A TWENTY-FIVE POINT GEOMETRY

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A TWENTY-FIVE POINT GEOMETRY

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. CURVES AND CONGRUENCE</td>
<td>20</td>
</tr>
<tr>
<td>III. GENERAL CONGRUENCES</td>
<td>30</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>39</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Consider a geometry based on the following postulates where the words "point" and "line" will remain undefined.

Postulate I. There exists at least one line.

Postulate II. A line is a set of exactly five points.

Postulate III. Not all points are points of the same line.

Postulate IV. For any two distinct points A and B there is exactly one line \( \kappa \) such that \( \{A, B\} \subset \kappa \).

Postulate V. For any line \( \kappa \) and point \( Z \notin \kappa \), there is exactly one line \( \kappa_1 \), such that \( Z \in \kappa_1 \) and \( \kappa \cap \kappa_1 = \emptyset \). This will be called the parallel postulate.

Remark: Throughout this paper postulates will be referred to by P(I), P(II), ...

Theorem 1.0. There exist exactly twenty-five points.

Proof. By P(I) there is at least one line \( \kappa_1 \). By P(II) there are exactly five distinct points, denoted by \( A, B, C, D, E \), in \( \kappa_1 \). Hence \( \kappa_1 = \{A, B, C, D, E\} \). Thus, there are at least five points. By P(III) there exists a point F such that \( F \notin \kappa_1 \) and \( F \neq A, F \neq B, F \neq C, F \neq D, \) and \( F \neq E \). Hence there are at least six points. By P(IV) there is a line \( \kappa_2 \supset \{A, F\} \). By P(II) there
exist three distinct points which may be denoted G, H, I without loss of generality and \( \{G, H, I\} \in \mathcal{A}_2 \). By P(IV) \( G \neq A, G \neq B, G \neq C, G \neq D, G \neq E, \) and \( G \neq F \). Likewise for H and I. Hence there are at least nine points. By P(IV) there is a line \( \mathcal{A}_3 \supset \{B, F\} \).

By P(II) there exist three distinct points which may be denoted J, K, L without loss of generality and \( \{J, K, L\} \in \mathcal{A}_3 \). By P(IV) \( J \neq A, J \neq B, J \neq C, J \neq D, J \neq E, J \neq F, J \neq G, J \neq H, \) and \( J \neq I \). Likewise for K and L. Hence there are at least twelve points.

By P(IV) there is a line \( \mathcal{A}_4 \supset \{C, F\} \). By P(II) there exist three distinct points which may be denoted M, N, O without loss of generality and \( \{M, N, O\} \in \mathcal{A}_4 \). By P(IV) \( M \neq A, M \neq B, M \neq C, M \neq D, M \neq E, M \neq F, M \neq G, M \neq H, M \neq I, M \neq J, \) and \( M \neq K \). Likewise for N and O. Hence there are at least fifteen points. By P(IV) there is a line \( \mathcal{A}_5 \supset \{D, F\} \). By P(II) there exist three distinct points which may be denoted P, Q, R without loss of generality and \( \{P, Q, R\} \in \mathcal{A}_5 \). By P(IV) \( P \neq A, P \neq B, P \neq C, P \neq D, P \neq E, P \neq F, \) and \( P \neq G, P \neq H, P \neq I, P \neq J, P \neq K, P \neq L, P \neq M, P \neq N, \) and \( P \neq O \). Likewise for Q and R. Hence there are at least eighteen points. By P(IV) there is a line \( \mathcal{A}_6 \supset \{E, F\} \). By P(II) there exist three distinct points which may be denoted S, T, U without loss of generality and \( \{S, T, U\} \in \mathcal{A}_6 \). By P(IV) \( S \neq A, S \neq B, S \neq C, S \neq D, S \neq E, S \neq F, S \neq G, S \neq H, S \neq I, S \neq J, S \neq K, S \neq L, S \neq M, S \neq N, S \neq O, S \neq P, S \neq Q, \) and \( S \neq R \). Likewise for T and U. Hence there are at least twenty-one points.

Examination of the lines \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \) and \( \mathcal{A}_6 \) shows that the point F has been paired with each of the
determine a line. Consider the point A and the points B, C,
D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, and U. It is noted that
A has been paired with each of B, C, D, E, F, G, H, and I in $\alpha_1$
and $\alpha_2$. Thus J is the first point which has not been paired
with A. Hence, by P(IV), line $\alpha_7 \ni \{A, J\}$ is determined.

Consider the points A and J. It is noted that each of the
points B, C, D, E, F, G, H, I, K, L, and M has been paired with either
A or J. Consider the point M. The point M has not been
paired with either A or J, hence let $\pi \in \alpha_7$. Each of the
points B, C, D, E, F, G, H, I, K, L, M, and N has been paired with
either A, J, or M. Consider P. Examination of $\alpha_1$, $\alpha_2$, $\alpha_3$,
$\alpha_4$, $\alpha_5$, and $\alpha_6$ shows that P has not been paired with A, J, or
M, hence let $P \in \alpha_7$. Consider the points B, C, D, E, F, G, H, I, K,
L, M, N, O, Q, and R. Each of these points has been paired with
either A, J, M, or P. Now consider point S. The point S has
not been paired with either A, J, M, or P, thus let $S \in \alpha_7$. Hence
$\alpha_7 = \{A, J, M, P, S\}$ is determined which satisfies all of the
postulates stated. In like manner $\alpha_8 = \{A, K, N, Q, T\}$ and
$\alpha_9 = \{A, L, O, R, U\}$ exist and satisfy the given postulates. Hence
A is paired with each of the twenty points B, C, D, E, F, G, H, I,
is a line $\alpha_{10}$ determined by B and G. By a pairing, similar
to that used to determine $\alpha_7$, it can be shown that $\alpha_{10} =
\{B, G, M, Q, U\}$. By P(IV) there is a line $\alpha_{11} \ni \{B, H\}$. Since N
and P are the only points not paired with B, then $\{N, P\} \in \alpha_{11}$. 
By P(II) and P(IV) there exists a point which may be denoted \( V \) without loss of generality such that \( V \in \alpha_{11} \). By P(IV)
\[ \{V \neq A, V \neq B, V \neq C, V \neq D, V \neq E, V \neq F, V \neq G, V \neq H, V \neq I, V \neq J, V \neq K, V \neq L, \\
V \neq M, V \neq N, V \neq O, V \neq P, V \neq Q, V \neq R, V \neq S, V \neq T, \text{ and } V \neq U. \]
Hence \( \alpha_{11} = \{B, H, N, P, V\} \), and there are at least twenty-two points.
Consider the point \( A \). Since \( A \) has not been paired with \( V \), by P(IV), there is a line \( \alpha_{12} = \{A, V\} \). By P(II) there exist three distinct points which may be denoted \( W, X, Y \) without loss of generality and \( \{W, X, Y\} \in \alpha_{12} \). By P(IV) \( W \neq A, W \neq B, W \neq C, \\
W \neq D, W \neq E, W \neq F, W \neq G, W \neq H, W \neq I, W \neq J, W \neq K, W \neq L, W \neq M, W \neq N, W \neq O, \\
W \neq P, W \neq Q, W \neq R, W \neq S, W \neq T, W \neq U, \text{ and } W \neq V. \) Likewise for \( X \) and \( Y \). Hence there are at least twenty-five points. Assume there exists a distinct twenty-sixth point \( Z \). By P(IV) there is a line \( \alpha_{13} = \{A, Z\} \). By P(II) \( \alpha_{13} \) would contain exactly five points. Hence \( \alpha_{13} = \{A, Z, Z_1, Z_2, Z_3\} \) where \( Z_1, Z_2, Z_3 \) denote the other three points of the line. By P(IV) there is a line \( \alpha_{14} = \{F, Z\} \). Denote the other points of \( \alpha_{14} \) by \( Z_4, Z_5, Z_6 \) such that \( Z_4 \neq Z_5, Z_5 \neq Z_6, \text{ and } Z_4 \neq Z_6. \) Hence \( \alpha_{14} = \{F, Z, Z_4, Z_5, Z_6\} \). By P(IV) there is a line \( \alpha_{15} = \{G, Z\} \). Denote the other points of \( \alpha_{15} \) by \( Z_7, Z_8, Z_9 \) such that \( Z_7 \neq Z_8, Z_7 \neq Z_9, \text{ and } Z_8 \neq Z_9. \) Hence \( \alpha_{15} = \{G, Z, Z_7, Z_8, Z_9\} \). Consider the lines \( \alpha_1 = \{A, B, C, D, E\} \) and \( \alpha_{14} = \{F, Z, Z_4, Z_5, Z_6\} \). By P(IV) \( Z_4 \neq A, Z_4 \neq B, Z_4 \neq C, Z_4 \neq D, Z_4 \neq E. \) Likewise for \( Z_5 \) and \( Z_6. \) Hence \( \alpha_1 \cap \alpha_{14} = \emptyset. \) Consider lines \( \alpha_1 = \{A, B, C, D, E\} \) and \( \alpha_{15} = \{G, Z, Z_7, Z_8, Z_9\}. \) By P(IV) \( Z_7 \neq A, Z_7 \neq B, \\
Z_7 \neq C, Z_7 \neq D, Z_7 \neq E. \) Likewise for \( Z_8 \) and \( Z_9. \) Hence \( \alpha_1 \cap \alpha_{15} = \emptyset. \)
This contradicts P(V) since \( Z \notin \alpha_1 \) and \( \alpha_{14} \neq \alpha_{15}. \) Hence there
does not exist a twenty-sixth point. Therefore there are exactly twenty-five points.

Theorem 1.1. There exist exactly thirty lines.

Proof. By theorem 1.0 there are exactly twenty-five points. By P(IV) two distinct points determine exactly one line. Consider all the possible pairings of these twenty-five points. Since the twenty-five points are taken two at a time there are exactly 300 possible pairings of these points. Choose an arbitrary set of exactly five points, which is a line by P(II). Denote this set \{Z_1,Z_2,Z_3,Z_4,Z_5\}. There are exactly ten pairings of points in this set. By P(IV) for any two distinct points there is exactly one line. Therefore since any set of five points determines exactly one line the 300 pairings are decreased by a factor of ten. Hence exactly thirty lines.

