frac{e}{c^4} \tilde{A}_a} \right\}$$

$$+ \frac{e}{c} \frac{\tilde{v}_a (\tilde{p}_a - \frac{e}{c} \tilde{A}_a) \cdot (\tilde{p}_a - \frac{e}{c} \tilde{A}_a)}{1 + \frac{e}{c^4} \tilde{A}_a} + \sum_a \frac{e}{c} \frac{\tilde{v}_a (\tilde{p}_a - \frac{e}{c} \tilde{A}_a)}{1 + \frac{e}{c^4} \tilde{A}_a}$$

$$+ \sum_a \frac{e}{c} \frac{\tilde{v}_a (\tilde{p}_a - \frac{e}{c} \tilde{A}_a) \cdot (\tilde{p}_a - \frac{e}{c} \tilde{A}_a)}{1 + \frac{e}{c^4} \tilde{A}_a}.$$
The following notation will simplify some of the equations to be used in later development. Let

\[ \vec{J}_i = \vec{p}_i - \frac{e}{c} \vec{A}_{\text{ext}} \]  

(II-5)

Then the Hamiltonian has the form

\[ \mathcal{H} = \sum_i \left\{ \frac{\pi_i^2}{2m_i} - \frac{1}{8} \frac{\pi_i^2}{m_i c^2} \sum_{k(<i)} \left[ \frac{\vec{\pi}_k \cdot \vec{J}_k}{\epsilon_{\text{ext}k}} + \frac{\vec{A}_{\text{ext}} \cdot \vec{\pi}_i}{\epsilon_{\text{ext}i}} \frac{\vec{q}_{\text{ext}k}}{\epsilon_{\text{ext}k}} \frac{\vec{q}_{\text{ext}i}}{\epsilon_{\text{ext}i}} \frac{\vec{J}_k}{\epsilon_{\text{ext}k}} \right] + \sum_{k(<i)} \frac{e^2}{4 \epsilon_{\text{ext}i}^2} \right\} \]

(II-6)

The third term was first obtained by Darwin (7); so this expression is often referred to as the Darwin Hamiltonian. Note that for \( \vec{A}_{\text{ext}} = 0 \), \( \vec{J}_i = \vec{p}_i \).
CHAPTER BIBLIOGRAPHY


CHAPTER III

DEPARTURE FROM EQUILIBRIUM AND DISPERSION RELATIONS

IN THE ABSENCE OF EXTERNAL FIELDS

The Hamiltonian of a plasma consisting of \( N \) charged, weakly interacting particles in the absence of any external fields is found from a special case of equation II-6.

\[
\mathcal{H} = \sum_n \left\{ \frac{\mathbf{p}_n^2}{2m} - \frac{1}{8} \frac{\mathbf{p}_n^4}{m^2c^4} - \frac{e^2}{4\pi \varepsilon_0 c^2} \sum_{\lambda_{\alpha\beta}} \left[ \frac{\mathbf{p}_{\alpha} \cdot \mathbf{p}_{\beta}}{g_{\alpha\beta}} + \frac{\mathbf{p}_{\alpha} \cdot \mathbf{p}_{\beta}}{g_{\alpha\beta}^3} \right] + \sum_{\lambda_{\alpha\beta}} \frac{e^2}{2g_{\alpha\beta}^2} \right\}
\]

The kinetic equation is then formed by inserting this Hamiltonian into equation II-14 to obtain

\[
\frac{1}{N} \frac{\partial \tilde{\xi}}{\partial t} = \frac{i}{\hbar} \left\{ \frac{\mathbf{p}_n^2 - \mathbf{p}_n^4}{2m} - \frac{1}{8} \frac{\mathbf{p}_n^4}{m^2c^4} - \frac{e^2}{8\pi \varepsilon_0 c^2} \sum_{\lambda_{\alpha\beta}} \left[ \frac{\mathbf{p}_{\alpha} \cdot \mathbf{p}_{\beta}}{g_{\alpha\beta}} + \frac{\mathbf{p}_{\alpha} \cdot \mathbf{p}_{\beta}}{g_{\alpha\beta}^3} \right] + \sum_{\lambda_{\alpha\beta}} \frac{e^2}{2g_{\alpha\beta}^2} \right\} \times
\]

\[
\sum_{\lambda_{\alpha\beta}} \left[ \frac{\mathbf{p}_n^2}{g_{\alpha\beta}^2} \tilde{\xi} + \frac{\mathbf{p}_n^4}{g_{\alpha\beta}^4} \tilde{\xi} \right] + \sum_{\lambda_{\alpha\beta}} \left[ \frac{\mathbf{p}_n^2}{g_{\alpha\beta}^2} \tilde{\xi} + \frac{\mathbf{p}_n^4}{g_{\alpha\beta}^4} \tilde{\xi} \right] \times
\]

\[
f_n(\{\tilde{\xi}_n, \tilde{\xi}_\beta, \mathbf{r}_n\}) e^{i(\tilde{\xi}_n \cdot \tilde{\mathbf{r}}_n - \tilde{\xi}_\beta \cdot \mathbf{r}_\beta)} \frac{e^2}{\frac{1}{2} \varepsilon_0 c \hbar} \int d\mathbf{r}_n. d\mathbf{r}_\beta \delta_{\tilde{\xi}_n, \tilde{\xi}_\beta} \delta_{\mathbf{r}_n, \mathbf{r}_\beta}
\]

In terms of the two-body distribution, since the equation involves only sums of pair interactions at the most, it can be written
\[
\frac{\partial f}{\partial t} = \frac{1}{\hbar (2\pi)^3} \left\{ \frac{\left(\mathbf{p} + \frac{e}{c} \mathbf{A}\right)^2 - \left(\mathbf{p} - \frac{e}{c} \mathbf{A}\right)^2}{2m(N-1)} - \frac{e^2}{2mc^2} \left[ \frac{\left(\mathbf{p} + \frac{e}{c} \mathbf{A}\right) \cdot \mathbf{p} - \left(\mathbf{p} - \frac{e}{c} \mathbf{A}\right) \cdot \mathbf{p}}{\mathbf{p} \cdot \mathbf{p} + \mathbf{A} \cdot \mathbf{A}} \right] \right. \\
+ \frac{\left(\mathbf{p} + \frac{e}{c} \mathbf{A}\right) \cdot \left(\mathbf{q} + \frac{e}{c} \mathbf{A}\right) - \left(\mathbf{p} - \frac{e}{c} \mathbf{A}\right) \cdot \left(\mathbf{q} - \frac{e}{c} \mathbf{A}\right)}{\mathbf{q} \cdot \mathbf{q} + \mathbf{A} \cdot \mathbf{A}} \\
\left. - \frac{e^2}{\mathbf{q} \cdot \mathbf{q} + \mathbf{A} \cdot \mathbf{A}} \right\} f_0(\mathbf{q}, \mathbf{p}, t) e^{i\mathbf{k} \cdot \mathbf{q}} d\mathbf{q} d\mathbf{p} d\mathbf{p} d\mathbf{q} d\mathbf{r} d\mathbf{r}.
\]

(III-1)

If the particles are not strongly interacting, a first approximation of \( f_2 \) is a product of one-body distribution functions

\[
f_2(\mathbf{q}, \mathbf{p}, t) = f(\mathbf{q}, \mathbf{p}, t) f_0(\mathbf{p}, t)
\]

where

\[
f_0(\mathbf{p}, t) = f_0(\mathbf{p}) + f(\mathbf{p}, t)
\]

The term \( f_0(\mathbf{p}) \) is the equilibrium distribution function and \( f(\mathbf{q}, \mathbf{p}, t) \) is a small correction measuring the departure from equilibrium which is assumed to be small, so \( \mathcal{A} \approx f_0 \). Retaining first-order corrections,

\[
f_2(\mathbf{q}, \mathbf{p}, t) = f_0(\mathbf{p}) f(\mathbf{p}, t) + f(\mathbf{q}, \mathbf{p}, t)
\]

The left-hand side of equation III-1 will then contain only \( f \). On the other side both \( f_0(\mathbf{p}) f_0(\mathbf{p}) \) and \( f_0(\mathbf{p}) f(\mathbf{q}, \mathbf{p}, t) \) terms vanish, as shown later.

The equation III-1 can be transformed into a partial algebraic form which is easier to work with than the integro-differential equation as it appears. The new variables will also have some useful physical significance. The procedure is to calculate the Fourier transform \( F(k, p, t) \) defined in the following manner:

\[
F(k, p, t) = \frac{i}{(2\pi)^3} \int f(q, p, t) e^{ik \cdot q} dq.
\]
The transformation is obtained by multiplying the equation by \( \frac{1}{(2\pi)^N} e^{ik\cdot\vec{q}} \) and integrating over \( \vec{q} \). Each of the terms is evaluated below.

\[
I_1 = \frac{1}{(2\pi)^N} \int \frac{2}{\epsilon} \tilde{S}(\vec{q},\vec{p},t) e^{ik\cdot\vec{q}} d\vec{q} = \frac{\partial}{\partial\epsilon} F(k,\vec{p},t)
\]

\[
I_2 = \frac{i}{m(2\pi)^N} \int \left( \frac{2}{\epsilon} \frac{1}{\epsilon} \right) \tilde{S}(\vec{q},\vec{p},t) e^{ik\cdot\vec{q}} d\vec{q}
\]

\[
= \frac{i}{m(2\pi)^N} \int \tilde{S}(\vec{q},\vec{p},t) e^{ik\cdot\vec{q}} \frac{\partial}{\partial\epsilon} \tilde{S}(\vec{q},\vec{p},t) d\vec{q}
\]

It should be noted that unless specified otherwise, the term obtained in integrating by parts which is evaluated at the infinite limits will be zero due to either boundary conditions on the distribution function or evaluation of a delta function where its argument does not vanish.

