PERIPHERALLY CONTINUOUS FUNCTIONS, GRAPH MAPS AND CONNECTIVITY MAPS

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PERIPHERALLY CONTINUOUS FUNCTIONS, GRAPH MAPS AND CONNECTIVITY MAPS

THESIS

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PREFACE

The purpose of this paper is to investigate some of the more basic properties of peripherally continuous functions, graph maps and connectivity maps. As it turns out, continuous functions are peripherally continuous, as well as connectivity maps. This is pointed out in two later theorems.

Although continuous functions are peripherally continuous and are connectivity maps, the converse is not necessarily true. Each chapter will contain examples of this.

An understanding of the concepts presented in a basic course in topology is essential in following the proofs presented here. The first chapter presents these concepts in a more or less organized form. The second chapter deals with peripherally continuous functions, ending with a few theorems stating conditions under which peripheral continuity will imply continuity.

The third chapter deals with connectivity maps and graph maps. The definition of a connectivity map depends on the notion of a graph map, and so the first part of the chapter is devoted to developing the idea of a graph map, and ends with theorems stating under what conditions a connectivity map will be continuous.

LIST OF ILLUSTRATIONS

CHAPTER I

TOPOLOGICAL CONCEPTS

The proofs in this paper assume a familiarity with a basic course in topology. This chapter will set forth the definitions and theorems, without proofs, which are referred to in later proofs. Since this chapter deals with topological spaces, the following definition is made.

DEFINITION 1.1 Let S be a set. A collection of subsets T of S is called a topology for S if and only if

a) the union of each subcollection of T is a member of T, and

b) the intersection of each finite subcollection of T is a member of T.

The set S, with its topology T, is denoted by (S,T).

DEFINITION 1.2 If (S,T) is a topological space, p is an element of S, U is an element of T such that p is in U, then U is called an open neighborhood of p.

DEFINITION 1.3 A subset V of S is said to be closed if and only if its complement, denoted by S - V, is open.

DEFINITION 1.4 Let U be a subset of a topological space S. Then a point p is called a limit point of U if and only if each open neighborhood of p contains a point of U distinct from p. The set of limit points of U is denoted by U', and this set may or may not share points with U.

The above definition gives rise to a basic theorem dealing with a set and its limit points.

THEOREM 1.1 A subset H of S is closed if and only if it contains all of its limit points.

The next theorem follows almost directly from Definitions 1.1 and 1.3.

THEOREM 1.2 If (S,T) is a topological space, then

a) the intersection of each subcollection of closed sets is a closed set, and

b) the union of each finite subcollection of closed sets is a closed set.

THEOREM 1.3 If (S,T) is a topological space, and U and V are subsets of S with $U \subset V$, then $U' \subset V'$.

DEFINITION 1.5 The closure of a set A, denoted by $\overline{A}$, is the union of A with its set of limit points.

DEFINITION 1.6 The interior of A, denoted by A^{o}, is the largest, in the sense of containment, open set contained in A, and is obtained by taking the union of all open sets contained in A.

DEFINITION 1.7 A space (S,T) is called T_1 if and only if for each point p in S, the set $\{p\}$ is a closed set.

The next definition is equivalent to the one above.

DEFINITION 1.7.1 A space (S,T) is called T_1 if and only if for each pair of distinct points p and q in S, there is an open set which contains one point but does not contain the other.

DEFINITION 1.8 A space (S,T) is called Hausdorff or T_2 if and only if for each pair of distinct points p and q in S, there exist disjoint open neighborhoods U and V containing p and q respectively.

The two previous definitions give rise to a very important theorem, although a simple one.

THEOREM 1.4 If a space (S,T) is T_2, then it is T_1.

DEFINITION 1.9 A space (S,T) is called regular if and only if for each point p in S, and each open neighborhood U of p, there exists an open neighborhood V of p and contained in U such that $\overline{V}$ is also contained in U.

Sequences

DEFINITION 1.10 Let $\{a_n\}$ be a sequence of points in a topological space (S,T). Let p be an element of S. Then the sequence $\{a_n\}$ converges to p if and only if for each open neighborhood U of p, there is an integer $m > 0$ such that if $n > m$, then a_n is in U.

Notice that the above definition implies that if U is any arbitrarily chosen open neighborhood of p, then then there can exist at most a finite number of points of the sequence outside of U.

The notation $\{a_n\} \rightarrow p$ will mean that the sequence $\{a_n\}$ converges to the point p. Under this definition, sequences will behave the same as do those defined on the real number line, i.e. E_1, with the usual topology defined on E_1, in which open sets take the form of unions of open intervals. This is pointed out by the following theorem.

THEOREM 1.5 If (S,T) is a T_1 topological space, then

a) if the sequence $\{a_n\}$ has a limit, then this limit is unique, and

b) every subsequence of $\{a_n\}$ has the same limit.

DEFINITION 1.11 Let (S,T) be a space, and let C be a collection of open neighborhoods of the point p in S.

a) The set C is a local base at p if and only if for every open neighborhood U of p there is an open neighborhood V in C such that V is a subset of U.

b) The space (S,T) is first countable if and only if each point in S has a countable local base.

The notion of a countable local base at a point is a powerful one, but is a little difficult to apply in the proofs of this paper. Therefore the following definition is made.

DEFINITION 1.12 A sequence $\{A_n\}$ of sets is monotone descending if and only if $A_{n+1} \subset A_n$ for every n in N, the positive integers.

On the basis of the previous definition, the following theorem is stated.

THEOREM 1.6 If $\{U_n\}$ is a countable local base at the point p, then there is a monotone decreasing sequence of open neighborhoods $\{V_n\}$ which is a countable local base at p.

This theorem is a very powerful and useful one, and in the course of this paper, whenever it is assumed that there is a countable local base at a point p, it will be understood that it is the monotone descending sequence of open neighborhoods which is guaranteed by the above theorem.

THEOREM 1.7 In a first countable space, the point p is a limit point of the set A if and only if there is a sequence of points in A - $\{p\}$ which converges to p.

Theorem 1.7 will be of invaluable use in a later theorem which will give conditions for a function F to be continuous at a point.

DEFINITION 1.13 A family C of sets is called a cover for a set A if and only if A is a subset of the union of the members of C. The set C is called an open cover if and only if each member of C is an open set. A subfamily D of C is called a subcover for A if and only if D covers A.

Another property of topological spaces, compactness, is defined in terms of open coverings, and a definition and theorems are stated later in this chapter.

Connectedness

DEFINITION 1.14 Two subsets A and B of a space (S,T) are said to be separated if and only if A and B are nonempty, and $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

DEFINITION 1.15 A set U is said to be open in a set V if and only if U is a subset of V, and U is the intersection of V and an open set in (S,T). Likewise, U is said to be closed in V if and only if $U = C \cap V$ where C is closed in (S,T).

THEOREM 1.8 Let A and B be nonempty disjoint subsets of a space (S,T). Then

a) A and B are separated if and only if each is open in their union.

b) A and B are separated if and only if each is closed in their union.

DEFINITION 1.16 A subset H of a space (S,T) is said to be connected if and only if H is not the union of two separated sets.

In order to prove that a set U is connected, the approach in this paper will be to take two arbitrary disjoint subsets of U whose union is U, and show that they cannot be separated.

THEOREM 1.9 A subset Y of a space (S,T) is connected if and only if Y contains no nonempty proper subset which is both open and closed in Y.

THEOREM 1.10 Let C be a connected set, and let D be a set such that $C \subset D \subset \overline{C}$. Then D is connected.

From the above theorem, it is seen that the closure of a connected set is connected. Furthermore, if any number of the limit points of a connected set C are added to C, then the result is still connected.

DEFINITION 1.17 A space S is said to be locally connected if and only if for each element p in S and each open neighborhood U of p there is a connected open neighborhood V of p such that $V \subset U$.

DEFINITION 1.18 A subset C of a set A is a component of A if and only if C is a connected set that is not a subset of another connected set in A.

Notice that if C is a component of a set A, then C is a maximal connected subset of A.

THEOREM 1.11 Let (S,T) be a topological space.

a) Each connected subset of S is contained in a unique component of S.

b) Each component of S is closed.

c) If A and B are different components of S, then A and B are separated.

If the set of all components of a space is denoted by C, then it is immediately seen that C is a partition of the space. If any two components of the space are chosen, these components will be disjoint. Also, since a single point is obviously connected, then it must be contained in some unique component of the space.

Boundary Points

DEFINITION 1.19 If p is a boundary point of a set U, then every open neighborhood of p contains a point of U and a point of the complement of U, both distinct from p.

It is clear from the definition that if p is a limit point of a set U such that p is not in U, then p is also a boundary point of U. Also notice that if p is a boundary point of a set U, then it is also a boundary point of the complement of U. Therefore, from Theorem 1.1, it is seen that an open set cannot contain any of its boundary points, and that a closed set must contain all of its boundary points.