By P(I) there is a line \( r_1 = \{Z_1,Z_2,Z_3,Z_4,Z_5\} \). By P(III) there is a point \( Z_6 \) and by P(V) there is a line \( r_2 \) such that \( Z_6 \in r_2 \) and \( r_1 \cap r_2 = \emptyset \). Thus \( r_2 = \{Z_6,Z_7,Z_8,Z_9,Z_{10}\} \) and \( Z_1,Z_2,\ldots,Z_{10} \) are distinct points. By P(IV) there is a line \( c_1 \) determined by \( Z_1 \) and \( Z_6 \). Let \( c_1 = \{Z_1,Z_6,Z_{11},Z_{16},Z_{21}\} \). By P(II) these are distinct points. By P(IV), \( Z_{11},Z_{16}, \) and \( Z_{21} \) are distinct from \( Z_2,Z_3,Z_4,Z_5,Z_7,Z_8,Z_9, \) and \( Z_{10} \). By P(V) there is a unique line \( r_3 \) such that \( Z_{11} \in r_3 \) and \( r_2 \cap r_3 = \emptyset \). Let \( r_3 = \{Z_{11},Z_{12},Z_{13},Z_{14},Z_{15}\} \). Suppose \( Z_{12} = Z_1 \). Then for the line \( r_2 \) and point \( Z_1 \) there would be two lines, \( r_1 \) and \( r_3 \), that contain the point \( Z_1 \) and such that \( r_1 \cap r_2 = \emptyset \) and \( r_2 \cap r_3 = \emptyset \).
Therefore $Z_{12} \neq Z_1$. The same argument establishes that $Z_{12}$ is distinct from $Z_2, Z_3, Z_4, Z_5$. In a like manner $Z_{13}, Z_{14}$, and $Z_{15}$ are distinct from $Z_1, Z_2, Z_3, Z_4$, and $Z_5$. Suppose $Z_{12} = Z_{16}$. This would be a contradiction of P(IV). A similar argument establishes that $Z_{12}, Z_{13}, Z_{14}$, and $Z_{15}$ are distinct from both $Z_{16}$ and $Z_{21}$. By the same argument the lines $r_4 = \{Z_{16}, Z_{17}, Z_{18}, Z_{19}, Z_{20}\}$ and $r_5 = \{Z_{21}, Z_{22}, Z_{23}, Z_{24}, Z_{25}\}$ can be shown to exist and each of the points $Z_1, Z_2, \ldots, Z_{25}$ are distinct. For line $c_1$, and the point $Z_2 \notin c_1$, there is a line $c_2$ such that $Z_2 \notin c_2$ and $c_1 \cap c_2 = \emptyset$. Let $c_2 = \{Z_2, Z_7, Z_{12}, Z_{17}, Z_{22}\}$. For line $c_2$ and point $Z_3 \notin c_2$ there is a line $c_3$ such that $Z_3 \notin c_3$ and $c_2 \cap c_3 = \emptyset$. Let $c_3 = \{Z_3, Z_8, Z_{13}, Z_{18}, Z_{23}\}$. Suppose $Z_3 = Z_6$. There would be two lines, $c_1$ and $c_3$, that contained $Z_3$ and such that $c_1 \cap c_2 = \emptyset$ and $c_2 \cap c_3 = \emptyset$. This is a contradiction to P(V). Therefore $Z_3 \neq Z_6$. Likewise $Z_3, Z_8, Z_{13}, Z_{18}$, and $Z_{23}$ are each distinct from $Z_1, Z_6, Z_{11}, Z_{16}, Z_{21}$. In a like manner it can be shown that lines $c_4 = \{Z_4, Z_9, Z_{14}, Z_{19}, Z_{24}\}$ and $c_5 = \{Z_5, Z_{10}, Z_{15}, Z_{20}, Z_{25}\}$ exist and that each of the points $Z_1, Z_2, \ldots, Z_{25}$ is distinct.

These can be represented in the following array:

\[
\begin{array}{cccccc}
Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\
Z_6 & Z_7 & Z_8 & Z_9 & Z_{10} \\
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\
Z_{16} & Z_{17} & Z_{18} & Z_{19} & Z_{20} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25}
\end{array}
\]
where the points in a row or a column will be a line. This will be referred to as the fundamental array.

It is apparent that $Z_1$ has been paired with each of $Z_2, Z_3, Z_4, Z_5, Z_6, Z_{11}, Z_{16}, Z_{21}$. There is a line determined by $Z_1$ and each of $Z_7, Z_8, Z_9, Z_{10}, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{17}, Z_{18}, Z_{19}, Z_{20}, Z_{22}, Z_{23}, Z_{24}, Z_{25}$. To determine the other twenty lines arrays similar to the fundamental array will be constructed consistent with the postulates. Consider point $Z_1$ to be paired with each of $Z_7, Z_8, Z_9, Z_{10}$ to determine lines.

Consider the pairing $Z_1$ and $Z_7$. By P(IV) the choice of the other three points must be one of $Z_{13}, Z_{14}, Z_{15}$ and one of $Z_{18}, Z_{19}, Z_{20}$ and one of $Z_{23}, Z_{24}, Z_{25}$. Suppose $Z_{13}$ is selected, then one of $Z_{19}$ and $Z_{20}$ and one of $Z_{24}$ and $Z_{25}$ must be selected. Therefore, the possibilities for a line determined by $Z_1$ and $Z_7$ are as follows:

\[
\{Z_1, Z_7, Z_{13}, Z_{19}, Z_{25}\} \\
\{Z_1, Z_7, Z_{13}, Z_{20}, Z_{24}\} \\
\{Z_1, Z_7, Z_{14}, Z_{18}, Z_{25}\} \\
\{Z_1, Z_7, Z_{14}, Z_{20}, Z_{23}\} \\
\{Z_1, Z_7, Z_{15}, Z_{18}, Z_{24}\} \\
\{Z_1, Z_7, Z_{15}, Z_{19}, Z_{23}\}
\]

All but one of these must be rejected to satisfy P(IV).

Consider the pairing of $Z_1$ and $Z_8$. By P(IV) the choice of the other three points must be one of $Z_{12}, Z_{14}, Z_{15}$ and one of $Z_{17}, Z_{19}, Z_{20}$ and one of $Z_{22}, Z_{24}, Z_{25}$. By a similar
argument as before the possibilities for a line determined by $Z_1$ and $Z_0$ are as follows:

$\{Z_1, Z_8, Z_{12}, Z_{19}, Z_{25}\}$
$\{Z_1, Z_8, Z_{12}, Z_{20}, Z_{24}\}$
$\{Z_1, Z_8, Z_{14}, Z_{17}, Z_{25}\}$
$\{Z_1, Z_8, Z_{14}, Z_{20}, Z_{23}\}$
$\{Z_1, Z_8, Z_{15}, Z_{17}, Z_{24}\}$
$\{Z_1, Z_8, Z_{15}, Z_{19}, Z_{23}\}$

All but one of these must be rejected to satisfy $P(IV)$.

In a similar manner the possibilities for a line determined by $Z_1$ and $Z_9$ are as follows:

$\{Z_1, Z_9, Z_{12}, Z_{18}, Z_{25}\}$
$\{Z_1, Z_9, Z_{12}, Z_{20}, Z_{23}\}$
$\{Z_1, Z_9, Z_{13}, Z_{17}, Z_{25}\}$
$\{Z_1, Z_9, Z_{13}, Z_{20}, Z_{22}\}$
$\{Z_1, Z_9, Z_{15}, Z_{17}, Z_{23}\}$
$\{Z_1, Z_9, Z_{15}, Z_{18}, Z_{22}\}$

All but one of these must be rejected to satisfy $P(IV)$.

In a similar manner the possibilities for a line determined by $Z_1$ and $Z_{10}$ are as follows:

$\{Z_1, Z_{10}, Z_{12}, Z_{18}, Z_{24}\}$
$\{Z_1, Z_{10}, Z_{12}, Z_{19}, Z_{23}\}$
$\{Z_1, Z_{10}, Z_{13}, Z_{17}, Z_{24}\}$
$\{Z_1, Z_{10}, Z_{13}, Z_{19}, Z_{22}\}$
$\{Z_1, Z_{10}, Z_{14}, Z_{17}, Z_{23}\}$
$\{Z_1, Z_{10}, Z_{14}, Z_{18}, Z_{22}\}$

All but one of these must be rejected to satisfy $P(IV)$.
It can be shown that to satisfy the postulates
the lines \( \{Z_1, Z_7, Z_{13}, Z_{19}, Z_{25}\}, \{Z_1, Z_8, Z_{15}, Z_{17}, Z_{24}\}, \{Z_1, Z_9, Z_{12}, Z_{20}, Z_{23}\}, \text{ and } \{Z_1, Z_{10}, Z_{14}, Z_{18}, Z_{22}\} \) must be selected.
It will be noted that \( Z_1 \) has been paired with every other point.