The next term in III-1 to consider involves laborious expansions and combining. In order to save some tedium in the next chapter, it will be evaluated with an external magnetic potential which can then be set equal to zero for this chapter.
It is sufficient to choose \( A(q) \cdot \frac{1}{2} \hat{B} \times \vec{c} \) where \( \hat{B} \) is the external constant magnetic field. It is easy to show that the condition \( \hat{B} \cdot \nabla \times A \) is then satisfied. The following symbols will be used:

\[
\begin{align*}
\vec{\alpha} &= \frac{\hbar}{2} \hat{\rho} \\
\vec{\beta} &= \frac{\hbar}{2} \hat{\tau} \\
\vec{A} &= \frac{1}{2} \hat{B} \times \vec{q} \\
\vec{C} &= \frac{1}{2} \hat{B} \times \vec{p} \\
\vec{A}_1 &= \frac{1}{2} \hat{B} \times (\vec{q} - \frac{1}{2} \vec{r}) \times \vec{A} - \vec{C} \\
\vec{A}_2 &= \frac{1}{2} \hat{B} \times (\vec{q} + \frac{1}{2} \vec{r}) \times \vec{A} + \vec{C}
\end{align*}
\]

\[
\begin{align*}
\left[ \vec{p} \cdot \vec{z} - \frac{\varepsilon}{C} \vec{A} \right]^2 - \left[ \vec{p} \cdot \vec{z} - \frac{\varepsilon}{C} \vec{A}_1 \right]^2 &= \left( \vec{p} \cdot \vec{z} \right)^2 - \left( \vec{p} \cdot \vec{z} \right) + \frac{4e^2}{c^2} \left[ \vec{A} \cdot \left( \vec{p} + \vec{z} \right) \right]^2 - \frac{4e^2}{c^2} \left[ \vec{A} \cdot \left( \vec{p} - \vec{z} \right) \right]^2 \\
- \frac{4e^2}{C} \left( \vec{p} \cdot \vec{z} \right) \vec{A} \cdot \left( \vec{p} + \vec{z} \right) + \frac{4e^2}{C} \left( \vec{p} \cdot \vec{z} \right) \vec{A} \cdot \left( \vec{p} - \vec{z} \right) + \frac{2e^2}{c^2} \vec{A} \cdot \left( \vec{p} + \vec{z} \right) - \frac{2e^2}{c^2} \vec{A} \cdot \left( \vec{p} - \vec{z} \right) \\
- \frac{4e^2}{C} \vec{A} \cdot \vec{A} \left( \vec{p} + \vec{z} \right) + \frac{4e^2}{C} \vec{A} \cdot \vec{A} \left( \vec{p} - \vec{z} \right) \\
= p^2 + 4p^2 \vec{p} \cdot \vec{z} + 4(p \cdot \vec{z})^2 + 2p^2 \vec{a}^2 + 4(p \cdot \vec{a})^2 + \alpha^2 + \frac{4e^2}{c^2} \left\{ \left[ \vec{A} \cdot \vec{p} - \vec{A} \cdot \vec{p} - \vec{A} \cdot \vec{z} + \vec{A} \cdot \vec{z} \right]^2 \\
- \left[ \vec{A} \cdot \vec{p} - \vec{C} \cdot \vec{p} - \vec{A} \cdot \vec{z} - \vec{C} \cdot \vec{z} \right]^2 \right\} + \frac{e^2}{C} \left\{ \left[ \vec{A} \cdot 2 \vec{A} \cdot \vec{z} + \vec{C} \cdot \vec{C} \right]^2 - \left[ \vec{A} \cdot 2 \vec{A} \cdot \vec{z} + \vec{C} \cdot \vec{C} \right]^2 \right\} \\
+ \frac{4e^2}{C} \left\{ (p^2 - 2p \cdot \vec{z} + \alpha^2) \left( \vec{A} \cdot \vec{p} - \vec{C} \cdot \vec{p} - \vec{A} \cdot \vec{z} - \vec{C} \cdot \vec{z} \right) - (\vec{A} \cdot 2 \vec{A} \cdot \vec{z} + \vec{C} \cdot \vec{C}) (p^2 - 2p \cdot \vec{z} + \alpha^2) \right\} \\
+ \frac{2e^2}{C} \left\{ (A^2 - 2A \cdot \vec{C} + \vec{C}^2) (p^2 - 2p \cdot \vec{z} + \alpha^2) - (p \cdot 2p \cdot \vec{z} + \alpha^2) (\vec{A} \cdot \vec{p} - \vec{C} \cdot \vec{p} + \vec{A} \cdot \vec{z} - \vec{C} \cdot \vec{z}) \right\} \\
+ \frac{4e^2}{C} \left\{ (A^2 + 2A \cdot \vec{C} + \vec{C}^2) (\vec{A} \cdot \vec{p} + \vec{C} \cdot \vec{p} - \vec{A} \cdot \vec{z} - \vec{C} \cdot \vec{z}) - (A^2 - 2A \cdot \vec{C} + \vec{C}^2) (\vec{A} \cdot \vec{p} - \vec{C} \cdot \vec{p} + \vec{A} \cdot \vec{z} - \vec{C} \cdot \vec{z}) \right\} \\
= 8(p \cdot \vec{z})[p^2 + \alpha^2] + \frac{8e^2}{C} [\vec{C} \left\{ (p \cdot \vec{p})(\vec{C} \cdot \vec{p}) + (p \cdot \vec{a})(\vec{C} \cdot \vec{a}) + (\vec{C} \cdot \vec{p})(\vec{C} \cdot \vec{a}) - (\vec{A} \cdot \vec{a})(\vec{C} \cdot \vec{z}) \right\} \\
- \frac{e^2}{C} \left\{ (\vec{A} \cdot \vec{z})(\vec{A} \cdot \vec{C}) \right\}] + \left\{ (p \cdot \vec{z})[p^2 + \alpha^2] - (\vec{A} \cdot \vec{a})(\vec{A} \cdot \vec{z}) + 2(\vec{p} \cdot \vec{z})(\vec{C} \cdot \vec{z}) - 2(p \cdot \vec{z})(\vec{p} \cdot \vec{p}) \right\} \]
Each of these twenty-four integrals can be evaluated using appropriate vector identities. One example will be given and the others will be quoted in the order in which they appear. The Fourier transform is taken later.

\[
I_{3,6} = \frac{1}{2m^2c} \rho \cdot \frac{\partial}{\partial \rho} \left\{ \left( 2 \pi \right)^3 \delta \left( \rho - \rho' \right) \right\} f_{(q, \tilde{p}, t)} e^{i \left[ (\tilde{q} \cdot \tilde{p}) + f \left( \rho \right) \right]}
\]

\[
= \frac{-i \hbar e^2}{16m^2c^{2}(2\pi)^3} \left[ \left( \partial \cdot \partial \right) \left( \tilde{p} \cdot \partial \right) \right] f_{(q, \tilde{p}, t)} e^{i \left[ (\tilde{q} \cdot \tilde{p}) + f \left( \rho \right) \right]}
\]

\[
= \frac{-i \hbar e^2}{16m^2c^{2}(2\pi)^3} \left[ \left( \partial \cdot \partial \right) \left( \tilde{p} \cdot \partial \right) \right] f_{(q, \tilde{p}, t)} e^{i \left[ (\tilde{q} \cdot \tilde{p}) + f \left( \rho \right) \right]}
\]

\[
= \frac{-i \hbar e^2}{16m^2c^{2}(2\pi)^3} \left[ \left( \partial \cdot \partial \right) \left( \tilde{p} \cdot \partial \right) \right] f_{(q, \tilde{p}, t)} e^{i \left[ (\tilde{q} \cdot \tilde{p}) + f \left( \rho \right) \right]}
\]

\[
= \frac{-i \hbar e^2}{16m^2c^{2}(2\pi)^3} \left[ \left( \partial \cdot \partial \right) \left( \tilde{p} \cdot \partial \right) \right] f_{(q, \tilde{p}, t)} e^{i \left[ (\tilde{q} \cdot \tilde{p}) + f \left( \rho \right) \right]}
\]

\[
= \frac{-i \hbar e^2}{16m^2c^{2}(2\pi)^3} \left[ \left( \partial \cdot \partial \right) \left( \tilde{p} \cdot \partial \right) \right] f_{(q, \tilde{p}, t)} e^{i \left[ (\tilde{q} \cdot \tilde{p}) + f \left( \rho \right) \right]}
\]

\[
I_{3,1} = \frac{1}{2m^2c} \rho \cdot \frac{\partial}{\partial \rho} \left\{ \left( 2 \pi \right)^3 \delta \left( \rho - \rho' \right) \right\} f_{(q, \tilde{p}, t)}
\]
\[ I_{3,2} = -\frac{\hbar^2}{8m^2c} \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \]

\[ I_{3,3} = \frac{e^2}{4m^2c^2} \left[ (\mathbf{q} \times \mathbf{p}) \cdot (\mathbf{q} \times \mathbf{q}) \mathbf{F}(\mathbf{q}, \mathbf{p}, t) + (\mathbf{q} \times \mathbf{q}) \cdot (\mathbf{q} \times \mathbf{p}) \cdot \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \right] \]

\[ I_{3,4} = \frac{e^2}{4m^2c^2} \left[ -(\mathbf{q} \times \mathbf{q}) \cdot (\mathbf{q} \times \mathbf{p}) \mathbf{F}(\mathbf{q}, \mathbf{p}, t) + (\mathbf{q} \times \mathbf{q}) \cdot (\mathbf{q} \times \mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \right] \]

\[ I_{3,5} = \frac{\hbar^2 e^2}{16m^2c^2} \left[ (\mathbf{q} \times \mathbf{p}) \cdot (\mathbf{q} \times \mathbf{q}) \mathbf{F}(\mathbf{q}, \mathbf{p}, t) - \mathbf{p} \times \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{p}} (\mathbf{q} \cdot \mathbf{q} \times \mathbf{q}) \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \right] \]

\[ I_{3,7} = \frac{e^2}{32m^2c^2} (\mathbf{q} \times \mathbf{q}) \cdot \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \times (\mathbf{q} \times \mathbf{q}) \]

\[ I_{3,8} = \frac{\hbar^2 e^2}{128m^2c^2} (\mathbf{q} \times \mathbf{q}) \times \mathbf{B} \cdot (\mathbf{q} \times \mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \]

