HOMOGENEOUS CANONICAL FORMALISM

AND RELATIVISTIC WAVE EQUATIONS

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CHAPTER I

INTRODUCTION

The fundamental problem is that of prediction; one knows what the physical state is now, can one predict what it will be in the future? This problem arises in classical dynamics, classical quantum dynamics, relativistic dynamics and relativistic quantum dynamics.

What is it that one wishes to predict? Classically one usually is most interested in the coordinates of some physical entity at some given time. For quantum systems the wave function gives one the pertinent physical information one wishes at a certain given time. Therefore it is necessary to have the equation that gives the change in the wave function with respect to some parameter, usually the time. The Schroedinger equation is well-known to be that equation for non-relativistic wave functions and the Dirac equation for relativistic (spin one-half) wave functions. Since it is possible to "derive" the Schroedinger equation from a correspondence between infinitesimal canonical transformations of classical physics and infinitesimal unitary transformations of quantum physics, one might expect to be able to derive the Dirac, Klein-Gordon, etc., equations by a similar correspondence.

One should, of course, recognize that some quantum physics may not have a classical description, but the object of this endeavor is to find to what extent the relativistic wave equations have a classical counterpart, or more accurately, can be derived from a classical correspondence. Since the above-mentioned correspondence is the one desired, it is best to look at that formulation of classical dynamics that will be best suited for a transition to quantum dynamics. The formulation in phase space is the one of most interest for this purpose. The coordinates in this space are the coordinates x_r, and the momenta p_r, both as functions of time. The time development of a physical system in this formulation is given as a trajectory in phase space; these trajectories are of immediate interest.

How does one generate trajectories in phase space? To answer this question one has to go to a representative space, that is, the space of coordinates which are the most familiar. A curve in three dimensional Euclidian space, E_3, gives useful physical information. Now there are a large number of curves in this space which could serve as states of physical interest. How does one decide which ones to use? If a number could be attached to each curve in E_3 and then a way found for sorting out the curve that really fits the situation, the question could be answered.

Let then a number be attached to curves by integrating the function $L(x, \dot{x})$ along them, where

$$\dot{x} = \frac{dx}{dt}. \qquad \text{(I-1)}$$

That is, consider the definite integral

$$\int_{t_0}^{t} L(x, \dot{x})dt. \qquad \text{(I-2)}$$

The sorting process will be: insert in the integral every curve of suitable mathematical nature, take the difference between the number obtained and one preceding and the one following until the changes show a change in direction. Rather, one would say, differentiate with respect to the class of curves, or functions under consideration and set equal to zero:

$$\delta\int_{t}^{t_0} L(x, \dot{x})dt = 0. \qquad \text{(I-3)}$$

The process is well known and results in the differential equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_r} - \frac{\partial L}{\partial x_r} = 0 \; . \qquad \text{(I-4)}$$

Hence it is seen that the answer to the question of generating curves of physical interest is in the form of differential equations, the solutions of which are the required curves. The question of whether these are absolute or relative extremums is not considered here.

Since the interest was in phase space, it would do better to modify the action principle slightly and take as the integral

$$\int_{x_0}^{x} p_i \, dx^i - \int H \, dt \tag{I-5}$$

along the trajectories, where H is defined by

$$H(x, p, t) = p_i \, \dot{x}_i - L(x, \dot{x}) \tag{I-6}$$

and

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \; . \tag{I-7}$$

The variation of this integral gives the sorting,

$$\delta \left[\int_{x}^{x_0} p_i \, dx^i - \int_{t_0}^{t} H \, dt \right] = 0 \tag{I-8}$$

or

$$\int_{x}^{x_0} \left(\delta p_i \, dx_i + p_i \delta \, dx^i \right) - \int_{t_0}^{t} (\delta H \, dt + H \delta \, dt) = 0; \tag{I-9}$$

after integration by parts

$$\left[p_i\ \delta x_i\right]_{x_0}^{x} - \left[H\ \delta t\right]_{t_0}^{t} + \int\left[\left(\delta p_i\ dx^i - \delta x_i\ dp_i\right) - (\delta H\ dt + \delta t\ dH)\right] = 0. \tag{I-10}$$

For fixed end-points the first term on the right vanishes, and using

$$\delta H = \frac{\partial H}{\partial x_i}\,dx_i + \frac{\partial H}{\partial t}\,\delta t + \frac{\partial H}{\partial p_i}\ \delta p_i \tag{I-11}$$

gives

$$\int\left[\delta p_i\left(dx_i - \frac{\partial H}{\partial p_i}\,dt\right) - \delta x_i\left(dp_i + \frac{\partial H}{\partial x_i}\ dt\right) + \delta t\left(dH - \frac{\partial H}{\partial t}\ dt\right)\right] = 0. \tag{I-12}$$

Since δp_i, δx_i, δt are all arbitrary variations, the only way for the integral to be zero is to have

$$dx_i = \frac{\partial H}{\partial p_i}\,dt, \quad dp_i = -\frac{\partial H}{\partial x_i}\,dt, \quad dH = \frac{\partial H}{\partial t}\,dt \tag{I-13}$$

or

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{H} = \frac{\partial H}{\partial t}. \tag{I-14}$$

For conservative systems $H(x, t, p)$ is independent of t, so that

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \tag{I-15}$$

These are called the canonical equations of motion. They give the trajectories in phase space that are of interest.

Thus, within the bounds of interest, the fundamental equation of classical dynamics has been solved. To get into a better position for a transition to quantum dynamics, it is necessary to extend the results somewhat. Thus, we consider the subject of canonical transformations.

If one applies arbitrary transformation of coordinates

$$(x, p) \rightarrow (\bar{x}, \bar{p}), \tag{I-16}$$

then those transformations under which

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{I-17}$$

transform into

$$\frac{d\bar{x}}{dt} = \frac{\partial \bar{H}}{\partial \bar{p}_i}, \quad \frac{d\bar{p}_i}{dt} = -\frac{\partial \bar{H}}{\partial \bar{x}_i} \tag{I-18}$$

are called <u>canonical transformations</u> (2).

Since the above come from an action principle, one is interested in the equivalence of functionals, that is, the functionals

$$\int \left[p_i \dot{x}_i - H(x, p, t) \right] dt \tag{I-19}$$

and

$$\int \left[\bar{p}_i \dot{\bar{x}}_i - \bar{H}(\bar{x}, \bar{p}, t) \right] dt. \tag{I-20}$$

One can state the following relation without proof (4): two functionals of the form

$$\int F(t, x, p) dt \tag{I-21}$$

are said to be equivalent if their integrands differ by a function of the form

$$g(t, x, p) = \frac{\partial G}{\partial t} + \sum_{i=1}^{n} \frac{\partial G}{\partial x_i} p_i = dG(t, x, p) \tag{I-22}$$

where dG is a total differential.

