ON BOUNDED VARIATION

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CHAPTER I

INTRODUCTION

1. Content and Scope

1.1. This paper is primarily concerned with developing the theory of real-valued functions of bounded variation and those ideas which are closely related to this main topic. In addition to this, some emphasis has been placed on the relationship of the theory of functions of bounded variation to specific areas of analysis. In particular, integration theory has been chosen as the vehicle to demonstrate this connection.

In this thesis the notion of bounded variation has been treated primarily with respect to functions of a real variable. Furthermore, the domain of such functions is usually taken to be an interval. Unquestionably this somewhat limits the scope of this study, but the fundamental theorems of §7 indicate that the class of functions which admits of this property is indeed quite large. For those readers who might desire a more abstract approach, certain sections of Chapter III might be of particular interest. In addition, Saks' Theory of the Integral (1) indicates in considerable detail how this idea might be generalized.

1.2. In Chapter I the saltus function and functions which possess certain monotonicity properties are described, and several theorems are proved concerning the nature of such functions. Then in §5 a close connection between these concepts and Riemann integration is demonstrated.

In Chapter II the notions of absolute continuity and bounded variation are defined and discussed in considerable detail. Complete characterizations of functions having these properties are given in this chapter. In addition, certain decomposition theorems and convergence theorems are included here. Then in §8 the Riemann-Stieltjes Integral is defined, and some significant theorems are proved indicating how the notion of bounded variation might apply to a study of integration.

In Chapter III the ideas of bounded variation and absolute continuity are generalized. Based on one of these generalizations, a short discussion is given of a class of functions which is Riemann-Stieltjes integrable in a broader context than that usually considered. In addition, the Burkill integral of a real-valued interval function defined in Euclidean n-space is discussed; and several theorems are proved indicating the relationship of this integral to some of the generalizations given in this chapter.

2. Definitions

1.3. The term <u>point set</u> will be used interchangeably with <u>set of real numbers</u>.

1.4. The <u>closed interval</u> $[a,b]$ will denote the set of all real numbers x so that $a \leq x \leq b$; the <u>open interval</u> (a,b)

will denote the set of all real numbers x so that $a < x < b$; $[a,b)$ will denote the set of all x so that $a \leq x < b$; and $(a,b]$ will denote the set of all x so that $a < x \leq b$.

1.5. If $I = [a,b]$ is a closed interval, then I_1 will be an <u>open subinterval with respect to</u> I (<u>open relative to</u> I) if, and only if, I_1 is the intersection of an open interval with I.

1.6. If S is a point set, then S is <u>bounded</u> if, and only if, there exists a real number $K > 0$ so that if $x \in S$, then $|x| < K$.

1.7. δ is a <u>limit point</u> of the point set S means that if I is an open interval containing δ, then I contains at least one point of S other than δ.

1.8. The statement that the point set S is <u>closed</u> means that S contains all of its limit points; the <u>closure</u> of S is the set S itself together with all of its limit points.

1.9. The statement that K is an <u>upper bound</u> for the point set S means that if $x \in S$, then $x < K$; K is a <u>lower bound</u> for S means that if $x \in S$, then $K < x$.

1.10. The statement that K is the <u>least upper bound</u> (<u>greatest lower bound</u>) for the point set S means that

i) K is an upper bound (lower bound), and

ii) if $\epsilon > 0$, then there exists an $x \in S$ so that $x > K - \epsilon$ ($x < K + \epsilon$). Least upper bound will be denoted by lub or max, and greatest lower bound will be denoted by glb or min.

1.11. If S is a set, then S is <u>countable</u> if, and only if, S can be mated biuniquely with a subset of the positive integers.

1.12. The statement that f is a <u>function from</u> (<u>on</u>) <u>a set</u> A <u>into a set</u> B means that f is a correspondence which associates with each element of the set A a unique element of the set B. A is called the domain of f, and the collection of mates of elements of A is called the range of f.

1.13. If the function f is defined on the set A, then to say that f is <u>bounded</u> on A means that the range of f is bounded.

1.14. If $A \subset B$, then A is <u>everywhere dense</u> in B means that B is a subset of the closure of A.

1.15. Two intervals I and J are said to be <u>abutting</u> if, and only if, they have only one point in common and that point is an end point.