In a similar manner \( Z_2 \) can be paired with \( Z_6, Z_8, Z_9, \) and \( Z_{10} \) respectively to determine four more lines and \( Z_2 \) will be paired with every other point. If \( Z_3 \) is paired with \( Z_6, Z_7, Z_9, Z_{10}; Z_4 \) with \( Z_6, Z_7, Z_8, Z_{10}; \) and \( Z_5 \) with \( Z_6, Z_7, Z_8, Z_9, \) twelve more lines will be shown and each point has been paired with every other point. For convenience and ease of reference the points \( Z_1, Z_2, \ldots, Z_{25} \) shall be denoted by \( A, B, C, D, E, \ldots, Y \) and the order of the points so arranged that the thirty lines can be exhibited in the following arrays.

\[
\begin{array}{cccccc}
#1 & A & B & C & D & E \\
F & G & H & I & J \\
K & L & M & N & O \\
P & Q & R & S & T \\
U & V & W & X & Y \\
#2 & A & I & L & T & W \\
S & V & E & H & K \\
G & O & R & U & D \\
Y & C & F & N & Q \\
M & P & X & B & J \\
#3 & A & H & O & Q & X \\
N & P & W & E & G \\
V & D & F & M & T \\
J & L & S & U & C \\
R & Y & B & I & K \\
#4 & A & E & D & C & B \\
U & Y & X & W & V \\
P & T & S & R & Q \\
K & O & N & M & L \\
F & J & I & H & G
\end{array}
\]
An examination shows that array #1 is equivalent to array #4, array #2 is equivalent to array #5, and array #6 is equivalent to array #3. Hence arrays #1, #2, and #6 will be chosen as the arrays for study (2, p. 21).

Definition 1.0. Two lines $\alpha_1$ and $\alpha_2$ are said to be parallel if and only if $\alpha_1 \cap \alpha_2 = \emptyset$, denoted by $\alpha_1 \parallel \alpha_2$, conversely $\alpha_2 \parallel \alpha_1$ since $\alpha_2 \cap \alpha_1 = \alpha_1 \cap \alpha_2$.

Definition 1.1. If $\alpha_1$ is any line, then the set of lines consisting of $\alpha_1$ and all lines parallel to $\alpha_1$ is called the parallel class of $\alpha_1$, denoted by $[\alpha_1]$ (1, p. 431).

Theorem 1.2. If $\alpha$ is a line, then there are exactly five lines in $[\alpha]$ (i.e., there are exactly four lines parallel to $\alpha$).

Proof. Let $\alpha$ be a line. By P(III) there is a point $Z \in \alpha$. If $Z_1 \in \alpha$, then by P(IV) there is a line $\alpha_1$ such that $\{Z, Z_1\} \subset \alpha_1$. By P(II) there are distinct points $Z_2, Z_3, Z_4$ such that $\alpha_1 = \{Z, Z_1, Z_2, Z_3, Z_4\}$ and $Z_2 \notin \alpha$, $Z_3 \notin \alpha$, and $Z_4 \notin \alpha$. By P(V) there is a line $\alpha_2$ such that $Z \in \alpha_2$, and $\alpha \cap \alpha_2 = \emptyset$. Hence $\alpha_2 \parallel \alpha$.

Since $Z_2 \notin \alpha$ by P(V) there is a line $\alpha_3$ such that $Z_2 \in \alpha_3$ and $\alpha \cap \alpha_3 = \emptyset$. Hence $\alpha_3 \parallel \alpha$. Since $Z_3 \notin \alpha$ by P(V) there is a line $\alpha_4$ such that $Z_3 \in \alpha_4$ and $\alpha \cap \alpha_4 = \emptyset$. Hence $\alpha_4 \parallel \alpha$. Also by P(V) there
is a line $\alpha_5$ such that $Z_4 \in \alpha_5$ and $\alpha_5 \cap C = \emptyset$, since $Z_4 \not\in C$. Hence $\alpha_5 \parallel \alpha$. Thus there are at least four lines ($\alpha_2, \alpha_3, \alpha_4, \alpha_5$) parallel to $\alpha$.

Assume there exists a line $\alpha_6$ such that $\alpha_6 \parallel \alpha$ and $\alpha_6 \neq \alpha_2, \alpha_6 \neq \alpha_3, \alpha_6 \neq \alpha_4, \alpha_6 \neq \alpha_5$. Since $\alpha_2, \alpha_3, \alpha_4$, and $\alpha_5$ each contain five distinct points and no two of these have a point in common, twenty of the twenty-five points of this geometry are not in either $\alpha_6$ or $\alpha$. Hence there are only five points which could compose $\alpha_6$ and $\alpha$. Since $P(II)$ says a line is a set of exactly five points, $\alpha_6$ would contain each of these five remaining points and $\alpha$ would contain each of the five points. Hence $\alpha$ and $\alpha_6$ would contain exactly the same five points and hence $\alpha = \alpha_6$. Thus there are exactly four lines parallel to $\alpha$. Therefore $[\alpha]$ would contain exactly five lines.

**Theorem 1.3.** If $\alpha_1 \parallel \alpha_2$ and $\alpha_2 \parallel \alpha_3$, then $\alpha_1 \parallel \alpha_3$.

**Proof.** Let $\alpha_1, \alpha_2, \alpha_3$ be three distinct lines such that $\alpha_1 \parallel \alpha_2$ and $\alpha_2 \parallel \alpha_3$. Either $\alpha_1 \cap \alpha_3 = \emptyset$ or $\alpha_1 \cap \alpha_3 \neq \emptyset$. If $\alpha_1 \cap \alpha_3 \neq \emptyset$, then there exists a $Z \in \alpha_1 \cap \alpha_3$. Hence $Z \not\in \alpha_1$ and $Z \not\in \alpha_3$. Since $Z \not\in \alpha_1$ and $\alpha_1 \parallel \alpha_2$ and $Z \not\in \alpha_3$ and $\alpha_3 \parallel \alpha_2$, then by $P(V)$ it follows that $\alpha_1 = \alpha_3$, a contradiction that $\alpha_1$ and $\alpha_3$ are distinct lines. Therefore $Z \not\in \alpha_1 \cap \alpha_3$, hence $\alpha_1 \cap \alpha_3 = \emptyset$, thus $\alpha_1 \parallel \alpha_3$. If $\alpha_1 \cap \alpha_3 = \emptyset$, the theorem follows.

**Theorem 1.4.** If $\alpha_2$ is an element of $[\alpha_1]$, then $[\alpha_1] = [\alpha_2]$ and conversely $[\alpha_2] = [\alpha_1]$ (1, p. 432).
Proof. Let \( L_1, L_2 \) be two distinct lines. If \( L_2 \parallel L_1 \), then \( L_2 \parallel L_1 \) since \( L_1 \) is the class of all lines parallel to \( L_1 \).

If \( L_3 \parallel L_2 \), then \( L_3 \parallel L_2 \). Since \( L_2 \parallel L_1 \), by theorem 1.3, \( L_3 \parallel L_1 \). Hence \( L_3 \parallel L_1 \) and \( [L_2] \subseteq [L_1] \).

Since \( L_1 \parallel L_2 \), \( L_1 \parallel L_2 \). If \( L_4 \) is any line in \( [L_1] \), then \( L_4 \parallel L_1 \) and by theorem 1.3 it follows that \( L_4 \parallel L_2 \). Hence \( L_4 \parallel L_2 \) and \( [L_2] \subseteq [L_1] \).

Therefore \( [L_1] = [L_2] \).

Since \( [L_1] \subseteq [L_2] \) and \( [L_2] \subseteq [L_1] \), it follows that \( [L_2] = [L_1] \).

Theorem 1.5. Two parallel classes, \( [L_1] \) and \( [L_2] \), are equivalent or have no elements in common (1, p. 432).

Proof. Let \( [L_1] \) and \( [L_2] \) be two parallel classes of lines. Consider \( [L_1] \cap [L_2] \). Either \( [L_1] \cap [L_2] = \emptyset \) or \( [L_1] \cap [L_2] \neq \emptyset \).

If \( [L_1] \cap [L_2] = \emptyset \), the theorem follows.

If \( [L_1] \cap [L_2] \neq \emptyset \), then there exists a line \( L \in ([L_1] \cap [L_2]) \) which implies \( L \in [L_1] \) and \( L \in [L_2] \). Since \( L \in [L_1] \), \( L \parallel L_1 \). Also, \( L \in [L_2] \) implies \( L \parallel L_2 \). Since \( L \parallel L_1 \) and \( L \parallel L_2 \), by theorem 1.3, \( L_2 \parallel L_1 \). Hence, \( L_2 \in [L_1] \) and, by theorem 1.4, \( [L_1] = [L_2] \).

Theorem 1.6. Each point is a point of exactly six lines.

Proof. Let \( Z \) be a point. By P(I) and P(III) there is a line \( L \in [L_1] \) of this geometry such that \( Z \in L \). Let the points of \( L \) be designated by \( \{Z_1, Z_2, Z_3, Z_4, Z_5\} \). By P(IV) there is
exactly one line $\alpha_1 \{Z, Z_1\}$, exactly one line $\alpha_2 \{Z, Z_2\}$,
exactly one line $\alpha_3 \{Z, Z_3\}$, exactly one line $\alpha_4 \{Z, Z_4\}$, and
exactly one line $\alpha_5 \{Z, Z_5\}$. By $P(V)$ there is exactly one
line $\alpha_6$ such that $Z \notin \alpha_6$ and $\alpha \cap \alpha_6 = \emptyset$. Hence, there are at
least six lines containing $Z$. Since $\alpha_1 \cap \alpha_2 \cap \alpha_3 \cap \alpha_4 \cap \alpha_5 \cap \alpha_6 = \{Z\}$,
the lines $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ contain exactly twenty-five distinct points. Hence all the points of the geometry
are in these lines. Therefore a point is contained in
exactly six lines.

**Corollary 1.60.** For each point $Z$ there exists at least
one line $\alpha$ such that $Z \notin \alpha$.

**Proof.** Let $Z$ be a point in this geometry. By theorem
1.6, $Z$ is a point of exactly six lines. By theorem 1.1 there
are thirty lines in this geometry. Hence there is at least
one line which does not contain $Z$.

**Definition 1.2.** If $\alpha_1$ and $\alpha_2$ are two distinct lines,
then the statement that $\alpha_1$ intersects $\alpha_2$ means $\alpha_1 \cap \alpha_2$ con-
tains exactly one element.

**Theorem 1.7.** If $\alpha_1$ and $\alpha_2$ are any two lines then one
and only one of the following holds:

1) $\alpha_1$ is parallel to $\alpha_2$.