\[ I_{3,9} = \frac{e}{4m^2c^2} \mathbf{B} \cdot \mathbf{p} = \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \times \mathbf{p} \]

\[ I_{3,10} = \frac{-\hbar^2 e^2}{16m^2c^2} \mathbf{p} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \]

\[ I_{3,11} = \frac{-e}{4m^2c^2} \mathbf{p} \times (\mathbf{q} \times \mathbf{q}) \cdot \mathbf{B} \times \mathbf{q} \]

\[ I_{3,12} = \frac{-\hbar^2 e^2}{16m^2c^2} \mathbf{B} \times \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \]

\[ I_{3,13} = \frac{\hbar^2 e^2}{8m^2c^2} \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{q}} \left( \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \right) \]

\[ I_{3,14} = \frac{-e}{2m^2c^2} \mathbf{p} \cdot (\mathbf{q} \times \mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \]

\[ I_{3,15} = \frac{e^2}{8m^2c^2} \mathbf{p} \cdot \left[ \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \frac{\partial}{\partial \mathbf{q}} (\mathbf{q} \times \mathbf{q}) + (\mathbf{q} \times \mathbf{q}) \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \right] \]

\[ I_{3,16} = \frac{-\hbar^2 e^2}{32m^2c^2} \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \]

\[ I_{3,17} = \frac{e}{8m^2c^2} \mathbf{B} \times (\mathbf{q} \times \mathbf{q}) \cdot [\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t)] \]

\[ I_{3,18} = \frac{-\hbar^2 e^2}{32m^2c^2} \frac{\partial}{\partial \mathbf{q}} \left[ \mathbf{B} \times (\mathbf{q} \times \mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{q}, \mathbf{p}, t) \right] \]
These integrals are then added together and combined to form the expression

\[ I_3 = \frac{c^3}{2m} \left\{ -\left( \vec{p} - \vec{c} \vec{A} \right) \cdot \vec{B} - \frac{c^3}{2} \left( \vec{p} - \vec{c} \vec{A} \right) \left( \vec{B} \times \vec{A} \right) \cdot \vec{D} \right\} + \frac{c^5}{2} \left( \vec{p} - \vec{c} \vec{A} \right) \left( \vec{B} \times \vec{A} \right) \cdot \vec{D} \]

The change of variables given in equation II-5 will further simplify this equation by eliminating \( \vec{p} \). The chain rule of differentiation gives

\[
\frac{d}{dt} \rightarrow \frac{d}{dt} - \frac{c}{\epsilon} \frac{d}{dt} \vec{A} \cdot \frac{d}{dt} - \frac{d}{dt}
\]

\[
\vec{J} \cdot \frac{d}{dt} \rightarrow \vec{J} \cdot \frac{d}{dt} - \frac{c}{\epsilon} \left[ \left( \vec{J} \cdot \frac{d}{dt} \right) \vec{A} \right] \cdot \frac{d}{dt}
\]
\begin{align*}
\frac{\partial}{\partial \phi} & \rightarrow \frac{\partial}{\partial \phi^*} \\

The integral then reduces to the compact form
\begin{equation}
I_3 = \frac{1}{2m^2 c^2} \vec{J} \cdot \frac{\partial}{\partial \phi^*} \left[ \mathcal{P}^2 - \mathcal{P}^2_c \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi^*} \right)^2 \right] \mathcal{J}_3 (\vec{r}, \vec{r}, t)
\end{equation}
\end{align*}

In the case of no external field, \( \vec{E} \rightarrow 0 \) and the integrals \( I_3 \) and \( I_{3,1} \) are retrieved, since this equation contains \( \vec{q} \) only in the function \( \mathcal{J}_3 \) and in derivatives, the Fourier transform effectively replaces derivatives with respect to \( \vec{q} \) by \(-i\vec{k} \). The final transformed form is then
\begin{equation}
I_3 = \frac{i \vec{k} \cdot \vec{J}^*}{2m^2 c^2} \left\{ \mathcal{P}^2 - \frac{i}{4} \left( -i\vec{k} + \frac{\partial}{\partial \phi^*} \right)^2 \right\}.
\end{equation}

The terms in brackets in equation III-1 can be divided into two groups, those which are multiplied by \( \vec{r} \) and allow immediate integration over \( \vec{r} \) and \( \vec{r}_1 \), and those that are multiplied by \( \vec{r}_1 \). Replacing \( \vec{r} \) by a derivative with respect to the exponential and integrating by parts, these terms are
\begin{align*}
&= \frac{i e^2}{2m^2 c^2} \frac{1}{h(2\pi)^6} \left\{ \frac{\vec{q}}{2} \left( \frac{\vec{q}}{2} \cdot \vec{r} - \vec{q} \cdot \vec{r}_1 \right) + \frac{\vec{q} \cdot \vec{q} - \vec{r} \cdot \vec{r}_1}{\vec{q} \cdot \vec{q} - \vec{r} \cdot \vec{r}_1} \right\} e^{i \vec{r}_1 \cdot (\vec{q} - \vec{r})}, \\
&= e^{i \vec{r}_1 \cdot (\vec{q} - \vec{r})} f_2 (\vec{q}, \vec{r}_1, \vec{r}_2, t) d\vec{q}_1 d\vec{q}_2 d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4 \\
&= e^{i \vec{r}_1 \cdot (\vec{q} - \vec{r})} \frac{1}{4m^2 c^2 (2\pi)^6} \left\{ \frac{\vec{q}}{2} \left( \frac{\vec{q}}{2} \cdot \vec{r} - \vec{q} \cdot \vec{r}_1 \right) + \frac{\vec{q} \cdot \vec{q} - \vec{r} \cdot \vec{r}_1}{\vec{q} \cdot \vec{q} - \vec{r} \cdot \vec{r}_1} \right\} f_2 (\vec{q}, \vec{r}_1, \vec{r}_2, t) e^{i \vec{r}_1 \cdot (\vec{q} - \vec{r})} d\vec{q}_1 d\vec{q}_2 d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4.
\end{align*}

In each of the integrals arising from the terms in brackets of III-1, \( f_2 \) can be replaced by \( f_2 (\vec{r}_1, \vec{r}_2) \) since one group of terms cancels in pairs when integrated over \( \vec{q}_1 \) with its infinite limits because the form of the terms is the same, and the others vanish because the terms can be combined into the form of the divergence of \( \vec{A} \), which is zero.
The \( \mathcal{I}_4 \) and \( \mathcal{Q}_4 \) integrals have already been performed. Using the substitutions

\[
\bar{\mu} = \left( \frac{-1}{2} \right) \bar{\mu} - \bar{\alpha}_2
\]

which is common in the following integrals, it becomes

\[
\mathcal{I}_4 = \frac{-ie^x}{2m^2 c^2 \hbar (2\pi)^3} \int \frac{1}{|\bar{P} + \bar{P} + \bar{P} + \bar{P}|} e^{i\bar{P} \cdot \bar{P} + \bar{P} \cdot \bar{P}} \mathcal{F}(\bar{\mu}, \bar{\mu}, \bar{\mu}, \bar{\mu}) e^{i\bar{P} \cdot \bar{P} + \bar{P} \cdot \bar{P}} d\bar{\mu}_1 d\bar{\mu}_2 d\bar{\mu}_3 d\bar{\mu}_4 d\bar{\mu}_5 d\bar{\mu}_6 d\bar{\mu}_7 d\bar{\mu}_8 d\bar{\mu}_9 d\bar{\mu}_{10}
\]

The \( \bar{\chi} \)-integration is performed as follows:

\[
\int \frac{1}{|\bar{x}|} e^{i\bar{P} \cdot \bar{x}} d\bar{x} = \int \int \int \frac{1}{\bar{x}} e^{i\bar{P} \cdot \bar{x} \sin \theta} d\bar{x} d\phi d\theta
\]

\[
= 2\pi \int \int \int e^{ik\mu} \bar{x} d\mu d\bar{x} = \frac{4\pi}{k} \int _0 ^{\infty} \sin kx \, dx
\]

\[
= \lim _{a \to 0} \frac{4\pi}{k} \int _0 ^{\infty} e^{-ax} \sin kx \, dx = \lim _{a \to 0} \frac{4\pi}{k} \frac{a^2 + k^2}{a^2 + k^2}
\]

The term \( e^{-ax} \) is a convergence factor. It can be shown by more rigorous means that this gives the correct answer.

\[
\mathcal{I}_4 = \frac{-2nie^x}{m^2 c^2 \hbar^2 (2\pi)^3} \int \bar{\mu} \bar{\mu} \mathcal{F}(\bar{\mu}, \bar{\mu}, \bar{\mu}, \bar{\mu}) \mathcal{F}(\bar{\mu}, \bar{\mu}, \bar{\mu}, \bar{\mu}) e^{-\bar{\mu} \cdot \bar{\mu} + \bar{\mu} \cdot \bar{\mu}} \, d\bar{\mu}_1 d\bar{\mu}_2 d\bar{\mu}_3 d\bar{\mu}_4 d\bar{\mu}_5 d\bar{\mu}_6 d\bar{\mu}_7 d\bar{\mu}_8 d\bar{\mu}_9 d\bar{\mu}_{10}
\]

The \( \bar{\chi} \)-integration is performed as follows:
\[
\int \frac{1}{\partial x} \frac{1}{|x|} e^{ikx} \, dx = -\int i \xi \frac{1}{|\xi|} e^{i\xi x} \, d\xi = -i \xi \frac{4m}{\hbar^2} 
\]

\[
I_4 = \frac{-ie^\xi}{m^2 c^2 \hbar^2 (2\pi)^3} \int \tilde{\rho}_4 \cdot \tilde{\rho}_4 \cdot \delta(\tilde{\rho}_4 - \tilde{\rho}_4 + \frac{1}{2} \hbar \nu) \cdot f(\tilde{\rho}_4, \tilde{\rho}_4, \nu) \cdot e^{i\nu t} \cdot d\tilde{\rho}_4 
\]

\[
I_5 = \frac{-ie^\xi}{2m^2 c^2 \hbar^2 (2\pi)^3} \int \frac{\tilde{\rho}_5 \cdot \tilde{\rho}_5 \cdot \tilde{\rho}_5 \cdot \tilde{\rho}_5 \cdot \delta(\tilde{\rho}_5 - \tilde{\rho}_5 + \frac{1}{2} \hbar \nu) \cdot f(\tilde{\rho}_5, \tilde{\rho}_5, \nu) \cdot e^{i\nu t} \cdot d\tilde{\rho}_5 \cdot d\tilde{\rho}_5 \cdot d\tilde{\rho}_5 \cdot d\tilde{\rho}_5 
\]