1.16. If the function f is defined, except possibly for the point ξ, in some open interval I containing ξ, then $\lim_{x \to \xi} f(x) = \alpha$ means that if $\epsilon > 0$, then there exists a $\delta > 0$ so that if $0 < |x - \xi| < \delta$ and $x \in I$, then $|f(x) - \alpha| < \epsilon$. $\lim_{x \to \xi^+} f(x) = \alpha$ means that if $\epsilon > 0$, then there exists a $\delta > 0$ so that if $0 < x - \xi < \delta$, then $|f(x) - \alpha| < \epsilon$. $\lim_{x \to \xi^-} f(x) = \alpha$ means that if ϵ is chosen positive, then there exists a $\delta > 0$ so that if $0 < \xi - x < \delta$, then $|f(x) - \alpha| < \epsilon$. It should be understood that if f is defined on $[a, b]$ and $\xi = a$ or $\xi = b$, then $\lim_{x \to \xi} f(x)$ will be interpreted in terms of the appropriate one-sided limit. $\lim_{x \to \xi^+} f(x)$ will be denoted by $f(\xi+)$, and $\lim_{x \to \xi^-} f(x)$ will be denoted by $f(\xi-)$.

1.17. If f is defined at $x = \xi$, then f is <u>continuous</u> at $x = \xi$ if, and only if, $\lim_{x \to \xi} f(x) = f(\xi)$, where it is understood that the limit does exist. If f is defined on the interval I, the statement that f is continuous on I means that f is continuous at each point of I.

1.18. The statement that the function f is <u>uniformly continuous</u> on $I = [a,b]$ means that if $\epsilon > 0$, then there exists a $\delta > 0$ so that if x_1 and x_2 are in I and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.

1.19. A <u>sequence</u> $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ is merely a function f defined on the set of positive integers so that $f(1) = a_1$, $f(2) = a_2$, ... $f(n) = a_n$, ...

1.20. Let $\{f_n\}$ be a sequence of functions, all of which are defined on $[a,b]$. If f is a function defined on $[a,b]$, then the statement that the sequence $\{f_n\}$ <u>converges</u> to f on $[a,b]$ means that if $x \in [a,b]$ and $\epsilon > 0$, then there exists a positive integer N so that if $n > N$, then $|f_n(x) - f(x)| < \epsilon$. The sequence $\{f_n\}$ is said to <u>converge uniformly</u> to f on $[a,b]$ provided that for each $\epsilon > 0$ there exists a positive integer N (depending on ϵ only) so that if $n > N$ and $x \in [a,b]$, then $|f_n(x) - f(x)| < \epsilon$.

1.21. A <u>descending, infinitesimal sequence of closed intervals</u> $\{I_n\} = \{[a_n, b_n]\}$ is a sequence of intervals so that

i) for each positive integer n, $I_{n+1} \subset I_n$, and

ii) the length of (I_n), denoted by $l(I_n)$, approaches zero as n increases without bound, i.e. $\lim_{n \to \infty} l(I_n) = 0$.

1.22. If S is a point set, then S has Jordan-Content-0 if, and only if, for each $\epsilon > 0$ there exists a finite collection $I = \{I_1, I_2, \ldots, I_n\}$ of non-overlapping open intervals so that

i) $S \subset \bigcup_{i=1}^{n} I_i$, and

ii) $\sum_{i=1}^{n} l(I_i) < \epsilon$, where $l(I_i)$ denotes the length of I_i.

The term exterior Jordan-Content-0 may be used interchangeably with Jordan-Content-0.

1.23. If S is a set of real numbers, then S has exterior Lebesgue-Measure-0 if, and only if, for each $\epsilon > 0$ there exists a countable collection J of non-overlapping open intervals I so that

i) $S \subset \bigcup_{I \in J} I$, and

ii) $\sum_{I \in J} l(I) < \epsilon$.

It should be noted that Lebesgue-Measure-0 is equivalent to exterior Lebesgue-Measure-0 and that any countable set is of Lebesgue-Measure-0.