2) $\alpha_1$ intersects $\alpha_2$.

3) $\alpha_1 = \alpha_2$.

**Proof.** Let $\alpha_1$ and $\alpha_2$ be any two lines. Either
$\alpha_1 \cap \alpha_2 = \emptyset$ or $\alpha_1 \cap \alpha_2 \neq \emptyset$. 
If \( \alpha_1 \cap \alpha_2 = \emptyset \), then \( \alpha_1 \parallel \alpha_2 \). Since \( \alpha_1 \cap \alpha_2 = \emptyset \), \( \alpha_1 \) does not intersect \( \alpha_2 \). Since \( \alpha_1 \cap \alpha_2 = \emptyset \), there are no elements in \( \alpha_1 \cap \alpha_2 \), thus \( \alpha_1 \neq \alpha_2 \). Therefore case one holds.

If \( \alpha_1 \cap \alpha_2 \neq \emptyset \), then either \( \alpha_1 \cup \alpha_2 \) contains exactly one element or \( \alpha_1 \cap \alpha_2 \) contains at least two elements. In either case \( \alpha_1 \) is not parallel to \( \alpha_2 \).

If \( \alpha_1 \cap \alpha_2 \) contains exactly one element, then \( \alpha_1 \) intersects \( \alpha_2 \). Since \( \alpha_1 \cap \alpha_2 \) has only one element by P(IV) it follows that \( \alpha_1 \neq \alpha_2 \). Therefore case two holds.

If \( \alpha_1 \cap \alpha_2 \) contains at least two elements then, by P(IV), \( \alpha_1 = \alpha_2 \). Since \( \alpha_1 \cap \alpha_2 \) does not contain exactly one element, \( \alpha_1 \) does not intersect \( \alpha_2 \). Therefore case three holds.

Definition 1.3. A segment, denoted \((Z_1, Z_2)\), is a point pair.

Definition 1.4. The measure of a segment \((Z_1, Z_2)\), denoted \(m(Z_1, Z_2)\), is the least number of steps along the line from one point to the other, where the first letter of a row or a column is considered as following the last letter of the row or column respectively.

Definition 1.5. Two point pairs or segments \((Z_1, Z_2)\) and \((Z_3, Z_4)\) are congruent, denoted \(Z_1Z_2 \cong Z_3Z_4\), if both \{\(Z_1, Z_2\)\} and \{\(Z_3, Z_4\)\} are subsets of row lines or column lines, but not both, and if \(m(Z_1, Z_2) = m(Z_3, Z_4)\). "Congruent" will remain undefined throughout the geometry.

Definition 1.6. A line \( \alpha_1 \) is perpendicular to a line \( \alpha_2 \), denoted \( \alpha_1 \perp \alpha_2 \), if there exist two distinct points \( Z' \) and
Z'' of \( \kappa_2 \) such that for each \( Z_i \in \kappa_1 \), \( i = 1, 2, 3, 4, 5 \), \( Z_i Z'' \) is an equivalence.

Consider the arrays of the geometry. According to definition 1.4, the measure of \((A, B)\) is one row step, denoted \( m(A, B) = l_r \). Likewise, the measure of \((A, F)\) is one column step, denoted \( m(A, F) = l_c \). It is noted here that \( m(A, B) \) is not equal to \( m(A, F) \), denoted \( m(A, B) \neq m(A, F) \).

Recalling that the arrays were set up in such a way that the first letter of a row or column is considered to follow the last, then \( m(S, K) = l_r \) and \( m(H, P) = l_c \).

Theorem 1.6 states that each point is a point of exactly six lines. Recall that each array was obtained in such a way that a point occurs exactly one time in each array, and a line is a set of exactly five points which lie in either a row or a column of an array. Let \( Z \) be any point of the geometry. Thus, \( Z \) is a point of the fundamental array. Examination of this array shows that \( Z \) is a point of exactly one row line and exactly one column line. Now \( Z \) is also a point of each of the other two arrays. Similar examination shows that \( Z \) is a point of exactly one row line in each array and that \( Z \) is a point of exactly one column line in each array. From this examination it follows that \( Z \) is a point of exactly two lines in each array. Hence, \( Z \) is a point of exactly three row lines, and \( Z \) is a point of exactly three column lines.
It can be verified that in a given array two distinct lines are either perpendicular to each other or are parallel to each other. In a given array the only possibilities for two distinct lines are as follows:

1) two row lines,
2) two column lines,
3) one row line and one column line.

If two distinct row lines are chosen from the same array, say \( \alpha_1 \) and \( \alpha_2 \), then \( \alpha_1 \cap \alpha_2 = \emptyset \) and \( \alpha_1 \parallel \alpha_2 \). There do not exist points \( Z', Z'' \in \alpha_2 \) such that for each \( Z_i \in \alpha_1 \), \( i = 1, 2, 3, 4, 5 \), \( Z_iZ' \neq Z_iZ'' \), hence \( \alpha_1 \) is not perpendicular to \( \alpha_2 \). In a similar manner it can be verified that \( \alpha_2 \) is not perpendicular to \( \alpha_1 \). Therefore \( \alpha_1 \parallel \alpha_2 \), if \( \alpha_1 \) and \( \alpha_2 \) are distinct row lines in the same array.

In like manner, if \( \alpha_1 \) and \( \alpha_2 \) are two distinct column lines in the same array, it can be verified that \( \alpha_1 \parallel \alpha_2 \), that \( \alpha_1 \) is not perpendicular to \( \alpha_2 \), and that \( \alpha_2 \) is not perpendicular to \( \alpha_1 \). Therefore two distinct column lines in the same array are parallel.

If a row line, \( \alpha_1 \), and a column line, \( \alpha_2 \), are chosen then \( \alpha_1 \cap \alpha_2 \neq \emptyset \) and hence, \( \alpha_1 \) is not parallel to \( \alpha_2 \). It can be verified that there exist points \( Z' \) and \( Z'' \) in \( \alpha_2 \) such that for each \( Z_i \in \alpha_1 \), \( i = 1, 2, 3, 4, 5 \), \( Z_iZ' = Z_iZ'' \). Hence, \( \alpha_1 \parallel \alpha_2 \). Likewise there exist two points \( Z_6 \) and \( Z_7 \) of \( \alpha_1 \) such that for each \( Z_j \in \alpha_2 \), \( j = 1, 2, 3, 4, 5 \), \( Z_jZ_6 = Z_jZ_7 \). Hence \( \alpha_2 \parallel \alpha_1 \).
Theorem 1.9. For a given point, Z, and a given line \( \ell \), there is a unique line containing Z which is perpendicular to \( \ell \).

Proof. Let \( \ell_1 \) be a line and Z a point. If \( Z \in \ell_1 \), then there exists a line \( \ell_2, \ell_2 \neq \ell_1 \), containing Z and in the same array as \( \ell_1 \). Since a point is in exactly one row line and exactly one column line of each array and any row line and column line of the same array are perpendicular, it follows that \( \ell_1 \parallel \ell_2 \). Hence, there is at least one line containing Z which is perpendicular to the given line \( \ell_1 \). Assume there is a line \( \ell_3, \ell_3 \neq \ell_2 \) and \( \ell_3 \neq \ell_1 \), such that \( Z \in \ell_3 \) and \( \ell_3 \perp \ell_1 \). Thus there are three lines in the array that contain Z. Since Z is contained in three lines of this array and two lines of each of the other two arrays, Z would be a point of at least seven lines, a contradiction to theorem 1.6. Hence there is exactly one line containing Z which is perpendicular to the given line.

If \( Z \notin \ell_1 \), then there exist two distinct lines which contain Z and are in the same array as \( \ell_1 \). Let \( \ell_2 \) be the row line containing Z and \( \ell_3 \) the column line containing Z. Thus \( \ell_2 \neq \ell_3 \), and since \( Z \notin \ell_1 \), \( \ell_1 \neq \ell_2 \) and \( \ell_1 \neq \ell_3 \). If \( \ell_1 \) is a row line then \( \ell_3 \parallel \ell_1 \). If \( \ell_1 \) is a column line then \( \ell_2 \) is perpendicular to \( \ell_1 \). Hence there is at least one line containing Z which is perpendicular to a given line. Assume there exists a line \( \ell_4 \), \( \ell_4 \neq \ell_3 \) and \( \ell_4 \neq \ell_2 \), and \( \ell_4 \neq \ell_1 \) such that \( Z \in \ell_4 \) and \( \ell_4 \perp \ell_1 \). Since \( Z \in \ell_2 \), \( Z \in \ell_3 \), and \( Z \in \ell_4 \), and \( \ell_2, \ell_3, \ell_4 \) are all in the
same array, this implies Z is in three lines of this array. Since Z is in four lines of the remaining two arrays this contradicts theorem 1.6. Thus, there is exactly one line containing Z and perpendicular to $\alpha_1$.

Therefore, in either case, there is a unique line containing the given point and perpendicular to a given line.

Theorem 1.10. A line perpendicular to one of two parallel lines is perpendicular to the other line also.

Proof. Let $\alpha_1, \alpha_2, \alpha_3$ be three distinct lines such that $\alpha_1 \perp \alpha_2$ and $\alpha_2 \parallel \alpha_3$. Since $\alpha_1 \perp \alpha_2$, $\alpha_1$ and $\alpha_2$ are both in the same array. Suppose $\alpha_1$ is a row line, then $\alpha_2$ is a column line. It follows that $\alpha_3$ is a column line in the same array as $\alpha_2$, since $\alpha_2 \parallel \alpha_3$. Since $\alpha_1$ is a row line, $\alpha_3$ a column line, and both lines in the same array, then $\alpha_1 \perp \alpha_3$.

If $\alpha_1$ is a column line, then $\alpha_2$ is a row line in the same array. Since $\alpha_2 \parallel \alpha_3$, $\alpha_3$ is a row line in the same array as $\alpha_2$. Since $\alpha_1$ is a column line in the same array as the row line $\alpha_3$, it follows that $\alpha_1 \perp \alpha_3$. 
CHAPTER BIBLIOGRAPHY


CHAPTER II

CURVES AND CONGRUENCE

Theorem 2.0. If $Z_1$ and $Z_2$ are two distinct points, then $Z_1Z_2 \cong Z_1Z_2$.

