LATTICES

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LATTICES

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CHAPTER I

INTRODUCTION

Lattice theory as it is known today was first introduced by E. Schroder in 1890, even though in 1824, while researching in mathematical logic, G. Boole introduced an important class of lattices called Boolean Algebras. Several years later, in 1897, R. Dedekind arrived at the same conclusions as Schroder did. Dedekind was credited with the discovery of distributive and modular lattices. As a result of this early work and large contributions made later by G. Birkhoff, lattice theory became recognized as a substantial branch of abstract algebra.

Lattice theory has applications in various areas of meth- ematics. For example, in the study of lattices of subgroups of groups, lattices have been helpful in studying the structure of a group. Birkhoff, in his book on lattice theory, discusses applications of lattice theory to the areas of logic, prob- ability, functional analysis, and topology. Lattice theory also has applications in theoretical physics, particularly in the areas of quantum mechanics and relativity.

Because lattice theory is so vast, the primary purpose of this paper will be to present some of the general properties of lattices, exhibit examples of lattices, and discuss the prop- erties of distributive and modular lattices. Chapter I, in addition to containing a brief history of lattice theory, will contain the basic definitions from abstract algebra needed to develop lattices. Chapter II will be devoted to the development of some of the general properties of lattices, and Chapter III will be primarily concerned with distributive and modular lattices.

The undefined notions in this paper will be those of an "ordered pair" and a "set."

Let each of A and B be a set.

Definition 1.1. The Cartesian product of the set A by the set B, denoted by A x B, is $\{(a,b): a \in A,\ b \in B\}$.

Definition 1.2. A relation R on A,B is any subset of A x B.

If A and B are the same set, the definition will read as follows: a relation defined on A is any subset of A x A. Throughout the paper, $(a,b) \in R$ and aRb will mean the same and will be used interchangeably.

Definition 1.3. The domain of a relation R on A,B is $\{a \in A:$ there exists a $b \in B$ such that $aRb\}$.

Definition 1.4. The range of a relation R on A,B is $\{b \in B:$ there exists an $a \in A$ such that $aRb\}$.

Definition 1.5. The relation f on A,B is said to be a function on A,B if $(a,b) \in f$ and $(a,c) \in f$ implies $b = c$.

Definition 1.6. The function f on A,B is said to be reversible if $(a,b) \in f$ and $(c,b) \in f$ implies $a = c$.

Definition 1.7. A binary operation 0 on a set L is a function whose domain is L x L and whose range is a subset of L.

Definition 1.8. The relation R on the set L is said to be an ordering relation if

i) for each $a \in L$, $(a,a) \in R$;

ii) for each pair $a,b \in L$, $(a,b) \in R$ and $(b,a) \in R$ imply $a = b$; and if

iii) for each triplet of elements a, b and c in L, $(a,b) \in R$ and $(b,c) \in R$ imply $(a,c) \in R$.

For the relation "$\leq$," $a \leq b$ will be written instead of $(a,b) \in \leq$.

Definition 1.9. The relation $\overline{R}$ on a set L is said to be the converse of the relation R on L if for x, $y \in L$, then $(x,y) \in \overline{R}$ if and only if $(y,x) \in R$.

Definition 1.10. Suppose that R is a relation defined on the set L. Then L is said to be partially ordered if R is an ordering relation.

Theorem 1.1. The converse of an ordering relation is an ordering relation.

Proof. Let R be an ordering relation defined on the set L, and let $\overline{R}$ be the converse of $\overline{R}$ on L. Let $x \in L$. Then $(x,x) \in R$ and, by Definition 1.8, $(x,x) \in \overline{R}$. Thus, property (i) of Definition 1.8 is satisfied. If x, y is any pair in L such that $(x,y) \in \overline{R}$ and $(y,x) \in \overline{R}$, then $(y,x) \in R$ and $(x,y) \in R$. Since R is an ordering relation, $x = y$. Thus, property (ii) is satisfied. If x, y, z is any triplet in L such that $(x,y) \in \overline{R}$ and $(y,z) \in \overline{R}$, then $(y,x) \in R$ and $(z,y) \in R$. Thus, $(z,x) \in R$, but this implies $(x,z) \in \overline{R}$. Hence, $\overline{R}$ is an ordering relation.

Let P denote the set of positive integers, and let R be a relation defined by the following: for each pair of elements x, y P, xRy if and only if x is a divisor of y. Clearly, R is an ordering relation. Hence, the set P is partially ordered by R.

CHAPTER II

GENERAL PROPERTIES OF LATTICES

In this chapter, lattices in general, various types of complemented lattices, and chains are discussed.

<u>Definition 2.1</u>. A set L is called a lattice if there are defined two binary operations, meet and join, which assign to every pair a, b of elements of L, uniquely an element $a \wedge b$ (meet of a and b) and an element $a \vee b$ (join of a and b) such that the following lattice axioms are satisfied. Let a, b and $c \in L$. Then

L1: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, L2: $(a \vee b) \vee c = a \vee (b \vee c)$

L3: $a \wedge b = b \wedge a$, L4: $a \vee b = b \vee a$

L5: $a \wedge (a \vee b) = a$, L6: $a \vee (a \wedge b) = a$.

<u>Theorem 2.1</u>. If L is a lattice, then $a \wedge a = a$ and $a \vee a = a$ for all a in L.

<u>Proof</u>. Let L be a lattice and $1 \in L$. Then by Axiom L5, $1 = 1 \wedge (1 \vee 1)$ and $1 \vee 1 = 1 \vee [1 \wedge (1 \vee 1)]$. By Axiom L6, $1 \vee [1 \wedge (1 \vee 1)] = 1$. Hence, $1 \vee 1 = 1$. To show that $1 \wedge 1 = 1$, note that $1 = 1 \vee (1 \wedge 1)$. Then $1 \wedge 1 = 1 \wedge [1 \vee (1 \wedge 1)] = 1$.

<u>Corollary</u>. If a and b are elements of the lattice L, then $a \wedge b = a \vee b$ if and only if $a = b$.

<u>Proof</u>. Let a and b be elements of the lattice L such that $a \wedge b = a \vee b$. Then

$$\begin{aligned} a &= a \vee (a \wedge b) \\ &= a \vee (a \vee b) \\ &= (a \vee a) \vee b \\ &= a \vee b \\ &= b \vee a \\ &= (b \vee b) \vee a \\ &= b \vee (b \vee a) \\ &= b \vee (b \wedge a) \\ &= b. \end{aligned}$$

Hence, $a = b$.

Let a and b be elements of L such that $a = b$. By Theorem 1.1, $a = a \wedge a$ and $a = a \vee a$. Note that $a \wedge a = a \wedge b$ and $a \vee a = a \vee b$. Hence, $a \wedge b = a \vee b$.

<u>Theorem 2.2</u>. If a and b are elements of the lattice L, then $a \wedge b = b$ if and only if $b \vee a = a$.

<u>Proof</u>. Let a and b be elements of the lattice L such that $a \wedge b = b$. By Axioms L6 and L4,

$$a = a \vee (a \wedge b) = a \vee b = b \vee a.$$

Hence, $a \wedge b = b$ implies $a = b \vee a$.

Suppose that $b \vee a = a$. Then $b = b \wedge (b \vee a) = b \wedge a = a \wedge b$. Hence, $b \vee a = a$ implies $a \wedge b = b$.

<u>Theorem 2.3</u>. If a, b, c and d are four arbitrary elements of a lattice, then

1) $(a \wedge c) \vee (b \wedge d)$

$$= \left[(a \wedge c) \vee (b \wedge d)\right] \wedge \left[(a \vee b) \wedge (c \vee d)\right]; \text{ and}$$

2) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

$$= \left[(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)\right] \wedge \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a)\right].$$

<u>Proof</u>. Let a, b, c, and d be arbitrary elements of the lattice L. By Axioms L1, L3, and L5,

$$\begin{aligned} &(a \wedge c) \wedge \left[(a \vee b) \wedge (c \vee d)\right] \\ &= \left[a \wedge (a \vee b)\right] \wedge \left[c \wedge (c \vee d)\right] \\ &= a \wedge c \end{aligned}$$

and

$$\begin{aligned} &(b \wedge d) \wedge \left[(a \vee b) \wedge (c \vee d)\right] \\ &= \left[b \wedge (a \vee b)\right] \wedge \left[d \wedge (c \vee d)\right] \\ &= b \wedge d. \end{aligned}$$

Using Theorem 1.2, $(a \wedge c) \wedge \left[(a \vee b) \wedge (c \vee d)\right] = a \wedge c$ implies $(a \wedge c) \vee \left[(a \vee b) \wedge (c \vee d)\right] = (a \vee b) \wedge (c \vee d)$ and

$$(b \wedge d) \wedge \left[(a \vee b) \wedge (c \vee d)\right] = b \wedge d$$

implies $(b \wedge d) \vee \left[(a \vee b) \wedge (c \vee d)\right] = (a \vee b) \wedge (c \vee d)$ so that

$$\begin{aligned} &\left[(a \wedge c) \vee (b \wedge d)\right] \vee \left[(a \vee b) \wedge (c \vee d)\right] \\ &= (a \wedge c) \vee \left\{(b \wedge d) \vee \left[(a \vee b) \wedge (c \vee d)\right]\right\} \\ &= (a \wedge c) \vee \left[(a \vee b) \wedge (c \vee d)\right] \\ &= (a \vee b) \wedge (c \vee d). \end{aligned}$$

Hence, by Theorem 1.2,

$$\left[(a \wedge c) \vee (b \wedge d)\right] \wedge \left[(a \vee b) \wedge (c \vee d)\right] = (a \wedge c) \vee (b \wedge d).$$

To prove part (2), note that (i),

$$\begin{aligned} &(a \wedge b) \wedge (a \vee b) \wedge (b \vee c) \wedge (c \vee a) \\ &= a \wedge \left[b \wedge (a \vee b)\right] \wedge (b \vee c) \wedge (c \vee a) \\ &= (a \wedge b) \wedge (a \vee b) \wedge (b \vee c) \\ &= a \wedge \left[b \wedge (a \vee b)\right] \wedge (b \vee c) \end{aligned}$$

$$= (a \wedge b) \wedge (b \vee c)$$
$$= a \wedge \left[b \wedge (b \vee c) \right]$$
$$= a \wedge b.$$

Similarly, (ii), $(b \wedge c) \wedge (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = b \wedge c$ and (iii), $(c \wedge a) \wedge (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = c \wedge a$. By Theorem 1.2 and (i),

$$(a \wedge b) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (a \vee c) \right]$$
$$= (a \vee b) \wedge (b \vee c) \wedge (a \vee c),$$

and (ii),

$$(b \wedge c) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right]$$
$$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a),$$

and (iii),

$$(c \wedge a) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right]$$
$$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Thus,

$$\left[(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \right] \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right]$$
$$= (a \wedge b) \vee (b \wedge c) \vee \left\{ (c \wedge a) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right] \right\}$$
$$= (a \wedge b) \vee (b \wedge c) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right]$$
$$= (a \wedge b) \vee \left\{ (b \wedge c) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right] \right\}$$
$$= (a \wedge b) \vee \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right]$$
$$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Hence, by Theorem 1.2,

$$\left[(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \right] \wedge \left[(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \right]$$
$$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a).$$

<u>Theorem 2.4</u>. If a and b are elements of a lattice L, then the relation " $\leq$," defined by $a \leq b$ if and only if $a \wedge b = a$, is an order relation.

Proof. Let a and b be any pair of elements in the lattice L, and let the relation "$\leqq$" be defined by $a \leqq b$ if and only if $a \wedge b = a$. Since $a \wedge a = a$ implies $a \leqq a$, property (i) of Definition 1.8 is satisfied. Suppose that x, y is any pair of elements in L such that $x \leqq y$ and $y \leqq x$. Then $x \wedge y = x$ and $y \wedge x = y$, but $x \wedge y = y \wedge x$. Thus, $x = y$, and property (ii) of the definition is satisfied. Suppose that x, y, z is any triplet in L such that $x \leqq y$ and $y \leqq z$. Now $x \wedge y = x$ and $y \wedge z = y$ so that

$$x = x \wedge y = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z.$$

Thus, $x \leqq z$, and property (iii) of the definition is satisfied. Hence, "$\leqq$" is an order relation.