Taking difference of the integrands of interest,

$$\left(p_i \dot{x}_i - H\right) - \left(\bar{p}_i \dot{\bar{x}}_i - \bar{H}\right) = \frac{d}{dt} G(t, x, p) \tag{I-23}$$

or by using

$$\bar{x}_i = \bar{x}_i(x, p, t), \quad \bar{p}_i = \bar{p}_i(x, p, t) \tag{I-24}$$

one can write $G(x, p, t)$ variously as

$$G_1 = G_1(x, \bar{x}, t), \quad G_2 = G_2(x, \bar{p}, t), \quad G_3 = G_3(p, \bar{x}, t), \quad G_4 = G_4(p, \bar{p}, t).$$

Here, of most interest is G_2, so that

$$\frac{dG_2}{dt} = \frac{\partial G_2}{\partial x_i} \dot{x}_i + \frac{\partial G_2}{\partial \bar{p}_i} \dot{\bar{p}}_i + \frac{\partial G_2}{\partial t}. \tag{I-25}$$

From (I-23)

$$p_i = \partial G_2/\partial x_i \tag{a}$$

$$\bar{x}_i = \partial G_2 / \partial \bar{p}_i \tag{b}$$

$$\bar{H} = H + \frac{\partial G_2}{\partial t} \,. \tag{c}$$

(I-26)

One says that G_2 is a <u>generating function</u>, such that if it is known, the transformation

$$(x, p) \rightarrow (\bar{x}, \bar{p})$$

is determined.

Of particular interest is the generating function

$$G_1 = x_i \bar{p}_i . \tag{I-27}$$

Then (I-17a) and (I-17b) become

$$p_i = \bar{p}_i \tag{I-28}$$

$$\bar{x}_i = x_i \tag{I-29}$$

and G_1 is just the identity transformation.

Introduce now the infinitesimal contact transformation. One writes

$$(x, p) \rightarrow (\bar{x}, \bar{p})$$

as

$$\bar{x}_i = x_i + \delta x_i, \quad \bar{p}_i = p_i + \delta p_i.$$

It can be said here that interest will be in those transformations which shift the coordinates (x_i, p_i) an amount $(x_i + dx_i, \; p_i + dp_i)$ with respect to the same coordinate frame, rather than a relabeling of coordinates, such as changing to polar coordinates, etc. (5).

Consider the special generating function, G_2, where ϵ is some infinitesimal parameter of transformation,

$$G_2 = G_1 + \epsilon K(x, p) \tag{I-30}$$

or

$$G_2 = x_i \bar{p}_i + \epsilon K(x, \bar{p}). \tag{I-31}$$

One has a generating function that differs by an infinitesimal amount from the identity transformation. Then (I-26a) and (I-26b) give

$$\delta p_i = -\epsilon \frac{\partial K}{\partial x_i} \tag{I-32}$$

$$\delta x_i = \epsilon \frac{\partial K}{\partial \left[p_i + \delta p_i\right]} \tag{I-31}$$

and to first order in p_i

$$\delta x_i = \epsilon \frac{\partial K}{\partial p_i}. \tag{I-32}$$

An important observation is that if one takes $\epsilon = dt$, and K as the Hamiltonian H,

$$\delta p_i = -dt \frac{\partial H}{\partial x_i} = dt \frac{dp_i}{dt} = dp_i, \tag{I-33}$$

and

$$\delta x_i = dt \frac{\partial H}{\partial p_i} = dt \frac{dx_i}{dt} = dx_i. \tag{I-34}$$

One comes to the following conclusion: if one takes K, the generator of the infinitesimal canonical transformation, to be the Hamiltonian H, then one can change the coordinates and momenta at time t to those at time $t + dt$,

$$\left(x_i, p_i\right) \to \left(x_i + \delta x_i, \; p_i + \delta p_i\right).$$

This is then the dynamical postulate (1, 3, 4): the class of all trajectories of a physical system is determined by the unfolding-in-time of a canonical transformation.

Lastly consider the definition of the Poisson bracket for μ and ν functions of x and p,

$$[\mu, \nu]_c = \left(\frac{\partial\mu}{\partial x_i}\frac{\partial\nu}{\partial p_i} - \frac{\partial\mu}{\partial p_i}\frac{\partial\nu}{\partial x_i}\right) \tag{I-35}$$

and notice that

$$\dot{x}_i = \left[x_i, H\right], \quad \dot{p}_i = \left[p_i + H\right]_c; \tag{I-36}$$

from (I-35) it follows that

$$\left[x_i, x_j\right]_c = 0, \quad \left[p_i, p_j\right] = 0, \quad \left[x_i, p_j\right]_c = \delta_{ij}, \tag{I-37}$$

and

$$\delta p_i = dt\left[p_i H\right]_c, \quad \delta x_i = dt\left[x_i, H\right]_c. \tag{I-38}$$

These relations are very useful for the analogies to be drawn in quantum mechanics.

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CHAPTER II

THE FUNDAMENTAL PROBLEM OF NON-RELATIVISTIC QUANTUM DYNAMICS

Classical mechanics was developed in the last chapter in such a way that it would lead to an analogy with quantum mechanics. Dirac (1) makes the connection between classical and quantum dynamics very explicit. That is, dynamical variables such as the velocity and momentum can be given quantum descriptions, namely through changing the Poisson brackets into Commutator brackets. One thus assumes the following general mathematical framework of classical quantum mechanics as given: linear operators, state vectors, and their adjoints, and the probability interpretation of these quantities (1, 2).

Momentum and position can be considered physical observables in quantum theory and are represented by Hermitian operators operating in a linear vector space. The analogy with classical mechanics, set up by Dirac, comes from

$$[A, B]_c = (i\hbar)^{-1} [\hat{A}, \hat{B}] \qquad \text{(II-1)}$$

where $\hat{A}$ and $\hat{B}$ are Hermitian operators.

If one writes in analogy to equation (I-38)

$$\delta\hat{F} = \frac{i\epsilon}{\hbar}[\hat{F},\hat{G}] \tag{II-2}$$

where $\hat{G}$ is the generator of the infinitesimal unitary transformations, $\hat{F}$ becomes

$$\hat{\hat{F}} = \hat{F} + \delta\hat{F} = F + \frac{i\epsilon}{\hbar}[\hat{F},\hat{G}] = \hat{U}\hat{F}\hat{U}^{+}, \tag{II-3}$$

where

$$\hat{U} = 1 - \frac{i}{\hbar}\epsilon\hat{G}, \quad U^{+} = 1 + \frac{i}{\hbar}\epsilon\hat{G}^{+}, \tag{II-4}$$

with the conditions

$$\hat{U}U^{+} = I; \quad \hat{G} = \hat{G}^{+}. \tag{II-5}$$

The connection to the state vectors comes through the operators. Let $\hat{a}_i$ be a complete set of commuting Hermitian operators, subject them to a unitary transformation

$$\hat{\bar{a}}_i = U\hat{a}_i U^{+} \tag{II-6}$$

and their eigenvectors to

$$|\bar{a}_i{}'> = U|a_i{}'> \tag{II-7}$$

where $|a_i{}'>$ are orthogonal eigenvectors belonging to the operators a_i.

U is required to be unitary in order to preserve the probability interpretation. If

$$U = I - \frac{i}{\hbar}\epsilon\widehat{G}$$

then

$$\delta|a'\rangle = -\frac{i}{\hbar}\epsilon\widehat{G}|a'\rangle. \tag{II-8}$$

The statement of the fundamental problem of quantum dynamics and its solution in two equivalent forms is given below.

(a) Given the operators representing the observables of the system at a certain time, how does one find them at a later time?