Proof. Let $Z_1$ and $Z_2$ be two distinct points. The segment $(Z_1, Z_2)$ is this point pair. It follows, by definition 1.4, that $m(Z_1, Z_2) = m(Z_1, Z_2)$. By P(IV), $\{Z_1, Z_2\}$ determines a unique line. Hence $(Z_1, Z_2)$ is a subset of either a row or a column line, but not both. Therefore, by definition 1.5, $Z_1Z_2 \cong Z_1Z_2$.

Corollary 2.00. If $Z_1$ and $Z_2$ are points then $Z_1Z_2 \cong Z_2Z_1$.

Proof. Let $Z_1$ and $Z_2$ be two distinct points. This point pair is the segment $(Z_1, Z_2)$. By definition 1.4, the measure of $(Z_1, Z_2)$ is the least number of steps from one point to the other. Since measure is the least number of steps from one point to the other $m(Z_1, Z_2) = m(Z_2, Z_1)$. By P(IV), $\{Z_1, Z_2\}$ is a subset of exactly one line. By the notation for the arrays, this line is either a row line or a column line, but not both. Therefore, $Z_1Z_2 \cong Z_2Z_1$.

Theorem 2.1. If $Z_1, Z_2, Z_3, Z_4$ are four distinct points and if $Z_1Z_2 \cong Z_3Z_4$, then $Z_3Z_4 \cong Z_1Z_2$. 
Proof. Let \( Z_1, Z_2, Z_3, Z_4 \) be four distinct points such that \( Z_1Z_2 \cong Z_3Z_4 \). Since \( Z_1Z_2 \cong Z_3Z_4 \), by definition 1.5, \( m(Z_1, Z_2) = m(Z_3, Z_4) \) and \( \{Z_1, Z_2\} \) and \( \{Z_3, Z_4\} \) are subsets of either row lines or column lines, but not both. Hence \( \{Z_3, Z_4\} \) and \( \{Z_1, Z_2\} \) are subsets of either row lines or column lines, but not both. Consider \( m(Z_1, Z_2) = m(Z_3, Z_4) \). By definition 1.4, \( m(Z_1, Z_2) \) means the least number of steps along the line from one point to the other. Since \( m(Z_1, Z_2) = m(Z_3, Z_4) \), the number of steps from \( Z_1 \) to \( Z_2 \) (or \( Z_2 \) to \( Z_1 \)) must be the same as the number of steps from \( Z_3 \) to \( Z_4 \) (or \( Z_4 \) to \( Z_3 \)). Thus \( m(Z_3, Z_4) = m(Z_1, Z_2) \). Therefore, by definition 1.5, \( Z_3Z_4 \cong Z_1Z_2 \).

Theorem 2.2. If \( Z_1, Z_2, Z_3, Z_4, Z_5, Z_6 \) are six distinct points and if \( Z_1Z_2 \cong Z_3Z_4 \) and \( Z_3Z_4 \cong Z_5Z_6 \), then \( Z_1Z_2 \cong Z_5Z_6 \).

Proof. Let \( Z_1, Z_2, Z_3, Z_4, Z_5, Z_6 \) be six distinct points such that \( Z_1Z_2 \cong Z_3Z_4 \) and \( Z_3Z_4 \cong Z_5Z_6 \). Since \( Z_1Z_2 \cong Z_3Z_4 \), by definition 1.5, it follows that \( m(Z_1, Z_2) = m(Z_3, Z_4) \) and that \( \{Z_1, Z_2\} \) and \( \{Z_3, Z_4\} \) are subsets of either row lines or column lines, but not both. Since \( m(Z_1, Z_2) = m(Z_3, Z_4) \), by definition 1.4, the number of steps from \( Z_1 \) to \( Z_2 \) (or \( Z_2 \) to \( Z_1 \)) is the same as the number of steps from \( Z_3 \) to \( Z_4 \) (or \( Z_4 \) to \( Z_3 \)).

Since \( Z_3Z_4 \cong Z_5Z_6 \), by definition 1.5, \( m(Z_3, Z_4) = m(Z_5, Z_6) \) and \( \{Z_3, Z_4\} \) and \( \{Z_5, Z_6\} \) are subsets of either row lines or column lines, but not both. By definition 1.4, since \( m(Z_3, Z_4) = m(Z_5, Z_6) \), the number of steps from \( Z_3 \) to \( Z_4 \) (or
$Z_4$ to $Z_3$) is the same as the number of steps from $Z_5$ to $Z_6$
(or $Z_6$ to $Z_5$).

Consider $(Z_1, Z_2)$. By P(IV) and the array notation, 
${Z_1, Z_2}$ is a subset of exactly one row line or exactly one
column line, but not both.

If $\{Z_1, Z_2\}$ is a subset of a row line, then $\{Z_3, Z_4\}$ is
a subset of a row line and, since $\{Z_3, Z_4\}$ is a subset of a
row line, $\{Z_5, Z_6\}$ is a subset of a row line. Hence $\{Z_1, Z_2\}$
and $\{Z_5, Z_6\}$ are subsets of row lines. Since the number of
steps from $Z_1$ to $Z_2$ is the same as the number of steps from
$Z_3$ to $Z_4$ which is the same as the number of steps from $Z_5$ to
$Z_6$, it follows that $m(Z_1, Z_2) = m(Z_5, Z_6)$. Therefore $Z_1 Z_2 \cong Z_5 Z_6$,
by definition 1.5.

If $\{Z_1, Z_2\}$ is a subset of a column line, then $\{Z_3, Z_4\}$
is a subset of a column line and, since $\{Z_3, Z_4\}$ is a subset
of a column line, $\{Z_5, Z_6\}$ is a subset of a column line.
Hence $\{Z_1, Z_2\}$ and $\{Z_5, Z_6\}$ are subsets of column lines. Also,
the number of steps from $Z_1$ to $Z_2$ is the same as the number
of steps from $Z_3$ to $Z_4$ which is the same as the number of
steps from $Z_5$ to $Z_6$. Hence, by definition 1.4, $m(Z_1, Z_2) =
m(Z_5, Z_6)$. Therefore, by definition 1.5, $Z_1 Z_2 \cong Z_5 Z_6$.

By theorem 2.0, congruence for segments is a reflexive
relation. Also, by theorem 2.1, congruence for segments is
a symmetric relation, and theorem 2.2 establishes the transi-
tive property of congruence for segments. Thus congruence
for segments, as defined in definition 1.5, is an equivalence relation.

Definition 2.0. Given \{Z_1, Z_2, \cdots, Z_n\}, n an integer and $2 \leq n \leq 25$, a curve is $(Z_1, Z_2) \cup (Z_2, Z_3) \cup \cdots \cup (Z_{n-1}, Z_n)$.

Definition 2.1. Given \{Z_1, Z_2, \cdots, Z_n\}, n an integer and $3 \leq n \leq 25$, ordered so that no three consecutive points are points of the same line, then a closed curve is $(Z_1, Z_2) \cup (Z_2, Z_3) \cup \cdots \cup (Z_n, Z_1)$.

Definition 2.2. A simple closed curve, denoted $s(Z_1, Z_2, \cdots, Z_n)$, is a closed curve determined by \{Z_1, Z_2, \cdots, Z_n\} such that $Z_i$, $i=1, 2, \cdots, n$, is the intersection of at most two lines which contain $(Z_1, Z_2)$, $(Z_2, Z_3)$, $\cdots$, $(Z_n, Z_1)$.

Definition 2.3. A trilateral, denoted $\Delta(Z_1, Z_2, Z_3)$, is a simple closed curve which is the union of exactly three segments.

Definition 2.4. A scalene trilateral is a trilateral in which no two of the segments of the union are congruent.

Consider the trilateral composed of the segments (A,B), (B,F), and (F,A), denoted $\Delta(A,B,F)$. Examination of the arrays shows $m(A,B) = 1r$, $m(B,F) = 2c$, and $m(F,A) = 1c$. By definition 1.5, no two of these segments are congruent. Hence, $\Delta(A,B,F)$ is a scalene trilateral, by definition 2.4.

Definition 2.5. An isosceles trilateral is a trilateral such that at least two of the segments of the union are congruent.
Examine the trilateral composed of the segments (C,G),
(G,I), and (I,C). Examination shows m(C,G)=2c, m(G,I)=2r,
and m(I,C)=2c. By definition 1.5, CG ≅ IC, thus Δ(C,G,I) is
an isosceles trilateral, by definition 2.5.

Definition 2.6. An equilateral trilateral is a tri-
lateral such that each of the segments of the union is
congruent to each of the other two segments.

Consider Δ(I,T,F), composed of segments (I,T), (T,F),
and (F,I). By definition 1.4, m(I,T)=2r, m(T,F)=2r, and
m(F,I)=2r. By definition 1.5, IT ≅ TF, IT ≅ FI, and TF ≅ FI.
Thus, by definition 2.6, Δ(I,T,F) is an equilateral
trilateral.

Definition 2.7. A quadrilateral, denoted q(Z_1,Z_2,Z_3,Z_4),
is a simple closed curve which is the union of exactly four
segments.

Examine the points A,B,Q,F and (A,B) U (B,Q) U (Q,F) U
(F,A). These four points and the union of exactly four
segments compose a quadrilateral denoted q(A,B,Q,F).

Definition 2.8. Two segments, (Z_1,Z_2) and (Z_3,Z_4), are
parallel segments if and only if the line determined by
{Z_1,Z_2} is parallel to the line determined by {Z_3,Z_4}.