The dual of this theorem would read as follows: if a and b are elements of a lattice L, then the relation "$\geqq$," defined by $a \geqq b$ if and only if $a \vee b = a$, is an order relation. Throughout the remainder of this paper, the statement that "$a < b$" will mean $a \wedge b = a$ and $a \neq b$. Also, $a \leqq b$ and $b \geqq a$ will be used interchangeably.

Definition 2.2. The element a of a lattice L is said to be an upper (lower) bound of the elements x and y, $y, x \in L$, if $x \wedge a = x$ and $y \wedge a = y$ ($x \vee a = x$ and $y \vee a = y$).

Definition 2.3. The element a of a lattice L is the supremum (infimum) of the elements x and y in L if

i) a is an upper (lower) bound for x and y, and

ii) if b is any upper (lower) bound, then

$$a \wedge b = a \ (a \vee b = a).$$

Definition 2.4. If R is a subset of the lattice L, then $a \in R$ is said to be a minimal (maximal) element of R if for each $x \in R$ such that $x \wedge a = x (x \vee a = x)$, then $x = a$.

Definition 2.5. The statement that 0 (i) is a zero (unity) of a lattice L means that $0 \wedge x = 0$ $(i \vee x = i)$ for all x in L.

Definition 2.6. A lattice is said to be bounded below if it has a zero, bounded above if it has a unity, and bounded if it has both a unity and a zero.

Theorem 2.5. Every finite subset of a lattice has an infimum and supremum.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of the lattice L. By induction, $a_1 \wedge a_2 \wedge a_3 \ldots \wedge a_n = \bigwedge_{i=1}^{n} a_i$ is an element in L, where $\bigwedge_{i=1}^{n} a_i$ denotes the meet of the n elements of A. Let $\bigwedge_{i=1}^{n} a_i = a$ and $j \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned} a_j \wedge a &= a_j \wedge (\bigwedge_{j=1}^{n} a_j) \\ &= a_j \wedge (a_1 \wedge a_2 \wedge \ldots \wedge a_j \wedge a_{j+1} \cdots \wedge a_n) \\ &= (a_j \wedge a_j) \wedge (a_1 \wedge \ldots \wedge a_{j-1} \wedge a_{j+1} \wedge \ldots \wedge a_n) \\ &= a_j \wedge (a_1 \wedge \ldots \wedge a_{j-1} \wedge a_{j+1} \wedge \ldots \wedge a_n) \\ &= a_1 \wedge \ldots \wedge a_n = a. \end{aligned}$$

Thus, a is the infimum of the set A. By a dual proof, $b = \bigvee_{i=1}^{n} a_i$, where $\bigvee_{i=1}^{n} a_i$ denotes the join of all the elements in A, is the supremum of the set A.

Theorem 2.6. Every lattice has at most one minimal and one maximal element, which are the zero and the unity, respectively.

Proof. Let L be a lattice with minimal elements m and m', and let $m' \wedge m = c$. Since m and m' are in L, then $c \in L$. Note that $m \wedge c = m \wedge (m \wedge m') = (m \wedge m) \wedge m' = m \wedge m' = c$. The fact that m is minimal implies that $m = c$. Thus, $m' \wedge m = m$, which means that $m = m'$. Hence, there is at most one minimal element. By a dual proof there is at most one maximal element. To show that m is the zero of L, let $x \in L$ and let $m \wedge x = c \in L$. Note that $m \wedge c = m \wedge (m \wedge x) = m \wedge x = c$. By the same reasoning as before, $m = c$ and $m \wedge x = m$. Therefore, m is the zero of the lattice L. By a dual proof, if there is a maximal element in L, it is the unity of L.

Definition 2.7. Suppose that a and b are elements of the lattice L such that $a < b$. Then the statement that "b covers a" means that there is no $x \in L$ such that $a < x < b$, and will be denoted by $b \succ a$.

The notation "$a \prec b$" means that a is covered by b, which is synonymous with "b covers a," and will be used interchangeably with it.

Let $M = \{x_1, x_2, x_3\}$ be a set of three elements. Then the set of all subsets of M forms a lattice with respect to the operations of union and intersection. To illustrate this, let $a_1 = \{x_1\}$, $a_2 = \{x_2\}$, $a_3 = \{x_3\}$, $a_4 = \{x_1, x_2\}$, $a_5 = \{x_1, x_3\}$, $a_6 = \{x_2, x_3\}$, $0 = \emptyset$ (empty set), $i = M$. By observing Tables I (a) and I (b), it is seen that

$$L = \{0, i, a_1, a_2, a_3, a_4, a_5, a_6\}$$

satisfies the lattice axioms.

TABLE I (a)

MEET OPERATION

$\wedge$	i	0	a_1	a_2	a_3	a_4	a_5	a_6
i	i	0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	0	0	0	0
a_1	a_1	0	a_1	0	0	a_1	a_1	0
a_2	a_2	0	0	a_2	0	a_2	0	a_2
a_3	a_3	0	0	0	a_3	0	a_3	a_3
a_4	a_4	0	a_1	a_2	0	a_4	a_1	a_2
a_5	a_5	0	a_1	0	a_3	a_1	a_5	a_3
a_6	a_6	0	0	a_2	a_3	a_2	a_3	a_6

TABLE I (b)

JOIN OPERATION

$\vee$	i	0	a_1	a_2	a_3	a_4	a_5	a_6
i	i	i	i	i	i	i	i	i
0	i	0	a_1	a_2	a_3	a_4	a_5	a_6
a_1	i	a_1	a_1	a_4	a_5	a_4	a_5	i
a_2	i	a_2	a_4	a_2	a_6	a_4	i	a_6
a_3	i	a_3	a_5	a_6	a_3	i	a_5	a_6
a_4	i	a_4	a_4	a_4	i	a_4	i	i
a_5	i	a_5	a_5	i	a_5	i	a_5	i
a_6	i	a_6	i	a_6	a_6	i	i	a_6

The elements of L can be represented graphically by Figure 1, which follows. Note that the elements are represented by small circles; an element x is higher than an element $y(x,y \in L)$ if $y < x$. A line segment then is drawn from y to x whenever y is covered by x. This type of representation, called a diagram, is used mainly to represent finite lattices.

Note that by Theorem 2.5 L has an infimum and a supremum. Throughout the rest of this paper, the infimum of a lattice D will be denoted by inf D, and the infimum of a subset R of D will be denoted by $\inf_D R$. The same type of notation, i.e. sup D and $\sup_D R$, will be be used for the supremum. Then the $\inf L = a \wedge i \wedge (\bigwedge_{j=1}^{6} a_j) = 0$ and $\sup L = 0 \vee i \vee (\bigvee_{j=1}^{6} a_j) = i$. This lattice is also an example of a bounded lattice. In fact, the zero, minimal, and inf L are all the same element. The same is true about the unity, maximal, and sup L.

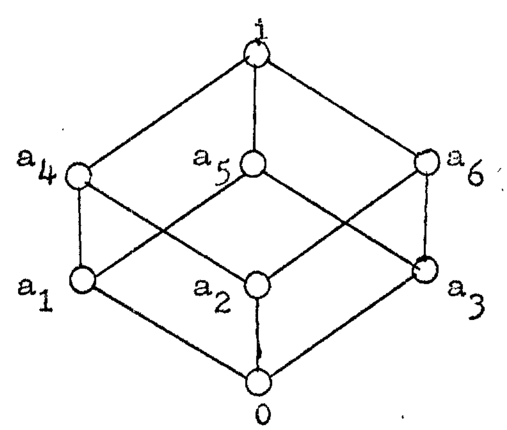

Fig. 1--Lattice of subsets

Definition 2.8. An element p of a lattice L bounded above is called an atom if for each $x \in L$, either $p \wedge x = p$ or $p \wedge x = 0$, where 0 is the zero of L and $p \neq 0$.

An example of an atom would be the element a_2 in Figure 1.

Definition 2.9. An element q of a lattice L bounded above is called a dual atom if for each $x \in L$, either $q \vee x = q$ or $q \vee x = i$, where i is the unity of L and $q \neq i$.

An example of a dual atom would be the element a_5 shown in Figure 1.

Definition 2.10. Suppose that L is a bounded lattice and that x is an element of L. Then x' is said to be a complement of x if $x \wedge x' = 0$ and $x \vee x' = i$. A lattice L is said to be complemented (uniquely complemented) if each element in L has a complement (exactly one complement).

Figure 2a below is an example of a complemented lattice. Note that for each x in the lattice there is an x' such that $x \wedge x' = 0$ and $x \vee x' = i$. For example, $a \wedge b = 0$ and $a \vee b = i$, Figure 2b below is an example of a uniquely complemented lattice. Throughout the remainder of this paper, the lattice in Figure 2a will be called Lattice A.

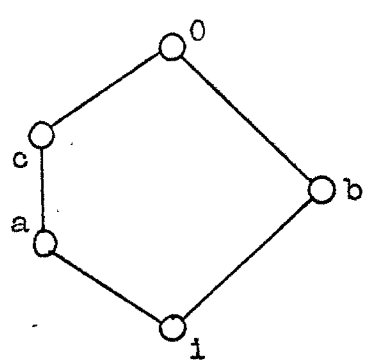

Fig. 2a--
Lattice A.

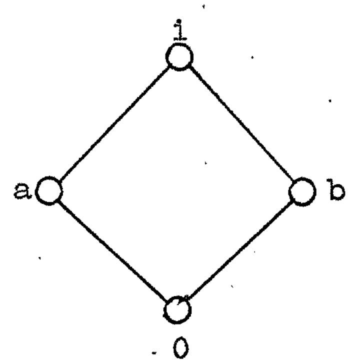

Fig. 2b--
Uniquely complemented lattice.

<u>Theorem 2.7</u>. If p and q are distinct atoms in a uniquely complemented lattice L, if $p' \in L$ is the complement of p, and if p' is also a dual atom, then $p' \wedge q = q$.

<u>Proof</u>. Let L be a uniquely complemented lattice, p and q be distinct atoms, p' be the complement of p, and p' be a dual atom. By Definition 2.8, $p \neq 0$, and either $p \wedge q = p$ or $p \wedge q = 0$. Suppose that $p \wedge q = p$. Since q is an atom, either $p \wedge q = q$ or $p \wedge q = 0$, $q \neq 0$. If $p \wedge q = q$, then $q = p$, which is a contradiction to the hypothesis. If $p \wedge q = 0$, then $p = 0$, which again is a contradiction. Thus, the assumption that $p \wedge q = p$ is false so that $p \wedge q = 0$. Using the fact that q is an atom, either $p' \wedge q = q$ or $p' \wedge q = 0$. Suppose that $p' \wedge q = 0$. The fact that p' is a dual atom implies that either $p' \vee q = p'$ or $p' \vee q = i$, $p' \neq i$. If $p' \vee q = p'$, then, by Theorem 2.2, $p' \wedge q = q$ and $q = 0$, which is a contradiction since q is an atom. If $p' \vee q = i$, then q would be a complement of p', but p' has only one complement, which is p. Thus, $q = p$, which is a contradiction. Hence, $p' \wedge q = q$.

<u>Theorem 2.8</u>. If L is a uniquely complemented lattice and p' is the complement of $p(p', p \in L)$, then p' is a dual atom if and only if p is an atom.