Solution: The temporal behavior of the operators representing the observables of a physical system is determined by the unfolding-in-time of a unitary transformation

$$\dot{\widehat{F}} = (i\hbar)^{-1}[\widehat{F},\widehat{H}] + \frac{\partial\widehat{F}}{\partial t} \tag{II-9}$$

where the step has been taken of identifying H, the Hamiltonian, as the generator of the infinitesimal unitary transformation. Since H is a function of x_i and p_i, it is useful to recall that

$$\left[\widehat{x}_i,\widehat{x}_j\right] = 0, \tag{II-10}$$

$$\left[\hat{x}_i, \hat{p}_i\right] = i\hbar\delta_{ij} \quad \text{and} \tag{II-11}$$

$$\left[\hat{p}_i, \hat{p}_j\right] = 0. \tag{II-12}$$

(b) Given the initial state $|a, t>$ of the system, how is the state at time $t, |a, t>$ determined from this?

To find a solution to this question, one takes a linear operator T, the application of which gives

$$|a, t> = \hat{T}(t, t_0)\,|a, t_0>. \tag{II-13}$$

T has the properties

$$\hat{T}(t, t) = 1$$

$$\hat{T}(t_2, t_1)\hat{T}(t_1, t_0) = \hat{T}(t_2, t_0). \tag{II-14}$$

From what has been developed one can write T as an infinitesimal unitary operator with the help of the infinitesimal generator $\hat{H}(t)$ and δt the infinitesimal parameter, that is,

$$\hat{T}(t + \delta t, t) = 1 - \frac{i}{\hbar}\delta t\hat{H}(t). \tag{II-15}$$

Consider (II-13)

$$\hat{T}(t + \delta t, t)\,|a, t> = |a, t + \delta t> \tag{II-16}$$

or

$$\frac{|a, t + \delta t> - |a, t>}{\delta t} = -\frac{i}{\hbar}\hat{H}|a, t>. \tag{II-17}$$

Taking the limit as $t \rightarrow 0$,

$$\lim_{\delta t \rightarrow 0} \frac{i\hbar|a, t + \delta t> - |a, t>}{\delta t} = \hat{H}|a, t> \tag{II-18}$$

whence

$$i\hbar \frac{d|a, t>}{dt} = \hat{H}|a, t> . \tag{II-19}$$

What has been done is to identify $\hat{H}(t)$, the Hamiltonian of the system, as the generator of the time transformation. This is possible because of the analogy one is able to draw between canonical transformations in phase space and unitary transformations in Hilbert space. In essence, this leads to the way state vectors behave and hence to equation (IV-19), which is Schroedinger's equation, or to (II-9), which is Heisenberg's equation.

Consequently, (II-9) or (II-13) with (II-15) will be taken as the fundamental dynamical equations in this work.

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CHAPTER III

THE RELATIVISTIC SCHROEDINGER EQUATION

The two foregoing chapters have been presented to give a possible development of non-relativistic quantum dynamics to this point. The usual presentation of relativistic quantum dynamics starts with the development of a quasi-classical Hamiltonian; and this development is given here for contrast with the chapters to follow.

The expression for the relation between energy and momentum from special relativity is

$$E = c\sqrt{m_0 c^2 + p_x{}^2 + p_y{}^2 + p_Z{}^2} \qquad \text{(III-1)}$$

and it is "natural" to let the Hamiltonian, since it is usually equal to the energy of the system, be

$$\hat{H} = c\sqrt{m_0 c^2 + \hat{p}^2}. \qquad \text{(III-2)}$$

Then (II-20) becomes

$$i\hbar \frac{\partial \Psi}{\partial t} = c\sqrt{m_0 c^2 + \hat{p}^2}\, \Psi. \qquad \text{(III-3)}$$

It is to be noted that the positive square root was taken to avoid problems of negative energy.

There is great difficulty in dealing with (III-2) as a square-root operator (5). Further, equation (III-3) is so unsymmetrical in the order of differentiation that it cannot be generalized in a relativistic way when fields are present. The space and time coordinates also appear in an unsymmetrical form.

If holding to the first order nature of the time derivative is dispensed with, one can write

$$E^2 = c^2 \vec{p}^{\,2} + m_0^{\,2} c^4 \qquad \text{(III-4)}$$

and use the correspondence

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \ \vec{p} = -i\hbar \vec{\nabla}$$

and one gets the more symmetrical form

$$-\hbar^2 \frac{\partial^2 \Psi(x,t)}{\partial t^2} = \left(\hbar^2 c^2 \nabla^2 - m^2 c^2\right) \Psi(x,t) \qquad \text{(III-5)}$$

or the covariant form,

$$\left[\Box + \frac{m^2 c^2}{\hbar^2}\right] \Psi(x,t) = 0 \qquad \text{(III-6)}$$

where

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \; . \tag{III-7}$$

This is the Klein-Gordon equation which is sometimes known as the relativistic Schroedinger equation (3). Although every solution of (III-3) is also a solution of (III-6), the converse is not true (2). This equation is a correct one, describing spin-zero particles, if $\Psi(x, t)$ is a scalar wave function.

Here one should take note of the fact that the procedure was from quantum mechanics to relativistic quantum mechanics. Remembering that it is possible to go over to non-relativistic (classical) quantum mechanics by way of classical mechanics, one might ask whether it is possible to transform from relativistic mechanics to relativistic quantum mechanics. Historically it has not been done this way. The relation between energy and momentum is derived in the usual texts on quantum physics, but the canonical formalism which was followed above is usually not presented.

The most objectionable feature, however, is that equation (II-19) is carried, with no further assumptions, over into relativistic quantum mechanics. Equation (II-19) definitely puts time and space on an unequal footing. Relativistic mechanics treats space and time symmetrically. Even from what one knows about non-relativistic quantum mechanics, it is easy to see the symmetry cropping up, such as in the principle of uncertainty, that is

$$\Delta x \, \Delta p \geq i h$$

and

$$\Delta t \, \Delta E \geq h.$$

Time, an absolute in Newtonian dynamics, gives way to absolute "intervals," defined by

$$ds^2 = dx^2 + dy^2 + dz^2 + dx^{0^2}$$

or

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \; x^0 = ict. \qquad \text{(III-8)}$$

One can call s the "proper time." The motion of a particle is now described by a "world line" in four-dimensional space, where the coordinates are the following functions of proper time (1), where

$$x = x(s), \; y = y(s), \; z = z(s), \; t = t(s), \qquad \text{(III-9)}$$

or in four-vector forms

$$x^\mu = x^\mu(s), \; \mu = 0, 1, 2, 3.$$

One also invokes covariance of any dynamical equations. That is, the form of the equations of motion should remain the same in any frame of reference. More explicitly, the equations are covariant under Lorentz transformations.

For these reasons one can raise one small objection to equation (III-6); that is, it is just an ad hoc equation. Equation (III-6) was not derived like (II-20), yet it is important in that it is covariant. Setting as a guiding principle the fact our dynamical equation should show "manifest" covariance, a formulation will now be developed that exhibits Lorentz covariance at each step of the development (1).

A few aspects of relativistic particle dynamics should be summarized here. The four-velocity is defined by

$$x'^{\mu} = \frac{dx^{\mu}}{ds} \qquad \text{(III-10)}$$

and, calling m_0 the rest mass, one defines the four-momentum

$$\underline{P}^{\mu} = m_0 x_{\mu}' \qquad \text{(III-11)}$$

which leads to the invariant

$$\underline{P}_{\mu}\underline{P}_{\mu} = m_0^{\,2}c^2 = \underline{P}^{0^2} - \vec{\underline{P}}^2 \qquad \text{(III-12)}$$

where $\bar{p}$ is the ordinary momentum, and P^0 turns out to be the ordinary total energy, thus the relation (III-1).