DEFINITION 1.20 If U is a subset of a space S, then the set of boundary points of U will be denoted by $B(U)$.

THEOREM 1.12 If S is a connected space, and U is a subset of S, then $B(U) \neq \emptyset$.

The proof of the above theorem follows from the remarks preceding Definition 1.20 above, and from Theorem 1.9. If U were a subset of S such that $B(U) = \emptyset$, then there certainly cannot be any limit points of U which do not lie in U, and thus U must be closed. But the same thing can be said for the complement of U, and thus the complement of U must also be closed. Thus U and its complement are both open and closed, which is a contradiction to Theorem 1.9, since S is connected. This is a very important observation, since many of the later theorems will be dealing with the boundary points of a set, and it would be useful to know that this set of boundary points is nonempty. Therefore it will always be assumed that the underlying space is connected, so that whenever the set B(U) is considered, where U is a nonempty proper subset of the space, then $B(U) \neq \emptyset$.

Metric Spaces

DEFINITION 1.21 Let S be a set. A mapping $d: S \times S \to R$, where R is the set of reals, is a metric on S if and only if for all x, y, and z in S,

a) $d(x,y) \geq 0$,

b) $d(x,y) = 0$ if and only if $x = y$,

c) $d(x,y) = d(y,x)$,

d) $d(x,y) \leq d(x,z) + d(z,y)$. (triangle inequality)

DEFINITION 1.22 Let p be an element of S and let r be a positive real number. Then the set denoted by $S_r(p) = \{x \mid x \text{ is in } S,\ d(x,p) < r\}$ is called an r-sphere about p, p is called the center and r the radius of $S_r(p)$.

The topology induced by $B = \{S_r(x) \mid x \text{ in } S,\ r \text{ in } R\}$ i.e. the set of all possible unions of elements of B, is called the metric topology for S, and open sets in S take the form of unions of elements of B. Then the space S, with the metric topology, is called a metric space. A metric space is fairly well behaved, and has most of the basic properties that will be required in later proofs.

THEOREM 1.13 Every metric space is a T_2 space.

THEOREM 1.14 Every metric space is first countable.

Continuity

DEFINITION 1.23 Let (X,T) and (Y,F) be topological spaces, and let F be a mapping from X into Y.

a) Let p be an element of X. The mapping F is said to be continuous at p if and only if for each open neighborhood U of F(p) there is an open neighborhood V of p such that $F(V) \subset U$.

b) The mapping F is said to be continuous on X if and only if it is continuous at each point in X.

The following theorems are made use of throughout this paper, for they give necessary and sufficient conditions that a mapping F be continuous on a space.

THEOREM 1.15 Let F be a mapping of a space X into a space Y. Then the following statements are equivalent.

a) The function F is continuous.

b) For each open subset G of Y, $F^{-1}(G)$ is open in X.

c) For each closed subset A of Y, $F^{-1}(A)$ is closed in X.

d) For each subset A of X, $F(\overline{A}) \subset \overline{F(A)}$.

e) For each subset B of Y, $\overline{F^{-1}(B)} \subset F^{-1}(\overline{B})$.

THEOREM 1.16 Let F be a mapping of a first countable space X into a space Y and let p be an element of X. Then F is continuous at p if and only if the sequence $\{F(x_n)\}$ converges to F(p) for each sequence $\{x_n\}$ that converges to p.

Since a metric space is first countable, then both of the preceding theorems apply to metric spaces.

There are many properties of spaces and subsets of spaces which are preserved under continuous mappings in their continuous images. One is given by the next theorem, and others will be presented as needed throughout this paper.

THEOREM 1.17 Let X and Y be topological spaces, and let F be a continuous mapping from X into Y. Let C be a connected subset of X. Then F(C) is connected in Y.

Compactness

DEFINITION 1.24 Let X be a topological space, and let A be a subset of X.

a) The set A is said to be compact if and only if every open cover for A has a finite subcover for A.

b) The set A is said to be countably compact if and only if every infinite subset of A has a limit point in A.

DEFINITION 1.25 A space S is said to be locally compact if and only if for each x in S and each open neighborhood U of x there is an open subneighborhood V of U and containing x such that $\overline{V}$ is compact.

THEOREM 1.18 If A is a compact subset of a space S, then A is countably compact.

THEOREM 1.19 A closed subset of a compact set is compact.

THEOREM 1.20 A closed subset of a countably compact set is countably compact.

There are other definitions which will be presented as they are needed. These deal mainly with peripherally continuous functions, connectivity maps, and graph maps, and thus do not really belong in this chapter, since, as already pointed out, the preceding material is merely intended to provide basic material which is necessary to understand the proofs in later chapters.

CHAPTER II

PERIPHERAL CONTINUITY

For purposes of simplification, it will be assumed that, unless otherwise stated, the topological spaces dealt with in the remainder of this paper will be at least Hausdorff, regular and connected, as defined in Chapter I.

DEFINITION 2.1 Let X and Y be topological spaces. Let p be an element of X. Then a mapping F from X into Y is said to be peripherally continuous at p if and only if for each pair of open neighborhoods U and V of p and F(p) respectively, there exists an open subneighborhood W of U and containing p such that $F(B(W)) \subset V$. The mapping F is said to be peripherally continuous on X if and only if it is so at each point of X.

A natural question arises at this point, whether or not continuous functions are peripherally continuous. Thus the following theorem is stated.

THEOREM 2.1 If F is a continuous mapping of a space X into a space Y, then F is peripherally continuous.

PROOF: Let x be an element of X, and let U and V be open neighborhoods of x and F(x) respectively. Since F is continuous, there is an open neighborhood W of x such that $F(W) \subset V$. Let $W \cap U = D$. Then D is a subset of W, and therefore $F(D) \subset F(W) \subset V$. Since D is also an open neighborhood of x, then there is a subneighborhood H of D and containing x, such that $\overline{H} \subset D$, since all spaces have been assumed regular. Then the boundary of H, B(H), is contained in $\overline{H}$ and thus in D, and therefore is contained in W, so that $F(B(H)) \subset F(W) \subset V$. Then $F(B(H)) \subset V$, and F is peripherally continuous at x, and thus on X.

Although a continuous function is always peripherally continuous, the converse is not necessarily true. The following is an example of a one to one peripherally continuous mapping of the unit interval onto itself, which is not continuous.

EXAMPLE 2.1 Let F be a mapping of the open interval (0,1) onto itself, defined as follows. Let x be an element of the interval. If x is rational, then define $F(x) = x$. If x is irrational, define $F(x) = 1 - x$. It is clear from Figure 2.1 that F is continuous only at the point $\frac{1}{2}$, and is discontinuous everywhere else. Let p be an element of the interval, $p \neq \frac{1}{2}$. Let V be an open neighborhood about p, such that V is properly contained in the interval, in the range. If p is rational, then the irrational points of any open neighborhood about $F^{-1}(p)$ will not be thrown entirely into V. Similarly, if p is irrational, the set of rational points of any open neighborhood about $F^{-1}(p)$ will not be thrown entirely into V. However, given any open neighborhood U of $F^{-1}(p)$, there can always be found an open subneighborhood W of U, containing $F^{-1}(p)$, and which has rational endpoints if p is rational, and irrational endpoints if p is irrational.

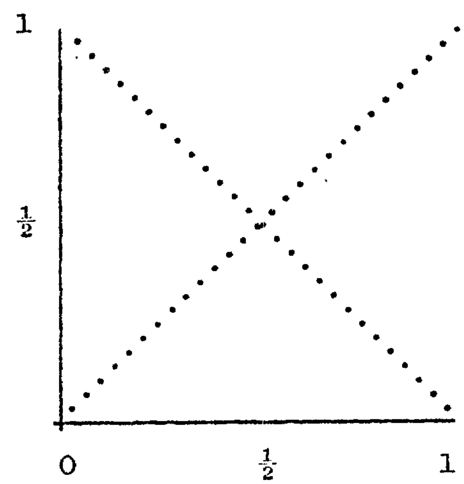

Fig. 2.1--A peripherally continuous mapping which is not continuous.

The endpoints of these open neighborhoods are mapped into V, and since these endpoints are the boundary points of the open neighborhoods, it follows from the definition that the mapping F is peripherally continuous.

The next example is also an example of a one to one peripherally continuous mapping which is not continuous. In this case, however, the mapping is not onto, and the image of a bounded space is unbounded. It is defined in much the same way as was the function in the previous example.

EXAMPLE 2.2 Let F be a mapping of the open interval $(0, \pi/2)$ into the positive reals, defined as follows. Let x be an element of the interval. If x is rational, define $F(x) = \tan(x)$. If x is irrational, then define $F(x) = \cot(x)$. The function is continuous at the point 45° and discontinuous elsewhere. The function is, however, peripherally continuous. The mapping is not onto, for there are infinitely many points in the range which are neither the image of a rational under the tangent function, nor the image of an irrational under the cotangent function. The point $\tan(\pi/2 - 1/5)$ is an example.