<u>Proof</u>. Let p' be the complement of p in the uniquely complemented lattice L, and suppose that p' is a dual atom. Let $x \in L$. Then $x \wedge p \in L$. Since p' is a dual atom, either $(x \wedge p) \vee p' = i$ or $(x \wedge p) \vee p' = p'$, $p' \neq i$. If $(x \wedge p) \vee p' = p'$, then, by Theorem 2.2, $(x \wedge p) \wedge p' = x \wedge p$. Note that

$$(x \wedge p) \wedge p' = x \wedge (p \wedge p') = x \wedge 0 = 0.$$

Thus, $x \wedge p = 0$. If $(x \wedge p) \vee p' = 1$, then, by the uniqueness of the complements, $p = x \wedge p$. To finish the first part of the proof, suppose that $p = 0$. Since p' is the complement of p, $p' \wedge p = 0$ and $p' \vee p = 1$. Now $p' \wedge p = 0 = p$ and, by Theorem 2.2, $p' \vee p = p'$. Thus, $p' = 1$, which is a contradiction. Hence, $p \neq 0$ and p is an atom. By a dual proof, the converse is proven.

Definition 2.11. Suppose that L is a lattice bounded below and that x is an element of L. Then u is said to be a semicomplement of x if $x \wedge u = 0$. If $u \neq 0$, then u is a proper semicomplement.

Definition 2.12. Suppose that a and b are elements of a lattice L such that $a < b$. Then

i) $[a,b] = \{x \in L: \; a \leqq x \text{ and } x \leqq b\}$ ($[a,b]$ is said to be an interval of L);

ii) $(a,b) = \{x \in L: \; a < x < b\}$;

iii) $(a,b] = \{x \in L: \; a < x \text{ and } x \leqq b\}$; and

iv) $(a] = \{x \in L: \; a \leqq x\}$.

Definition 2.13. Suppose that L is a lattice and that R is a subset of L. Then the statement that R is a convex set means that $a, b \in R$ $(a < b)$ and $x \in [a,b]$ imply $x \in R$.

Theorem 2.9. If L is a lattice bounded below, then the set of all semicomplements of $x \in L$ forms a convex set.

Proof. Let L be a lattice bounded below, x be an element of L, and $C = \{u \in L: x \wedge u = 0\}$ be the set of semicomplements of

x. C is not empty since $0 \in C$; that is, $x \wedge 0 = 0$. Let $q \in C$. Then $x \wedge q = 0$. If $y \in L$ such that $y \wedge q = y$, $y \neq q$, then $y \wedge x = (y \wedge q) \wedge x = y \wedge (q \wedge x) = y \wedge 0 = 0$. Thus, $y \in C$ and C is convex.

Definition 2.14. An element x of a lattice L is said to be an inner element of L if x is not a bound of L.

Definition 2.15. A lattice L is said to be semicomplemented if every inner element of L has at least one proper semicomplement.

Definition 2.16. A lattice L bounded below is said to be weakly complemented if for each pair $a, b \in L$, $a < b$, there exists an $x \in L$ such that $a \wedge x = 0$ and $b \wedge x \neq 0$.

Theorem 2.10. Every weakly complemented lattice is semicomplemented.

Proof. Let a be an inner element of the weakly complemented lattice L. By Theorem 2.6, a is not maximal, so there exists an element $x \in L$ such that $x \vee a = x$ and $x \neq a$. By Definition 2.15, there exists an element $y \in L$ such that $a \wedge y = 0$ and $x \wedge y \neq 0$. To show that $y \neq 0$, suppose that $y = 0$. Then $x \wedge y = x \wedge 0 = 0$, which is a contradiction. Thus, every inner element of L has at least one proper semicomplement.

Theorem 2.11. Every uniquely complemented lattice is weakly complemented.

Proof. Let x and y be a pair of elements of the uniquely complemented lattice L such that $x < y$ and let $x' \in L$ be the unique complement of x; that is, $x \wedge x' = 0$ and $x \vee x' = i$. To show that $y \wedge x' \neq 0$, suppose that $y \wedge x' = 0$. Note that $y \vee x' = (y \vee x) \vee x' = y \vee (x \vee x') = y \vee i = i$. Using the fact that x is the complement of x' and the fact that $y \wedge x' = 0$, $x = y$, which is a contradiction since $x \neq y$. Thus, $x' \wedge y \neq 0$ and L is weakly complemented.

The converse of this theorem is not necessarily true, for Figure 3 is an example of a lattice which is weakly complemented but not uniquely complemented. Note that $a \wedge b = 0$, $a \vee b = i$, $b \wedge c = 0$, and $b \vee c = i$, but $a \neq c$; that is, b has two distinct complements a and c.

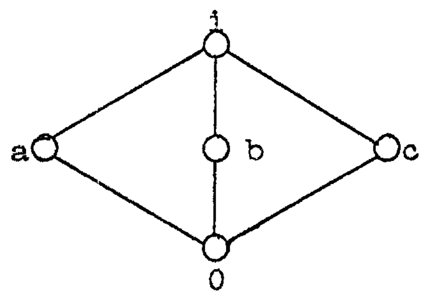

Fig. 3--Lattice B

<u>Definition 2.17</u>. A lattice L bounded below is said to be section complemented if for each pair x, $a \in L$ such that

$$x \wedge a = x,$$

there exists $u \in L$ such that $x \wedge u = 0$ and $x \vee u = a$.

An example of a section complemented lattice would be the lattice shown in Figure 2a.

<u>Theorem 2.12</u>. Every section complemented lattice bounded below is weakly complemented.

Proof. Let x and y $(x < y)$ be a pair of elements of a section complemented lattice bounded below. Since L is section complemented there exists an element $u \in L$ such that $x \wedge u = 0$ and $x \vee u = y$. To show that $u \wedge y \neq 0$, suppose that $u \wedge y = 0$. Note that $u \vee y = u \vee (u \vee x) = (u \vee u) \vee x = u \vee x = y$. By Theorem 2.2, $u \vee y = y$ implies $u \wedge y = u$. Thus, $u = 0$. But $y = x \vee u = x \vee 0 = x$, which is a contradiction since $x < y$. Hence, for each pair of elements x, y $(x < y)$ in L there exists an element u such that $u \wedge x = 0$ and $u \wedge y \neq 0$, which means that L is weakly complemented.

Theorem 2.13. A lattice L bounded below is weakly complemented if and only if there exists for every pair of distinct elements u and v in L an element x such that

$$(u \wedge v) \wedge x = 0, \ (v \vee u) \wedge x \neq 0.$$

Proof. Let x, y be a pair of elements of a lattice L such that $x \wedge y = x$ and $x \neq y$. Also, suppose that there exists an element $u \in L$ such that $(x \wedge y) \wedge u = 0$ and $(x \vee y) \wedge u \neq 0$. Then, since $x \wedge y = x$ implies $x \vee y = y$ (Theorem 2.2),

$$0 = (x \wedge y) \wedge u = x \wedge u$$

and $0 \neq (x \vee y) \wedge u = y \wedge u$. Thus, L is weakly complemented.

Let x and y be distinct elements of the weakly complemented lattice L such that $x \wedge y = x$. There exists an element $u \in L$ such that $x \wedge u = 0$ and $y \wedge u \neq 0$. Since $x \wedge y = x$ implies $x \vee y = y$, $(x \wedge y) \wedge u = 0$ and $(x \vee y) \wedge u \neq 0$.

Definition 2.18. Suppose that L is a lattice and that R is a subset of L. Then R is said to be a sublattice of L if for each pair $x, y \in L$, then $x \wedge y \in L$ and $x \vee y \in L$.

The subset $\{i, a_4, a_6, a_2\} = R$ of the lattice L shown in Figure 1 is an example of a sublattice of L because for each pair x, y in R, $x \wedge y$ and $x \vee y$ are also in R. Note that not every subset of a lattice is a sublattice. For example, the subset $S = \{i, a_4, a_6, 0\}$ of the lattice L shown in Figure 1 is a lattice with respect to the ordering of L, but note that $a_4 \wedge a_6 \notin S$. Thus, it is not a sublattice.

<u>Theorem 2.14</u>. A sublattice of a lattice L is convex if and only if a, $b \in R$ implies $[a \wedge b, a \vee b] \subset R$.

<u>Proof</u>. Let R be a sublattice of the lattice L. Suppose that a, $b \in R$ implies $[a \wedge b, a \vee b] \subset R$. Let a and b be a pair of elements in R. By Definition 2.12,

$$[a \wedge b, a \vee b] = \{x \in L: a \wedge b \leqq x \text{ and } x \leqq a \vee b\}.$$

Let $x \in [a,b]$ and show that $x \in R$. Note that $x \in [a,b]$ implies $a \leqq x$ and $x \leqq b$; that is, $a \wedge x = a$ and $b \vee x = b$. Now

$$(a \wedge b) \wedge x = (b \wedge a) \wedge x = b \wedge (a \wedge x) = b \wedge a = a \wedge b$$

and $(a \vee b) \vee x = a \vee (b \vee x) = a \vee b$. By Theorem 2.3, $a \wedge b \leqq x$ and $x \leqq a \vee b$. Thus, $x \in [a \wedge b, a \vee b] \subset R$. Hence, R is convex.

Let a, b be any distinct pair of elements of R such that $a \wedge b = a$. Suppose that R is convex. Note that a, $b \in R$ implies $a \wedge b$, $a \vee b \in R$. If $x \in [a \wedge b, a \vee b]$, then $x \in R$ from the convexity. Hence, $[a \wedge b, a \vee b] \subset R$.

<u>Definition 2.19</u>. The function φ is a homomorphism of lattice L_1 into (onto) lattice L_2 if the domain of φ is L_1, the range is a subset of L_2 (is L_2), and for every pair of

elements x, y in L_1, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y).$$

Another way of stating this definition would be as follows: the lattice L_1 is homomorphic with the lattice L_2 if there exists a function φ with the properties stated above. The two forms of the definition will be used interchangeably.

Definition 2.20. If φ is a homomorphism of lattice L_1 into lattice L_2 where L_2 is bounded below by 0_2, then the kernel of the homomorphism φ is $K_\varphi = \{x \in L_1 : \varphi(x) = 0_2\}$.

Definition 2.21. A subset I of a lattice L is said to be an ideal if I satisfies the following two conditions:

i) a, b $\in$ I implies that $a \vee b \in I$; and

ii) for any element x in L, $a \in I$ implies that $a \wedge x \in I$.

The subset $I = \{i, b, 0\}$ of the lattice L shown in Figure 3 would be an example of an ideal. Note that I is also a sublattice of L. This is true for any ideal of a lattice.

Theorem 2.15. If a homomorphism of a lattice has a kernel, then the kernel is an ideal of the lattice.

Proof. Let φ be a homomorphism of the lattice L_1 into the lattice L_2, K_φ be the kernel of the homomorphism φ, and a, b be an arbitrary pair of elements in K_φ. Then $\varphi(a) = 0_2$ and $\varphi(b) = 0_2$. Note that $\varphi(a \vee b) = \varphi(a) \vee \varphi(b) = 0_2 \vee 0_2 = 0_2$. This implies that $a \vee b$ is an element of K_φ which satisfies condition (i) in Definition 2.21. Let $x \in L_1$. Note that

$\varphi(a \wedge x) = \varphi(a) \wedge \varphi(x) = 0_2 \wedge \varphi(x) = 0_2$. This implies that $a \wedge x \in K_\varphi$, which satisfies condition (ii) of the definition. Since, by definition, $K_\varphi \subset L_1$, K is an ideal of L_1.

Theorem 2.16. If φ is a homomorphism of lattice L_1 onto lattice L_2 and L_1 is bounded, then L_2 is bounded.

Proof. Let φ be a homomorphism of the lattice L_1, bounded above by i and below by 0, onto the lattice L_2. By the homomorphism there exist elements k and 1 in L_2 such that $\varphi(i) = k$ and $\varphi(0) = 1$. Let $x \in L_2$. By the onto property there is an element $y \in L_1$ such that $\varphi(y) = x$. Since

$$x \vee k = \varphi(y) \vee \varphi(i) = \varphi(y \vee i) = \varphi(i) = k$$

and $x \wedge 1 = \varphi(y) \wedge \varphi(0) = \varphi(y \wedge 0) = \varphi(0) = 1$, k and 1 are upper and lower bounds, respectively, for L_2.