The integral involving \( \chi \) can be written in dyadic form.

\[
\int \frac{(\tilde{\rho}_5 \cdot \tilde{\rho}_5 \cdot \tilde{\rho}_5 \cdot \tilde{\rho}_5 \cdot \delta(\tilde{\rho}_5 - \tilde{\rho}_5 + \frac{1}{2} \hbar \nu) \cdot f(\tilde{\rho}_5, \tilde{\rho}_5, \nu) \cdot e^{i\nu t} \cdot d\tilde{\rho}_5 \cdot d\tilde{\rho}_5 \cdot d\tilde{\rho}_5 \cdot d\tilde{\rho}_5 
\]

In order to evaluate this integral, the matrix form of a dyad is used. Choose a coordinate system as shown in Fig. 1 where \( \hat{k} \) points along the z-axis. Then in terms of cartesian coordinates,

\[
\hat{x} = \chi \hat{e} \\
\hat{e} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = \text{unit vector}
\]

Fig. 1—Coordinate system with \( \hat{k} \) along the z-axis
\[ \vec{r} \times \vec{X} = \vec{k} \hat{e}_z \cdot \vec{E} = \vec{k} \times \cos \theta \]

\[ \int \frac{\vec{x} \cdot \vec{x}}{|\vec{x}|^2} e^{i \vec{k} \cdot \vec{x}} \, d\vec{x} = \int \frac{\vec{x}}{|\vec{x}|} e^{i \vec{k} \cdot \vec{x}} \, x^2 \sin \theta \, d\theta \, d\varphi \, dx \]

\[ \int e^{i \vec{k} \cdot \vec{x}} \begin{pmatrix} \sin^2 \theta \cos^2 \phi & \sin^2 \theta \cos \phi & \sin^2 \theta \sin \phi & \sin \theta \cos \phi \cos \theta \\ \sin^2 \theta \cos \phi & \sin^2 \theta \sin \phi & \sin^2 \theta \cos \phi \cos \theta & \sin \phi \cos \theta \\ \sin^2 \phi & \sin \phi \cos \phi & \cos^2 \phi & \sin \phi \cos \phi \\ \sin \phi \cos \phi & \sin \phi \cos \phi & \sin \phi \cos \phi & \cos^2 \phi \end{pmatrix} \]

The integrations involving off-diagonal elements are zero due to the \( \varphi \)-integration.

\[ I_{s,1} = \pi \int_0^\infty x \, dx \int_0^\pi \sin^2 \theta \, e^{i k x \cos \theta} \, d\theta \int_0^{2\pi} \cos^2 \varphi \, d\varphi \]

\[ = \pi \int_0^\infty x \, dx \left[ \int_0^1 (1 - \mu^2) \, e^{i k x \mu} \, d\mu \right] \]

\[ = \frac{4\pi}{k^2} \left[ \int_0^\infty \frac{\sin k x}{k x^2} \, dx - \int_0^\infty \frac{\cos k x}{x} \, dx \right] \]

\[ = \frac{4\pi}{k^2} \]

The other dyadic integrals are evaluated similarly.

\[ I_{s,22} = \frac{4\pi}{k^2} \]

\[ I_{s,13} = -\frac{4\pi}{k^2} \]

\[ I_3 = -\frac{2n e^2}{m^2 c^3 \hbar^2 \kappa^2} \int \vec{P}_1 \cdot \left( \frac{1 \cdot \vec{0} \cdot \vec{0}}{\vec{P}_2 \cdot \vec{0} \cdot \vec{0}} \right) f_0(\vec{P}_1) \, e^{i k \cdot \vec{x}} \, S(\vec{P}_1, \vec{P}_2, \vec{P}_2) \, d\vec{P}_1 \, d\vec{P}_2 \]

\[ = -\frac{2n e^2}{m^2 c^3 \hbar^2 \kappa^2} f_0(\vec{P}_2 = \vec{P}_1) \left( \frac{1}{\vec{P}_2 \cdot \vec{P}_2} \right) \int \vec{P}_1 \cdot \left( \frac{1 \cdot \vec{0} \cdot \vec{0}}{\vec{P}_2 \cdot \vec{0} \cdot \vec{0}} \right) F(\vec{k}, \vec{P}_2, t) \, d\vec{P}_2 \]
These integrals can now be inserted into equation III-1 to obtain the kinetic equation. It can be simplified somewhat by recalling that a vector may be written as the product of that vector with the unit dyad. Then the $z$-$z$ elements cancel out. The net effect is the removal of the $z$-component of $\mathbf{P}$; i.e., $\mathbf{P}$ will have no component in the direction of propagation, which is the direction of the wavevector $\mathbf{k}$. The equation is now

$$\frac{d}{dt} F - i \mathbf{k} \cdot \mathbf{F} + \frac{i}{2m\epsilon^2} \left( \mathbf{p} \times \frac{\mathbf{k} \mathbf{l}}{c} \right) \mathbf{k} \cdot \mathbf{F}$$

$$= \frac{4\pi ie}{\hbar \epsilon} \left[ f_0(\mathbf{p} - \frac{\mathbf{k}}{c}) - f_0(\mathbf{p} + \frac{\mathbf{k}}{c}) \right] \left[ 1 + \frac{1}{m\epsilon^2} \mathbf{p} \cdot \mathbf{P} \left( \frac{\mathbf{100}}{c^2} \right) \right] F(k, \mathbf{p}, t) d\mathbf{p}$$
This equation can be made more algebraic by removing the partial time derivative through a Laplace transform. This is used instead of a Fourier transform because the lower limit will provide an initial-value form for the equation. The Laplace transform \( G \) is defined by

\[
G(k, \rho, \omega) = \int_{\infty}^{\infty} e^{i\omega t} F(k, \rho, t) \, dt
\]

When the equation just derived is multiplied by \( e^{-i\omega t} \) and integrated over \( t \), the first term is the only one which requires any manipulation.

\[
\int_{\infty}^{\infty} e^{-i\omega t} \frac{dF}{dt} \, dt = F(k, \rho, t) e^{-i\omega t} \bigg|_{-\infty}^{\infty} - \int_{\infty}^{\infty} F(k, \rho, t) \, e^{-i\omega t} \, dt
\]

\* \( -F(k, \rho, 0) + i\omega G(k, \rho, \omega) \)

Then the kinetic equation becomes

\[
i \left[ \omega - \frac{k^2}{m} \left\{ 1 - \frac{1}{2m}\left( \frac{\rho^2 + k^2}{4} \right) \right\} \right] G(k, \rho, \omega)
\]

\[= F(k, \rho, 0) + \frac{4\pi n e^2}{h k^2} \left( \delta \left( \rho - \frac{1}{2} k \right) - \delta \left( \rho + \frac{1}{2} k \right) \right) \left[ 1 + \frac{1}{m^2} \left( \frac{\rho \cdot \rho}{\alpha} \right) \right] G(k, \rho, \omega) \, d\rho
\]

This can be solved by dividing both sides by the coefficient of \( G \), but the same function still appears on the right-hand side. The following symbols will be used:

\[
\Delta f_0(\rho) = f_0(\rho - \frac{1}{2} k) - f_0(\rho + \frac{1}{2} k)
\]

\[
\Delta = \frac{4\pi n e^2}{h k^2} \quad \alpha = \frac{1}{(mc)^2} \quad F_0 = F(k, \rho, 0) \quad M_0 = \left( \begin{array}{c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)
\]

\[
H(\rho) = \omega - \frac{k^2}{m} \left\{ 1 - \frac{1}{2k} \left( \frac{\rho^2 + k^2}{4} \right) \right\}
\]

\[
\Delta(\rho) = \frac{\Delta f_0(\rho)}{H(\rho)} \quad \Xi(\rho) = \frac{F_0}{H(\rho)}
\]
\[ G(p) = \mathcal{E}(p) + A \Delta(p) \int G(p) \, dp + \mathcal{A} A(p) \int \varphi \cdot \bar{P} \cdot M \cdot G(p) \, dp \]

\[ \int G(p) \, dp = \int \mathcal{E}(p) \, dp + A \Delta(p) \int G(p) \, dp + \mathcal{A} A(p) \int \varphi \cdot \bar{P} \cdot M \cdot G(p) \, dp \]

Except for the dummy variable of integration, one of the factors on the right-hand side is the same as the left-hand side. Using the symbols

\[ J_1 = \int \Delta(p) \, dp \quad J_2 = \int \mathcal{E}(p) \, dp \]

this can be written

\[ \int G(p) \, dp = [J_2 + \frac{\mathcal{A} A(p) \int \varphi \cdot \bar{P} \cdot M \cdot G(p) \, dp}{1 - \mathcal{A} A(p)}] \int \varphi \cdot \bar{P} \cdot M \cdot G(p) \, dp \]

This is substituted into the equation at the top of this page to eliminate one of the troublesome integrals. A similar procedure is used to eliminate the other integral problem.

\[ G(p) = \mathcal{E}(p) + \frac{A_2}{1 - \mathcal{A} A(p)} \Delta(p) + \mathcal{A} A(p) \left[ \frac{A_2}{1 - \mathcal{A} A(p)} \int \varphi' \Delta(p) \, dp' + \bar{P}' \right] \int \varphi \cdot \bar{P} \cdot M \cdot G(p) \, dp \]

\[ \int \varphi \cdot \bar{P} \cdot M \cdot G(p) \, dp = \left\{ \int \varphi \cdot \bar{P} \cdot M \cdot \mathcal{E}(p) \, dp + \frac{A_2}{1 - \mathcal{A} A(p)} \int \varphi \cdot \bar{P} \cdot M \cdot \Delta(p) \, dp \right\} \times \]

\[ \left\{ \frac{1}{1 - \mathcal{A} A(p)} \int \varphi' \Delta(p) \, dp' + \bar{P}' \right\}^{-1} \]

\[ G(p) = \mathcal{E}(p) + \frac{A_2}{1 - \mathcal{A} A(p)} \Delta(p) - \mathcal{A} A(p) \left[ \frac{A_2}{1 - \mathcal{A} A(p)} \int \varphi' \Delta(p) \, dp' + \bar{P}' \right] \times \]

\[ \left\{ \frac{1}{1 - \mathcal{A} A(p)} \int \varphi' \Delta(p) \, dp' + \bar{P}' \right\}^{-1} \left\{ \int \varphi \cdot \bar{P} \cdot M \cdot \mathcal{E}(p) \, dp + \frac{A_2}{1 - \mathcal{A} A(p)} \int \varphi \cdot \bar{P} \cdot M \cdot \Delta(p) \, dp \right\} \]
This is the desired equation. It describes the behavior of $G$, which is the Fourier-Laplace transform of the function $\hat{s}$, which described the departure from equilibrium of the system. If the initial departure $F(k,p,o)$ is known, the integral can be evaluated explicitly.