The next theorem deals with the concept of a sequence of sets converging to a point. The definition given below reads very much like Definition 1.10 in Chapter I.

DEFINITION 2.2 If $\{A_n\}$ is a sequence of sets, then $\{A_n\}$ converges to the point p if and only if for each neighborhood U of p there is an integer $m > 0$ such that if $t > m$, then $A_t \subset U$.

THEOREM 2.2 A necessary and sufficient condition that a function F which maps a space X into a space Y be peripherally continuous is that for each p in X there exists a countable local base $\{K_n\}$ of p such that the sequence $\{F(B(K_i))\}$ converges to $F(p)$.

PROOF: Suppose that F is peripherally continuous. Let $\{U_n\}$ and $\{V_n\}$ be countable local bases for p and $F(p)$

respectively. Consider U_1 in $\{U_n\}$. There is an open neighborhood C_1 of p such that $C_1 \subset U_1$, and $\overline{C}_1 \subset U_1$. Let $W_1 = U_2 \cap C_1$. Note that W_1 contains p.

There is an open neighborhood C_2 of p such that $C_2 \subset W_1$, and $\overline{C}_2 \subset W_1$. Note that $\overline{C}_2 \subset C_1$, and $\overline{C}_2 \subset U_2$. Let $W_2 = U_3 \cap C_2$.

By continuing in this manner, a sequence $\{C_n\}$ of open neighborhoods of p is constructed which is a countable local base for p. This countable local base has the additional property that if $n < m$, then not only is it true that $C_m \subset C_n$, but also that $\overline{C}_m \subset C_n$. This inclusion is proper, or else C_n would be both open and closed in X, and X would not be connected, by Theorem 1.9. This is a contradiction to the original assumption that all spaces considered in this chapter are connected.

Consider C_1 in $\{C_n\}$. There is an open neighborhood K_1 of p, $K_1 \subset C_1$, such that $F(B(K_1)) \subset V_1$. There is an element C_m of $\{C_n\}$ such that $C_m \subset K_1$. Then there is an open neighborhood K_2 of p such that $K_2 \subset C_{m+1}$ and such that $F(B(K_2)) \subset V_2$. Note that K_2 together with all of its boundary points is properly contained in K_1, since $K_2 \subset C_{m+1}$ implies that $B(K_2) \subset \overline{C_{m+1}} \subset C_m \subset K_1$.

By continuing in this manner, another countable local base $\{K_n\}$ of p is constructed, which, from the method of construction, has the property required in the hypothesis, that the sequence of sets $\{F(B(K_i))\}$ converge to F(p).

Suppose there is a countable local base $\{K_n\}$ for each p in X such that $\{F(B(K_i))\}$ converges to F(p). Let p be an element of X. Let U and V be open neighborhoods of p and F(p) respectively. Then there is an $m > 0$ such that if $t > m$, then $F(B(K_t)) \subset V$. Also, there is a K_j such that $K_j \subset U$. Let $z = \max \{j, m+1\}$. Then $m < z$, and $K_z \subset K_j$, so that $K_z \subset U$, and $F(B(K_z)) \subset V$. Thus F is peripherally continuous at p, and thus on X.

The next theorem follows almost immediately from the previous theorem.

THEOREM 2.3 If $F:X \rightarrow Y$, and F is peripherally continuous, then, if p is in X, there is a sequence $\{x_n\}$ of distinct points converging to p such that the sequence $\{F(x_n)\}$ converges to F(p).

PROOF: In the proof of Theorem 2.2 there was constructed a sequence $\{C_n\}$ of sets about an arbitrary point p which had the property that if C_m and C_n were in $\{C_n\}$, where $m > n$, then $C_m \subset \overline{C}_m \subset C_n$, so that the boundaries of C_m and C_n were disjoint. Since $\{C_n\}$ was a countable local base for p, then, if one point x_m is chosen from the boundary of each C_m, the desired sequence is constructed, with the desired properties.

Theorem 2.3 states that if a function is peripherally continuous, then there must exist at least one convergent sequence such that the limit of the images of the members of the sequence converge to the image of the limit. Peripheral continuity does not, in fact, cannot, guarantee this for every convergent sequence, as continuity does, or else peripherally continuous functions would always be continuous by Theorem 1.16.

The converse of Theorem 2.3 is not necessarily true, obviously, and the next example illustrates this, as well as a one to one peripherally continuous function whose inverse is not peripherally continuous.

EXAMPLE 2.3 Let F be a one to one and onto mapping of the set $A = [0, \pi/2) \cup (\pi/2, \pi)$ onto the interval $(-1, +1)$ defined as follows. If x is an element of $[0, \pi/2)$, define $F(x) = \sin(x)$. If x is an element of the interval $(\pi/2, \pi)$, define $F(x) = \cos(x)$. The function is continuous on each of the intervals, and therefore on A, and thus is also peripherally continuous on A. The inverse of F is not peripherally continuous at the point 0 in the range. For example, let U be the interval $(-1/3, +1/3)$ about 0 in the domain. Then the endpoints, i.e., the boundary points, of any open interval containing F(0) will be mapped partly into, and partly outside of U by the function F^{-1}.

As for the convergent sequence whose image converges to the image of its limit, consider the sequence $\left\{\frac{1}{n}\right\}$, which converges to 0 in the range. The sequence $\left\{F^{-1}(\frac{1}{n})\right\}$ converges to $F^{-1}(0)$, and yet F^{-1} is not peripherally continuous at 0, as already noted.

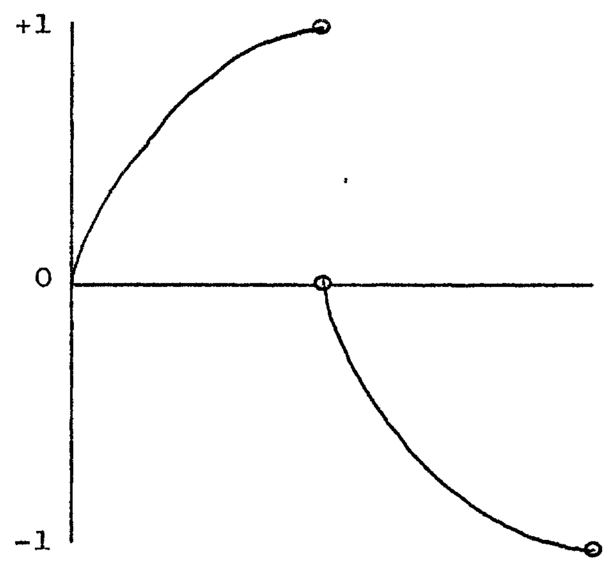

Fig. 2.2--A peripherally continuous mapping whose inverse is not peripherally continuous.

Although the next theorem does not deal specifically with peripherally continuous functions, it is useful in the proofs of several later theorems.

THEOREM 2.4 If C is a connected subset of a connected space X, and H is a subset of X such that $H \cap C \neq \emptyset$, and $H \cap C$ is a proper subset of C, then $B(H) \cap C \neq \emptyset$.

PROOF: Suppose, by way of contradiction, that $B(H) \cap C = \emptyset$. Let $H \cap C = J$, and $C - J = K$. Then $J \cap K = \emptyset$

and $J \cup K = C$. Note that K is a subset of the complement of H. Now J contains no limit points of K, or else these limit points would be boundary points of H lying in C. Likewise, K contains no limit points of J by the same reasoning. Then J and K are separated, and C is not connected. This is a contradiction. Thus $B(H) \cap C \neq \emptyset$.

The next several theorems deal with various properties of peripherally continuous functions.

THEOREM 2.5 If F is a peripherally continuous mapping from a space X into a space Y, and N is a closed subset of Y, then every component of $F^{-1}(N)$ is closed.

PROOF: Denote $F^{-1}(N)$ by A, and let H be a component of A. Suppose, by way of contradiction, that H is not closed. Then there must exist a limit point q of H such that q is not in H. Now q is not in A, or else $H \cup \{q\}$ is a subset of A and contains H, and since by Theorem 1.10 the set $H \cup \{q\}$ is connected, then H would not be a maximal connected subset of A, *i.e.*, a component of A.

Let $F(q) = z$. Since q is not in A, then z is not in N, so that, since N is closed, there must exist an open neighborhood V of z such that $V \cap N = \emptyset$.