Definition 2.22. A function ψ is said to be an order homomorphism of a lattice L_1 into (onto) a lattice L_2 if the domain of ψ is L_1, the range of ψ is a subset of L_2 (is L_2), and for every pair of elements $a, b \in L$, $a \wedge b = a$ implies $\psi(a) \wedge \psi(b) = \psi(a)$.

Just as in Definition 2.19, an equivalent way of stating this definition would be as follows: a lattice L_1 is order homomorphic with lattice L_2 if there exists a function ψ with the properties described above. Again, the two forms will be used interchangeably.

Definition 2.23 (a). Suppose that L is a lattice and that C is a subset of L. Then C is said to be a chain if for each pair $x, y \in C$, either $x \wedge y = x$ or $x \vee y = x$.

Definition 2.22 holds for chains since each chain of a lattice L is a sublattice of L and hence a lattice.

Theorem 2.17. If ψ is an order homomorphism of a chain C_1 into a chain C_2, then ψ is a homomorphism of the chain C_1 into the chain C_2.

Proof. Let ψ be an order homomorphism of the chain C_1 into the chain C_2. ψ is a function whose domain is C_1 and whose range is a subset of C_2. Let a, b be a pair of elements in C_1. By Definition 2.23 (a), either $a \wedge b = a$ or $a \wedge b = b$. If $a \wedge b = a$, then, since ψ is an order homomorphism, $\psi(a) \wedge \psi(b) = \psi(a)$. Note that

$$\psi(a \wedge b) = \psi(a) = \psi(a) \wedge \psi(b).$$

Using the fact that every chain is a lattice and Theorem 2.2, $a \wedge b = a$ implies $a \vee b = a$ and $\psi(a) \wedge \psi(b) = \psi(a)$ implies $\psi(a) \vee \psi(b) = \psi(b)$. Thus,

$$\psi(a \vee b) = \psi(b) = \psi(a) \vee \psi(b).$$

If $a \wedge b = b$, by an analogous argument, $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$ and $\psi(a \vee b) = \psi(a) \vee \psi(b)$. Hence, ψ is a homomorphism of C_1 into C_2.

Theorem 2.18. If φ is a homomorphism of lattice L_1 onto lattice L_2, and if a non-empty subset R of L_2 is an ideal of L_2, then the set $S = \{x \in L_1 : \varphi(x) \in R\}$ is an ideal of L_1.

Proof. Let φ be a homomorphism of lattice L_1 onto lattice L_2, let the non-empty subset R of L_2 be an ideal of L_2, and let $S = \{x \in L_1 : \varphi(x) \in R\}$. S is not empty since for each $h \in R$ there is an $x \in L_1$ such that $\varphi(x) = h$. Let a and b be a pair of elements in S. Note that $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ and $\varphi(a) \vee \varphi(b) \in R$ implies that $a \vee b \in S$. Let $z \in L_1$. Then $\varphi(z \wedge a) = \varphi(z) \wedge \varphi(a)$. Now $\varphi(z) \in L_2$ and R an ideal of L_2 imply $\varphi(z) \wedge \varphi(a) \in R_2$. This in turn implies that $z \wedge a \in S$. Thus, S is an ideal of L_1.

Definition 2.23 (b). If a, b is a pair of elements in a lattice, then a and b are said to be comparable if either $a \wedge b = a$ or $a \vee b = a$; otherwise, a and b are said to be incomparable, denoted by $a \| b$.

The pair of elements a_4 and a_6 in Figure 1 is an example of incomparable elements, and the pair a_1 and a_4 is an example of comparable elements.

Theorem 2.19. If a, b is a pair of elements in a lattice L, then a and b are incomparable if and only if $a \wedge b < a$ and $a < a \vee b$.

Proof. Let a, b be a pair of incomparable elements of the lattice L. Since $a, b \in L$, then $a \wedge b, a \vee b \in L$. Note that $(a \wedge b) \wedge a = b \wedge (a \wedge a) = b \wedge a = a \wedge b$ and $a = a \wedge (a \vee b)$. Now $a \| b$ implies $(a \wedge b) \neq a$ and $(a \vee b) \neq a$. Thus, $a \wedge b < a$ and $a < a \vee b$.

Suppose that a, b is a pair of elements of L such that $a \wedge b < a$ and $a < a \vee b$. Then $a \wedge b \neq a$ and $a \neq a \vee b$. Thus, $a \| b$.

Definition 2.24. An element a of a lattice L is said to be meet-reducible if there exist in L elements a_1, a_2 such that $a = a_1 \wedge a_2$, $a < a_1$, and $a < a_2$.

<u>Definition 2.25</u>. An element a of a lattice L is said to be meet-irreducible if for each pair b, c of L such that $b \wedge c = a$, either $b = a$ or $a = c$ or both.

The definitions for join-reducible and join-irreducible are duals of Definitions 2.24 and 2.25, respectively.

<u>Definition 2.26</u>. An element a of a lattice L is said to be meet-prime if for each pair of elements a_1 and a_2 of L, $a_1 \wedge a_2 \leqq a$ implies either $a_1 \leqq a$ or $a_2 \leqq a$ or both.

The dual of this definition is the definition for a join-prime element.

<u>Theorem 2.20</u>. If L is a complemented lattice, then every join-prime element p of L, $p \neq 0$, is an atom of L.

<u>Proof</u>. Let L be a complemented lattice with zero, 0, and unity, i. Let a be a join-prime element of L, suppose that $a \neq 0$, and let $x \in L$. Since L is complemented there exist elements x' and a' such that $a' \wedge a = 0$, $a' \vee a = i$, $x' \wedge x = 0$, and $x' \vee x = i$. Note that $(x' \vee x) \wedge a = i \wedge a = a$. Then, by Theorem 2.4, $a \leqq x' \vee x$, which is equivalent to $x' \vee x \geqq a$. By the dual of Definition 2.26, $x' \vee x \geqq a$ implies that either $x' \geqq a$ or $x \geqq a$. Suppose that $x \geqq a$. By the dual of Theorem 2.4, $x \geqq a$ implies $x \vee a = x$ and, by Theorem 2.2, $x \vee a = x$ implies $x \wedge a = a$. Suppose that $x' \geqq a$. Then $x' \geqq a$ implies $x' \vee a = x'$ which implies $x' \wedge a = a$. Thus, if $x' \geqq a$, then

$$x \wedge a = x \wedge (x' \wedge a) = (x \wedge x') \wedge a = 0 \wedge a = 0.$$

Hence, a is an atom of L.

By a dual proof, the dual of this theorem is true. The converse of this theorem is not always true. Consider the lattice shown in Figure 4. Note that a and b are the only join-prime elements. They are also atoms, but neither is complemented.

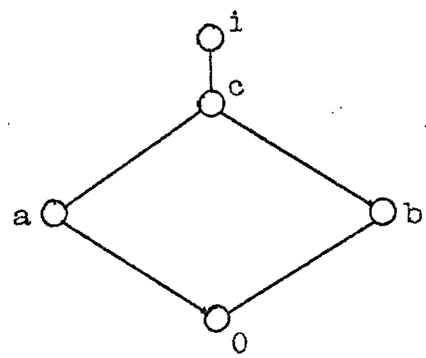

Fig. 4--Non-complemented lattice

Theorem 2.21. A lattice is a chain if and only if every one of its elements is meet-irreducible.

Proof. Let L be a lattice such that if $a \in L$, then a is meet-irreducible. Let x, y be any pair in L. Then $x \wedge y \in L$. Clearly, $x \wedge y = x \wedge y$. Since $x \wedge y$ is meet-irreducible, then either $x = x \wedge y$ or $y = x \wedge y$. By Theorem 2.2, $y = x \wedge y$ implies $x = x \vee y$. Hence, L is a chain.

Suppose that the lattice L is a chain. Let c be any element of L, and let a, b be any pair of elements in L such that $a \wedge b = c$. Since L is a chain, either $a \wedge b = a$ or $a \vee b = a$. If $a \wedge b = a$, then $a \wedge b = a = c$. If $a \vee b = a$, then, by Theorem 2.2, $a \wedge b = b$ and $b = c$. Hence, each element of L is meet-irreducible.

Definition 2.27. Suppose that L is a lattice and that $\bar{c}$ is a chain in L. Then $\bar{c}$ is said to be maximal if there exists no chain in L which would properly contain $\bar{c}$.

Chain Axiom. For any chain C of a lattice L there exists at least one maximal chain $\bar{c}$ such that $\bar{c}$ contains C.

Theorem 2.22. Every chain of a lattice L has an upper bound if and only if L contains a maximal element.

Proof. Let C be a chain of the lattice L. By the Chain Axiom there exists a maximal chain $\bar{c}$ which contains C. Let r be an upper bound of $\bar{c}$ and suppose that r is not a maximal element of L. Then there exists an x in L such that $x \vee r = x$ and $x \neq r$. If $x \in \bar{c}$, then $x \wedge r = x$ and, by the corollary to Theorem 2.1, $x = r$, which is a contradiction. Thus, $x \notin \bar{c}$. Note that for each $y \in \bar{c}$, $y \wedge r = y$. By Theorem 2.2, $x \vee r = x$ implies $x \wedge r = r$ so that

$$x \wedge y = x \wedge (y \wedge r) = (x \wedge r) \wedge y = r \wedge y = y \wedge r = y.$$

Thus, $\bar{c} \vee \{x\}$ is a chain which properly contains $\bar{c}$, but this contradicts the fact that $\bar{c}$ is maximal. Hence, r is the maximal element in L.

To prove the converse, suppose that the lattice L has a maximal element m. Then, by Theorem 2.6, m is the unity i. Let C be any chain in L and $y \in L$. Note that $y \wedge i = y$. Hence, i is an upper bound for C.

Definition 2.28. Suppose that L is a lattice. Then L satisfies the minimum (maximum) condition if for each $x \in L$, all the chains formed by letting x be the maximal (minimal) element are finite.

Theorem 2.23. If a lattice L satisfies the minimum (maximum) condition, then for each $x \in L$ there exists at least one minimal (maximal) element $m \in L$ such that $x \vee m = x$ $(x \wedge m = x)$.

Proof. Let x be an element of the lattice L. Either x is a minimal element in L or not. Suppose that x is not minimal. Then there exists an $x_1 \in L$ such that $x_1 \wedge x = x$ and $x_1 \neq x$. This implies $x_1 < x$. Either x_1 is minimal or not. Suppose it is not. Then there exists an $x_2 \in L$ such that

$$x_2 \wedge x_1 = x_2$$

and $x_1 \neq x_2$, which again implies $x_2 < x_1$. By a continuation of this process and the fact that L satisfies the minimum condition, a finite chain $x_K < x_{K-1} < \ldots < x_1 < x$, which has the property that if $y \in L$ and $y \wedge x_K = y$, then $y = x_K$, is obtained. Thus, $x_K = m$ is a minimal element, and if $x \in L$, then $m \wedge x = m$. Hence, by Theorem 2.3, $m \vee x = x$.

By a proof analogous to the preceding one there exists a maximal element $M \in L$ such that $x \wedge M = x$.

Corollary. If L is a lattice satisfying the maximum (minimum) condition, then every chain of L has a maximal (minimal) element.

Proof. Let L be a lattice satisfying the maximum condition, let R be any chain in L, and let x be an arbitrary element in R. Since any subset of L will satisfy the maximum condition, R, in particular, will. Thus, there exists a maximal element $m \in R$ such that $m \wedge x = x$. Hence, every chain has a maximal element.

By a similar proof, every chain has a minimal element.

<u>Theorem 2.24</u>. A lattice satisfies both the maximum and minimum conditions if and only if every one of its chains is finite.