It is also possible to give a quasi-classical treatment to derive the Hamiltonian (III-2) in this chapter (4). Starting with the usual assumption

$$\delta \int L dt = 0 \tag{III-13}$$

and taking

$$L = - m_0 c^2 \sqrt{1 - \beta^2}\ , \text{ where } \beta = \nu / c,$$

for a free particle, one obtains the Euler-Lagrange equations,

$$\frac{d}{dt}\left[\frac{m_0 x_i}{\sqrt{1 - \beta^2}}\right] = 0, \ P_i = \frac{m_0 x_i}{\sqrt{1 - \beta^2}}. \tag{III-14}$$

From the formal definition of the Hamiltonian

$$H = p_i \dot{x}_i - L, \tag{III-15}$$

$$H = \frac{m_0 \dot{x}_i \dot{x}_i}{\sqrt{1 - \beta^2}} + m_0 c^2 \sqrt{1 - \beta^2}\ , \tag{III-16}$$

or

$$H = \frac{m_0 c^2}{\left(1 - \beta^2\right)^{1/2}}. \qquad \text{(III-17)}$$

To get this into the "canonical" form, multiply the second equation of (II-4) by c^2, square it, square (III-17) and subtract this from it,

$$H^2 - c^2 \vec{p}^2 = \frac{m_0 c^4}{\left(1 - \beta^2\right)} - \frac{c^2 m_0 \nu^2}{\left(1 - \beta^2\right)} \qquad \text{(III-18)}$$

or

$$H^2 - c^2 p^2 = \frac{m_0 c^4 \left(1 - \beta^2\right)}{\left(1 - \beta^2\right)} \qquad \text{(III-19)}$$

or

$$H = \sqrt{c^2 \vec{p}^{\,2} + m^2 c^4}\,. \qquad \text{(III-20)}$$

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CHAPTER IV

THE HOMOGENEOUS CANONICAL FORMALISM

In relativistic mechanics the integral

$$L\big(x(t), \dot{x}(t)\big)\,dt \tag{IV-1}$$

has to be replaced by one that is invariant under the Lorentz group of linear homogeneous coordinate transformations.

In order to treat the four space-time coordinates on an equal footing, one writes the equations of the possible paths in parametric form as

$$x^{\mu} = x^{\mu}(s) \quad (\mu = 0, 1, 2, 3),$$

where s is an arbitrary scalar parameter, not necessarily the proper time.

The most logical generalization of (IV-1) is

$$\int F\big(x(s), x'(s), s\big)\,ds. \tag{IV-2}$$

The integral can have physical meaning only if it is invariant under transformation of the parameter s.

To investigate what properties F must have to make (IV-2) parameter invariant, consider another parameter σ related to s by

$$\sigma = \sigma(s), \ \sigma' > 0 \tag{IV-3}$$

and write the inverse transformation as

$$s = s(\sigma), \ s' > 0. \tag{IV-4}$$

Take

$$x^{\nu}\big(s(\sigma)\big) = \xi^{\nu}(\sigma), \ \sigma(s_1) = \sigma_1, \ \sigma(s_2) = \sigma_2, \tag{IV-5}$$

so that

$$x'^{\nu}\big(s(\sigma)\big)\, s'(\sigma) = \xi'^{\nu}(\sigma). \tag{IV-6}$$

One requires invariance in the sense that the transformed Lagrangian $\Phi\big(\sigma, \xi(\sigma), \xi'(\sigma)\big)$ should satisfy

$$\int_{s_1}^{s_2} F\big(s, x(s), x'(s)\big)\, ds = \int_{\sigma_1}^{\sigma_2} \Phi\big(\sigma, \xi(\sigma), \xi'(\sigma)\big)\, ds. \tag{IV-7}$$

Substituting s from (IV-3) and using (IV-5) and (IV-6), one obtains.

$$\int_{\sigma_1}^{\sigma_2} F\left(s(\sigma), \xi(\sigma), \frac{\xi'(\sigma)}{s'(\sigma)}\right) s'(\sigma)\, d\sigma = \Phi(\sigma, \xi(\sigma), \xi'(\sigma))\, d\sigma. \qquad \text{(IV-8)}$$

Thus it is seen that if Φ is defined as the integrand on the left of (IV-8), the invariance requirement is satisfied. (IV-8) gives the required transformation property of F upon a change in parameter:

$$\Phi(\sigma, \xi, \xi') = F\left(s(\sigma), \xi, \frac{\xi'}{s'(\sigma)}\right) s'(\sigma). \qquad \text{(IV-9)}$$

If F is positive homogeneous of degree 1 in the x_μ' and does not depend explicitly on s, then (IV-9) shows that Φ does not depend explicitly on σ, and

$$\Phi(\xi, \xi') = F(\xi, \xi'). \qquad \text{(IV-10)}$$

Thus in this case the transformed Lagrangian is the same function of the new variables as F is of the old ones (6).

There is the possibility of using a nonhomogeneous Lagrangian, but it will not be dealt with here (4, 5).

Following Synge (7), one may introduce the homogeneous Lagrangian function

$$\Lambda\left(x_0, x_1, x_2, x_3, x_0', x_1', x_2', x_3'\right).$$

For this function to have the proper dynamics, it should be positive homogeneous of degree 1 in the derivatives x_μ', i.e.,

$$\Lambda(x, kx') = k\Lambda(x, x') \qquad (k > 0) \tag{IV-11}$$

and, by Euler's Theorem for homogeneous functions,

$$x_\mu' \frac{\partial \Lambda}{\partial x'_\mu} = \Lambda. \tag{IV-12}$$

Also, one may define the Lagrangian action along any directed curve in four-space as

$$A_L = \int_{s_1}^{s_2} \Lambda(x, x')\, ds. \tag{IV-13}$$

As before, one requires the variation of this action to vanish, leading to

$$\delta A_L = \int_{s_1}^{s_2} \left(\frac{\partial \Lambda}{\partial x_\mu} \delta x_\mu + \frac{\partial \Lambda}{\partial x'_\mu} \delta x_\mu' \right) ds = 0. \tag{IV-14}$$

Integration by parts gives

$$\delta A_L = \left[\frac{\partial \Lambda}{\partial x_\mu'} \delta x_\mu \right]_{s_1}^{s_2} - \int_{s_1}^{s_2} \left(\frac{d}{ds} \frac{\partial \Lambda}{\partial x_\mu'} - \frac{\partial \Lambda}{\partial x_\mu} \right) \delta x_\mu \, ds. \tag{IV-15}$$

If the end points are held fixed, the first term vanishes and since δx_μ is arbitrary, the curve in space-time which satisfied (IV-14) is a solution of the equation

$$\frac{d}{ds}\frac{\partial \Lambda}{\partial x'_\mu} - \frac{\partial \Lambda}{\partial x_\mu} = 0. \qquad \text{(IV-16)}$$

Introduce the usual Hamiltonian formalism in the manner of classical dynamical formalism. If one defines the momentum in the usual manner,

$$\underline{P}_\mu = \frac{\partial \Lambda}{\partial x'_\mu}, \qquad \text{(IV-17)}$$

the definition of the "Hamiltonian" becomes by use of the relation (IV-11)

$$\Omega = x'^\mu \underline{P}_\mu - \Lambda = x'^\mu \frac{\partial \Lambda}{\partial \underline{P}_\mu} - \Lambda = \Lambda - \Lambda = 0. \qquad \text{(IV-18)}$$

Attention should also be drawn to the fact that in order to have a canonical formalism, equation (IV-17) must be solvable for the x^μ's in terms of all the other variables, i.e.,

$$x'^\mu = f\left(\underline{P}_\mu, x_\mu\right) \qquad \text{(IV-19)}$$

or P^μ and x^μ are canonically conjugate.