If H is degenerate, then H is a closed component of A, and the proof would be concluded. Then suppose H is not degenerate. Let p be an element of H. Then $q \neq p$, so that there is an open neighborhood D of q such that p is not in D. Thus $D \cap H$ is properly contained in H, and is not empty, since q is a limit point of H. Then there must exist an open neighborhood U of q such that $U \subset D$, and such that $F(B(U)) \subset V$. Then, since $V \cap N = \emptyset$, it must be true that $B(U) \cap A = \emptyset$, and thus $B(U) \cap H = \emptyset$. This is a contradiction to Theorem 2.4. Therefore H must be closed.

Notice the similarity between the above theorem and Theorem 1.15-c in Chapter I. A necessary and sufficient condition that a function F be continuous is that the inverse image of every closed set be closed. The above theorem states that a necessary condition that a function F be peripherally continuous is that the components of the inverse image of every closed set be closed.

Notice also the similarity between the following theorem and Theorem 1.15-d, which gave necessary and sufficient conditions that a function F be continuous.

THEOREM 2.6 If F is a peripherally continuous mapping of a space X into a space Y, and N is a connected subset of X, then $F(\overline{N}) \subset \overline{F(N)}$.

PROOF: The proof follows almost immediately from the previous theorem. Denote $F^{-1}(\overline{F(N)})$ by A. Then N is a subset of A, and since N is connected, it must be at least a subset of some component, say H, of A. Then, since H is closed by the previous theorem, the closure of the set N is a subset of H and thus of A, and $F(\overline{N})$ is a subset of $F(A) = \overline{F(N)}$.

THEOREM 2.7 Suppose F is a one to one peripherally continuous mapping of a space X onto a space Y. Suppose that M is a nondegenerate connected subset of X, such that X - M has only a finite number of components. Let x be in the boundary of M such that x is a limit point of X - M. Then F(x) is a boundary point of F(M).

PROOF: Let F(x) = y. It will be shown that for each open neighborhood V of y there is an open neighborhood of x whose boundary maps into V, such that there is a point of the boundary in M and a point in X - M. This will give a point of F(M) and a point of F(X - M) in V, since F is one to one, and this will imply that y is a boundary point of F(M).

It needs to be shown at this point that there is an open neighborhood H of x that does not contain any component of X - M. Let K be a set of points consisting of one point from each component of X - M. Then K must be finite, and for each point t in K there must exist an open neighborhood U_t of x such that t is not in U_t. If one such open neighborhood is constructed for each point in K, then the set of all such open neighborhoods must be finite. Denote their intersection by H. Then H is an open neighborhood of x which is disjoint from the set K. Thus if C is any component of X - M, then H cannot contain C. Since x is a limit point of X - M, then H must contain at least one point of X - M distinct from x. Therefore H must intersect at least one component of X - M and this intersection must be properly contained in the component.

Since M is nondegenerate, there is a point z of M such that $z \neq x$. There is an open neighborhood A of x such that z is not in A. Then $D = A \cap H$ is an open neighborhood of x that does not contain z. There is an open neighborhood W of x such that W is contained in D, and such that $F(B(W)) \subset V$. Then W does not contain z, W is a subset of H, and W clearly has the same properties attributed to H in the preceding paragraph.

Suppose, by way of contradiction, that $B(W) \cap (X - M)$ is empty, or that $B(W) \cap M$ is empty.

Case 1: Suppose that $B(W) \cap (X - M) = \emptyset$. Then there is a component C of X - M such that $W \cap C \neq \emptyset$, and such that $W \cap C$ is properly contained in C. But, since $B(W) \cap (X - M) = \emptyset$, then $B(W) \cap C = \emptyset$, and this is a contradiction to Theorem 2.4.

Case 2: Suppose that $B(W) \cap M = \emptyset$. Then $W \cap M \neq \emptyset$, since x is in the boundary of M, and since z is not in W, then $W \cap M$ is contained properly in M. This is a contradiction to Theorem 2.4.

Thus there must be a point of the boundary of W in M and a point of the boundary in X - M. Since the image of this boundary is contained in V, this is sufficient to show, as stated at the beginning of the proof, that x is a boundary point of F(M).

THEOREM 2.8 If F is a peripherally continuous mapping from a space X onto a space Y, M is a subset of Y, y is an element of the interior M^o of M, x is an element of $F^{-1}(y)$, then x is a limit point of $F^{-1}(M)$.

PROOF: Let U be an open neighborhood of x. Then there is an open subneighborhood V of U, containing x, such that $\overline{V} \subset U$. Then M^o is open about F(x) = y, so that there must exist an open neighborhood J of x and which is contained in V such that $F(B(J)) \subset M^o$.

Let p be an element of B(J). Then p ≠ x, F(p) is in M^o, p is an element of $F^{-1}(M^o)$ and thus is an element of $F^{-1}(M)$. In addition, since J is a subset of V, then p must be an element of $\overline{V}$, and thus of U. Therefore every open neighborhood of x contains a point of $F^{-1}(M)$ different from x, and thus x must be a limit point of $F^{-1}(M)$.

THEOREM 2.9 If X and Y are metric spaces, $\{F_n\}$ is a sequence of functions which approaches a function F uniformly, F and F_n for each n maps X into Y, and each F_n is peripherally continuous, then F is peripherally continuous.

PROOF: Let p be an element of X. Let U and V be open neighborhoods of p and F(p) respectively. Then there is an $e > 0$ such that $S_e(F(p)) \subset V$. Then $e/6 > 0$, and there is an $n > 0$ such that if $m > n$, then $d(F_m(p), F(p)) < e/6$. Now F_m is peripherally continuous, so that there is an open subneighborhood U_1 of U, containing p, such that $F_m(B(U_1)) \subset S_{e/6}(F_m(p))$. Hence, if z is a boundary point of U_1, then $d(F_m(z), F_m(p)) < e/6$. Then, using the triangle inequality for a metric space, the two previous distance inequalities give $d(F_m(z), F(p)) < e/3$. But the sequence is converging uniformly, so that $d(F_m(z), F(z)) < e/6$. This, together with the inequality immediately before, again using the triangle inequality, gives rise to the inequality $d(F(z), F(p)) < e/3 + e/6 = e/2 < e$. Therefore $F(z)$ is contained in the open neighborhood $S_e(F(p))$ which is a subset of V, and thus the image of the boundary of U_1 is contained in V. Thus F is peripherally continuous.

In the above theorem, it is necessary that the convergence of the sequence of functions be uniform. If the sequence does not converge uniformly, it is not necessarily true that F is peripherally continuous. This is pointed out in the following example.

EXAMPLE 2.4 Consider the sequence $\{F_n\}$ of functions defined by $F_m(x) = \tan^m(x)$ over the interval $[0, \pi/4]$. Each function F_m throws the closed interval onto the closed interval $[0,1]$. Each F_m is continuous, and thus is peripherally continuous by Theorem 1.1. Notice, however, that the sequence does converge, but not uniformly, to the function F which throws the closed interval $[0, \pi/4]$ onto the subset $[0,1]$ of the Y axis, and which is defined by $F(x) = 0$ if x is in the interval $[0, \pi/4)$, and $F(x) = 1$ if $x = \pi/4$. This function is clearly not peripherally continuous at the point 1 in the range. Consider an open interval V about 1 which does not include 0, in the range. The boundary points of any open interval containing $\pi/4$ will be thrown into 0, and thus cannot be thrown into V. Thus F is not peripherally continuous.

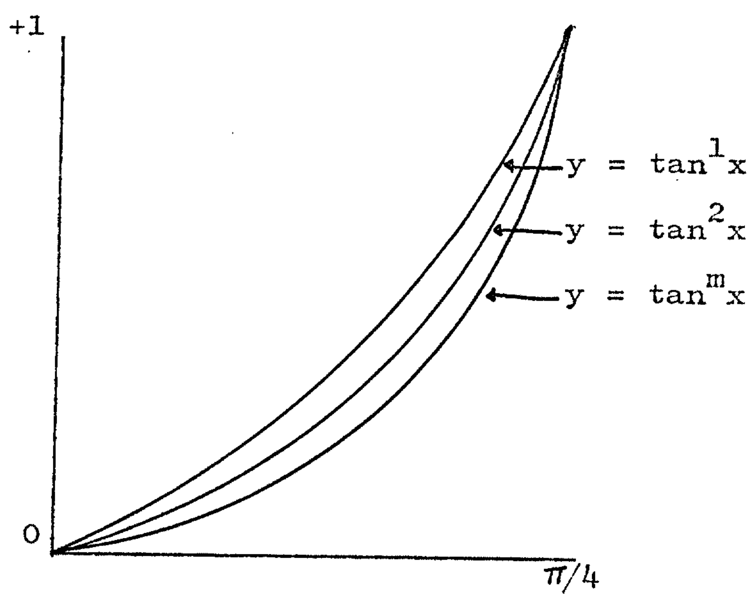

Fig. 2.3--A sequence of peripherally continuous functions which converges, but not uniformly.

The remainder of the theorems in this chapter give conditions under which a peripherally continuous function will be continuous.