<u>Proof</u>. Let L be a lattice satisfying the minimum and maximum conditions. Suppose that there exists a chain R in L such that R has an infinite number of elements. By the corollary to Theorem 2.23, R has a minimal element m. Let $R_1 = R - \{m\}$. Note that R_1 is a chain and also has a minimal element m_1 with $m < m_1$. In general, $R_K = R_{K-1} - \{m_{K-1}\}$ where m_{K-1} is the minimal element in R_{K-1}. Since

$$R \supset R_1 \supset R_2 \supset \ldots \supset R_{K-1} \supset R_K,$$

R_K is a chain with a minimal element m_K, and $m_{K-1} < m_K$. Thus, the infinite chain $m < m_1 < \ldots < m_K < \ldots$ is obtained. But this contradicts the fact that L satisfies the maximum condition. Hence, every chain in L is finite.

Suppose that L is a lattice and that each chain in L is finite. Let $x \in L$. Each chain formed by letting x be either a maximal or a minimal element must be finite. Thus, L satisfies both the minimum and maximum conditions.

<u>Theorem 2.25</u>. A lattice satisfying the minimum (maximum) condition has a zero (unity).

Proof. Let L be a lattice satisfying the minimum condition, and let $x \in L$. By Theorem 2.23 there exists at least one minimal element in L and, by Theorem 2.6, it is the zero. By a dual proof, a lattice satisfying the maximum condition has a unity.

Lemma. If the lattice L satisfies the maximum condition, then every subset of L has at least one maximal element.

Proof. Let L be a lattice satisfying the maximum condition, let R be a subset of L, and let $x \in R$. Either x is a maximal element of R or not. Suppose that it is not. Then there exists an $x_1 \in R$ such that $x_1 \vee x = x_1$ and $x_1 \neq x$; that is, $x < x_1$. Either x_1 is a maximal element of R or not. Suppose it is not. Then there exists an element $x_2 \in R$ such that $x_1 \vee x_2 = x_2$ and $x_1 \neq x_2$; that is, $x_1 < x_2$. Since L satisfies the maximum condition, after a finite number of steps, the chain $x < x_1 < x_2 < \ldots < x_r$ is obtained such that if $y \in R$ and $y \vee x_r = y$, then $y = x_r$. Thus, x_r is a maximal of R, and the lemma follows.

Theorem 2.26. In a lattice satisfying the maximum condition, every one of its elements can be represented as the meet of a finite number of meet-irreducible elements.

Proof. Let L be a lattice satisfying the maximum condition, and let S be the set of all elements in L which cannot be represented as the meet of a finite number of meet-irreducible elements. To show that S is empty, suppose that it is not. Let $K \in S$, and suppose that K is meet-irreducible.

Note that $K = K \wedge K$. Thus, K can be represented as the meet of a finite number of meet-irreducible elements, which contradicts the preceding assumption. Thus, K is not meet-irreducible. Furthermore, every element in S is not meet-irreducible. By the lemma, S contains a maximal element m. Since m is not meet-irreducible, there exist elements x and y in L such that $x \wedge y = m$, $x \neq m$ and $y \neq m$. Note that

$x \wedge m = x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y = m$ and

$$y \wedge m = y \wedge (x \wedge y) = x \wedge (y \wedge y) = x \wedge y = m;$$

that is, $m < x$ and $m < y$. But the fact that m is maximal, $m < x$, and $m < y$ together imply that x and y do not belong to S. Since $x, y \in L-S$, x and y can be represented as the meet of a finite number of meet-irreducible elements; that is, $x = \bigwedge_{i=1}^{r} x_i$ and $y = \bigwedge_{j=1}^{s} y_j$ where $x_i, y_j (i = 1, \ldots, r; j = 1, \ldots, s)$ are meet-irreducible. Thus, $m = x \wedge y = (\bigwedge_{i=1}^{r} x_i) \ (\bigwedge_{j=1}^{s} y_i)$, which is a contradiction since $m \in S$. Hence, S is empty, and the theorem follows.

<u>Theorem 2.27</u>. If $a_0 \prec a_1 \prec \cdots \prec a_r$ and $b_0 \prec b_1 \prec \cdots \prec b_s$ are two chains of join-irreducible elements in the lattice L, and if $a_j \| b_k (j = 0, 1, \ldots, r; k = 0, 1, \ldots, s)$, then $a_m \wedge b_n = a_0 \wedge b_0$ for each pair of indices

$$m, n (m = 0, 1, \ldots, r; n = 0, 1, \ldots, s).$$

<u>Proof</u>. Let $a_0 \prec a_1 \prec \cdots \prec a_r$ and $b_0 \prec b_1 \prec \cdots \prec b_s$ be two chains both belonging to the lattice L and both satisfying the hypothesis of the theorem. To prove this theorem, mathematical induction will be employed several times. First, it

will be used to show that $a_0 \wedge b_n = a_0 \wedge b_0$, for

$$n = 0, 1, \ldots, s;$$

then induction will be employed to show that $a_m \wedge b_n = a_0 \wedge b_0$, for $m = 0, 1, \ldots, r$ and $n = 0, 1, \ldots, s$. To show that

$$a_0 \wedge b_1 = a_0 \wedge b_0,$$

let $x = a_0 \wedge b_1$. Note that

$$x \wedge a_0 = (a_0 \wedge b_1) \wedge a_0 = (a_0 \wedge a_0) \wedge b_1 = a_0 \wedge b_1 = x.$$

If $x = b_1$, then $b_1 \wedge a_0 = b_1$, which is a contradiction since $a_0 \parallel b_1$. If $x = b_0$, then $a_0 \wedge b_0 = b_0$, which again is a contradiction. Thus, x, b_1, and b_0 are distinct. Now

$$(b_0 \vee x) \vee b_1 = (b_0 \vee b_1) \vee x = b_1 \vee (a_0 \wedge b_1) = b_1$$

and $(b_0 \vee x) \wedge b_0 = b_0$ imply $b_0 \leqq x \vee b_0 \leqq b_1$. Since $b_0 \prec b_1$, then either $b_0 \vee x = b_1$ or $b_0 \vee x = b_0$. If $b_0 \vee x = b_1$, then b_1 is not join-irreducible, which is a contradiction. Thus,

$$b_0 \vee x = b_0$$

and, by Theorem 2.2, $b_0 \wedge x = x$. Hence,

$$a_0 \wedge b_1 = x = b_0 \wedge x = b_0 \wedge (a_0 \wedge b_1) = (b_0 \wedge b_1) \wedge a_0 = b_0 \wedge a_0.$$

Suppose that $a_0 \wedge b_l = a_0 \wedge b_0$ $(l < s)$ is true, and show that $a_0 \wedge b_{l+1} = a_0 \wedge b_0$ is true. Let $a_0 \wedge b_{l+1} = y$. If x is replaced by y, b_1 by b_{l+1}, and b_0 by b_l in what has just been shown, then $a_0 \wedge b_{l+1} = a_0 \wedge b_0$. Hence, $a_0 \wedge b_n = a_0 \wedge b_0$ for

$$n = 0, 1, \ldots, s.$$

By an analogous proof, $a_m \wedge b_0 = a_0 \wedge b_0$ for $m = 0, 1, \ldots, r$. To show that $a_1 \wedge b_n = a_0 \wedge b_0$ for $n = 0, 1, \ldots, s$, let

$$a_1 \wedge b_1 = z.$$

By proofs analogous to the preceding one, $b_0 \wedge z = z$ and

$a_0 \wedge z = z$. Hence,

$$\begin{aligned} a_1 \wedge b_1 &= z \\ &= z \wedge z \\ &= (a_0 \wedge z) \wedge (b_0 \wedge z) \\ &= a_0 \wedge a_1 \wedge b_1 \wedge b_0 \wedge a_1 \wedge b_1 \\ &= a_0 \wedge b_0. \end{aligned}$$

Suppose that $a_1 \wedge b_l = a_0 \wedge b_0$, $l < s$. Then show that

$$a_1 \wedge b_{l+1} = a_0 \wedge b_0.$$

Let $a_1 \wedge b_{l+1} = u$. In the proof above, replace a_0 by a_1, b_1 by b_{l+1}, x by u, and b_0 by b_l. The result obtained will be $a_1 \wedge b_{l+1} = a_1 \wedge b_l$. But $a_1 \wedge b_l = a_0 \wedge b_0$. Hence,

$$a_1 \wedge b_n = a_0 \wedge b_0$$

for $n = 0, 1, \ldots, s$. Suppose that $a_K \wedge b_n = a_0 \wedge b_0$ for $n = 0, 1, \ldots, s$. Then show that $a_{K+1} \wedge b_n = a_0 \wedge b_0$ for $n = 0, 1, \ldots, s$. Let $a_{K+1} \wedge b_1 = v$. In a similar manner, $a_K \wedge v = v$ and

$$a_{K+1} \wedge b_1 = v = a_K \wedge v = a_K \wedge a_{K+1} \wedge b_1 = a_K \wedge b_1 = a_0 \wedge b_0.$$

Now suppose that $a_{K+1} \wedge b_l = a_0 \wedge b_0$, $l < s$, and show that $a_{K+1} \wedge b_{l+1} = a_0 \wedge b_0$. Let $a_{K+1} \wedge b_{l+1} = w$. Using preceding techniques, $a_K \wedge w = w$ and $a_l \wedge w = w$. Thus,

$$\begin{aligned} a_{K+1} \wedge b_{l+1} &= w \\ &= w \wedge w \\ &= a_K \wedge w \wedge b_l \wedge w \\ &= a_K \wedge a_{K+1} \wedge b_{l+1} \wedge b_l \wedge a_{K+1} \wedge b_{l+1} \\ &= a_K \wedge b_l \\ &= a_0 \wedge b_0. \end{aligned}$$

Hence, by the principle of induction, $a_m \wedge b_n = a_0 \wedge b_0$ for $m = 0, 1, \ldots, r$ and $n = 0, 1, \ldots, s$.

Note that if the condition

$$a_j \| b_K \; (j = 0, 1, \ldots, r; \; K = 0, 1, \ldots, s)$$

were removed, then $a_0 \wedge b_0$ may or may not be equal to $a_1 \wedge b_m$. In fact, if the chains were arranged in the following manner, $a_0 \prec a_1 \prec \cdots \prec a_r \prec b_0 \prec \cdots \prec b_s$, then

$$a_0 \wedge b_0 \neq a_1 \wedge b_m \; (1 \neq 0).$$

CHAPTER III

DISTRIBUTIVE AND MODULAR LATTICES

In this chapter, techniques for characterizing distributive and modular lattices will be presented.

Definition 3.1. A lattice L is said to be distributive if for each triplet of elements, x, y, z, the following conditions hold:

i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; and

ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Theorem 3.1. A lattice satisfies property (i) if and only if it satisfies property (ii).

Proof. Let L be a lattice such that each triplet satisfies property (i), and let x, y, z be a triplet in L. By Property (i),

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= \left[(x \vee y) \wedge x\right] \vee \left[(x \vee y) \wedge z\right] \\ &= x \vee \left[(x \vee y) \wedge z\right] \\ &= x \vee \left[z \wedge (x \vee y)\right] \\ &= x \vee \left[(z \wedge x) \vee (z \wedge y)\right] \\ &= \left[x \vee (z \wedge x)\right] \vee (z \wedge y) \\ &= x \vee (z \wedge y).\end{aligned}$$

Conversely, if each triplet of L satisfies property (ii), then by a dual proof, each triplet satisfies property (i).

As a result of this theorem, any lattice which satisfies either property (i) or property (ii) is distributive. Chains are examples of distributive lattices.

Theorem 3.2. Every sublattice of a distributive lattice is distributive, and if f is a homomorphism of the distributive lattice L onto a lattice L', then L' is also distributive.

Proof. To prove the first part of the theorem, suppose that L is a distributive lattice and that G is any sublattice of L. Let x, y, z be any triplet of G. Since

$$x,\ y,\ z \in L,$$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Note that x, y, $z \in G$ implies, by the definition of sublattice, $y \wedge z \in G$ and $x \wedge (y \vee z) \in G$. Hence, G is distributive.