In order to solve (IV-17) for the x'^{μ}s, it is required that

$$\left\| \frac{\partial \underline{P}_{\mu}}{\partial x^{\mu}} \right\| \neq 0. \qquad \text{(IV-20)}$$

(IV-20) comes from the following mathematical definition. Consider the equation

$$x_n = x_n(y_1 \ldots y_n)$$

and by definition, n functions of n variables are said to be independent when their functional determinant, Δ, does not vanish identically, where

$$\Delta = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} .$$

From (IV-12) differentiation with respect to x'^{ν} yields

$$x'^{\mu} \frac{\partial^2 \Lambda}{\partial x'^{\mu} \partial x'^{\nu}} = 0. \qquad \text{(IV-21)}$$

If not all the x'^{μ} vanish (IV-21) shows that

$$\left\| \frac{\partial^2 \Lambda}{\partial x'^{\mu} \partial x'^{\nu}} \right\| = 0. \tag{IV-22}$$

Therefore, not all the x^{μ}'s can be separately solved as functions of P and x. One needs a further relation

$$\Omega(x, \underline{P}) = 0. \tag{IV-23}$$

Following Synge (7), instead of imposing a Lagrangian $\Lambda(x, x')$, impose an entity that is called an energy equation,

$$\Omega(x, \underline{P}) = 0.$$

This equation arises as a necessary consequence of a search for a canonical formalism in four-space where one treats time on the same footing as the space coordinates, and the need to find a canonical formalism in analogy with the non-relativistic case (1, 2, 3).

Define a new kind of action, the Hamiltonian action, along curves in four-space,

$$A_H = \int P_{\mu}\, dx^{\mu}. \tag{IV-24}$$

Vary (IV-24) under the constraint (IV-23), i.e.

$$\delta A_H = \delta \int \underline{P}_{\mu}\, dx^{\mu} + \Omega\, dw \tag{IV-25}$$

where dw is an infinitesimal Lagrange multiplier. Variation, after partial integrals, gives

$$\delta A_H = \left[\underline{P}_\mu \delta x^\mu\right] + \int \left(\delta \underline{P}_\mu dx_\mu - \delta x_\mu d\underline{P}_\mu + dw \frac{\partial \Omega}{\partial x_\mu} \delta x_\mu + dw \frac{\partial \Omega}{\partial \underline{P}_\mu} \delta \underline{P}_\mu \right). \tag{IV-26}$$

For fixed end-points and arbitrary x_μ and $\underline{P}_\mu$

$$\frac{dx_\mu}{dw} = \frac{\partial \Omega}{\partial \underline{P}_\mu}, \quad \frac{d\underline{P}_\mu}{dw} = - \frac{\partial \Omega}{\partial x_\mu}. \tag{IV-27}$$

The curve which satisfies (IV-27),

$$x = x_\mu(w)$$

will have associated with it the vector field

$$\underline{P}_\mu = \underline{P}_\mu(w)$$

and one may take w as the parameter s, which is specified by the function Ω.

The equations (IV-27) which are a consequence of the conditions $\left(\delta A_H = 0,\ \Omega = 0\right)$ are called also the canonical equations of motion. These dynamics may be called Ω-dynamics.

Once again canonical transformations have the same meaning, i. e., for

$$\frac{dx_\mu}{ds} = \frac{\partial\Omega}{\partial \underline{P}_\mu} \qquad \frac{d\underline{P}_\mu}{ds} = -\frac{\partial\Omega}{\partial x_\mu}$$

$$\frac{d\overline{x}_\mu}{ds} = \frac{\partial\Omega}{\partial \overline{\underline{P}}_\mu} \qquad \frac{d\overline{\underline{P}}_\mu}{ds} = -\frac{\partial\Omega}{\partial \overline{x}_\mu} \tag{IV-28}$$

where Ω is treated as an invariant.

Examination of the integral (IV-25) in light of these transformations and the theorem of Chapter II gives the same conclusions. Thus one finds a generating function G for canonical transformations.

To get an infinitesimal contact transformation, introduce the generating function

$$G(x, \overline{\underline{P}}) = x^\mu \overline{\underline{P}}_\mu + F(x, \underline{P})\, ds, \tag{IV-29}$$

where F is an infinitesimal generating function and where ds is an infinitesimal parameter. According to

$$\underline{P}_\mu = \frac{\partial G}{\partial x_\mu}, \quad \overline{x}'^\mu = \frac{\partial G}{\partial \overline{\underline{P}}_\mu}$$

$$\overline{x}_\mu = x_\mu + \delta x_\mu, \quad \underline{P}_\mu = \underline{P}_\mu + \delta \underline{P}_\mu \tag{IV-30}$$

then to first order (IV-29) becomes

$$d\underline{P}_\mu = \overline{\underline{P}}_\mu - \underline{P}_\mu = -ds \frac{\partial F}{\partial \underline{P}_\mu}$$

$$dx_\mu = \overline{x}_\mu - x_\mu = ds \frac{\partial F}{\partial \overline{\underline{P}}_\mu}. \qquad \text{(IV-31)}$$

One sees that when $F(x, \underline{P}) = \Omega(x, \underline{P})$

$$\frac{dx_\mu}{ds} = \frac{\partial \Omega}{\partial \underline{P}_\mu}, \quad \frac{d\underline{P}_\mu}{ds} = -\frac{\partial \Omega}{\partial x_\mu}. \qquad \text{(IV-32)}$$

Considering (IV-32) in the light of Chapter II, and regarding (x, P) and $(\overline{x}, \overline{P})$ as two different points relative to the same set of coordinate axes, one may say that Ω generates an infinitesimal increment in the coordinates and the momenta, $(x, \underline{P}) \rightarrow (x + \delta x, \underline{P} + \delta \underline{P})$.

The canonical equations themselves may be written

$$x_\mu' = \left[x_\mu, \Omega\right]_c, \quad \underline{P}_\mu' = \left[\underline{P}_\mu, \Omega\right]_c,$$

where $[\ ]_c$ denotes the usual Poisson brackets.

A note about the equivalence of Ω-dynamics and Λ-dynamics will conclude this section (7). We have seen how (IV-23) gives a derivation of Ω from the Lagrangian. So start with

$$\Omega = 0 \qquad \text{(IV-33)}$$

and restrict P by imposing

$$\frac{dx_\mu}{ds} = \zeta \frac{\partial \Omega}{\partial \underline{P}_\mu} \tag{IV-34}$$

where ζ is an undetermined factor. Solve (IV-33) and (IV-34) for $\underline{P}_\mu$ and ζ as functions of the x^μ's and x'^μ's and define Λ as

$$\Lambda = \underline{P}_\mu x'^\mu$$

Then the Hamiltonian action may be written

$$A_H = \int \underline{P}_\mu dx^\mu = \int \underline{P}_\mu x'^\mu ds = \int \Lambda(x, x')\, ds \tag{IV-35}$$

and one sees that $A_H = A_L$ from the homogeneity of Λ; the equivalence is evident.