1.24. If the function f is defined on $[a,b]$, the statement that f has a bounded difference quotient on $[a,b]$ means that there exists a positive number M so that if $a \leq c < d \leq b$, then

$$\left| \frac{f(d)-f(c)}{d-c} \right| < M.$$

1.25. A <u>subdivision</u> σ of $[a,b]$ is a finite set of points $\{x_0, x_1, x_2, \ldots, x_n\}$ so that

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

The notation $\sigma : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is sometimes used to refer to the subdivision σ. The <u>norm</u> of the subdivision σ, denoted by $\|\sigma\|$, is defined to be the max of $\{(x_1-x_0), (x_2-x_1), \ldots, (x_n-x_{n-1})\}$.

1.26. To say that the subdivision σ_1 is a <u>refinement</u> of the subdivision σ means that if $x \in \sigma$, then $x \in \sigma_1$, <u>i.e.</u> each subdivision point of σ is a subdivision point of σ_1. The notation $\sigma_1 \supset \sigma$ will indicate that σ_1 is a refinement of σ.

1.27. If f is a bounded function defined on $I = [a,b]$, then $U(f;I)$ is by definition $\max\{f(x) | x \in I\}$, $L(f;I)$ is $\min\{f(x) | x \in I\}$, and $S(f;I)$ is $\max\{d | d = |f(x_1)-f(x_2)|$, where x_1 and x_2 are arbitrary points of $I\}$.

If $I = (a,b)$ is an open interval, the definitions are made in a similar fashion.

1.28. Let f be a bounded function defined on $[a,b]$. Let $\xi \in (a,b)$. The following definitions are then made: $U(f;\xi) = \min\{U(f;I) |$ where I is an open interval containing ξ and lying in $[a,b]\}$, $L(f;\xi) = \max\{L(f;I) |$ I is an open interval containing ξ and lying in $[a,b]\}$, and

$$S(f;\xi) = \min\{S(f;I) | \text{ I}$$

is an open interval containing ξ and lying in $[a,b]\}$. If $\xi = a$ or $\xi = b$, the intervals I are taken open relative to $[a,b]$.

1.29. The function f defined on $[a,b]$ is said to be <u>monotone non-decreasing</u> (<u>monotone non-increasing</u>) on $[a,b]$ provided that if x_1 and x_2 belong to $[a,b]$ and $x_1 < x_2$, then $f(x_1) \leq f(x_2)$ $(f(x_1) \geq f(x_2))$.

1.30. Let f be a bounded function defined on $[a,b]$. Let $\sigma : a = x_0 < x_1 < \ldots < x_n = b$ be an arbitrary subdivision of $[a,b]$. M_i will denote the max of f on $[x_{i-1}, x_i]$, and m_i will denote the min of f on $[x_{i-1}, x_i]$, where $i = (1, 2, \ldots, n)$. Then let

$$\overline{\sum_{\sigma}} f = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

and let

$$\underline{\sum_{\sigma}} f = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

These summations are called the <u>upper</u> and <u>lower sums</u>, respectively, of f with respect to σ. Then define the <u>upper Riemann integral</u> of f on $[a,b]$, denoted by $\overline{\int_a^b} f$, to be the min of $\overline{\sum_{\sigma}} f$ for all σ of $[a,b]$. Similarly, define the <u>lower Riemann integral</u> of f on $[a,b]$, denoted by $\underline{\int_a^b} f$, to be the max of $\underline{\sum_{\sigma}} f$ for all σ of $[a,b]$. Then f is said to be <u>Riemann integrable</u> on $[a,b]$ when $\underline{\int_a^b} f = \overline{\int_a^b} f$; and the common value of $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ is denoted by $\int_a^b f$ and called the <u>Riemann integral</u> of f on $[a,b]$.

3. Assumed Theorems

1.31. If $\{I_n\} = \{[a_n, b_n]\}$ is a descending, infinitesimal sequence of closed intervals, then there is a unique point which belongs to I_n for every n.

1.32. If the function f is continuous on $I = [a,b]$, then f is bounded on I.

1.33. If the function f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.

1.34. If the function f is continuous on $[a,b]$, then f assumes its max as a value, and assumes its min as a value.

1.35. If a and b are real numbers, then

$$\big|\,|a|-|b|\,\big| \le |a-b| \le |a|+|b|.$$

1.36. A necessary and sufficient condition that a function f be continuous at $x = \xi$ is that $f(\xi-)$ and $f(\xi+)$ exist and $f(\xi-) = f(\xi) = f(\xi+)$.

1.37. Any non-empty set S of real numbers that is bounded from above (below) has a least upper bound (greatest lower bound).