Suppose that f is a homomorphism of the distributive lattice L onto the lattice L'. Let x', y', z' be any triplet in L. Since the function f is onto, there exist elements x, y, and z in L such that f(x) = x', f(y) = y', and f(z) = z'. To show that L' is distributive, show that x', y', and z' satisfy property (i) of Definition 3.1. Note that

$$\begin{aligned}(x' \wedge y') \vee (x' \wedge z') &= \left[f(x) \wedge f(y)\right] \vee \left[f(x) \wedge f(z)\right]\\ &= f(x \wedge y) \vee f(x \wedge z)\\ &= f\left[(x \wedge y) \vee (x \wedge z)\right]\\ &= f\left[x \vee (y \wedge z)\right]\\ &= f(x) \vee f(y \wedge z)\\ &= x' \vee \left[f(y) \wedge f(z)\right]\\ &= x' \vee (y' \wedge z').\end{aligned}$$

Thus, L' is distributive.

Lemma 3.1. If a, b, and x are arbitrary elements of a lattice L such that $a \leqq b$, then $a \vee x \leqq b \vee x$ and $a \wedge x \leqq b \wedge x$.

Proof. Let a, b, and x be elements of a lattice L such that $a \leqq b$. By Theorem 2.4, $a \leqq b$ implies $a \wedge b = a$. Using lattice Axioms L1 and L3 and Theorem 2.1,

$$\begin{aligned}(a \wedge x) \wedge (b \wedge x) &= a \wedge (x \wedge (b \wedge x)) \\ &= a \wedge (b \wedge (x \wedge x)) \\ &= (a \wedge b) \wedge x \\ &= a \wedge x.\end{aligned}$$

Hence, by Theorem 2.4, $a \wedge x \leqq b \wedge x$. By Theorem 2.2, $a \wedge b = a$ implies $a \vee b = b$. Note that

$$\begin{aligned}(a \vee x) \vee (b \vee x) &= a \vee (x \vee (b \vee x)) \\ &= a \vee (b \vee (x \vee x)) \\ &= a \vee (b \vee x) \\ &= (a \vee b) \vee x \\ &= b \vee x.\end{aligned}$$

Now $(a \vee x) \vee (b \vee x) = b \vee x$ implies $(a \vee x) \wedge (b \vee x) = a \vee x$. Hence, $a \vee x \leqq b \vee x$.

Theorem 3.3. If for each triplet x, y, z of a lattice L $x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z)$, then $(x \vee y) \wedge (x \vee z) \leqq x \vee (y \wedge z)$, and conversely.

Proof. Suppose that for each triplet x, y, z of the lattice L that $x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z)$. Then let x, y, z be a triplet of L, and consider $(x \vee y) \wedge (x \vee z)$. Note that $(x \vee y) \wedge (x \vee z) \leqq \left[(x \vee y) \wedge x\right] \vee \left[(x \vee y) \wedge z\right] = x \vee \left[z \wedge (x \vee y)\right]$ and $z \wedge (x \vee y) \leqq (z \wedge x) \vee (z \wedge y)$. By Lemma 3.1,

$$(x \vee y) \wedge (x \vee z) \leqq x \vee \left[(z \wedge x) \vee (z \wedge y)\right]$$
$$= \left[x \vee (z \wedge x)\right] \vee (z \wedge y)$$
$$= x \vee (z \wedge y).$$

Conversely, suppose that for each triplet x, y, z of L, $(x \vee y) \wedge (x \vee z) \leqq x \vee (y \wedge z)$. Then, by a dual proof,

$$(x \wedge y) \vee (x \wedge z) \geqq x \wedge (y \vee z).$$

Thus, $x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z)$.

Note that the dual of the statement

$$"x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z)"$$

is the statement "$x \vee (y \wedge z) \geqq (x \vee y) \wedge (x \vee z)$." This fact was used to prove the converse of the previous theorem.

<u>Theorem 3.4</u>. The lattice L is distributive if and only if for each triplet x, y, z of its elements,

$$x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z).$$

<u>Proof</u>. Suppose that for each triplet x, y, z of elements in the lattice L that $x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z)$. Then let x, y, z be a triplet of L. By Theorems 2.3 and 2.4, $(x \wedge y) \vee (x \wedge z) \leqq (x \vee x) \wedge (y \vee z) = x \wedge (y \vee z)$ and, by Theorem 2.4 and the corollary to Theorem 2.1, $(x \wedge y) \vee (y \wedge z) = x \wedge (y \vee z)$. Thus, L is distributive.

Let x, y, z be any triplet of the distributive lattice L. By Definition 3.1, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Thus, by the corollary to Theorem 2.1 and Theorem 2.4,

$$x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z).$$

Note that by replacing $x \wedge (y \vee z) \leqq (x \wedge y) \vee (x \wedge z)$ with $x \vee (y \wedge z) \geqq (x \vee y) \wedge (x \vee z)$ in Theorem 3.4, the dual theorem is obtained and is proved by a dual proof.

Theorem 3.5. A lattice is distributive if and only if for each triplet x, y, z of its elements, $(x \vee y) \wedge z \leqq x \vee (y \wedge z)$.

Proof. Suppose that for each triplet x, y, z in the lattice L that $(x \vee y) \wedge z \leqq x \vee (y \wedge z)$. Let x, y, z be a triplet in L. Note that $(x \vee y) \wedge (x \vee z) \leqq x \vee [y \wedge (x \vee z)]$. Since $y \wedge (x \vee z) = (x \vee z) \wedge y \leqq x \vee (z \wedge y)$, then, by Lemma 3.1,

$$x \vee [(x \vee z) \wedge y] \leqq x \vee [x \vee (y \wedge z)]$$
$$\leqq (x \vee x) \vee (y \wedge z)$$
$$\leqq x \vee (y \wedge z).$$

Thus, $(x \vee y) \wedge (x \vee z) \leqq x \vee (y \wedge z)$. By Theorems 2.2 and 2.4, $x \vee (y \wedge z) \geqq (x \vee y) \wedge (x \vee z)$ and, by the dual of Theorem 3.4, L is distributive.

Let x, y, z be any triplet of the distributive lattice L. Since $z = z \wedge (z \vee x)$, then, by Theorem 2.4, $z \leqq z \vee x = x \vee z$. By Lemma 3.1, $(x \vee y) \wedge z \leqq (x \vee y) \wedge (z \vee y)$ and, by property (ii) of Definition 3.1, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$. Thus,

$$(x \vee y) \wedge z \leqq x \vee (y \wedge z).$$

Definition 3.2. A lattice L is said to be modualr if for each triplet x, y, z of L such that $x \wedge z = x$, then

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

Theorem 3.6. Every sublattice of a modualr lattice is modular, and if f is a homomorphism of a modular lattice L onto a lattice L', then L' is also modular.

Proof. Suppose that L is a modular lattice and that G is any sublattice of L. Let x, y, z be a triplet of G such that $x \wedge z = x$. Since x, y, z $\in$ L, $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Now x, y, $z \in G$ implies $y \wedge z \in G$ and $x \vee (y \wedge z)$ G. Hence, G is modular.

Suppose that f is a homomorphism of a modular lattice L onto a lattice L'. Let x', y', z' be any triplet of L' such that $x' \wedge z' = z'$. Since the function f is onto, there exist elements x, y, and z in L such that $f(x) = x'$, $f(y) = y'$, and $f(z) = z'$. Note that $x' \vee (y' \wedge z') = f(x) \vee \left[f(y) \wedge f(z) \right]$. Since f is a homomorphism, then

$$\begin{aligned} f(x) \vee \left[f(y) \wedge f(z) \right] &= f(x) \vee f(y \wedge z) \\ &= f\left[x \vee (y \wedge z)\right] \\ &= f\left[(x \vee y) \wedge z\right] \\ &= f(x \vee y) \wedge f(z) \\ &= \left[f(x) \vee f(y)\right] \wedge f(z) \\ &= (x' \vee y') \wedge z'. \end{aligned}$$

Hence, L' is modular.

<u>Theorem 3.7</u>. A lattice L is modular if and only if for each triplet x, y, z of L such that $x \wedge z = x$,

$$x \vee \left[y \wedge (x \vee z) \right] = (x \vee y) \wedge (y \vee z).$$

<u>Proof</u>. Let L be a lattice. Suppose that for each triplet x, y, z of L such that $x \wedge z = x$,

$$x \vee \left[y \wedge (x \vee z) \right] = (x \vee y) \wedge (x \vee z).$$

Let x, y, z be any triplet of L such that $x \wedge z = x$. By Theorem 2.2, $x \wedge z = x$ implies $x \vee z = z$. Note that

$$x \vee \left[y \wedge (x \vee z) \right] = x \vee (y \wedge z)$$

and $(x \vee y) \wedge (x \vee z) = (x \vee y) \wedge z$. Thus, $x \vee (y \wedge z) = (x \vee y) \wedge z$. Hence, L is modular.

To prove the converse, suppose that the lattice L is modular, and let x, y, z be any triplet in L. Note that $x \wedge (x \vee z) = x$. Thus, since L is modular,

$$x \vee [y \wedge (x \vee z)] = (x \vee y) \wedge (x \vee z).$$

<u>Theorem 3.8</u>. Suppose that L is a modular lattice and that x, y, z is any triplet of L. Then $(x \vee y) \wedge z = y \wedge z$ if and only if $(z \vee y) \wedge x = y \wedge x$.

<u>Proof</u>. Let L be a modular lattice, and let x, y, z be any triplet of L such that $(x \vee y) \wedge z = y \wedge z$. Note that $(x \vee y) \wedge y = y$. Since L is modular,

$$y \vee [z \wedge (x \vee y)] = (y \vee z) \wedge (x \vee y),$$

but $y \vee [z \wedge (x \vee y)] = y \vee (y \wedge z) = y$ so that $y = (y \vee z) \wedge (x \vee y)$. Hence,

$$\begin{aligned} y \wedge x &= [(y \vee z) \wedge (x \vee y)] \wedge x \\ &= (y \vee z) \wedge [(x \vee y) \wedge x] \\ &= (y \vee z) \wedge x \\ &= (z \vee y) \wedge x. \end{aligned}$$

To prove the converse, suppose that x, y, z is any triplet of L such that $y \wedge x = (z \vee y) \wedge x$. Note that $(z \vee y) \wedge y = y$. Since L is modular, $y \vee [x \wedge (z \vee y)] = (y \vee x) \wedge (z \vee y)$, but $y \vee [x \wedge (z \vee y)] = y \vee (y \wedge x) = y$ so that $y = (y \vee x) \wedge (z \vee y)$. Hence, $y \wedge z = [(y \vee x) \wedge (z \vee y)] \wedge z = (y \vee x) \wedge z = (x \vee y) \wedge z$.

<u>Definition</u>. Suppose that a, b, c is a triplet of elements of the lattice L. Then the statement that the triplet a, b, c has a median means that

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

<u>Theorem 3.9</u>. The lattice L is modular if and only if each triplet a, b, c such that $a \wedge c = a$ has a median.

<u>Proof</u>. Suppose that each triplet a, b, c in L such that $a \wedge c = a$ has a median. By lattice Axioms L5, L6, and L4,

$$a \vee (b \wedge c) = [a \wedge (a \vee b)] \vee [(b \wedge c) \vee [(b \wedge c) \wedge c]]$$
$$= [a \wedge (a \vee b)] \vee (b \wedge c) \vee [(b \wedge c) \wedge c].$$

Now, by hypothesis,

$$[a \wedge (a \vee b)] \vee (b \wedge c) \vee [(b \wedge c) \wedge c]$$
$$= [a \vee (a \vee b)] \wedge (b \vee c) \wedge [(b \wedge c) \vee c]$$
$$= (a \vee b) \wedge (b \vee c) \wedge c$$
$$= (a \vee b) \wedge c.$$

Hence, $a \vee (b \wedge c) = (a \vee b) \wedge c$, and L is modular.