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CHAPTER V

KLEIN-GORDON EQUATION FROM Ω-DYNAMICS

There is now available a manifestly covariant method of describing dynamics in four-dimensional space-time. As long as one deals with a point particle free from interaction with other particles (except for possible direct collisions), the four coordinates (x_0, x_1, x_2, x_3) specify the state of the system at a certain value of the parameter $s = s_0$, and a "later" state of the system may be determined at a monotonically increased value of the parameter s.

The dynamical problem takes the form: Given the operators of the system at a certain value of the parameter s, how does one find them at a monotonically increased value of the parameter?

One may take the solution as: The parametric development of the operators representing the observables of a physical system is determined by a parametric unfolding of a unitary transformation

$$\hat{F}' = (i\hbar)^{-1} [\hat{F}, \hat{\Omega}] + \frac{\partial \hat{F}}{\partial s}. \tag{V-1}$$

One also has

$$i\hbar \frac{d\hat{x}_\mu}{ds} = \left[\hat{x}_\mu, \hat{\Omega}\right], \quad i\hbar \frac{\hat{P}_\mu}{ds} = \left[\hat{P}_\mu, \hat{\Omega}\right] \tag{V-2}$$

and

$$\left[\hat{x}_\mu, \hat{x}_\nu\right] = 0, \left[\hat{\underline{P}}_\mu, \hat{\underline{P}}_\nu\right] = 0$$

$$\left[\hat{x}_\mu, \hat{\underline{P}}_\mu\right] = g_{\mu\nu} i \hbar. \tag{V-3}$$

One is more interested, however, in the alternate approach to the Heisenberg form of quantum dynamics, the Schroedinger picture. For this purpose, one supposes that the state vector $|x_\mu, s>$ is given at a certain value of the parameter, s_0, and it is wished to determine the state of the system at a later value of the parameter, s.

Take then a linear operator s, the application of which gives

$$|x_\mu, s> = \hat{S}(s, s_0)|x_\mu, s_0>. \tag{V-4}$$

$\hat{S}$ has the properties

$$\hat{S}(s, s) = 1,$$

$$\hat{S}(s_2, s_1)\,\hat{S}(s_1, s_0) = \hat{S}(s_2, s_0). \tag{V-5}$$

With the help of the infinitesimal generator $\hat{\Omega}$ one can write $\hat{S}$ as

$$\hat{S}(s + \delta s, s) = 1 - \frac{i}{\hbar}\,\delta s \hat{\Omega}. \tag{V-6}$$

So that

$$\hat{S}(s + \delta s)\,|x_\mu, s> = |x_\mu, s + \delta s>, \tag{V-7}$$

and from (V-6)

$$\frac{|x_\mu, s + \delta s> - |x_\mu, s>}{\delta s} = \frac{i}{\hbar}\hat{\Omega}\,|x_\mu, s>. \tag{V-8}$$

Taking the limit as $\delta s \to 0$,

$$\lim_{\delta s \to 0} \frac{i\hbar\,|x_\mu,\ s + \delta s> - |x_\mu,\ s>}{\delta s} = \hat{\Omega}\,|x_\mu,\ s> \tag{V-9}$$

giving

$$i\hbar\,\frac{\partial |x_\mu, s>}{\partial s} = \Omega |x_\mu, s>. \tag{V-10}$$

This last equation is now taken as the evolution equation for relativistic quantum particle dynamics. It has the interesting property that when one applies the energy relation Ω as an operator, one obtains the result that the state vector $|x_\mu, s>$ is not a function of the parameter s, that is, (V-10) becomes

$$\hat{\Omega}|x_\mu> = 0. \tag{V-11}$$

Let us now find an explicit energy relation, $\Omega = 0$, for a free spinless particle. In order to do this, one uses the free homogeneous Lagrangian (1),

$$\Lambda = m_0 c \sqrt{g_{\mu\nu} x_\mu{}' x_\nu{}'} \,.$$

The momentum from the usual definition is

$$\underline{P}_\mu = \frac{\partial \Lambda}{\partial x_\mu{}'}$$

$$= \frac{m_0 c x_\mu{}'}{\sqrt{g_{\mu\nu} x_\mu{}' x_\nu{}'}} \tag{V-14}$$

and, following the procedure of Chapter IV, one wishes to eliminate the velocities from this equation in order to find Ω and hence the canonical formulation. To do this just square (V-14),

$$\underline{P}_\mu \underline{P}_\mu = m_0{}^2 c^2 \frac{g_{\mu\nu} x_\mu{}' x_\nu{}'}{g_{\mu\nu} x_\mu{}' x_\nu{}'} \tag{V-15}$$

so the energy relation is

$$\Omega = \underline{P}_\mu \underline{P}_\mu - m_0{}^2 c^2 = 0 \,.$$

Thus the relativistic "Schroedinger" equation will become

$$\left[\hat{\underline{P}}_\mu \hat{\underline{P}}_\mu - m_0{}^2 c^2 \right] | x_\mu > = 0,$$

or a more usual form is given by

$$\underline{P}_\mu \underline{P}_\mu = -m_0^2 c^2 \frac{x_\mu' x_\mu'}{x_\mu' x_\mu'} \tag{V-15}$$

then the energy relation is

$$\Omega = \underline{P}_\mu \underline{P}_\mu + m_0^2 c^2. \tag{V-16}$$

Thus one goes over to the quantum mechanical relativistic "Schroedinger" equation by making (V-16) an operator equation, i.e.,

$$\hat{\Omega} | x_\mu > = 0. \tag{V-17}$$

One obtains a more usual form by using the correspondence

$$\underline{P}_\mu \rightarrow -i\hbar \frac{\partial}{\partial x_\mu}$$

which gives

$$\hat{\Omega} = \hbar^2 \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu} - m_0^2 c^2.$$

Letting

$$\varkappa^2 = \frac{m_0^2 c^2}{\hbar^2}$$

and

$$\Box = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}$$

(V-17) becomes

$$\left[\Box - \chi^2\right]|x_\mu> = 0.$$

This is, of course, the relativistic equation for scalar particles known as the Klein-Gordon equation. This equation was derived without placing time and space on different footings. Thus we have a "manifestly" covariant derivation of the Klein-Gordon equation.

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CHAPTER VI

RELATIVISTIC SPIN EQUATIONS

The question now arises that if the Klein-Gordon equation derives naturally from the homogeneous formalism, can one use this formalism to suggest relativistic particle equations with spin? It turns out that to ask this question implies the further question; is there a classical analog to spin?

One route to answering the above has been to suppose that one has a fluid droplet described relativistically with internal rotational motion (1). This formulation leads to complications, such as non-local properties.

Also one might just fix an intrinsic spin variable to a point in four-space (5), or one might use a Lagrangian with higher derivatives (2, 6). However, these apparently still do not have a good quantum transition scheme.