THEOREM 2.10 If F is a peripherally continuous mapping of a space X onto a space Y, and if for each closed subset M of Y, $F^{-1}(M)$ has only a finite number of components, then F is continuous.

PROOF: If M is any closed subset of Y, then each component of $F^{-1}(M)$ is closed, by Theorem 2.5. Since there is only a finite number of components of M, then their union is closed. But their union is $F^{-1}(M)$, and thus $F^{-1}(M)$ is closed. Then F is continuous on X, by Theorem 1.15-c.

THEOREM 2.11 If F is a peripherally continuous mapping of a space X into a space Y, and each x in X has the property that for any open neighborhood U of F(x), there is an open subneighborhood V of U, containing F(x), such that X - $F^{-1}(V)$ has only a finite number of components, then F is continuous.

PROOF: Let x be an element of X. Let U be an open neighborhood of F(x). Then there is an open neighborhood W of F(x), contained in U, such that X - $F^{-1}(W)$ has only a finite number of components. Now X - $F^{-1}(W)$ is clearly the same as $F^{-1}(Y - W)$. Since W is open, then Y - W is closed, and X - $F^{-1}(W)$ is the inverse image of a closed set. Then by the same reasoning as that in the proof of the previous theorem, X - $F^{-1}(W)$ is closed. Therefore $F^{-1}(W)$ must be open. Now $F^{-1}(W)$ contains x, and $F(F^{-1}(W))$ = W is a subset of the arbitrarily chosen neighborhood U of x. Then F is continuous on the space X by Definition 1.23.

THEOREM 2.12 Suppose that F is a peripherally continuous mapping of a space X into a space Y. Suppose that if N is a closed subset of Y and x is an element of $X - F^{-1}(N)$, then there is a neighborhood R of x such that R intersects at most a finite number of components of $F^{-1}(N)$. Then F is continuous.

PROOF: Let x be an element of X, and let V be an open neighborhood of F(x). Denote the complement of V by N, which is closed in Y. Then F(x) is not in N, and thus x is not in $F^{-1}(N)$, and x must therefore be in the complement of $F^{-1}(N)$. Then there is an open neighborhood R of x such that R intersects at most a finite number of components of $F^{-1}(N)$. Now from Theorem 2.5, each of these components is closed, since N is closed, and since there is only a finite number of them, their union, denoted by T, is closed and is a subset of $F^{-1}(N)$. Now x is not in $F^{-1}(N)$, and thus is not in this union, and thus is not in the intersection of R and T. Since x is in R, then $R \cap (X - T) \neq \emptyset$, and since $(X - T)$ is open, then $R \cap (X - T) = W$ is open. Since the union of the components of $F^{-1}(N)$ is exactly $F^{-1}(N)$, then the only part of R which lies in $F^{-1}(N)$ must lie in T. Therefore W lies in the complement of $F^{-1}(N)$, and thus F(W) lies in the image of the complement of $F^{-1}(N)$, which is V. Therefore F is continuous at x and thus on X.

CHAPTER III

GRAPH MAPS AND CONNECTIVITY MAPS

The definition of a connectivity map depends on the notion of a graph map. The definition of a graph map is therefore given first, and then the definition of a connectivity map. Properties of the graph map are developed first, and the remainder of the chapter will be devoted to connectivity maps. The definition of a connectivity map will be given again at that time.

It has already been mentioned that all spaces will be assumed at least Hausdorff, regular and connected. These and other special conditions for the spaces dealt with will be mentioned from time to time in the course of the chapter.

Graph Maps

DEFINITION 3.1 Let F be a mapping from a space S into a space T. Define $g(x) = (x, F(x))$ for each x in S. Then g is a mapping from S into SxT, and g is called the graph map of F. Then g(S) is called the graph of F, and g(S) is a subset of SxT.

DEFINITION 3.2 Let F be a mapping from a space S into a space T. Then F is a connectivity map if and only if for every connected subset C of S, g(C) is connected, where g is the associated graph map of F.

Since the statement of the theorems in this chapter will involve both the function F and the function g, then, for purposes of simplification, it will be assumed that whenever the functions F and g are mentioned, they will be exactly as defined in Definitions 3.1 and 3.2.

Therefore in the following theorems and definitions, F will be a mapping of a space S into a space T, and g will be its associated graph map. If F is required to be an onto mapping, it will be so stated.

In order to develop properties of the graph map, a discussion of the topology imposed on the space SxT is needed.

DEFINITION 3.3 If S* and T* are the topologies for the spaces S and T respectively, then B* is a topology for SxT, where B* is the set of all possible unions of subcollections of the set $B = \{UxV \ / \ U \text{ in } S^*, V \text{ in } T^*\}$.

In other words, an open set in the cross product space SxT will be the cross product of an open set in S and an open set in T, or is the union of such cross products. If SxT is E_2, then it has been established many times, in other publications and texts, that B* can be thought of as unions of open circles in the plane.

A very important point needs to be made at this time. If W is any open neighborhood about a point p = (x,y) in SxT, then W, it has been assumed, is the union of a collection of cross products of sets open in S and T.

Then there must exist open neighborhoods U and V of x and y respectively, where U is open in S and V is open in T, such that UxV is open in SxT and is a subset of W. It will be assumed that the open set UxV possesses the same degree of arbitrariness that was required of W. Therefore, whenever an arbitrary open neighborhood of a point p = (x,y) is required, it will be assumed that it takes the form of UxV.

Many times the set g(S) as defined in Definition 3.1 is thought of as a topological space in its own right. The topology for g(S) is made up of sets formed by the intersection of an open set in SxT with g(S). That is, if G is open in g(S), then G is the intersection of an open subset of SxT with g(S). This is what one would expect from Definition 1.15 in Chapter I. Therefore, it follows from the above paragraph that when an arbitrary open set W is required in g(S), it can be represented as $W = (UxV) \cap g(S)$, where U and V are open in S and T respectively.

Notice that the function g is one to one, since each x in S is matched with its image under F. Since F is a function, then two points of S could not be mapped to the same point in SxT.

THEOREM 3.1 The function F is continuous if and only if g is continuous.

PROOF: Suppose that F is continuous. Let $p = (a,b)$ be an element of $g(S)$. Then $b = F(a)$, a is in S, and $g(a) = (a, F(a)) = p$. Denote the relative topology for $g(S)$ by R. Let V be an open neighborhood about p. Then there are neighborhoods W and M of a and $F(a)$ respectively such that $V = (WxM) \cap g(S)$. Now there is an open neighborhood W_1 of p such that $F(W_1) \subset M$, since F is continuous. Let $A = W_1 \cap W$. Then A is an open neighborhood of a, and $F(A) \subset M$. Then $A \subset W$ and $F(A) \subset M$ implies that $AxF(A) \subset WxM$. Then $[AxF(A) \cap g(S)] \subset [WxM \cap g(S)] = V$. Thus, since $g(A) \subset [AxF(A) \cap g(S)]$, then $g(A) \subset V$, and g is continuous.

Suppose that g is continuous. Let t be in S. Then $F(t)$ is in T. Let V be an open neighborhood of $F(t)$. Let U be any open neighborhood of t. Then $(t, F(t))$ is in UxV, and is in $UxV \cap g(S) = A$. Then, since g is continuous, there is an open neighborhood U_1 of t such that $g(U_1)$ is a subset of A. Then $g(U_1) \subset [UxV \cap g(S)]$. Let x be any point in U_1. Then $(x, F(x))$ is in UxV, which implies that $F(x)$ is in V, so that $F(U_1) \subset V$. Thus F is continuous.

The function g possesses a very important property that is used many times in succeeding proofs.

THEOREM 3.2 The function g is an open and closed mapping.

PROOF: It will be shown first that g is an open mapping. Let U be an open subset of S. Now the space T is open in itself, so that $UxT \cap g(S)$ is open in $g(S)$, and clearly consists of all ordered pairs of the form $(x, F(x))$ for each x in U. Since this is the definition of $g(U)$, then $g(U) = U \times T \cap g(S)$, so that the image under g of an open subset of S is open, and thus g is an open mapping.

It will now be shown that g is a closed mapping. Let V be a closed subset of S. Then S - V is open, and $g(S - V)$ is open. But, since g is one to one, then $g(S - V) = g(S) - g(V)$, and since this is the open complement of $g(V)$, then $g(V)$ must be closed. Therefore g is a closed mapping.

The fact that g is one to one, and is an open and closed mapping, regardless of whether g or F is continuous, leads to a very interesting theorem.

THEOREM 3.3 If g is a graph map, then g^{-1} is always continuous.

PROOF: It has already been noted that g is one to one, so that g^{-1} is a function. From Theorem 1.15-b, a function f is continuous if and only if for each open subset U of the image space, the set $f^{-1}(U)$ is open. Since the inverse of g^{-1} is simply g, then g^{-1} must be continuous by virtue of the fact that its inverse maps open sets to open sets.