Consider the segments (N,T) and (C,V). The point pair
{N,T} determines a line α_1={T,H,U,N,B} while the point pair
{C,V} determines a line α_2={I,V,O,C,F}. Examination shows
α_1∩α_2=∅, hence, by definition 1.0, α_1∥α_2. Therefore (N,T)
and (C,V) are parallel segments.
Now consider \((B,G)\) and \((N,L)\). The point pair \(\{B,G\}\) determines a line \(\ell_1=\{B,G,L,Q,V\}\) and \(\{N,L\}\) determines a line \(\ell_2=\{K,L,M,N,0\}\). Examination shows \(\ell_1 \cap \ell_2 = \{L\}\). Hence, by definition 1.2, \(\ell_1\) intersects \(\ell_2\). By theorem 1.7, \(\ell_1\) is not parallel to \(\ell_2\). Thus, \((B,G)\) and \((N,L)\) are not parallel segments.

Definition 2.9. A parallelogram is a quadrilateral which has two pairs of parallel segments.

Examine the union of segments \((G,I) \cup (I,R) \cup (R,P) \cup (P,G)\). Examination of the lines determined by \(\{G,I\}\) and \(\{R,P\}\) shows that \((G,I)\) and \((R,P)\) are parallel segments. Also, segments \((I,R)\) and \((P,G)\) are parallel segments. Thus, by definition 2.9, \(q(G,I,R,P)\) is a parallelogram.

The quadrilateral \(q(A,B,Q,F)\) is not a parallelogram since there are not two pairs of parallel segments. The point pair \(\{A,B\}\) is a subset of \(\ell_1=\{A,B,C,D,E\}\), \(\{B,Q\}\) is a subset of \(\ell_2=\{B,G,L,Q,V\}\), \(\{Q,F\}\) is a subset of \(\ell_3=\{Y,C,F,N,Q\}\), and \(\{F,A\}\) is a subset of \(\ell_4=\{A,F,K,P,U\}\). Now, \(\ell_1 \cap \ell_2 = \{B\}\), \(\ell_2 \cap \ell_3 = \{Q\}\), \(\ell_3 \cap \ell_4 = \{F\}\), \(\ell_1 \cap \ell_4 = \{A\}\), \(\ell_1 \cap \ell_3 = \{C\}\), and \(\ell_2 \cap \ell_4 = \emptyset\). Since these are all the possible intersections of these four segments and there is only one pair of parallel segments, it follows, by definition 2.9, that \(q(A,B,Q,F)\) is not a parallelogram.

Definition 2.10. Two segments, \((Z_1,Z_2)\) and \((Z_3,Z_4)\), are perpendicular segments if and only if the line determined by \(\{Z_1,Z_2\}\) is perpendicular to the line determined by \(\{Z_3,Z_4\}\).
Consider segments \((V,T)\) and \((U,E)\). The point pair \(\{V,T\}\) is a subset of a row line \(\ell_1=\{V,T,M,F,D\}\), and \(\{U,E\}\) is a subset of a column line \(\ell_2=\{Q,I,U,M,E\}\). Examination shows \(\ell_1\) is a row line of array \#6 and \(\ell_2\) is a column line of array \#6. This is sufficient to make \(\ell_1\) perpendicular to \(\ell_2\). Therefore, by definition 2.10, \((V,T)\) and \((U,E)\) are perpendicular segments.

Recall that an earlier verification has shown that a row line and a column line of the same array are perpendicular and that two row lines or two column lines of the same array are parallel. It can also be verified that two lines from different arrays are neither parallel nor perpendicular.

Definition 2.11. A rectangle is a parallelogram such that two of the nonparallel segments are perpendicular.

Consider \(q(B,D,S,Q)\). The segments \((B,D)\) and \((S,Q)\) are parallel segments as are the segments \((D,S)\) and \((Q,B)\). Consider segments \((B,D)\) and \((D,S)\). These two segments are not parallel segments since the lines determined by \(\{B,D\}\) and \(\{D,S\}\) intersect at the point \(D\). The line determined by \(\{B,D\}\) is a row line in array \#1, while the line determined by \(\{D,S\}\) is a column line of array \#1. Thus \((B,D)\) and \((D,S)\) are perpendicular segments. Therefore, by definition 2.11, \(q(B,D,S,Q)\) is a rectangle.

Examine, again, \(q(G,I,R,P)\). The segments \((G,I)\) and \((R,P)\) are parallel segments, by definition 2.8, and \((I,R)\) and \((P,G)\) are also parallel segments. Consider \((G,I)\) and
(I,R). The line $x_1 = \{F,G,H,I,J\}$ contains $\{G,I\}$ as a subset, and $x_1$ is a row line of array #1. The line $x_2 = \{R,K,I,B,Y\}$ contains $\{I,R\}$ as a subset, and $x_2$ is a row line in array #6.

It follows, from previous discussion, that $x_1$ is not perpendicular to $x_2$. Therefore, by definition 2.10, $(G,I)$ and $(I,R)$ are not perpendicular segments. A similar argument shows that $(R,P)$ and $(P,G)$ are not perpendicular segments.

Therefore, by definition 2.11, $q(G,I,R,P)$ is not a rectangle.

It is noted at this point that the simple closed curve normally referred to as a square in Euclidean plane geometry does not exist in this geometry.

Definition 2.12. Two trilaterals, $\triangle(Z_1,Z_2,Z_3)$ and $\triangle(Z_4,Z_5,Z_6)$, are congruent if and only if the three segments of one trilateral are congruent respectively to the three segments of the other trilateral, denoted $\triangle Z_1Z_2Z_3 \cong \triangle Z_4Z_5Z_6$. This is known as the side side side (sss) postulate.

Consider two trilaterals, $\triangle(Z_1,Z_2,Z_3)$ and $\triangle(Z_4,Z_5,Z_6)$. To say $\triangle Z_1Z_2Z_3 \cong \triangle Z_4Z_5Z_6$ means that point $Z_1$ corresponds to point $Z_4$, point $Z_2$ corresponds to point $Z_5$, point $Z_3$ corresponds to point $Z_6$, and $Z_1Z_2 \cong Z_4Z_5$, $Z_2Z_3 \cong Z_5Z_6$, and $Z_3Z_1 \cong Z_6Z_4$, and in this order.

Examine $\triangle(P,R,I)$ and $\triangle(F,Q,A)$. By definition 1.4, $m(P,R) = 2r$, $m(R,I) = 2r$, $m(I,P) = 1c$, $m(F,Q) = 2r$, $m(Q,A) = 2r$, and $m(A,F) = 1c$. By definition 1.5, $PR \cong FQ$, $RI \cong QA$, and $IP \cong AF$. Therefore, by definition 2.12, $\triangle PRI \cong \triangle FQA$. 
Theorem 2.3. If $Z_1, Z_2, Z_3$ are three points such that
$(Z_1, Z_2) \cup (Z_2, Z_3) \cup (Z_3, Z_1)$ is $\Delta(Z_1, Z_2, Z_3)$, then $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_1 Z_2 Z_3$.

Proof: Let $Z_1, Z_2, Z_3$ be three points such that
$(Z_1, Z_2) \cup (Z_2, Z_3) \cup (Z_3, Z_1)$ is $\Delta(Z_1, Z_2, Z_3)$. It follows, by
theorem 2.0, that $Z_1 Z_2 \cong Z_1 Z_2$, $Z_2 Z_3 \cong Z_2 Z_3$, and $Z_3 Z_1 \cong Z_3 Z_1$.

Therefore, by definition 2.12, $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_1 Z_2 Z_3$.

Theorem 2.4. If $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_4 Z_5 Z_6$, then $\Delta Z_4 Z_5 Z_6 \cong \Delta Z_1 Z_2 Z_3$.

Proof. Let $Z_1, Z_2, Z_3$ be three points which determine
$\Delta(Z_1, Z_2, Z_3)$, and let $Z_4, Z_5, Z_6$ be three points which determine
$\Delta(Z_4, Z_5, Z_6)$ such that $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_4 Z_5 Z_6$.

Since $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_4 Z_5 Z_6$, by definition 2.12, it follows
that $Z_1 Z_2 \cong Z_4 Z_5$, $Z_2 Z_3 \cong Z_5 Z_6$, and $Z_3 Z_1 \cong Z_6 Z_4$. From theorem 2.1,
it follows that $Z_4 Z_5 \cong Z_1 Z_2$, $Z_5 Z_6 \cong Z_2 Z_3$, and $Z_6 Z_4 \cong Z_3 Z_1$. There-
fore, by definition 2.12, $\Delta Z_4 Z_5 Z_6 \cong \Delta Z_1 Z_2 Z_3$.

Theorem 2.5. If $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_4 Z_5 Z_6$ and $\Delta Z_4 Z_5 Z_6 \cong \Delta Z_7 Z_8 Z_9$, then $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_7 Z_8 Z_9$.

Proof. Let $Z_1, Z_2, Z_3$ be three points which determine
$\Delta(Z_1, Z_2, Z_3)$, let $Z_4, Z_5, Z_6$ be three points which determine
$\Delta(Z_4, Z_5, Z_6)$, and let $Z_7, Z_8, Z_9$ be three points which determine
$\Delta(Z_7, Z_8, Z_9)$ such that $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_4 Z_5 Z_6$ and $\Delta Z_4 Z_5 Z_6 \cong \Delta Z_7 Z_8 Z_9$.