If one, on the other hand, just places trust in the homogeneous formalism and makes some assumptions, it might be possible to derive some spin equations in a quasiclassical manner.

One can proceed by analogy with what was done for the K. G. energy relation. Suppose one has an energy relation that is linear in the momentum; say it is of the form

$$\Omega = \Gamma_{\mu} \underline{P}^{\mu} \pm m_0 = 0. \tag{VI-1}$$

This might be just guessed or might be considered a result of "factoring" the relativistic expression for energy, i. e., the Klein-Gordon energy relation

$$\Omega = \underline{P}_{\mu} \underline{P}_{\mu} \pm m_0^{2} = 0 \tag{VI-2}$$

in the following manner. Introduce some hypercomplex numbers, Γ_{μ}, such that

$$\left(\Gamma_{\mu} \underline{P}_{\mu} + m_0\right) \left(\Gamma_{\mu} \underline{P}_{\mu} - m_0\right) = \underline{P}_{\mu} \underline{P}_{\mu} - m_0^{2}. \tag{VI-3}$$

(VI-3) requires the Γ_{μ} to satisfy an expression of the form

$$\Gamma_{\mu} \Gamma_{\nu} + \Gamma_{\nu} \Gamma_{\mu} = 2 g_{\mu\nu} \tag{VI-4}$$

and hence one obtains two "linearized" energy relations

$$\Omega^{(1)} = \Gamma_{\mu} \underline{P}_{\mu} + m_0 = 0 \tag{VI-5}$$

and

$$\Omega^{(1)} = \Gamma_\mu \underline{P}_\mu - m_0 = 0. \tag{VI-6}$$

With the above new expressions (the plus sign will be used in what follows) and using the rules for deriving Λ when Ω is known, one has

$$x_\mu' = \frac{1}{4}\zeta \frac{\partial\Omega}{\partial \underline{P}_\mu} = \frac{1}{4}\zeta\Gamma_\mu, \tag{VI-7}$$

where ζ is a factor to be determined and the 1/4 is introduced for convenience. Solving for ζ with no summation implied over the μ's,

$$\frac{1}{4}\zeta = x_\mu{}'\Gamma_\mu{}^{-1} \tag{VI-8}$$

and then summing over μ

$$\frac{1}{4}\sum_\mu \zeta = x_\mu{}'\Gamma_\mu^{-1}. \tag{VI-9}$$

or

$$\frac{1}{4}(4)\zeta = \zeta = x_\mu \Gamma_\mu^{-1}. \tag{VI-10}$$

Accordingly, one writes

$$\Lambda^{(1)} = x_\mu{}' \underline{P}^\mu$$

$$= \zeta \Gamma_\mu \underline{P}^\mu, \qquad \text{(VI-11)}$$

or from (VI-1)

$$\Lambda^{(1)} = \zeta m_0. \qquad \text{(VI-12)}$$

Using ζ, $\Lambda = m_0 x_\mu \Gamma_\mu^{-1}$ gives

$$\Lambda^{(1)} = m_0 x'_\mu \Gamma_\mu^{-1} \qquad \text{(VI-13)}$$

In relativistic classical mechanics it is well known that the invariant action principle leads to a Lagrangian of the form

$$\Lambda = m_0 \sqrt{g_{\mu\nu} x_\mu{}' x_\nu{}'} \qquad \text{(VI-14)}$$

which follows from the Minkowski metric

$$s' = \left(\frac{ds}{d\mu}\right) = g_{\mu\nu} x_\mu{}' x_\mu{}' \qquad \text{(VI-15)}$$

which in turn is a special case of the general Riemannian metric

$$ds^2 = g_{\mu\nu}(x)\ dx^\mu\ dx^\nu. \qquad \text{(VI-16)}$$

Letting Γ_μ^{-1} be denoted by α_μ, equation (VI-6) can then be written

$$\Lambda^{(1)} = m_0\, \alpha_\mu x_\mu' . \tag{VI-17}$$

This has the appearance of a linearized form of the Minkowski or Riemannian metrics; i. e., if one wanted to obtain a linear metric from these "classical" metrics, one might attempt to write a "factored" form of Λ, thus let

$$\alpha_\mu x_\mu' = s' \tag{VI-18}$$

or

$$\frac{1}{s'}\, \alpha_\mu x_\mu' = s' \tag{VI-19}$$

where I is the unit matrix (3). The Minkowski metric (VI-15) can be written

$$\left(g_{\mu\nu} x'^\mu x'^\nu\right) I = s'^2 \tag{VI-20}$$

so that using (VI-19), (VI-20) becomes

$$g_{\mu\nu} x'^\mu x'^\nu \frac{1}{s'}\, \alpha_\sigma x_\sigma' = s'^2 \tag{VI-21}$$

or, letting $\mu = \lambda_1$, $\nu = \lambda_2$, $\sigma = \lambda_3$,

$$g_{\lambda_1 \lambda_2} x'^{\lambda_1} x'^{\lambda_2} x'^{\lambda_3} \left(\frac{1}{s'}\right) \alpha_{\lambda_3} = s'^2. \tag{VI-22}$$

If one continues iterating on the left with (VI-19),

$$\left(\frac{1}{s'}\right)^{n-2} \sum_{\lambda_1 \ldots \lambda_n} g_{\lambda_1 \lambda_2} \alpha_{\lambda_3} \ldots \alpha_{\lambda_n} x'^{\lambda_1} \ldots x'^{\lambda_n} = s'^2 \tag{VI-23}$$

so that dropping the sum and simplifying gives

$$g_{\lambda_1 \lambda_2} \alpha_{\lambda_3} \ldots \alpha_{\lambda_n} x'^{\lambda_1} \ldots x'^{\lambda_n} = s'^n. \tag{VI-24}$$

Summing over all $n!$ permutations of λ_n gives

$$\sum_{\{P\}} \left[g_{\lambda_1 \lambda_2} \alpha_{\lambda_3} \ldots \alpha_{\lambda_n} x'^{\lambda_1} \ldots x'^{\lambda_n} \right] = n! s^n. \tag{VI-25}$$

Now if one considers

$$\alpha_{\lambda_n} x_{\lambda_n}' = s' \tag{VI-26}$$

and multiplies through by

$$\alpha_{\lambda_1}\cdots\alpha_{\lambda_{n-1}} x_{\lambda_1}'\cdots x_{\lambda_{n-1}}' \tag{VI-27}$$

this gives

$$\alpha_{\lambda_1}\cdots\alpha_{\lambda_{n-1}} x'^{\lambda_1}\cdots x'^{\lambda_{n-1}} = s'\alpha_{\lambda_1}\cdots\alpha_{\lambda_{n-1}} x'^{\lambda_1}\cdots x'^{\lambda_n}$$

$$= s'^2\alpha_{\lambda_2}\cdots\alpha_{\lambda_{n-1}} x_{\lambda_2}'\cdots x_{\lambda_n}'$$

$$= s'^n. \tag{VI-28}$$

Summing over all permutations

$$\sum_{\{P\}}\left[\alpha_{\lambda_1}\cdots\alpha_{\lambda_{n-1}} x_{\lambda_1}'\cdots x_{n-1}'\right] = n!\,s^n. \tag{VI-29}$$