It turns out that if F is continuous, that is, what some call well behaved, then the associated graph map g is well behaved indeed.

THEOREM 3.4 If F is continuous, then g is a homeomorphism.

PROOF: For g to be a homeomorphism, g and g^{-1} must both be continuous, and g must be one to one and onto.

Since F is continuous, then g is continuous. It was shown in the preceding theorem that g^{-1} is continuous. It has already been shown that g is one to one, and g is certainly onto the space g(S).

Therefore g is a homeomorphism.

The following theorem gives sufficient conditions that F and g be continuous.

THEOREM 3.5 If g(S) is compact, S is T_1 and first countable, then F and g are continuous.

PROOF: Let p be an element of S. Let $\{p_n\}$ be a sequence of points converging to p. Suppose, by way of contradiction, that $\{g(p_n)\}$ does not converge to g(p), i.e., that g is not continuous, by Theorem 1.16.

Since g is one to one, the set $\{g(p_n)\}$ is an infinite subset of g(S). Let U be an open neighborhood of g(p) such that there is an infinite number of elements of $\{g(p_n)\}$ outside of U. This neighborhood must exist since $\{g(p_n)\}$ does not converge to g(p). Denote the complement of U by V. Then there is an infinite number of points of the set within V. Now g(S) is countably compact from Theorem 1.18, and since V is the complement of U, it is closed, and this is also countably compact, by Theorem 1.20. Then the infinite subset Z of $\{g(p_n)\}$ which lies in V must have a limit point q in V, and by Theorem 1.7, there must exist a sequence $\{q_n\}$ of points of Z which converges to q. Since q lies in the complement of U, and p is in U, then $q \neq p$. Notice that the sequence $\{q_n\}$ is a subsequence of $\{g(p_n)\}$. Then, since g is one to one, the sequence $\{q_n\}$ is the image of an infinite subsequence $\{p_n'\}$ of $\{p_n\}$. Or, to put it in another way, the sequence $\{p_n'\}$ is the continuous image under g^{-1} of the convergent sequence $\{q_n\}$. Then, from Theorem 1.16, $\{p_n'\}$ must converge to $g^{-1}(q) = t$ in S. Again, since g is one to one, $p \neq t$ since $g(p) \neq q$. This is a contradiction to Theorem 1.5. Therefore g is continuous, and thus so is F.

The following theorem is similar to Theorem 3.1.

THEOREM 3.6 The function F is peripherally continuous if and only if g is peripherally continuous.

PROOF: Suppose that F is peripherally continuous. Let x be an element of S. Let M and W be open neighborhoods of x and g(x) respectively. Then $W = UxV \cap g(S)$, where U is an open neighborhood of x and V is an open neighborhood of F(x). Denote $U \cap M$ by Z. Then Z is open about x, and $ZxV \subset W$. Now there is an open subneighborhood U_1 of Z and containing x such that $\overline{U_1} \subset Z$. There is an open neighborhood U_2 of x such that $U_2 \subset U_1$, and such that $F(B(U_2)) \subset V$. Since $B(U_2) \subset \overline{U_2} \subset \overline{U_1} \subset Z$, then $B(U_2)xV$ is a subset of ZxV which is a subset of W. Thus, if y is a boundary point of U_2, then $g(y) = (y, F(y))$ is an element of $B(U_2)xV$ and thus of W, so that $g(B(U_2)) \subset W$. Since U_2 is a subset of M, then g is peripherally continuous.

Suppose that g is peripherally continuous. Let x be an element of S. Let U and V be open neighborhoods of x and F(x) respectively. Then UxV is open in SxT, and $g(x) = (x, F(x))$ is an element of UxV. Then there is an open neighborhood U_1 of x, contained in U, such that $g(B(U_1)) \subset UxV$. Let y be a boundary point of U_1. Then $g(y) = (y, F(y))$ is in UxV, which shows that F(y) is in V for each y in $B(U_1)$, so that $F(B(U_1)) \subset V$. Thus F is peripherally continuous.

THEOREM 3.7 If K is a connected subset of g(S), then $g^{-1}(K)$ is connected in S.

PROOF: Since g^{-1} is continuous, then the image of a connected set under g^{-1} is connected by Theorem 1.17.

THEOREM 3.8 If g is continuous, S is locally connected, then g(S) is locally connected.

PROOF: Let p be an element of g(S). Then p is of the form (a, F(a)) for some a in S. Let W be an open neighborhood of p. Then g(a) = p is an element of W. Now g continuous implies that there is an open neighborhood M of a such that g(M) is a subset of W. Also, S locally connected implies that there is an open connected subset C of M, containing a. Then g(C) is a subset of W, and since g is an open continuous mapping, then $g(C)$ is an open connected subset of W which contains p. Thus $g(S)$ is locally connected.

THEOREM 3.9 If g is continuous, S is locally compact, then $g(S)$ is locally compact.

PROOF: Let p be an element of $g(S)$, and let W be an open neighborhood of p. Then $p = (a, F(a))$ for some a in S, and $W = U x V \cap g(S)$ for some open neighborhoods U and V of a and F(a) respectively. Since S is locally compact, there is an open subneighborhood U_1 of U, containing a, such that $\overline{U_1}$ is compact. Notice that $U_1 x V \cap g(S)$ is an open subset of W containing p. Denote $U_1 x V \cap g(S)$ by W_1.

Now suppose that Q is an open covering of $\overline{W_1}$. Then from Q can be constructed the set Q* which consists of the inverse images of the open elements of Q. It is clear that Q* is a covering for $\overline{U_1}$, and since g is continuous, each of the elements of Q* is open, so that Q* is in fact an open covering for $\overline{U_1}$. Then there is a finite subcovering of $\overline{U_1}$, and the open images of each of the elements of this subcovering is a finite subset of Q which covers $\overline{W_1}$. Thus $\overline{W_1}$ is compact, and $g(S)$ is locally compact.

It might be thought that in Theorem 3.8, a sufficient condition that $g(S)$ be locally connected when S is locally connected is that g map connected sets to connected sets. This is shown to be false in a later example. Connectivity maps will now be taken up, beginning with the restatement of the definition of a connectivity map.

Connectivity Maps

DEFINITION 3.4 Let F be a mapping from a space S into a space T. Then F is a connectivity map if and only if for every connected subset C of S, $g(C)$ is connected, where g is the associated graph map of F.

THEOREM 3.10 If F is continuous, then F is a connectivity map.

PROOF: The proof follows immediately from Theorem 3.1 which states that if F is continuous, then g is continuous. If g is continuous, then the image of connected sets under the mapping g are connected, by Theorem 1.17. Thus F is a connectivity map.

The next theorem is used several times in the explanation of examples.

THEOREM 3.11 If M is a connected subset of S, and $g(M)$ is not connected, that is, $g(M) = H \cup K$ where H and K are separated, then any limit points of $g^{-1}(H)$ that are contained in $g^{-1}(K)$ are points of discontinuity of the function F.

PROOF: Let $g^{-1}(H) = A$, and $g^{-1}(K) = B$. Then $A \cup B = M$, and A and B are disjoint, since g^{-1} is one to one. Since

M is connected, A and B cannot be separated, and thus at least one must contain a limit point of the other. Suppose, without loss of generality, that B contains a limit point x of A. Then g(x) is in g(B) = K, and since H and K are separated, there must exist an open neighborhood U of g(x) such that U and H are disjoint. Let V be an arbitrary open neighborhood of x. Then, since x is a limit point of A, then V must contain a point of A, and thus the image g(V) must contain a point of g(A) = H, and cannot be contained in U. Then the image of no open neighborhood of x can be mapped by g entirely into U, and thus x must be a point of discontinuity of g, by Definition 1.23, and therefore is a point of discontinuity of F by Theorem 3.1.

Theorem 3.10 states that if F is continuous, then it is a connectivity map. The converse is not necessarily true.

EXAMPLE 3.1 For a noncontinuous example of a connectivity map, consider the function $F(x) = \sin(\frac{1}{x})$ over the real numbers. The function is bounded, but oscillates near 0. If F(0) is defined to be 0, then the function is continuous everywhere but at 0. Consider any open neighborhood U of 0 in the range, properly contained in the interval $[-1, +1]$. Then any open neighborhood about 0 in the domain is thrown onto the interval $[-1, +1]$, since the function oscillates an infinite number of times between -1 and +1 inclusive in any interval about 0. Thus no open neighborhood about 0 could be thrown into U, since U is a proper subset of $[-1, +1]$.