Since $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_4 Z_5 Z_6$, it follows, by definition 2.12, that
$Z_1 Z_2 \cong Z_4 Z_5$, $Z_2 Z_3 \cong Z_5 Z_6$, and $Z_3 Z_1 \cong Z_6 Z_4$. However, since
$\Delta Z_4 Z_5 Z_6 \cong \Delta Z_7 Z_8 Z_9$, it follows that $Z_4 Z_5 \cong Z_7 Z_8$, $Z_5 Z_6 \cong Z_8 Z_9$, and $Z_6 Z_4 \cong Z_9 Z_7$. Applying theorem 2.2, $Z_1 Z_2 \cong Z_7 Z_8$, $Z_2 Z_3 \cong Z_8 Z_9$, and $Z_3 Z_1 \cong Z_9 Z_7$. Therefore, by definition 2.12, $\Delta Z_1 Z_2 Z_3 \cong \Delta Z_7 Z_8 Z_9$. 
The reflexive property for congruence of trilaterals is established by theorem 2.3. Theorem 2.4 establishes the symmetric property for congruence of trilaterals, and the transitive relation is established by theorem 2.5. Thus, congruence for trilaterals as defined by definition 2.12 is an equivalence relation.
CHAPTER III

GENERAL CONGRUENCES

Definition 3.0. Given a point Z, the set of all points $Z_i$, $i=1,2,\cdots,n$ where $n$ is an integer and $1 \leq n \leq 25$, such that $Z_1Z \equiv Z_2Z \equiv Z_3Z \equiv \cdots \equiv Z_nZ$ is called a circle, denoted $\Theta Z$. The point $Z$ will be called the center of the circle and $m(Z_i,Z)$ will be called the radius of the circle.

Let $A$ be the center of a circle of radius $2r$. In array #1 there are two points, $C$ and $D$, such that $m(A,C)=2r$ and $m(A,D)=2r$. In array #2 there are two points, $L$ and $T$, such that $m(A,L)=2r$ and $m(A,T)=2r$. Also, in array #6 there are two points, $Q$ and $O$, such that $m(A,Q)=2r$ and $m(A,O)=2r$. Examination of the three arrays will show there is no other point $Z$ such that $m(A,Z)=2r$. The set of points $\{C,D,L,T,Q,O\}$ is the circle with center $A$ and radius of $2r$.

Let $\alpha=\{Z_1,Z_2,Z_3,Z_4,Z_5\}$ be any line. Recall, from definition 1.4, that the measure of any segment of a line $\alpha$ is the least number of steps from one point to the other where $Z_1$ is considered to follow $Z_5$ in cyclic form. Consider the line $\alpha$. It can be verified that the measure of any segment determined by any two points of $\alpha$ is 0,1, or 2. Since, by P(IV), any two distinct points determine exactly
one line and the measure of a segment is recorded in terms of either row steps (r) or column steps (c), but not both, it can be shown that the measure of the radius of a circle is 0r, 1r, 2r, 0c, 1c, or 2c.

Theorem 3.0. A circle whose radius is of measure not equal to 0r or 0c is a set of exactly six points.

Proof. Let Z be the center of a circle. By theorem 1.6, Z is a point of exactly six lines. By a previous argument Z is a point of exactly three row lines and a point of exactly three column lines.

If the circle has a radius of measure not equal to 0r or 0c, then from an earlier discussion there are exactly four possible measures for the radius of the circle Z: 1r, 2r, 1c, and 2c.

Suppose the measure of the radius of the circle is 1r. For each row line containing Z, examination shows exactly two points such that the measure of the segment determined by each of the points and Z is of measure 1r. Since Z is a point of exactly three row lines and each row line contains exactly two points such that the measure of the segment determined by each of the points and Z is of measure 1r, there are exactly six points such that m(Zi, Z) = 1r, i = 1, 2, ..., 6. Hence a circle the measure of which is 1r is a set of exactly six points.

If the measure is 2r, 1c, or 2c, the proof follows in similar manner.
Therefore, a circle whose radius is of measure not equal to 0 or 0c is a set of exactly six points.

Definition 3.1. Two circles, ΩZ1 and ΩZ2 are congruent if and only if a radius of ΩZ1 is congruent to a radius of ΩZ2, denoted ΩZ1 ≤ ΩZ2.

Theorem 3.1. A circle is congruent to itself.

Proof. Let Z be the center of a circle with radius of measure p. Let Z1 and Z2 be two distinct points of the circle. Hence, m(Z1, Z) = p and m(Z2, Z) = p. Thus, by definition 1.5, Z1Z ≡ Z2Z. Therefore, by definition 3.1, Z1Z ≡ Z2Z.

Theorem 3.2. If ΩZ1 ≡ ΩZ2, then ΩZ2 ≡ ΩZ1.

Proof. Let Z1 be the center of a circle with radius of measure p and Z2 be the center of a circle with radius of measure q, Z1 ≠ Z2, such that ΩZ1 ≡ ΩZ2.

Since ΩZ1 ≡ ΩZ2, and each circle is a set of exactly six points, there exists a point Z3 ∈ ΩZ1 and a point Z4 ∈ ΩZ2 such that Z1Z3 ≡ Z2Z4, by definition 3.1. By theorem 2.1, Z2Z4 ≡ Z1Z3. Therefore, by definition 3.1, Z2Z4 ≡ Z1Z3.

Theorem 3.3. If ΩZ1 ≡ ΩZ2 and ΩZ2 ≡ ΩZ3, then ΩZ1 ≡ ΩZ3.

Proof. Let Z1 be the center of a circle, Z2 be the center of a circle, and Z3 the center of a circle, Z1 ≠ Z2 ≠ Z3, such that ΩZ1 ≡ ΩZ2 and ΩZ2 ≡ ΩZ3.

Since ΩZ1 ≡ ΩZ2, there exist points Z4 ∈ ΩZ1 and Z5 ∈ ΩZ2 such that Z4Z1 ≡ Z5Z2. Since ΩZ2 ≡ ΩZ3 there exist points Z6 ∈ ΩZ2 and Z7 ∈ ΩZ3 such that Z6Z2 ≡ Z7Z3. By definition 3.0, since Z5 ∈ ΩZ2 and Z6 ∈ ΩZ2 it follows that Z5Z2 ≡ Z6Z2. By
By theorem 3.1 congruence for circles is a reflexive relation. From theorem 3.2 it is evident that congruence of circles is also a symmetric relation. The transitive property is established by theorem 3.3. Hence, congruence for circles is an equivalence relation.

Definition 3.2. If Z is the center of a circle, OZ, and Z₁ is a point of the circle, the statement that \( \sim \) is tangent to the circle at Z₁ means (OZ ∩ \( \sim \)) = {Z₁} and \( \sim \) is perpendicular to the line determined by \{Z, Z₁\}.

Consider, again, the circle with center A and radius 2r, discussed on page 30, which is the set of points \{C, D, L, T, Q, O\}. Examination of the arrays shows that \( \sim = \{C, H, M, R, W\} \) is perpendicular to the line determined by \{A, C\} and (OA ∩ \( \sim \)) = \{C\}. Hence, by definition 3.2, \( \sim \) is tangent to OA at C. Similarly it can be demonstrated that \( \sim₁ = \{D, I, N, S, X\} \) is tangent to OA at D, \( \sim₂ = \{L, E, R, F, X\} \) is tangent to OA at L, \( \sim₃ = \{T, H, U, N, B\} \) is tangent to OA at T, \( \sim₄ = \{Q, I, U, M, E\} \) is tangent to OA at Q, and \( \sim₅ = \{O, B, S, F, W\} \) is tangent to OA at O (2, pp. 22-26).

The concept of congruence for trilaterals was introduced and discussed in Chapter II. Now that congruence for circles has been defined and discussed, consider the following concept of general congruency of two simple closed curves.
Definition 3.3. A one-to-one correspondence, $f^*$, from a simple closed curve $s(Z_1,Z_2,...,Z_n)$, $n$ an integer and $3 \leq n \leq 25$, onto a simple closed curve $s'(Z_1',Z_2',...,Z_n')$ means that each point $Z_i$, $i=1,2,...,n$, in $s(Z_1,Z_2,...,Z_n)$ corresponds to one and only one point $Z_i'$, $i=1,2,...,n$, in $s'(Z_1',Z_2',...,Z_n')$ and that each point $Z_i'$ in $s'(Z_1',Z_2',...,Z_n')$ corresponds to one and only one point $Z_i$ in $s(Z_1,Z_2,...,Z_n)$, denoted $Z_1 \leftrightarrow Z_1'$, $Z_2 \leftrightarrow Z_2'$, ..., $Z_n \leftrightarrow Z_n'$, and the segments determined by corresponding points correspond, denoted $(Z_1,Z_2) \leftrightarrow (Z_1',Z_2')$, $(Z_2,Z_3) \leftrightarrow (Z_2',Z_3')$, ..., $(Z_n,Z_1) \leftrightarrow (Z_n',Z_1')$.

Definition 3.4. A simple closed curve $s(Z_1,Z_2,...,Z_n)$ is congruent to a simple closed curve $s'(Z_1',Z_2',...,Z_n')$ if and only if there exists a one-to-one correspondence, $f^*$, from $s(Z_1,Z_2,...,Z_n)$ onto $s'(Z_1',Z_2',...,Z_n')$ such that $Z_1Z_2 \cong Z_1'Z_2'$, $Z_2Z_3 \cong Z_2'Z_3'$, ..., $Z_nZ_1 \cong Z_n'Z_1'$. This congruency will be denoted $sZ_1Z_2...Z_n \cong s'Z_1'Z_2'...Z_n'$.

Theorem 3.4. If $s(Z_1,Z_2,...,Z_n)$, $n$ an integer and $3 \leq n \leq 25$, is a simple closed curve, then $sZ_1Z_2...Z_n \cong sZ_1Z_2...Z_n$.

Proof. Let $s(Z_1,Z_2,...,Z_n)$, $n$ an integer and $3 \leq n \leq 25$, be a simple closed curve as defined in definition 2.2. Let $f^*$ be a correspondence defined on $s(Z_1,Z_2,...,Z_n)$ in such a way that $Z_1 \leftrightarrow Z_1$, $Z_2 \leftrightarrow Z_2$, ..., $Z_n \leftrightarrow Z_n$ and that $(Z_1,Z_2) \leftrightarrow (Z_1,Z_2)$, $(Z_2,Z_3) \leftrightarrow (Z_2,Z_3)$, ..., $(Z_n,Z_1) \leftrightarrow (Z_n,Z_1)$. By definition 3.3, $f^*$ is a one-to-one correspondence from $s(Z_1,Z_2,...,Z_n)$ onto
s(Z_1, Z_2, \ldots, Z_n). By theorem 2.0, Z_1Z_2 \cong Z_1Z_2, Z_2Z_3 \cong Z_2Z_3, \ldots, Z_nZ_1 \cong Z_nZ_1. Therefore, by definition 3.4, s(Z_1Z_2 \ldots Z_n) \cong s(Z_1Z_2 \ldots Z_n).