The equation describing departures from equilibrium may have singularities arising from two sources, the numerator or denominator. The former case will arise from particular initial conditions which give rise to transient oscillations damped by free motion of the particles. There is nothing fundamental about these oscillations. The latter case is more interesting since it depends on the system's parameters, being independent of the initial disturbance except in the way in which $k$ may be restricted.

These singularities may give rise to steady, growing, or damped oscillations at natural frequencies which are roots of the dispersion relation obtained by setting the denominator equal to zero. The resulting dispersion relation gives the relation between frequency $\omega$ and the wave vector $k$ in terms of the plasma frequency

$$\omega_p = \left(\frac{4\pi ne^2}{m}\right)^{1/2}$$

where $n$ is the particle density. The plasma frequency is a fundamental quantity found in all plasma descriptions, corresponding to the collective oscillation of a homogeneous plasma set up by motion to screen out a charge distribution.
The dispersion relation arising from the denominator is

\[ 1 \approx \frac{4 \pi e^2}{\hbar} \int \frac{f_0(\vec{p} - \frac{\hbar}{2} \vec{k}) - f_0(\vec{p} + \frac{\hbar}{2} \vec{k})}{\omega - \frac{\hbar^2}{m} \left\{ 1 - \frac{1}{2m c^2} (\vec{p} + \frac{\hbar}{2} \vec{k})^2 \right\} \} \, d\vec{p} \]

\[ = \frac{4 \pi e^2}{\hbar} \int \left\{ \frac{\hbar^2}{m c^2} \left[ 1 - \frac{1}{2m c^2} (\vec{p} + \frac{\hbar}{2} \vec{k})^2 \right]\right\} \frac{f_0(\vec{p}) \, d\vec{p}}{\left[ \omega - \frac{\hbar^2}{m} \left\{ 1 - \frac{1}{2m c^2} (\vec{p} + \frac{\hbar}{2} \vec{k})^2 \right\} \right] \} \left[ 1 - \frac{1}{2m c^2} (\vec{p} + \frac{\hbar}{2} \vec{k})^2 \right] \frac{1}{2m c^2} \frac{\hbar}{2m c^2} \right\} \]

The plasma under study is assumed to be composed of fermions. For small temperatures the Fermi-Dirac equilibrium distribution function can be approximated by

\[ f_0(\vec{p}) \approx \frac{2}{\hbar^3} H(|\vec{p} - \vec{p}_F|) \]

where

\[ H(|\vec{p} - \vec{p}_F|) = \begin{cases} 1 & \text{if } |\vec{p}| \leq |\vec{p}_F| \\ 0 & \text{if } |\vec{p}| > |\vec{p}_F| \end{cases} \]

and \( \vec{p}_F \) is the Fermi velocity.

The function is normalized such that

\[ \eta \cdot \int f_0(\vec{p}) \, d\vec{p} = \frac{2}{\hbar^3} \int_0^{\infty} p^3 \, dp = \frac{2}{3} \frac{p^3}{\hbar^3} \]

The integral will be evaluated in the long-wavelength approximation, i.e. for small \( \vec{k} \). The following relations will then hold:

\[ \frac{\hbar \vec{p}_F}{m \omega} < 1 \]

\[ \frac{\hbar \vec{k}}{2m \omega} < 1 \]

The denominator can then be approximated by a binomial expansion. Since \( \vec{k} \) is small, terms will be neglected which involve products such as

\[ k' \left( \frac{\vec{p}}{\hbar} \right)^r, \quad k^2 \left( \frac{\vec{p}}{\hbar} \right)^r. \]

\[ 1 \approx \frac{16 \pi e^2}{\hbar^3 m \omega} \int \int \left[ 1 - \frac{p^2}{m c^2} \frac{\hbar^2}{2m c^2} \right] \left[ 1 - \frac{2 \hbar \vec{p}_F}{m \omega} \right] \left[ 1 - \frac{\hbar^2 \vec{k}^2}{m \omega^2} \right] \left[ 1 - \frac{\hbar^2 \vec{k}^2}{4m \omega^2} \right] \, d\vec{p} \, d\vec{k} \]
where \( \mu = \cos \theta \).

\[ I = \frac{32\pi e^2}{\hbar^3 m_\omega^3} \int d^3p \left[ \beta \frac{k^2 p_x^2}{m \omega^4} + \frac{k^2 p_y^2}{m \omega^4} + \frac{k^2 p_z^2}{m \omega^4} + \frac{p^2}{3 m^2 c^2} - \frac{3 k^2 p_x^2}{5 m^2 c^2} - \frac{p^2}{2 m^2 c^2} - \frac{3 k^2 p_y^2}{5 m^2 c^2} - \frac{3 k^2 p_z^2}{5 m^2 c^2} \right] \]

\[ = \frac{32\pi e^2}{\hbar^3 m_\omega^3} \left[ \frac{p_x^2}{3} - \frac{3 k^2 p_x^2}{10 m^2 c^2} + \frac{k^2 p_y^2}{7 m^2 c^2} - \frac{3 k^2 p_z^2}{12 m^2 c^2} - \frac{p_x^2}{6 m^2 c^2} - \frac{3 k^2 p_z^2}{12 m^2 c^2} \right] \]

\[ = \left( \frac{\omega}{c} \right)^2 \left[ 1 + \frac{3}{5} \left( \frac{k p x}{m \omega} \right)^2 - \frac{1}{3} \left( \frac{k p y}{m \omega} \right)^2 + \frac{3}{7} \left( \frac{k p y}{m \omega} \right)^2 - \frac{1}{9} \left( \frac{k p z}{m \omega} \right)^2 - \frac{1}{10} \left( \frac{k p x}{m \omega} \right)^2 \right] \]

Since the terms in brackets which involve \( k \) are assumed to be small, \( \omega \) can be approximated by \( \omega_l \) on the right-hand side. The same result can be obtained in the same approximation by use of the quadratic formula.

\[ \omega^2 = \omega_l^2 \left[ 1 + \frac{3}{5} \left( \frac{k p x}{m \omega} \right)^2 + \frac{1}{3} \left( \frac{k p y}{m \omega} \right)^2 + \frac{3}{7} \left( \frac{k p y}{m \omega} \right)^2 - \frac{1}{9} \left( \frac{k p z}{m \omega} \right)^2 - \frac{1}{10} \left( \frac{k p x}{m \omega} \right)^2 \right] \]

The non-relativistic limit is obtained by letting \( c \to \infty \). This agrees with the dispersion relations obtained by other methods (1, p. 332; 2, p. 361).

Another dispersion relation can be obtained from the kinetic equation. This occurs when the inverse matrix becomes infinite, or when the determinant vanishes due to the matrix relations

\[ A \left( A_j A \right) = \det A \cdot I \quad \Rightarrow \quad A = \frac{a d_j A}{\det A} \]

where \( a d_j A \) is the transpose of the cofactor of \( A \). The required condition is then

\[ \det \left\{ \left( \frac{\omega}{c} \right) + \frac{U m c^2}{1} \int d^3p \frac{P \cdot \left( \frac{A p}{\omega} \right)}{P + \frac{A}{1 - \Lambda}} P' \Delta(p) dp \right\} = \frac{\frac{5 \omega \left( \omega - \frac{k p_1}{m} \right) - 3 \left( \omega - \frac{k p_2}{m} \right)}{2 m c^2}}{\omega - \frac{k p_1}{m} \left[ 1 - \frac{1}{2 m c^2} \left( \frac{k p_2}{m} \right)^2 \right]} \]

The second term in brackets makes no contribution since it has only a z-component due to integration over \( \varphi \) and is multiplied by a vector which has no z-component.
The small terms in the denominator have been neglected since there is already a small coefficient in front of the integral. This result will then apply to first-order corrections only. The dyad can be written

\[
\bar{\mathbf{p}} \cdot (L_{0,0}^0) = \bar{\mathbf{p}} \mathbf{\hat{a}} + \bar{\mathbf{p}} \mathbf{a}
\]

\[
\bar{\mathbf{p}} \cdot (L_{0,0}^0) \bar{\mathbf{p}} = \left( \begin{array}{ccc} p_x & p_y & p_z \\ p_y & p_x & 0 \\ p_z & 0 & p_z \end{array} \right) = \mathbf{p}^2 \left( \begin{array}{ccc} \sin^2 \theta \cos^2 \varphi & \sin \theta \sin \phi \cos \varphi & \sin \theta \cos \phi \\ \sin \theta \sin \phi \cos \varphi & \sin^2 \theta \sin^2 \varphi & \sin \theta \cos \phi \\ \sin \theta \cos \phi & \sin \theta \cos \phi & \sin^2 \theta \sin^2 \varphi \end{array} \right)
\]