Suppose that L is modular. Let a, b, c be any triplet of L such that $a \wedge c = a$. By Theorem 2.2, $a \wedge c = a$ implies $a \vee c = c$. Now $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \wedge b) \vee (b \wedge c) \vee a$. Rearranging the terms,

$$(a \wedge b) \vee (b \wedge c) \vee a = [(a \wedge b) \vee a] \vee (b \wedge c) = a \vee (b \wedge c),$$

but since L is modular,

$$a \vee (b \wedge c) = (a \vee b) \wedge c$$
$$= (a \vee b) \wedge [(b \vee c) \wedge c]$$
$$= (a \vee b) \wedge (b \vee c) \wedge (a \vee c).$$

Thus, each triplet a, b, c of L such that $a \wedge c = a$ has a median.

<u>Theorem 3.10</u>. A lattice is distributive if and only if every one of its triplets has a median.

<u>Proof</u>. Let a, b, and c be a triplet of the distributive lattice L. The fact that L is distributive implies

i) $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$, and

ii) $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$.

Show that $(a\wedge b)\vee(b\wedge c)\vee(c\wedge a) = (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$.

By commuting a with c, $a\wedge c$ with $b\wedge c$, and using (i),

$$\begin{aligned}(a\wedge b)\vee(b\wedge c)\vee(c\wedge a) &= (a\wedge b)\vee(a\wedge c)\vee(b\wedge c)\\ &= \left[a\wedge(b\vee c)\right]\vee(b\wedge c).\end{aligned}$$

By (ii),

$$\begin{aligned}&\left[a\wedge(b\vee c)\right]\vee(b\wedge c)\\ &=\left\{\left[a\wedge(b\vee c)\right]\vee b\right\}\wedge\left\{\left[a\wedge(b\vee c)\right]\vee c\right\}\\ &=\left\{b\vee\left[a\wedge(b\vee c)\right]\right\}\wedge\left\{c\vee\left[a\wedge(b\vee c)\right]\right\}\\ &=\left\{(b\vee a)\wedge\left[b\vee(b\vee c)\right]\right\}\wedge\left\{(c\vee a)\wedge\left[c\vee(b\vee c)\right]\right\}\\ &=\left[(a\vee b)\wedge(b\vee c)\right]\wedge\left[(c\vee a)\wedge(c\vee b)\right.\\ &=(a\vee b)\wedge(b\vee c)\wedge(c\vee a)\wedge(c\vee b)\\ &=(a\vee b)\wedge(b\vee c)\wedge(c\vee a)\wedge(b\vee c).\end{aligned}$$

Now, commuting the third and fourth terms,

$$(a\vee b)\wedge(b\vee c)\wedge(c\vee a)\wedge(b\vee c) = (a\vee b)\wedge(b\vee c)\wedge(b\vee c)\wedge(c\vee a).$$

Since $(b\vee c)\wedge(b\vee c) = b\vee c$, then

$$(a\vee b)\wedge(b\vee c)\wedge(b\vee c)\wedge(c\vee a) = (a\vee b)\wedge(b\vee c)\wedge(c\vee a).$$

Hence, the triplet a, b, c has a median.

Suppose that each triplet of the lattice L has a median. Let a, b, and c be a triplet of L. Since

$$(a\wedge b)\vee(b\wedge c)\vee(c\wedge a) = (a\vee b)\wedge(b\vee c)\wedge(a\vee c),$$

then, by Lemma 2.1,

$$a\vee\left[(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)\right] = a\vee\left[(a\vee b)\wedge(b\vee c)\wedge(a\vee c)\right].$$

By Axioms L1, L4, and L5,

$$\begin{aligned} & a\vee\left[(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)\right] \\ &= \left[a\vee(a\wedge b)\right]\vee\left[(b\wedge c)\vee(c\wedge a)\right] \\ &= a\vee\left[(b\wedge c)\vee(c\wedge a)\right] \\ &= \left[a\vee(c\wedge a)\right]\vee(b\wedge c) \\ &= a\vee(b\wedge c). \end{aligned}$$

Note that $a\vee(a\vee c) = (a\vee a)\vee c = a\vee c$ and, by Theorems 2.2 and 2.3, $a \leqq a\vee c$. By Theorem 2.9,

$$a\vee\left\{\left[(a\vee b)\wedge(b\vee c)\right]\wedge(c\vee a)\right\} = \left\{a\vee\left[(a\vee b)\wedge(b\vee c)\right]\right\}\wedge(c\vee a).$$

Similarly, $b \leqq b\vee c$ and, by Theorem 2.9,

$$\left[(a\vee b)\wedge(b\vee c)\right] = \left[(b\vee a)\wedge(b\vee c)\right] = b\vee\left[a\wedge(b\vee c)\right]$$

so that

$$\begin{aligned} & \left\{a\vee\left[(a\vee b)\wedge(b\vee c)\right]\right\}\wedge(c\vee a) \\ &= \left\{a\vee\left[b\vee(a\wedge(b\vee c))\right]\right\}\wedge(c\vee a) \\ &= \left\{(a\vee b)\vee\left[a\wedge(b\vee c)\right]\right\}\wedge(c\vee a) \\ &= \left\{(b\vee a)\vee\left[a\wedge(b\vee c)\right]\right\}\wedge(c\vee a) \\ &= \left\{b\vee\left[a\vee(a\wedge(b\vee c))\right]\right\}\wedge(c\vee a) \\ &= (b\vee a)\wedge(c\vee a) \\ &= (a\vee b)\wedge(a\vee c). \end{aligned}$$

Thus, $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$, and dually,

$$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c).$$

Hence, L is distributive.

Theorem 3.11. A lattice is modular if and only if no sublattice of it is isomorphic with the lattice A shown in Figure 2a.

<u>Proof</u>. Let L be a modular lattice, and suppose that there exists a sublattice Q of L which is isomorphic to the lattice A shown in Figure 2a. Let θ be a reversible function which maps Q onto A. For a, b, c, 0 and i in A there exist a', b', c', 0', and i' in Q such that $\theta(a') = a$, $\theta(b') = b$, $\theta(c') = c$, $\theta(0') = 0$ and $\theta(i') = i$. Since Q is isomorphic to A for each x and y in Q, $\theta(x \wedge y) = \theta(x) \wedge \theta(y)$ and

$$\theta(x \vee y) = \theta(x) \vee \theta(y).$$

To show that Q is not modular, consider $\theta(a' \wedge c')$, $\theta\left[a' \vee (b' \wedge c')\right]$ and $\theta\left[(a' \vee b') \wedge c'\right]$. Note that

$$\begin{aligned}
\theta(a' \wedge c') &= \theta(a') \wedge \theta(c') \\
&= a \wedge c \\
&= a \\
&= \theta(a'), \\
\theta\left[a' \vee (b' \wedge c')\right] &= \theta(a') \vee \left[\theta(b') \wedge \theta(c')\right] \\
&= a \vee (b \wedge c) \\
&= a \vee 0 \\
&= a \\
&= \theta(a'), \\
\theta\left[(a' \vee b') \wedge c'\right] &= \left[\theta(a') \vee \theta(b')\right] \wedge \theta(c') \\
&= (a \vee b) \wedge c \\
&= i \wedge c \\
&= c \\
&= \theta(c').
\end{aligned}$$

Since θ is a reversible function, $a' \wedge c' = a'$, $a' \vee (b' \wedge c') = a'$, $(a' \vee b') \wedge c' = c'$, and $a \neq c$ imply $a' \neq c'$ so that

$$a' \vee (b' \wedge c') \neq (a' \vee b') \wedge c'.$$

Therefore, Q is not modular, which is a contradiction to Theorem 3.6. Hence, the "only if part" of the theorem is proved.

Let L be a lattice such that no sublattice of it is isomorphic to the lattice A shown above. Suppose that L is not modular. Then there exists a triplet x, y, z in L such that $x \wedge z = x$ and $x \vee (y \wedge z) \neq (x \vee y) \wedge z$. Note that $x \neq z$, for if $x = z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$. To show that y and z are not comparable, suppose first that $y \wedge z = z$. Then $x \wedge y = (x \wedge z) \wedge y = x \wedge (z \wedge y) = x \wedge z = x$. Now

$$x \vee (y \wedge z) = x \vee z = z$$

and $(x \vee y) \wedge z = y \wedge z = z$. Thus, $x \vee (y \wedge z) = (x \vee y) \wedge z$, which contradicts the preceding assumption, and so $y \wedge z \neq z$. Now suppose that $y \wedge z = y$. By Theorem 2.2, $y \wedge z = y$ implies $y \vee z = z$. Note that $x \vee (y \wedge z) = x \vee y$ and

$$(x \vee y) \vee z = x \vee (y \vee z) = x \vee z = z.$$

By Theorem 2.2, $(x \vee y) \vee z = z$ implies $(x \vee y) \wedge z = x \vee y$. Thus, $x \vee (y \wedge z) = (x \vee y) \wedge z$, which again contradicts the initial assumption. Hence, y and z are incomparable. To show that x and y are incomparable, suppose first that $x \wedge y = x$. By Theorem 2.2, $x \wedge y = x$ implies $x \vee y = y$. Note that

$$(x \vee y) \wedge z = y \wedge z$$

and $x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z = x$. By Theorem 2.2,

$$x \vee (y \wedge z) = y \wedge z.$$

Thus, $x \vee (y \wedge z) = (x \vee y) \wedge z$, which is a contradiction.

Consequently, $x \wedge y \neq x$. Suppose then that $x \wedge y = y$. Note that $(x \vee y) \wedge z = x \wedge z = x$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z = y \wedge z$. By Theorem 2.2, $x \vee (y \wedge z) = x$. Hence, $x \vee (y \wedge z) = (x \vee y) \wedge z$, which again is a contradiction. Hence, x and y are not comparable.

Let $(x \vee y) \wedge z = c'$, $x \vee (y \wedge z) = a'$, $x \vee y = i'$, $y \wedge z = 0'$ and $y = b'$. Now it must be shown that a', b', c', $0'$, and i' are all distinct. By the initial assumption, $a' \neq c'$. It has previously been shown that $i' \neq b'$ and $0' \neq b'$. Suppose then that $a' = i'$, i.e. $x \vee (y \wedge z) = x \vee y$. Note that

$$(x \vee y) \wedge z = \left[x \vee (y \wedge z) \right] \wedge z.$$

Since $\left[x \vee (y \wedge z) \right] \vee z = x \vee \left[(y \wedge z) \vee z \right] = x \vee z = z$ and, by Theorem 2.2, $\left[x \vee (y \wedge z) \right] \wedge z = x \vee (y \wedge z)$, then

$$(x \vee y) \wedge z = x \vee (y \wedge z),$$

but this a contradiction, so $a' \neq i'$. Suppose that $c' = i'$, i.e. $(x \vee y) \wedge z = x \vee y$. By Theorem 2.2, $(x \vee y) \wedge z = x \vee y$ implies that $(x \vee y) \vee z = z$, but $(x \vee y) \vee z = (x \vee z) \vee y = z \vee y$ so that $z = z \vee y$. Thus, $y = z \wedge y$, a contradiction since y and z are incomparable. Hence, $c' \neq i'$. To show that $0' \neq i'$, suppose that $0' = i'$. Then $x \vee (y \wedge z) = x \vee (x \vee y) = x \vee y$ and $(x \vee y) \wedge z = (y \wedge z) \wedge z = y \wedge z$. Thus, $x \vee (y \wedge z) = (x \vee y) \wedge z$, which is a contradiction. Hence, $0' \neq i'$. Suppose that $c' = 0'$, i.e. $(x \vee y) \wedge z = y \wedge z$. Then $x \vee (y \wedge z) = x \vee \left[(x \vee y) \wedge z \right]$. But $x \wedge \left[(x \vee y) \wedge z \right] = (x \wedge z) \wedge (x \vee y) = x \wedge (x \vee y) = x$ and, by Theorem 2.2, $x \vee \left[(x \vee y) \wedge z \right] = (x \vee y) \wedge z$. Thus,

$$x \vee (y \wedge z) = (x \vee y) \wedge z,$$

which is a contradiction. Hence, $c' \neq 0'$.