Equating (VI-29) and (VI-25)

$$\sum_{\{P\}}\left[\left(\alpha_{\lambda_1}\alpha_{\lambda_2} - g_{\lambda_1\lambda_2}\right)\alpha_{\lambda_3}\cdots\alpha_{\lambda_n}\right] x_{\lambda_1}'\cdots x_{\lambda_n}' = 0 \tag{VI-30}$$

which implies

$$\sum_{\{P\}}\left[\alpha_{\lambda_1}\alpha_{\lambda_2} - g_{\lambda_1\lambda_2}\right]\alpha_{\lambda_3}\cdots\alpha_{\lambda_n} = 0. \tag{VI-31}$$

These are the conditions on the α's that follow from factoring the metric of special relativity in a general way. It should be noted that for only two α's, equation (VI-30) becomes

$$\alpha_\mu \alpha_\nu + \alpha_\upsilon \alpha_\mu = 2 g_{\mu\nu}. \tag{VI-32}$$

Which is the same as the condition on the Γ's. One can then see that these two conditions (VI-4) and (VI-32) together with (VI-13) implies that $\alpha_\mu^{\ 2} = 1$, $\Gamma_\mu^{\ 2} = 1$ or $\Gamma_\mu = \Gamma_\mu^{\ -1}$ so that $\alpha_\mu = \Gamma_\mu$.

One now can obtain Ω from either Λ or $\Lambda^{(1)}$. Using i. e.,

$$\Lambda = m_0 \sqrt{g_{\mu\nu} x'^{\mu} x'^{\nu}} \tag{VI-33}$$

with the defining relation

$$\underline{P}_\mu = \frac{\partial \Lambda}{\partial x_\mu'} \tag{VI-34}$$

gives

$$\underline{P}_\mu = \frac{m_0 x_\mu'}{\sqrt{g_{\mu\nu} x'^{\mu} x'^{\nu}}}$$

and if one introduces Γ_μ at this point by multiplying through by Γ_μ and summing over μ

$$\underline{P}_\mu \Gamma_\mu = \frac{m_0 x_\mu{}' \Gamma_\mu}{\sqrt{g_{\mu\nu} x'^\mu x'^\nu}} . \tag{VI-35}$$

One can "cut off" here if one assumes that the Γ_μ's introduced the factoring of

$$\sqrt{g_{\mu\nu} x_\mu{}' x_\nu{}'}$$

or that

$$\frac{m_0 x_\mu{}' \Gamma_\mu}{\sqrt{g_{\mu\nu} x'^\mu x'^\nu}} = m_0 \tag{VI-36}$$

which gives

$$\Omega = \underline{P}_\mu \Gamma_\mu - m_0 = 0, \tag{VI-37}$$

where one has eliminated the velocities and thus has obtained an energy relation. This is to be contrasted with the derivation of $\Omega = \underline{P}_\mu \underline{P}_\mu \pm m_0{}^2$ from $\Lambda = \sqrt{g_{\mu\nu} x'^\mu x'^\nu}$ in the previous chapter.

Using

$$\Lambda^{(1)} = \pm m_0 \alpha_\mu x_\mu'$$

with the following definition of $\underline{P}_\mu$

$$\underline{P}_\mu = \pm\frac{1}{4}\frac{\partial\Lambda^{(1)}}{\partial x_\mu} = \frac{1}{4} m_0 \alpha_\mu$$

or using $\alpha_\mu = \Gamma_\mu^{-1}$ and summing gives

$$\Gamma^\mu \underline{P}_\mu = \pm\frac{1}{4}\Sigma m_0 = \pm\frac{1}{4}(4) m_0$$

or

$$\Omega = \Gamma_\mu \underline{P}_\mu \pm m_0 = 0, \qquad \text{(VI-38)}$$

since the energy relation is the equation one obtains when the velocities are eliminated.

Using the dynamical relation from Chapter V, that is,

$$\hat{\Omega}\ |x_\mu, q_\gamma\rangle = 0, \qquad \text{(VI-39)}$$

where q_γ is an "intrinsic" variable, the wave equation can be obtained.

Using (VI-38), there results

$$i\hbar \frac{\partial |x_\mu, q_\gamma>}{\partial s} = \left[\Gamma_\mu \underline{P}_\mu + m_0\right] |x_\mu q_\gamma> = 0, \tag{VI-40}$$

and if the q_γ label that state vector, such as

$$|x_\mu, q_\gamma> = |x_\mu>_\gamma,$$

the wave functions will become column vectors. Hence the Γ's would become matrices. It is well known that matrices are usually used to represent hypercomplex numbers.

For the case of $n = 2$ and letting $\lambda_1 = \mu$, $\lambda_2 = \nu$ then (VI-31) becomes, with $\alpha_\mu = \Gamma_\mu$

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2g_{\mu\nu},$$

which are the conditions on the Dirac Γ_μ's, and (VI-40) becomes the Dirac equation for the electron. For $n = 3, 4, \ldots$ one will have the relations between the Γ's for the higher spin Duffin-Kemmer equations for mesons (3).

In closing this chapter the parallel between the "linear" metric (VI-17) and that of "Wave Geometry" should be drawn (4).

The fundamental idea is that just as

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

characterizes the macroscopic space-time, perhaps the metric ds and wave function Ψ characterize microscopic space-time, as

$$ds\,\Psi = \gamma_\mu\, dx^\mu\, \Psi$$

which is a possible kind of linearization of the general metric.

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CHAPTER VII

CONCLUSION

Classical canonical formalism was developed and the usual transition schema to quantum dynamics presented. The question of transition from relativistic mechanics to relativistic quantum dynamics was answered by developing a homogeneous formalism which is relativistically invariant; using this formalism the Klein-Gordon equation was derived as the relativistic analog of the Schroedinger equation. Using this formalism further, a method of generating other relativistic equations (with spin) was presented.

It would be desirable now to study the Heisenberg formulation of quantum dynamics, in particular, the fact that for the case of the Dirac equation, the velocity and momentum are not proportional to each other, a fact that was not really used in developing the "linearized" metric.

The "linearized" metric itself is of interest. Just what kind of geometry does it represent? What other possible kinds of metrics might be explored with the formalism that has been developed?

The question of the relationship of this approach to formalisms which use Lagrangians with higher derivatives to represent equations with spin is also of interest (1, 2).

Lastly, zero mass particles might be studied using null geodesics, though a new parameter different from ds would have to be introduced.

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SYMBOLS

The following symbols are used throughout the discussion of the problem.

$\hat{a}$	Hermitian operator
c	speed of light in vacuum
E	total energy
G	generating function
$g_{\mu\nu}$	metric tensor
H	classical Hamiltonian
$\hbar$	$\frac{h}{2\pi}$ = Planck constant
i	imaginary unit
L	classical Lagrangian
m_0	proper mass
$\underline{P}$	four-momentum
p	classical momentum
q	intrinsic quantum variable
$\hat{S}$	parametric operator
s	element of arc length
$\hat{T}$	time operator

t	time
x	coordinate
α	hyper complex number
Γ	hyper complex number
γ	hyper complex number
ζ	undetermined factor
Λ	homogeneous Lagrangian
ξ	coordinate
σ	arbitrary parameter
Φ	integrand in action integral
Ψ	wave function
Ω	energy relation

Subscripts:

μ	0, 1, 2, 3
ν	0, 1, 2, 3
λ	1, . . . , n

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