The off-diagonal components vanish due to the \( \varphi \)-integration.

\[
\mathbf{J}_{xx} = \frac{2}{h\omega^2 \sin \omega} \left[ \int_0^1 p_x^2 dp \int_0^{2\pi} d\theta \left[ \frac{1}{2} \left( \frac{\omega}{\omega_0} \right)^2 \sin^2 \theta \cos^2 \varphi + \frac{2h\omega}{m\omega_0} \sin \theta \cos \varphi \right] + \frac{3(h\omega)^2}{m\omega_0} \sin \theta \cos \varphi \right]
\]

\[
= \frac{1}{2h\omega^2 \sin \omega} \left[ \int_0^1 p_x^2 dp \int_0^{2\pi} d\theta \left[ 1 + \frac{(\omega_0 \omega)^2}{2m \omega_0} \right] \right]
\]

The other integral is performed in a similar manner.

\[
\mathbf{J}_{yy} = \frac{1}{h\omega^2 \sin \omega} \left[ \int_0^1 p_y^2 dp \int_0^{2\pi} d\theta \left[ \frac{1}{2} \left( \frac{\omega}{\omega_0} \right)^2 \sin^2 \theta \cos^2 \varphi + \frac{h\omega}{m\omega_0} \sin \theta \cos \varphi \right] \right]
\]

Solution of the determinant equation yields a degenerate pair of solutions which correspond to

\[
\omega^2 = \omega_p^2 \left( 1 + \frac{2}{7} \frac{(k\omega)^2}{m\omega_0} + \frac{1}{4} \frac{(k\omega)^2}{m\omega_0} \right)
\]
Drummond indicates a procedure by which damping terms can be obtained by assuming the frequency to be complex (3, pp. 19-21). This Landau damping is expressed by a term $e^{\gamma t}$ where

$$\gamma = \frac{\omega}{k^2 \lambda_0^2} \frac{\pi}{8} e^{-\frac{1}{2} k^2 \lambda_0^2}$$

and $\lambda_0$ is the Debye screening length. In the approximation of small $k$ the damping term becomes insignificant. Since the limit of small $k$ is taken above, damping effects are not expected to contribute to the waves. For larger values of $k$, damping becomes predominant.


DEPARTURE FROM EQUILIBRIUM AND DISPERSION RELATIONS
IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

The inclusion of an external magnetic field complicates
the situation somewhat since the Hamiltonian now includes
not only the canonical momenta \( \vec{p} \) but also the magnetic
vector potential in the form of equation II-5.

The terms which eventually are placed on the left-
hand side of the kinetic equation are not too difficult to
determine. One method is to expand \( \tilde{H} \) in terms of \( \vec{p} \) and \( \vec{A} \)
and substitute this into the original kinetic equation.

\[
\frac{\tilde{H}}{2m} = \frac{p^2}{2m} - \frac{e}{2mc} \vec{p} \cdot \vec{B} + \frac{e^2}{8mc} (\vec{\nabla} \times \vec{A})^2 = E
\]

\[
E_1 = \frac{1}{2m} \frac{i}{\hbar} \frac{1}{2\pi i} \int \left[ (\vec{p} + \frac{e}{c} \vec{A})^2 - (\vec{p} - \frac{e}{c} \vec{A})^2 \right] \frac{e^2}{16mc^2} \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] d^4x \frac{e^2}{16mc^2} \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] d^4x
\]

\[
E_2 = \frac{e}{2mc} \frac{i}{\hbar} \frac{1}{\lambda_n} \int \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] \frac{e}{2mc} \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] d^4x \frac{e}{2mc} \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] d^4x
\]

\[
E_3 = \frac{e^2}{4mc} \frac{i}{\hbar} \frac{1}{\lambda_n} \int \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] \frac{e^2}{4mc} \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] d^4x \frac{e^2}{4mc} \left[ (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))^2 \right] d^4x
\]
These terms can then be regrouped into the form

\[ E = -\frac{1}{m} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{c}{\xi} \frac{\partial}{\partial \xi} \cdot \vec{B} \times \frac{\partial}{\partial \eta} \right] \]

The relativistic correction to the kinetic energy was worked out in general in Chapter III. Combining these, the left-hand side of the kinetic equation becomes

\[ \left( \frac{\partial}{\partial t} + \frac{1}{m} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{c}{\xi} \frac{\partial}{\partial \xi} \cdot \vec{B} \times \frac{\partial}{\partial \eta} \right] - \frac{1}{2m^2c^2} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \right) \chi(\xi, \eta, t) \]

Application of the Fourier and Laplace transforms eliminates derivatives with respect to \( \xi \) and \( t \). The use of these transforms is straightforward and leads to

\[ -F(\xi, \eta, \omega) + \left( i \omega + \frac{1}{m} \left[ -i \frac{\partial}{\partial \xi} + \frac{c}{\xi} \frac{\partial}{\partial \eta} \cdot \vec{B} \times \frac{\partial}{\partial \eta} \right] \right) G(\xi, \eta, \omega) \]

The right-hand side involves integrals of the Darwin interaction terms. Some simplification will be gained by the transformation \( \{ \vec{0}, \vec{\omega}, \vec{r}, \vec{\eta} \} \rightarrow \{ \vec{0}, \vec{0}, \vec{J}, \vec{J} \} \). Since the Jacobian of this transformation is one, \( \vec{P} \) can be replaced by \( \vec{J} + \frac{c}{\xi} \vec{B} \times \vec{\eta} \) in the Wigner equation:

\[ \frac{\partial}{\partial t} \chi(\xi, \eta, t) = \frac{i}{\hbar} \left( \frac{1}{2m^2c^2} \right) \left[ \mathcal{H}(\vec{0} + \frac{i}{c} \vec{r} + \vec{J}, \vec{0} + \frac{i}{c} \vec{r} + \vec{J} + \frac{c}{\xi} \vec{B} \times \vec{\eta}, \vec{0} + \frac{c}{\xi} \vec{B} \times \vec{\eta}) \right] \]

In each case the only term of the two-body approximation which does not produce a vanishing integral is \( \chi(\vec{0}, \vec{0}) \mathcal{H}(\vec{0}, \vec{0}) \). The integrals therefore have the form

\[ \frac{\partial}{\partial t} \chi(\vec{0}, \vec{0}, \vec{r}, \vec{\eta}) = \frac{-i e}{2mc^2} \left[ \frac{\vec{J} \cdot (\vec{0} + \frac{i}{c} \vec{r} - \vec{0}) \cdot (\vec{J} + \frac{c}{\xi} \vec{B} \times \vec{\eta})}{\vec{0} + \frac{i}{c} \vec{r} - \vec{0}} \right] \chi(\vec{0}, \vec{0}, \vec{r}, \vec{\eta}) \]
A similar expression applies to the Coulomb terms. This is just the previously derived equation in the last chapter with $\vec{p}$ replaced by $\vec{F}$. By taking advantage of this correspondence the kinetic equation for a plasma subject to an external magnetic field $\vec{B}$ can be written as

$$
(i\omega - \frac{q}{m_c} \vec{F} \times \vec{B} \cdot \frac{\partial}{\partial t} - \frac{e}{m} \vec{F} \cdot \vec{J}) + \frac{e^2 q}{2m^2 c^2} \left[ \vec{J} \times \vec{B} - \frac{1}{4} \left( \frac{\sigma \vec{B} \times \partial}{\partial t} - \vec{E} \right) \right] C(\vec{k}, \vec{p}, \omega) = F(\vec{k}, \vec{p}, \omega) + \frac{4\pi i e^2}{\hbar} \left[ f_0 (\vec{p} - \frac{1}{2} \vec{k}) - f_0 (\vec{p} + \frac{1}{2} \vec{k}) \right] \left[ 1 - \frac{\sigma}{m_0 c} \vec{J} \cdot \vec{F} - \frac{4\pi \delta}{\sigma c} \right] C(\vec{k}, \vec{p}, \omega) d\vec{p}
$$

Since this equation is rather difficult to solve in general, several special cases will be considered.

The Classical Limit

The first special case is the classical limit of $\hbar \to 0$. Equation IV-1 can be made into a first-degree equation by assuming the magnetic field to be small so that terms involving $B^2$ can be neglected. The coordinate system is shown below.

![Fig. 2--Coordinate system with $\vec{F}$ in $x$-$z$ plane](image-url)
The magnetic field will be directed along the z-axis. Without loss of generality $\hbar$ can be chosen to lie in the x-z-plane.

The equilibrium distribution function can be expanded in a Taylor's series. Recalling that $\hbar \to 0$,

$$\frac{1}{\hbar} \left[ f_0(k, (\mathbf{n} - \frac{\mathbf{a}}{2}), k) - f_0(k, (\mathbf{n} + \frac{\mathbf{a}}{2}), k) \right] \to -\mathbf{a} \cdot \frac{\partial}{\partial \mathbf{n}} f_0(\mathbf{n})$$

In this case equation IV-1 can be written

$$(i\omega + \frac{e}{m_c} \mathbf{n} \times \mathbf{E} - \frac{2}{\hbar^2} \frac{\partial}{\partial \mathbf{n}} - \frac{i}{\hbar} \mathbf{n} \cdot \mathbf{B} + \frac{i}{2m_e} \mathbf{B} \cdot \mathbf{A}) G(k, \mathbf{n}, \omega) =$$

$$F(k, \mathbf{n}, \omega) - \frac{\hbar \mathbf{n} \cdot \mathbf{A}}{\hbar^2} \frac{\partial}{\partial \mathbf{n}} \int \left[ 1 - \frac{1}{\hbar^2} \mathbf{B} \cdot \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{A} \right] G(k, \mathbf{n}, \omega) d\mathbf{n}$$