Theorem 3.5. If s(Z_1, Z_2, \ldots, Z_n) and s'(Z'_1, Z'_2, \ldots, Z'_n) are two simple closed curves and if s(Z_1Z_2 \ldots Z_n) \cong s(Z'_1Z'_2 \ldots Z'_n), then s'(Z'_1Z'_2 \ldots Z'_n) \cong s(Z_1Z_2 \ldots Z_n).

Proof. Let s(Z_1, Z_2, \ldots, Z_n) and s'(Z'_1, Z'_2, \ldots, Z'_n) be two simple closed curves such that s(Z_1Z_2 \ldots Z_n) \cong s(Z'_1Z'_2 \ldots Z'_n). Since s(Z_1Z_2 \ldots Z_n) \cong s(Z'_1Z'_2 \ldots Z'_n), by definition 3.4, there exists a one-to-one correspondence, f*, from s(Z_1, Z_2, \ldots, Z_n) onto s'(Z'_1, Z'_2, \ldots, Z'_n) such that Z_1 \leftrightarrow Z'_1, Z_2 \leftrightarrow Z'_2, \ldots, Z_n \leftrightarrow Z'_n and (Z_1, Z_2) \leftrightarrow (Z'_1, Z'_2), (Z_2, Z_3) \leftrightarrow (Z'_2, Z'_3), \ldots, (Z_n, Z_1) \leftrightarrow (Z'_n, Z'_1) and Z_1Z_2 \cong Z'_1Z'_2, Z_2Z_3 \cong Z'_2Z'_3, \ldots, Z_nZ_1 \cong Z'_nZ'_1. Since f* is a one-to-one correspondence from s(Z_1, Z_2, \ldots, Z_n) onto s'(Z'_1, Z'_2, \ldots, Z'_n), there exists a one-to-one correspondence, (f*)^{-1}, from s'(Z'_1, Z'_2, \ldots, Z'_n) onto s(Z_1, Z_2, \ldots, Z_n) such that Z'_1 \leftrightarrow Z_1, Z'_2 \leftrightarrow Z_2, \ldots, Z'_n \leftrightarrow Z_n and (Z'_1, Z'_2) \leftrightarrow (Z_1, Z_2), (Z'_2, Z'_3) \leftrightarrow (Z_2, Z_3), \ldots, (Z'_n, Z'_1) \leftrightarrow (Z_n, Z_1) (1, p. 23). By theorem 2.1, Z'_1Z'_2 \cong Z_1Z_2, Z'_2Z'_3 \cong Z_2Z_3, \ldots, Z'_nZ'_1 \cong Z_nZ_1. Therefore, by definition 3.4, s'(Z'_1Z'_2 \ldots Z'_n) \cong s(Z_1Z_2 \ldots Z_n).

Theorem 3.6. If s(Z_1, Z_2, \ldots, Z_n), s'(Z'_1, Z'_2, \ldots, Z'_n), and s''(Z_1'', Z_2'', \ldots, Z_n'') are three simple closed curves and if s(Z_1Z_2 \ldots Z_n) \cong s'(Z'_1Z'_2 \ldots Z'_n) and s''(Z_1''Z_2'' \ldots Z_n'') \cong s'(Z_1'Z_2' \ldots Z_n') and s'(Z'_1Z'_2' \ldots Z'_n') \cong s(Z_1''Z_2'' \ldots Z_n''), then s(Z_1Z_2 \ldots Z_n) \cong s(Z_1''Z_2'' \ldots Z_n'').
Proof. Let \( s(Z_1, Z_2, \cdots, Z_n) \), \( s'(Z_1', Z_2', \cdots, Z_n') \), and \( s''(Z_1'', Z_2'', \cdots, Z_n'') \) be three simple closed curves such that
\[ s_1Z_2\cdots Z_n \cong s'Z_1'Z_2'\cdots Z_n' \quad \text{and} \quad s'Z_1'Z_2'\cdots Z_n' \cong s''Z_1''Z_2''\cdots Z_n''. \]
Since \( sZ_1Z_2\cdots Z_n \cong s'Z_1'Z_2'\cdots Z_n' \), by definition 3.4, there exists a one-to-one correspondence, \( f^* \), from \( s(Z_1, Z_2, \cdots, Z_n) \) onto \( s'(Z_1', Z_2', \cdots, Z_n') \) such that \( Z_1 \leftrightarrow Z_1' \), \( Z_2 \leftrightarrow Z_2' \), \( \cdots \), \( Z_n \leftrightarrow Z_n' \) and \( (Z_1, Z_2) \leftrightarrow (Z_1', Z_2') \), \( (Z_2, Z_3) \leftrightarrow (Z_2', Z_3') \), \( \cdots \), \( (Z_n, Z_1) \leftrightarrow (Z_n', Z_1') \) and such that \( Z_1Z_2\cdots Z_n \cong Z_1'Z_2'\cdots Z_n' \), \( Z_2'Z_3\cdots Z_n' \cong Z_2''Z_3''\cdots Z_n''' \), \( \cdots \). Also, by definition 3.4 and the fact that \( s'Z_1'Z_2'\cdots Z_n' \cong s''Z_1''Z_2''\cdots Z_n'' \), there exists a one-to-one correspondence, \( f^{**} \), from \( s'(Z_1', Z_2', \cdots, Z_n') \) onto \( s''(Z_1'', Z_2'', \cdots, Z_n'') \) such that \( Z_1' \leftrightarrow Z_1'' \), \( Z_2' \leftrightarrow Z_2'' \), \( \cdots \), \( Z_n' \leftrightarrow Z_n'' \) and \( (Z_1', Z_2') \leftrightarrow (Z_1'', Z_2'') \), \( (Z_2', Z_3') \leftrightarrow (Z_2'', Z_3'') \), \( \cdots \), \( (Z_n', Z_1') \leftrightarrow (Z_n'', Z_1'') \) and such that \( Z_1'Z_2'\cdots Z_n' \cong Z_1''Z_2''\cdots Z_n'' \), \( Z_2'Z_3'\cdots Z_n' \cong Z_2''Z_3''\cdots Z_n'' \), \( \cdots \). Since one-to-one correspondence is a transitive relation (1, pp. 23-26), there exists a one-to-one correspondence, \( f^{***} \), from \( s(Z_1, Z_2, \cdots, Z_n) \) onto \( s''(Z_1'', Z_2'', \cdots, Z_n'') \) such that \( Z_1 \leftrightarrow Z_1'' \), \( Z_2 \leftrightarrow Z_2'' \), \( \cdots \), \( Z_n \leftrightarrow Z_n'' \) and \( (Z_1, Z_2) \leftrightarrow (Z_1'', Z_2'') \), \( (Z_2, Z_3) \leftrightarrow (Z_2'', Z_3'') \), \( \cdots \), \( (Z_n, Z_1) \leftrightarrow (Z_n'', Z_1'') \). By theorem 2.2, \( Z_1Z_2\cdots Z_n \cong Z_1''Z_2''\cdots Z_n'' \), \( Z_2Z_3\cdots Z_n \cong Z_2''Z_3''\cdots Z_n'' \), \( \cdots \), \( Z_nZ_1 \cong Z_n''Z_1'' \). Therefore, by definition 3.4, \( sZ_1Z_2\cdots Z_n \cong s''Z_1''Z_2''\cdots Z_n'' \).

Congruence for simple closed curves as defined by definition 3.4 is an equivalence relation since theorem 3.4 establishes the reflexive property, theorem 3.5 the symmetric property, and theorem 3.6 the transitive property.
Consider two simple closed curves \( s(Z_1, Z_2, Z_3) \) and \( s'(Z_1', Z_2', Z_3') \) such that \( sZ_1Z_2Z_3 \cong s'Z_1'Z_2'Z_3' \) by definition 3.4. Since \( sZ_1Z_2Z_3 \cong s'Z_1'Z_2'Z_3' \), there exists a one-to-one correspondence, \( f^* \), from \( s(Z_1, Z_2, Z_3) \) onto \( s'(Z_1', Z_2', Z_3') \) such that \( Z_1 \leftrightarrow Z_1' \), \( Z_2 \leftrightarrow Z_2' \), \( Z_3 \leftrightarrow Z_3' \) and \( (Z_1, Z_2) \leftrightarrow (Z_1', Z_2') \), \( (Z_2, Z_3) \leftrightarrow (Z_2', Z_3') \), \( (Z_3, Z_1) \leftrightarrow (Z_3', Z_1') \) and such that \( Z_1Z_2 \cong Z_1'Z_2' \), \( Z_2Z_3 \cong Z_2'Z_3' \), \( Z_3Z_1 \cong Z_3'Z_1' \). The simple closed curve \( s(Z_1, Z_2, Z_3) \) is the union of exactly three segments, \( (Z_1, Z_2) \cup (Z_2, Z_3) \cup (Z_3, Z_1) \), and hence, by definition 2.3, is the trilateral, \( \Delta(Z_1, Z_2, Z_3) \). Likewise, the simple closed curve \( s'(Z_1', Z_2', Z_3') \) is the trilateral \( \Delta(Z_1', Z_2', Z_3') \). Since the points of \( \Delta(Z_1, Z_2, Z_3) \) correspond respectively to the points of \( \Delta(Z_1', Z_2', Z_3') \) and the respective segments are congruent, it follows that \( \Delta Z_1Z_2Z_3 \cong \Delta Z_1'Z_2'Z_3' \), by definition 2.12.

Hence, if two trilaterals are congruent according to definition 3.4, they are also congruent according to the earlier concept.
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