If $c' = b'$, then

$$\begin{aligned} a' &= x \vee (y \wedge z) \\ &= x \vee \left[(x \vee y) \wedge z \wedge z \right] \\ &= x \vee \left[(x \vee y) \wedge z \right] \\ &= x \vee y \\ &= i', \end{aligned}$$

a contradiction. Thus, $c' \neq b'$. In a similar manner it can be shown that $b' \neq a'$ and $a' \neq 0'$ so that a', b', c', $0'$, and i' are all distinct.

Let $B = \{a', b', c', 0', i'\}$. Next it will be shown that B is a sublattice of L, i.e. for any pair x, y in B, $x \wedge y \in B$ and $x \vee y \in B$. By Theorem 2.3,

$$\left[(x \wedge x) \vee (y \wedge z)\right] \wedge \left[(x \vee y) \wedge (x \vee z)\right] = (x \wedge x) \vee (y \wedge z).$$

Since $(x \wedge x) \vee (y \wedge z) = x \vee (y \wedge z)$ and

$$(x \vee y) \wedge (x \vee z) = (x \vee y) \wedge z,$$

then $\left[x \vee (y \wedge z)\right] \wedge \left[(x \vee y) \wedge z\right] = x \vee (y \wedge z)$. Thus, $a' \wedge c' = a'$. Note that $a' \vee b' = \left[x \vee (y \wedge z)\right] \vee y = x \vee y = i'$ and

$$\begin{aligned} &(a' \vee b') \vee (c' \vee b') \\ &= \left\{\left[x \vee (y \wedge z)\right] \vee y\right\} \vee \left\{\left[(x \vee y) \wedge z\right] \vee y\right\} \\ &= \left[x \vee (y \wedge z)\right] \vee \left[(x \vee y) \wedge z\right] \vee y \\ &= \left[(x \vee y) \wedge z\right] \vee y \\ &= c' \vee b' \end{aligned}$$

implies that $i' \vee (c' \vee b') = c' \vee b'$. Note also that

$$(c' \vee b') \vee i' = \{[(x \vee y) \wedge z] \vee y\} \vee (x \vee y)$$
$$= [(x \vee y) \wedge z] \vee (x \vee y)$$
$$= x \vee y$$
$$= i'.$$

Hence, $i' = c' \vee b' = a' \vee b'$. In a similar manner,

$$0' = c' \wedge b' = a' \wedge b'.$$

It is easily seen then that B is a sublattice of L.

Define $\varphi = \{(c',c),\ (a',a),\ (0',0),\ (i',i),\ (b',b)\}$. The domain of φ is B and the range is the lattice

$$A = \{a,\ b,\ c,\ 0,\ i\}$$

shown in the figure. φ is a reversible function since each element in B has one and only one image in A and each element in A has one and only one image in B. Now

$$\varphi(a' \wedge b') = \varphi(0') = 0 = a \wedge b = \varphi(a') \wedge \varphi(b')$$

and $\varphi(a' \vee b') = \varphi(i') = i = a \vee b = \varphi(a') \vee \varphi(b')$.

Taking all possible combinations in this manner, for any $x, y \in B$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. Thus, the lattice B is isomorphic to the lattice A. But this is a contradiction; hence, L is modular.

Corollary 1. No element of a bounded modular lattice has two comparable complements.

Proof. Let L be a bounded modular lattice bounded above by i and below by 0, and suppose that there does exist an element y such that y has two distinct, comparable complements x and z. By definition, $x \wedge y = 0$, $y \wedge z = 0$, $x \vee y = i$, and $y \vee z = i$. Since x and z are comparable, then either $x \wedge z = x$

or $x \wedge z = z$. The proof by supposing $x \wedge z = x$ is the same as the proof by supposing $x \wedge z = z$, so suppose that $x \wedge z = x$. The remainder of the proof is the same as the proof of the theorem. Thus, the sublattice formed from the elements a', b', c' o', i', where $a' = x \vee (y \wedge z)$, $b' = y$, $c' = (x \vee y) \wedge z$, $i' = x \vee y$, $o' = y \wedge z$, is isomorphic to the lattice A shown in Figure 2a, which contradicts the theorem. Hence, Corollary 1 follows.

<u>Corollary 2</u>. For the elements x, y and z of a modular lattice, $x \wedge z = x \wedge y$, $x \vee z = y \vee z$, and $x \wedge y = x$ imply $x = y$.

<u>Proof</u>. Let L be a modular lattice, and suppose that x, y and z are elements of L such that $x \wedge z = y \wedge z$, $x \vee z = y \vee z$, and $x \wedge y = x$. Since L is modular,

$$x \vee (z \wedge y) = (x \vee z) \wedge y,$$

but $x \vee (z \wedge y) = x \vee (x \wedge z) = x$ and $(x \vee z) \wedge y = (y \vee z) \wedge y = y$. Hence, $x = y$.

<u>Theorem 3.12</u>. Every distributive lattice is modular.

<u>Proof</u>. Let x, y, z be any triplet in the distributive lattice L such that $x \leqq z$. Then, by property (ii) of Definition 3.1, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. By Theorem 2.4, $x \leqq z$ implies $x \wedge z = x$ and, by Theorem 2.2, $x \wedge z = x$ implies $x \vee z = z$. Thus, $x \vee (y \wedge z) = (x \vee y) \wedge z$. Hence, L is modular.

Note that the converse of this theorem is not necessarily true, for consider the lattice B shown in Figure 3. Since no sublattice of B is isomorphic with lattice A, then B is modular, but B is not distributive since

$$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c).$$

Hence, this exhibits an example of a modular lattice which is not distributive. The following theorem shows that a distributive lattice can be characterized by examining its sublattices.

Theorem 3.13. A lattice is distributive if and only if it has no sublattice isomorphic with either lattice A (the lattice in Figure 2a) or lattice B (the lattice shown in Figure 3).

Proof. Let L be a distributive lattice. By Theorem 3.12, the fact that L is distributive implies that L is modular. Then, by Theorem 3.11, there is no sublattice which is isomorphic to lattice A. Note that lattice B is not distributive since $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$ and, by Theorem 3.2, every sublattice of a distributive lattice is distributive. Thus, there is no sublattice of L isomorphic with lattice B.

To prove the converse, suppose that L is a lattice such that no sublattice of it is isomorphic with either lattice A or lattice B. Suppose that L is not distributive. Either L is modular or it is not. If L is not modular, then, by Theorem 3.11, there is a sublattice isomorphic to lattice A, which is a contradiction. Thus, L is modular. By the contrapositive of Theorem 3.10 there exists a triplet x, y, z in L such that $(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) \neq (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$. Furthermore, by Theorems 2.3 and 2.4,

$$(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) < (x \vee y) \wedge (y \vee z) \wedge (x \vee z).$$

Let $0' = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$, $i' = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$, $a' = 0' \vee (x \wedge i')$, $b' = 0' \vee (y \wedge i')$ and $c' = 0' \vee (z \wedge i')$. Since $0' < i'$ and L is modular, $0' \vee (x \wedge i') = (0' \vee x) \wedge i'$ and $0' \vee (y \wedge i') = (0' \vee y) \wedge i'$. Note that

$$\begin{aligned} a' \wedge b' &= \left[0' \vee (x \wedge i')\right] \wedge \left[0' \vee (y \wedge i')\right] \\ &= (0' \vee x) \wedge i' \wedge (0' \vee y) \wedge i' \\ &= \left[(0' \vee x) \wedge (0' \vee y)\right] \wedge i'. \end{aligned}$$

Now $0' \vee x = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) \vee x = x \vee (y \wedge z)$, and $0' \vee y = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) \vee y = y \vee (x \wedge z)$ so that

$$a' \wedge b' = \left\{\left[x \vee (y \wedge z)\right] \wedge \left[y \vee (x \wedge z)\right]\right\} \wedge i'.$$

Since $y \wedge z \leq y \leq y \vee (x \wedge z)$ and $x \wedge z \leq x$, then

$$\begin{aligned} a' \wedge b' &= i' \wedge \left\{(y \wedge z) \vee \left[x \wedge \left[y \vee (x \wedge z)\right]\right]\right\} \\ &= i' \wedge \left[(y \wedge z) \vee (x \wedge z) \vee (x \wedge y)\right] \\ &= i' \wedge 0' \\ &= 0'. \end{aligned}$$

In a similar manner, $b' \wedge c' = a' \wedge c' = 0'$ and, by a dual proof, $a' \vee c' = b' \vee c' = a' \vee b' = i'$. Next it will be shown that $0'$, i', a', b' and c' are all distinct. Suppose that $a' = 0'$. Then $a' \wedge b' = a'$ and $a' \wedge c' = a'$. By Theorem 2.2, $a' \vee b' = b'$ and $a' \vee c' = c'$, but $a' \vee b' = i'$ and $a' \vee c' = i'$. Thus, $b' = i'$ and $c' = i'$. Note that $i' = i' \wedge i' = b' \wedge c' = 0'$, which is a contradiction. Hence, $a' \neq 0'$. Similarly, $b' \neq 0'$ and $c' \neq 0'$ and, by a dual proof, $i' \neq a', b', c'$. Suppose that $a' = b'$. Then $a' = a' \wedge a' = a' \wedge b' = 0'$, which is a contradiction. Thus, $a' \neq b'$. In a similar manner, $a' \neq c'$ and

$c' \neq b'$. Hence, $0'$, i', a', b', and c' are all distinct. Clearly, the set $G = \{0', i', a', b', c'\}$ is a sublattice of L. By a proof analogous to the proof to Theorem 3.11, the lattice G is isomorphic with the lattice B. But this is a contradiction. Hence, L is distributive.

<u>Corollary 1</u>. Every element of a bounded distributive lattice has at most one complement.

<u>Proof</u>. Let L be a distributive lattice bounded above by i and below by 0. Let $x \in L$ such that x has two complements x' and x'', and suppose that $x' \neq x''$. Then either $x'' \parallel x'$ or not. If x' and x'' are comparable, then either $x' \wedge x'' = x'$ or $x' \wedge x'' = x''$, but in either case, the set of elements

$$\{x,\ x',\ x'',\ 0,\ i\}$$

forms a sublattice of L isomorphic with the lattice A, a contradiction to Theorem 3.13. If $x' \parallel x''$, then $x' \wedge x'' < x' < x' \vee x''$. Since x' and x'' are both complements of x, then $x \wedge x' = 0$, $x \vee x' = i$, $x \wedge x'' = 0$ and $x \vee x'' = i$. Note that

$$(x' \wedge x'') \wedge x = x' \wedge (x'' \wedge x) = x' \wedge 0 = 0$$

and $(x' \wedge x'') \vee x = x \vee (x' \wedge x'') = (x \vee x') \wedge (x \vee x'') = i \wedge i = i$. Similarly, $(x' \vee x'') \vee x = i$ and $(x' \vee x'') \wedge x = 0$. Thus, the set of elements $\{x, x' \ \ x'', x' \ \ x'', 0, i\}$ forms a sublattice of L isomorphic with lattice A, a contradiction. Hence, $x' = x''$.

<u>Corollary 2</u>. For the elements x, y, z of a distributive lattice, $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$ imply $x = y$.

<u>Proof</u>. Let x, y, z be any triplet of the distributive lattice L such that $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$. Thus,

$$
\begin{aligned}
x &= x \vee (x \wedge z) \\
&= x \vee (y \wedge z) \\
&= (x \vee y) \wedge (x \vee z) \\
&= (y \vee x) \wedge (y \vee z) \\
&= y \vee (x \wedge z) \\
&= y \vee (y \wedge z) \\
&= y.
\end{aligned}
$$

Theorems 3.11 and 3.13 appear to be the most important tools for determining whether or not a lattice is distributive or modular. Note that the lattice shown in Figure 3 (lattice B) has no sublattice isomorphic to lattice A. By Theorem 3.11, B is modular. Then, by Theorem 3.13, B is not distributive. This example illustrates the use of Theorems 3.11 and 3.13.

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