The matrix term appears different in this case since it was defined to have the effect of removing the component of $\mathbf{B}$ parallel to $\mathbf{k}$. Now introduce the cyclotron frequency

$$\omega_c = \frac{eB}{mc}$$

The kinetic equation can be written in the form

$$\left[ i\omega - \frac{\mathbf{n} \cdot \mathbf{A}}{\hbar} (k_n \mathbf{n} + k_L \mathbf{n} \cdot \mathbf{A} \cdot \mathbf{n}) (1 - \frac{1}{2m_e}) - \omega_c \frac{\partial}{\partial \phi} \right] G(k, \mathbf{n}, \omega) =$$

$$F(k, \mathbf{n}, \omega) - \frac{\hbar \mathbf{n} \cdot \mathbf{A}}{\hbar^2} (k_n \frac{\partial^2}{\partial \mathbf{n}^2} + k_L \mathbf{n} \cdot \mathbf{A} \cdot \frac{\partial}{\partial \mathbf{n}} \mathbf{A} \cdot \mathbf{n}) \int \left[ 1 - \frac{1}{\hbar^2} \mathbf{B} \cdot \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{A} \right] G(\mathbf{n}) d\mathbf{n}$$

This can be solved by introducing the integrating factor

$$e^{-\frac{\mathbf{n} \cdot \mathbf{A}}{\hbar} \int [\omega - \frac{\mathbf{n} \cdot \mathbf{A}}{\hbar} \int (1 - \frac{1}{2m_e}) d\phi] d\mathbf{n} = e^{-\frac{\mathbf{n} \cdot \mathbf{A}}{\hbar} \int [\omega - \frac{\mathbf{n} \cdot \mathbf{A}}{\hbar} \int \mathbf{n} \cdot \mathbf{A} \cdot \mathbf{n} \cdot \mathbf{A} \cdot \mathbf{n}] d\mathbf{n}$$

Now multiply the equation by the integrating factor, divide by $-\omega_c$ and integrate with respect to $\phi$ from $-\infty$ to $\infty$. This
will make the lower limit on the left-hand side vanish upon evaluation. Define

$$q(j) \equiv 1 - \frac{1}{j^{m+1}}$$

When both sides are multiplied by the inverse of the integrating factor after integration, the equation for $G$ is

$$G(k, \pi, \omega) = -\int_{-\infty}^{\infty} \frac{F(k, \pi, \omega)}{\omega} e^{i\omega \phi} \left[ q(\phi) - \frac{1}{m} \left\{ k_1 \int_{-\infty}^{\infty} \left[ k_1 \phi + k_1 \phi_2 \right] \right\} \right] d\phi$$

The solution of this equation proceeds in the same manner as in Chapter III. With the definition of

$$\left[ \cdots \right] = \omega(\phi, \phi') - \frac{1}{\omega} \left\{ k_1 \int_{-\infty}^{\infty} \phi_1 \phi_2 \right\} q(\phi)$$

$$B = \frac{4\pi k_z}{\omega k_x^2}$$

the solution for $G$ can be written

$$G(k, \pi, \omega) = -E(\pi) - \frac{B}{1-B_1} \int E(\pi') d\pi' \int_{-\infty}^{\infty} \left( k_1 \phi_2 + k_1 \phi_2 \phi_2 \right) \phi_1(\pi) e^{i\omega \phi - \frac{1}{\omega} \phi^2} d\phi$$

The dispersion relations are found by setting the denominator equal to zero and the inverse matrix infinity.
In the classical case \( f_0(\varphi) \) is the Maxwell-Boltzmann distribution. Since the interaction term is considered small, use the non-relativistic function

\[
f_0(\varphi) = n \left( \frac{m}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{\beta n \tilde{m} \varphi}{2}}
\]

where \( n \) is the particle density and \( \beta = \frac{1}{kT} \) is the reciprocal of the product of Boltzmann's constant and the absolute temperature of the system.

\[
(k_x \partial_x + k_i \cos \varphi \partial_i) n \left( \frac{m}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{\beta n \tilde{m} (k_x \varphi + \tilde{m})}{2}} = -\beta m s_0(\varphi) (k_x \partial_x + k_i \cos \varphi)
\]

With this relation the dispersion relation from the denominator is

\[
1 = \frac{4\pi e^2 \beta m}{\omega c k^2} \int d\varphi \int_{-\infty}^{\infty} f_0(\varphi) (k_x \partial_x + k_i \cos \varphi \partial_i) e^{i \omega t (k_x \psi + \tilde{m} \sin \varphi)} \frac{1}{\omega c} (\cos \varphi \partial_x - \sin \varphi \partial_i) g(\varphi, \psi) d\varphi d\psi
\]

Let \( \gamma = \varphi - \psi \rightarrow d\gamma = d\varphi \)

\[
1 = \frac{4\pi e^2 \beta m}{\omega c k^2} \int f_0(\gamma) e^{i \omega c \tilde{m} \sin \gamma} (k_x \partial_x + \frac{m \omega_c}{\tilde{m} \omega_c} \partial_i) e^{i \tilde{m} \omega \partial_x - \frac{m \omega_c}{\tilde{m} \omega_c} (k_x \psi + k_i \sin \varphi \psi) g(\varphi, \psi)} d\gamma d\psi
\]

The exponentials can eventually be simplified somewhat by expanding them in terms of Bessel functions.

\[
e^{i a \sin \psi} = \sum_{n=-\infty}^{\infty} e^{in\psi} J_n(a)
\]

\[
1 = \frac{4\pi e^2 \beta m}{\omega c k^2} \int f_0(\gamma) e^{i \omega c \tilde{m} \sin \gamma} (k_x \partial_x + \frac{m \omega_c}{\tilde{m} \omega_c} \partial_i) \left( \sum_{n} e^{in\psi} J_n \left( \frac{k_x \psi + \frac{m \omega_c}{\tilde{m} \omega_c}}{\tilde{m} \omega_c} \right) \right) e^{i \tilde{m} \omega \partial_x - \frac{m \omega_c}{\tilde{m} \omega_c} (k_x \psi + k_i \sin \varphi \psi) g(\varphi, \psi)} d\gamma d\psi
\]

\[
1 = \frac{4\pi e^2 \beta m}{\omega c k^2} \sum_{i=0}^{\infty} \frac{1}{i!} \omega_c \tilde{m} \sin \gamma \left( k_x \partial_x + \frac{m \omega_c}{\tilde{m} \omega_c} \partial_i \right) \left( \sum_{n} e^{in\psi} J_n \left( \frac{k_x \psi + \frac{m \omega_c}{\tilde{m} \omega_c}}{\tilde{m} \omega_c} \right) \right) e^{i \tilde{m} \omega \partial_x - \frac{m \omega_c}{\tilde{m} \omega_c} (k_x \psi + k_i \sin \varphi \psi) g(\varphi, \psi)} d\gamma d\psi
\]

\[
1 = \frac{4\pi e^2 \beta m}{k^2} \sum_{n=0}^{\infty} J_n \left( \frac{k_x \psi + \frac{m \omega_c}{\tilde{m} \omega_c}}{\tilde{m} \omega_c} \right) \left( \frac{2 \pi m}{\tilde{m} \omega_c} \right) \left( \frac{\omega}{\omega_c - \omega + \frac{k_i \omega_c}{m}} \right) \left( \frac{\omega_c}{\tilde{m} \omega_c} \right) \left( \frac{\tilde{m} \omega_c}{\omega} \right) J_n(d\gamma d\psi)
\]
The sum is not easy to carry out, but in the long-wavelength approximation of small $\kappa$ the Bessel function can be expanded in second order in $\kappa$.

\[
J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{n!} \left[1 - \frac{x^2}{(n+1)^2} + \cdots\right]
\]

\[
J_n^2\left(\frac{\kappa \omega \tau}{m\omega_c}\right) = \frac{1}{(n+1)^2} \left[\frac{\kappa \omega \tau J_n(x)}{2m\omega_c}\right]^{2n} \left[1 - \frac{\kappa^2 \omega_c^2}{2(n+1)m^2\omega_c^2}\right]
\]

The only contributing terms to the order of approximation assumed are the values for $n = -1, 0, 1$. Using this with the relation

\[
J_n(x) = (-1)^n J_n(x)
\]

the integral becomes

\[
\int \frac{\sqrt{m} e^{\omega t} \rho}{\kappa^2} \int J_n \frac{\omega J_n(x)}{\omega_c} \left[1 - \frac{\kappa^2 \omega_c^2}{2m^2\omega_c^2}\right] \left[1 - \frac{\omega_c^2 - \omega^2}{\omega_c^2 - \omega^2}\right] J_n \frac{\kappa^2 \omega_c^2}{2m^2\omega_c^2} \left[\frac{\kappa^2 \omega_c^2}{2m^2\omega_c^2}\right] J_n \frac{\kappa^2 \omega_c^2}{2m^2\omega_c^2} \left[\frac{\kappa^2 \omega_c^2}{2m^2\omega_c^2}\right] \omega_c^2 \left[\frac{\kappa^2 + \kappa_0^2}{\omega_c^2} + \frac{\omega_c^2}{\omega_c^2}\right] \left[1 - \frac{\kappa^2}{2m^2\omega_c^2}\right]
\]

where the symbol $\Omega = \frac{\omega_c^2 + \omega^2}{\omega_c^2}$ was used. The dispersion relation can now be written as

\[
\omega^2 = \omega_p^2 \left(1 + \frac{\omega_c^2}{\omega^2} \sin^2 x\right) \left(1 - \frac{\omega_c^2}{2m^2\omega_c^2}\right)
\]

The wave dispersion for waves parallel to $\vec{k}$ is found by setting $\kappa_z = 0$.

\[
\omega^2 = \omega_p^2 \left(1 - \frac{\omega_c^2}{2m^2\omega_c^2}\right)
\]
The waves perpendicular to \( \vec{B} \) have the relation

\[
\omega^2 = (\omega_p^2 + \omega_2^2) \left(1 - \frac{5}{2m^2c^4} \right)
\]

There will also be dispersion relations obtained by setting the determinant of the inverse matrix in the kinetic equation equal to zero.