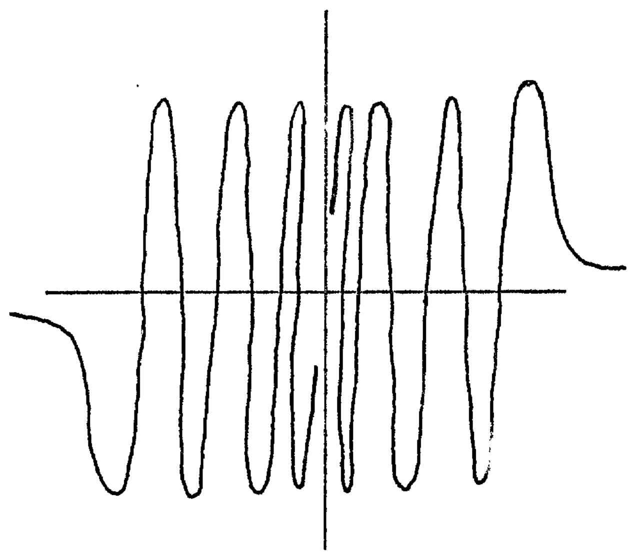

Fig. 3.1--A noncontinuous connectivity map

The function is, however, a connectivity map. Consider any connected subset C of E_1 not containing 0. Since F is continuous everywhere but at 0, then g is discontinuous only at 0, is therefore continuous on C and thus g(C) is connected. Now consider any connected subset U of the domain containing 0. The connected set U must take the form of an interval, and therefore the image g(U) consists of an unbroken line in the second and third quadrants, an isolated point (0,0), and an unbroken line in the first and fourth quadrants. It is evident from Theorem 3.11 that any division of the image g(U) into two disjoint separated sets must result in the point (0,0) being the one and only "division" point, which would clearly be a limit point of the set in which it is not contained. Thus the image of a connected set under the mapping g is connected, and therefore F is a connectivity map. Notice that F is not one to one, a condition which a later theorem points out is sufficient for a connectivity map from a metric space into E_1 to be continuous.

The idea for the next example is found in a paper by O. H. Hamilton, entitled "Fixed Points for Certain Noncontinuous Transformations." The paper is found in the Proceedings of the American Mathematical Society, volume eight, number four, pages 754-755.

EXAMPLE 3.2 As an example in E_2 of a connectivity map which is not continuous, consider the following mapping of a closed 2-cell I of radius 1 and center at the origin into itself.

Let (r,θ) denote a point of I, where $0 < r < 1$. Let $q = e^{\theta - \frac{1}{2}}$. Let C be a topological ray defined by the polar equation $f(\theta) = e^q/(1 + e^q)$ if $\frac{1}{2} \leq \theta < \infty$, and $(\theta, \frac{1}{2})$ if $0 \leq \theta < \frac{1}{2}$. As θ increases without bound, the ratio $e^q/(1 + e^q)$ approaches 1. The ray C is depicted in Figure 3.2. The ray has every point of the boundary G of I as a limit point. Clearly, any circle with radius less than 1 will intersect C in one and only one point. Then define a mapping F from I into itself as follows. If $x = (r,\theta)$ is in I, $0 < r \leq 1$, define $F(x)$ to be the point of C which lies a distance (1 - r) from the origin, i.e., the point of intersection of C with a circle of radius (1 - r). It has already been noted that this point is unique. If x = (0,0), the origin, henceforth denoted simply by O, let F(x) be a particular point O' of the boundary of I.

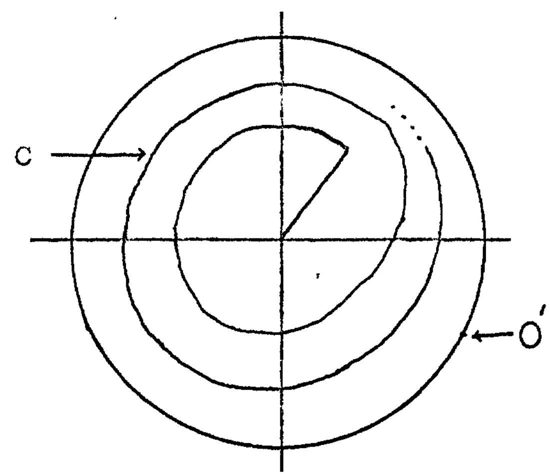

Fig. 3.2--A noncontinuous connectivity map

Notice that all points of I which are the same distance from the origin are thrown by F to the same point of C. It turns out that F is continuous on I everywhere but at the origin, and therefore the image under g of any connected subset of I which does not contain the origin will be connected. Consider, then, a connected subset W of I which contains the origin. Suppose, by way of contradiction, that g(W) is not connected. Then $g(W) = H \cup K$ where H and K are separated. Let $A = g^{-1}(H)$, and let $B = g^{-1}(K)$. Then $A \cup B = W$, and A and B are disjoint. But, since W is connected, then one of A and B must contain a limit point of the other, and, in light of Theorem 3.11

there is only one such limit point, and this point is O. Suppose, without loss of generality, that O is an element of B and thus a limit point of A. Then $g(O)$ is an element of K, and since H and K are separated, there must exist open sets U and V of O and O' respectively, such that $U \times V = D$ is open in $g(I) = C$, D contains $g(O)$, a point of K, but no point of H. Notice that V can be chosen so that $F^{-1}(V)$ is a subset of U, which a consideration of the construction of F will show. Then V cannot contain any points of $F(A)$. But, since O' is a limit point of C, as it lies in the boundary of I, then V must contain a point $(1 - t, \theta)$ of $C - F(A)$. Then the set $P = F^{-1}(1 - t, \theta)$ is a circle with radius t, and since $(1 - t, \theta)$ does not lie in $F(A)$, then $P \cap A = \emptyset$. Then A is divided into two sets A_1 and A_2, where A_1 is outside of the circle P and A_2 is inside the circle P. Clearly, the sets $B \cup A_2$ and A_1 are separated, nonempty, and W is the union of $B \cup A_2$ and A_1, which is a contradiction. Thus $g(W)$ must be connected, and F is a connectivity map.

Theorem 3.8 states that if g is continuous, and S is locally connected, then $g(S)$ is locally connected. Theorem 3.9 states that if g is continuous, S is locally compact, then $g(S)$ is locally compact. It is not sufficient that F be a connectivity map.

EXAMPLE 3.3 In Example 3.1 a noncontinuous connectivity map was defined. The graph map g was not continuous, the space S was locally connected and locally compact, but g(S) was not locally connected, and not locally compact.

To show that g(S) is locally connected, in Example 3.1, let U be an open circle with center at 0 and radius less than 1. If any open subcircle V with center at 0 is considered, it is seen that V contains infinitely many disjoint segments of g(S), so that $V \cap g(S)$ cannot be connected.

To show that g(S) is not locally compact, it will be shown that g(S) is not locally compact at 0. Consider the open circles U and V as defined in the preceding paragraph. Let $A = V \cap g(S)$. It will be shown that $\overline{A} = \overline{V} \cap g(S)$ is not compact, by showing that $\overline{A}$ is not countably compact, and using Theorem 1.18. The closure of V clearly includes a point q in the interval (0,1) on the Y axis such that $q \neq 0$. Since 0 is the only point that is in g(S), then q is not in $\overline{A}$. As seen in the preceding paragraph, V contains an infinite number of disjoint segments of g(S). Therefore $\overline{A}$ must consist of this infinite number of segments, together with their common points with the boundary of V. Construct a sequence of points, one from each segment, which converges to q. Each of these points is in $\overline{A}$, but the point q is not. Then this sequence is an infinite subset of $\overline{A}$ which has no limit point in $\overline{A}$. Therefore $\overline{A}$ is not countably compact, and therefore is not compact. Then $g(S)$ is not locally compact at 0, and hence is not locally compact.

EXAMPLE 3.4 An example of a one to one connectivity map whose inverse is not a connectivity map is the function defined in Example 2.3 in the second chapter, and depicted in Figure 2.2. The function F is continuous, and thus is a connectivity map, but the inverse mapping is not a connectivity map. No nondegenerate connected set containing 0, that is, no interval containing 0 has a connected image under g^{-1}. The image of an interval containing 0 under g^{-1} is split into two disjoint line segments, and thus cannot be connected.

THEOREM 3.12 If F is a connectivity map, then F maps connected sets to connected sets.

PROOF: Let C be a connected subset of S. Now, since F is a connectivity map, then $g(C) = (C \times F(C)) \cap g(S)$ is connected. Suppose, by way of contradiction, that $F(C)$ is not connected. Then there exist disjoint sets A and B such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, and $A \cup B = F(C)$. From Theorem 1.8, A and B are open in $F(C)$, so that there are open sets U and V such that $A = U \cap F(C)$ and such that $B = V \cap F(C)$. Since S and U are open, then $(S \times U) \cap g(C) = W$ is open in $g(C)$. Likewise, $Z = (S \times V) \cap g(C)$ is open in $g(C)$. Now, since A and B are disjoint, then U and V are disjoint, and hence Z and W are disjoint. Then Z and W are both disjoint and open in $g(C)$, and $Z \cup W = g(C)$. Thus $g(C)$ is not connected, by Theorem 1.8, and this is a contradiction. Therefore $F(C)$ must be connected.