\[
\det \left\{ \begin{pmatrix} 0 & \omega_p^2 \frac{1}{\omega_2^2} \\ \omega_2^2 & 0 \end{pmatrix} + \omega_p^2 \frac{1}{\omega_2^2} \frac{1}{\omega_c^2} \right\} \pi^*(k_0, \omega_2^2 \omega_0^2 + k_0^2 \omega_2^2 \omega_4^2) \approx \pi_*(\tilde{\rho}^2) \times
\]

\[
-\frac{\omega_p^2}{\omega_2^2} \left[ \omega_2^2 (\omega_0^2 + k_0^2 \omega_0^2 + k_0 \omega_2^2 \omega_4^2) \right] \left\{ \left[ \frac{\partial^2}{\partial \rho^2} \left( \frac{\omega_2^2}{\omega_0^2 + k_0^2 \omega_0^2 + k_0 \omega_2^2 \omega_4^2} \right) \right] \left( \tilde{\rho}_0 \cos \theta \cos \phi \right) + \frac{\omega_2^2}{\omega_0^2 + k_0^2 \omega_0^2 + k_0 \omega_2^2 \omega_4^2} \right\} = 0
\]

Nothing new is involved in evaluating the integrals outside of the fact that dyads are formed and care must be taken to keep track of which variables go with which integrations when the integrals are compounded. The results of the integrations are

\[
\det \left\{ \begin{pmatrix} 0 & \omega_p^2 \frac{1}{\omega_2^2} \\ \omega_2^2 & 0 \end{pmatrix} + \omega_p^2 \frac{1}{\omega_2^2} \frac{1}{\omega_c^2} \right\} \pi^*(k_0, \omega_2^2 \omega_0^2 + k_0^2 \omega_2^2 \omega_4^2) \approx \pi_*(\tilde{\rho}^2) \times
\]

\[
-\frac{\omega_p^2}{\omega_2^2} \left[ \omega_2^2 (\omega_0^2 + k_0^2 \omega_0^2 + k_0 \omega_2^2 \omega_4^2) \right] \left\{ \left[ \frac{\partial^2}{\partial \rho^2} \left( \frac{\omega_2^2}{\omega_0^2 + k_0^2 \omega_0^2 + k_0 \omega_2^2 \omega_4^2} \right) \right] \left( \tilde{\rho}_0 \cos \theta \cos \phi \right) + \frac{\omega_2^2}{\omega_0^2 + k_0^2 \omega_0^2 + k_0 \omega_2^2 \omega_4^2} \right\} = 0
\]

This corresponds to three roots obtained by multiplying diagonal elements when the three dyads are added together. In this case the degeneracy of a double root is removed. The first two frequencies will differ by a factor of \( \cos \alpha \). If \( \alpha \neq 0 \) they will not coincide. These frequencies are

\[
\omega_\alpha^2 = \frac{\omega p^2}{k^2 \omega_0^2 + 2k_0^2 \omega_4^2} \left(1 - \frac{7}{2m^2c^4} \right) \left[ k_0^2 + 2k_0^2 \omega_4^2 \frac{\omega_0^4 }{\omega_c^2} \right]
\]

\[
\approx \frac{\omega_p^2 + \omega_2^2 \sin \alpha}{m c^2 \omega_4^2}, \quad \omega_3 = \omega_1 \cos \alpha
\]
These frequencies differ little from the correction terms of the original dispersion frequency for the system.

If $\alpha=0$, these are the only frequencies obtained from the inverse dyad. If $\alpha=0$, there is a non-trivial root from the z-z element of the dyad. It can be written approximately as

$$\omega^2 = \omega_0^2 + \omega_c^2 \sin^2 \chi \approx \frac{\omega_0^2}{c^2} \frac{k_0^2 \sin^2 \chi}{k_*^2} + \frac{1}{k_*^2} \left( 3 \omega_0^2 + \omega_c^2 \right) \sin \chi$$

Care must be exercised with the apparent discontinuity with which these frequencies appear as $\alpha \to 0$.

The Case of $\mathbf{k} = \mathbf{a}$

Another special case which can be handled in explicit form is where $\alpha=0$ and quantum effects are retained. For a good degree of accuracy compared with other terms which will arise, the first two non-vanishing terms of the Taylor expansion of the equilibrium distribution function will be retained.

$$\frac{1}{\hbar} \left[ f_k (\mathbf{k} - \frac{1}{2} \mathbf{a}) \cdot f_k (\mathbf{k} + \frac{1}{2} \mathbf{a}) \right] = -\mathbf{k} \cdot \frac{\partial f_k}{\partial \mathbf{a}} + \frac{1}{2\hbar} \mathbf{k}^2 \frac{\partial^2 f_k}{\partial \mathbf{a}^2} \mathbf{a}$$

If the magnetic field is again weak, then waves which travel parallel to the field satisfy the kinetic equation

$$\left( i\omega + \frac{\mathbf{e}}{m_c} \mathbf{a} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{a}} - \frac{1}{\hbar} \mathbf{k} \cdot \mathbf{a} + \frac{1}{2m_c} \mathbf{k}^2 \left[ \mathbf{a} \cdot \mathbf{a} \mathbf{H} \right] \right) G(k, \mathbf{a}, \omega)$$

$$= F(k, \mathbf{a}, \omega) - \frac{\mathbf{e} k c^2}{\hbar} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{a}} \left[ 1 - \frac{\mathbf{a}^2}{mc^2} \mathbf{a} \cdot \mathbf{a} \right] G(k, \mathbf{a}, \omega)$$
$G$ can be solved for in the same systematic manner as in the previous problems. The only difference will be that the components will be a little different and that

$$g(\tilde{p}) = 1 - \left( \frac{\tilde{p}^2 + \frac{\hbar^2}{4m^2c^2}}{2m^2c^2} \right)$$

If $\Delta \tilde{p}$ is used as a symbol for the two expanded terms of the equilibrium distribution function, the solution is

$$G(\tilde{k}, \tilde{\tau}, \omega) = -\frac{\partial}{\partial \omega} \left[ \frac{\overline{\omega}}{1 - \overline{\omega}} \right] \int_{-\infty}^{\infty} \Delta \tilde{p} e^{i \omega \tau} d\omega$$

The main dispersion relation is found by setting the recurring denominator equal to zero. The Fermi-Dirac function is again approximated by a step function for the integration.

$$1 = \frac{4\pi e^2}{\omega k^3} \int_{-\infty}^{\infty} \Delta \tilde{p} e^{i \omega \tau} \left[ \omega - \frac{\hbar^2}{m^2} g(\tilde{p}, k) \right] d\omega$$

The integration variables may be changed from cylindrical to spherical coordinates to make some of the integrations easier. The first integral becomes

$$I_1 = \frac{16\pi e^2}{k^3} \int_0^1 \frac{r \mu d\mu}{\omega - \frac{\hbar^2}{m^2} g(\tilde{p}, k)} = \frac{\omega k^2}{2\mu^2} \left[ g + \frac{3\hbar^4}{2m^4\omega^4} g^3 + \cdots \right]$$
The second integral can be written

\[ I_2 = \frac{\gamma n e^2}{24} \int \frac{\mu_k}{\omega - \omega_k} \frac{\partial^2 \mathcal{H}}{\partial \omega^2} d\omega d\mu_k \]

This forms the second derivative of a delta-function in the numerator. It is evaluated by two successive integrations by parts. The result is

\[ I_2 = \omega_p^2 \left[ \frac{3}{20} \frac{\hbar^4 k^4}{\mu_k^3} - \frac{\hbar^4 k^4}{4\mu_k^3} \right] \]

These two solutions are then added together to obtain the dispersion relation

\[ \omega^2 = \omega_p^2 \left[ 1 + \frac{2}{5} \left( \frac{\hbar^4 f^4}{\mu_\omega} \right)^2 + \frac{5}{7} \left( \frac{\hbar^4 f^4}{\mu_\omega} \right)^4 - \frac{3}{7} \left( \frac{\hbar^4 f^4}{\mu_\omega} \right)^6 + \frac{3}{35} \frac{\hbar^4 f^4}{\mu_\omega} \right] \]

This result is approximately the same as was obtained in Chapter III for the case of \( \mathcal{H} n 2 \). The slight discrepancy arises somewhere in the second term of the expansion of \( \mathcal{H}_n \) when used to evaluate the integral.

Again, there is another dispersion relation obtained from the inverse dyad. It will have components only along the diagonal with the value one in the z-z position as was found in the corresponding case in Chapter III. The integration is straightforward if the techniques of evaluating the previous integrals have been mastered. The result is

\[ \omega^3 = \frac{\omega_p^3}{\mu_k^3} \left[ \frac{3}{5} \frac{\hbar^4 f^4}{\mu_\omega} + \frac{5}{2} \frac{\hbar^4 k^4}{\mu_\omega} + 4 \frac{\hbar^4 k^4}{\mu_\omega} \right] \]

There are no magnetic effects on the frequencies in this case, which is not surprising.
CHAPTER V

CONCLUSION

There appear to be no essential difficulties in applying the Wigner distribution function to a plasma in the first-order self-consistent approximation of two-body correlations. Equations of departure from equilibrium both with and without an external magnetic field were derived in the form of an initial-value problem. If the form of an initial disturbance $F(\mathbf{r}, \mathbf{p}, \mathbf{c})$ is specified, these equations can be solved to obtain an explicit expression for the non-equilibrium behavior of the plasma. The theory is not complete since no mechanism is employed to cause return to equilibrium, i.e. the equations are time reversible and no $H$-theorem applies.

The dispersion relations agree with accepted results in the appropriate limits. No published dispersion relations were found which included relativistic effects in an explicit form. This is apparently the first attempt to derive such relations from the Darwin viewpoint. Besides the expected dispersion relations, other resonant conditions arose from the inverse tensor obtained from the equation for equilibrium departure. They appear to be low-frequency terms of relativistic origin. In the limit $c \to \infty$ the terms vanish.
Dispersion relations of this form have not been previously reported.

A magnetic field does not alter the frequencies of waves parallel to it, but increases the frequencies of perpendicular waves. Relativistic corrections lower the frequencies. Such an effect is due in part to the more sluggish motion of the particles as their mass increases with velocity.
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