THEOREM 3.13 If F is a connectivity map from a T_1 space S onto a space T, p is an element of S, U and V are open neighborhoods of p and $F(p)$ respectively, then every nondegenerate connected subset of S containing p contains a point q in U, unequal to p, such that $F(q)$ is in V.

PROOF: Let C be a nondegenerate connected subset of S containing p. Since F is a connectivity map, then $g(C)$ is connected. Let $W = (U\times V) \cap g(S)$, which is open and contains $g(p)$. Suppose, by way of contradiction, that C does not contain a point q in U and unequal to p such that $F(q)$ is in V, i.e., that $g(C) \cap W = \{g(p)\}$.

Notice that since C is nondegenerate, then C must have at least a countably infinite number of points. If C consisted of only a finite number of points, then, since each point is a closed set in itself, any nonempty proper subset of C would be closed, as well as its complement with respect to C. Then C could not be connected, by Theorem 1.8. Since g is one to one, then $g(C)$ must also have at least a countably infinite number of points.

Let $A = g(C) - g(p)$, and let $B = \{g(p)\}$. Then $A \cap B = \emptyset$, and $A \cup B = g(C)$. Also, A and B are nonempty proper subsets of $g(C)$. Since W is an open neighborhood of $g(p)$ which is disjoint from A, then B contains no limit points of A, and thus $\bar{A} \cap B = \emptyset$. But, since the degenerate set B is closed, being the image under the closed mapping g of a closed set $\{p\}$, then $A \cap \bar{B} = A \cap B = \emptyset$. Therefore A and B are separated, and hence $g(C)$ is not connected, which is a contradiction.

Then there must exist another point q in U such that $q \neq p$, q is contained in C, $g(q)$ is in $g(C)$ and hence that $F(q)$ is in V.

The above theorem merely states that there must exist at least one point q other than p in both C and U such that $F(q)$ is in V. It turns out that there must exist at least a countably infinite number of such points. This is easily seen when it is pointed out that the set B as defined above would still be closed if it were composed of only a finite number of points, and the proof would be almost exactly the same.

The only difference between the following theorem and Theorem 2.5 is the substitution of "connectivity map" for "peripherally continuous function." Notice again the similarity to Theorem 1.15-c, which gives necessary and sufficient conditions that a function be continuous.

THEOREM 3.14 If F is a connectivity map from a T_1 space S into a space T, and C is a closed subset of T, then each component of $F^{-1}(C)$ is closed.

PROOF: Denote $F^{-1}(C)$ by D, and let W be a component of D. Suppose, by way of contradiction, that W is not closed. Then there is a limit point q of W that is not in W. Note that q is therefore not in D, or else the set $W \cup \{q\}$ would be connected by Theorem 1.10, and W would not be a component of D.

Consider the set $E = (\overline{W} x C) \cap g(S)$, which is closed in g(S) since $\overline{W}$ and C are closed. It will now be shown that g(W) = E. Since $F(W) \subset C$, then from the definition of g, $g(W) = (WxC) \cap g(S)$. Now the only difference between W and the closure $\overline{W}$ of W is the set of limit points of W which are not in W. But it was shown above that any limit points which are not in W cannot be in D, and thus the image under F of these limit points cannot be in C, so that $(WxC) \cap g(S) = (\overline{W}xC) \cap g(S)$. Therefore g(W) = E, which was shown above to be closed.

It has already been shown that $W \cup \{q\}$ is connected, and since F is a connectivity map, $g(W \cup \{q\}) = g(W) \cup \{g(q)\}$ is connected. Thus $g(W \cup \{q\})$ is a connected set which is the union of two disjoint, closed, and thus separated sets, which is a contradiction. Hence W must contain all of its limit points, and therefore is closed.

DEFINITION 3.5 A space S is said to be semi-locally connected at a point p if and only if for every open neighborhood U of p there is an open neighborhood V of p such that V is a subset of U, and such that X - V has only a finite number of components. A space S is said to be semi-locally connected if it is so at each of its points.

The next two theorems give sufficient conditions that a connectivity map be continuous.

THEOREM 3.15. If the function F is a connectivity map, S is T_1 and g(S) is semi-locally connected, then F is continuous.

PROOF: Let p be an element of S. Let U be an open neighborhood of g(p). Then, since g(S) is semi-locally connected, there is a subneighborhood V of U, containing g(p), such that the complement W = g(S) - V of V has only a finite number of components. Since V is open in g(S), then W is closed. Then from Theorem 3.3, g^{-1} is continuous, so that the inverse image of each of these components is also connected, and is a subset of $g^{-1}(W)$. It will now be shown that these inverse images are components of $g^{-1}(W)$.

Let C be a component of W. Then $g^{-1}(C)$ is a connected subset of $Q = g^{-1}(W)$. Suppose that $A = g^{-1}(C)$ were not a component of Q. Then there is a connected set B which is a component of Q and which contains A properly. Then g(A) = C is a proper subset of g(B), since g is one to one, and since F is a connectivity map, then g(B) is connected and is a subset of W. This contradicts the original assumption that C was a component of W. Thus A must be a component of Q. Thus Q has only a finite number of components, since each is the inverse image under g of a component of W.

Consider the component C of W. Since W is closed, then the closure of C must be contained in W, and since the closure of a connected set is connected, and C is a component of W, then clearly C must be closed. Therefore the inverse image of C, $g^{-1}(C) = A$, must be closed. If A were not closed, then the set $A \cup \{q\}$, where q is a limit point of A which is not contained in A, would be connected. Then by the same reasoning as in the proof of Theorem 3.14, the set $g(A \cup \{q\})$ would be connected. Then, since $g(A \cup \{q\}) = g(A) \cup \{g(q)\} = C \cup \{g(q)\}$, and if q is not in A then g(q) is not in C, then there would exist a connected set which was the union of two disjoint closed sets, which is a contradiction. Thus each component of Q must be closed, and since there is only a finite number of them, their union, which is Q, is closed.

Then the complement Z of Q must be open, and the image under g must then be open. But, since g is one to one, then $g(Z) = g(S - Q) = g(S) - g(Q) = g(S) - W = V$ which is a subset containing g(p) of the original arbitrarily chosen open neighborhood U of g(p). Then g is continuous, and so is F.

THEOREM 3.16 If F is a one to one real valued connectivity map from a locally connected metric space X onto E_1, then F is continuous.

PROOF: The fact that the image space is E_1 shows that connected sets take the form of intervals. Let p be an element of X. Let $\{U_n\}$ be a countable local base for p. Since X is locally connected, there is an open and connected subneighborhood V_1 of U_1 which contains p. There is a U_m in $\{U_n\}$ such that $U_m \subset V_1$, and there is an open and connected subneighborhood V_2 of U_m which contains p. Then $V_2 \subset V_1$. Continue in this manner, getting a countable local base $\{V_n\}$ of open connected sets about p.

Let W be an open neighborhood about F(p). Then there is an open neighborhood W_1 contained in W and containing F(p) such that $W_1 = (F(p) - e, F(p) + e)$ where e is a positive real number. Denote the inverse image of the right endpoint of W_1 by x, i.e., let $x = F^{-1}(F(p) + e)$. Likewise, let $y = F^{-1}(F(p) - e)$. Since F is one to one, then $x \neq p \neq y$, so that there is a V_m in $\{V_n\}$ such that V_m contains p but not x. Likewise, there is a V_t in $\{V_n\}$ such that V_t contains p but not y. Let V_p denote the smaller of V_m and V_t, in the sense of containment. Then neither x nor y is in V_p. Also, since V_p is connected, then $F(V_p)$ is connected in E_1, and takes the form of an interval, and clearly is a subset of W_1. Then $F(V_p)$ is a subset of W, and thus F is continuous at p and thus on X.

Notice that the previous theorem implies that any one to one connectivity map from a locally connected first countable space into the reals is continuous. Notice also that the function g was never used in the proof. The proof merely made use of the fact that if F is a connectivity map, then F as well as g maps connected sets to connected sets.

This paper is by no means a thorough investigation of peripherally continuous functions, graph maps or connectivity maps. For example, many more similarities could be demonstrated between the properties of peripherally continuous functions, graph maps or connectivity maps and continuous functions.

There are certainly other conditions which could be required of the spaces so that peripherally continuous functions and connectivity maps would be continuous, conditions likely to be simpler than those presented here. A very interesting investigation might be to try to set up Theorem 1.15 in Chapter I for peripherally continuous functions, and for connectivity maps.

A subject not touched upon in this paper is when peripherally continuous functions might also be connectivity maps, and *vice versa*, without necessarily being continuous. This could be a very interesting subject.