1.38. If T is a collection of open intervals covering the closed, bounded set S, then there exists a finite sub-collection T' of T which also covers S.

1.39. If the functions f and g are bounded on a,b and α and β are real numbers, then

i) if σ and σ_1 are subdivisions of $[a,b]$ so that $\sigma \subset \sigma_1$, then $\overline{\sum_{\sigma}} f \ge \overline{\sum_{\sigma_1}} f$ and $\underline{\sum_{\sigma_1}} f \ge \underline{\sum_{\sigma}} f$;

ii) if each of σ and σ_1 are subdivisions of $[a,b]$, then $\underline{\sum_{\sigma}} f \le \overline{\sum_{\sigma_1}} f$;

iii) $\overline{\int_a^b} f \ge \underline{\int_a^b} f$;

iv) if f and g are Riemann integrable on $[a,b]$, then $\alpha f+\beta g$ is Riemann integrable on $[a,b]$.

1.40. A necessary and sufficient condition that the bounded function f be Riemann integrable on $[a,b]$ is that for each $\epsilon > 0$ there exists a subdivision σ of $[a,b]$ so that $\overline{\sum_\sigma} f - \underline{\sum_\sigma} f < \epsilon$.

4. Monotonicity and the Saltus Function

1.41. __If the function__ f __is bounded on__ $I = [a,b]$, __then__ $S(f;I) = U(f;I)-L(f;I)$.

__Proof__. Suppose that $S(f;I) > U(f;I)-L(f;I)$. Hence there exists an $\epsilon > 0$ so that $S(f;I) - \epsilon = U(f;I)-L(f;I)$. Then by 1.27 there exist points x_1 and x_2 in I so that

$$|f(x_1)-f(x_2)| > S(f;I) - \epsilon .$$

Thus $|f(x_1)-f(x_2)| > U(f;I)-L(f;I)$. But then either

$$|f(x_1)-f(x_2)| = f(x_1)-f(x_2)$$

or $|f(x_1)-f(x_2)| = f(x_2)-f(x_1)$. Without loss of generality, suppose that $|f(x_1)-f(x_2)| = f(x_1)-f(x_2)$. But $f(x_1) \leq U(f;I)$ by 1.27. Similarly, $f(x_2) \geq L(f;I)$. Therefore

$$f(x_1)-f(x_2) \leq U(f;I)-L(f;I),$$

a contradiction of our assumption.

Hence $S(f;I) \leq U(f;I)-L(f;I)$. The assumption that $S(f;I) < U(f;I)-L(f;I)$ leads to a similar contradiction.

Thus $S(f;I) = U(f;I)-L(f;I)$ and the theorem is proved.

1.42. __If the function__ f __is bounded on the interval__ I (__open or closed__), __then__ f __is continuous at__ $\xi \in I$ __if, and only if,__ $S(f;\xi) = 0$.

__Proof__. Denote $S(f;\xi)$ by $S(\xi)$. Without loss of generality, suppose that $\xi \in I = (a,b)$.

Let f be continuous at $x = \xi$. Choose $\epsilon > 0$. Therefore there exists a $\delta > 0$ so that if $x \in I$ and $|x - \xi| < \delta$, then

$$|f(x) - f(\xi)| < \frac{\epsilon}{4}.$$

Then let $I_1 = (\xi - \delta, \xi + \delta)$. It follows that if x_1 and x_2 belong to I_1, then

$$|f(x_1) - f(x_2)| < \frac{\epsilon}{2}.$$

Hence $S(f;I_1) < \epsilon$. Clearly $S(\xi) \leq S(f;I_1)$. Therefore $S(\xi) = 0$.

Now suppose that $S(\xi) = 0$. Then choose $\epsilon > 0$. By Definition 1.28 there exists an interval $I_1 = (c,d)$ lying in I and containing ξ so that $S(f;I_1) < \epsilon$. Let

$$\delta = \min(\xi - c, d - \xi).$$

Then choose $x \in I$ so that $|x - \xi| < \delta$. Hence $x \in I_1$. Thus $|f(x) - f(\xi)| \leq S(f;I_1) < \epsilon$. The continuity of f at ξ is now apparent.

1.43. It is well known that if a function f is monotone non-decreasing (monotone non-increasing) on $[a,b]$ and $x \in (a,b)$, then the one-sided limits $f(x-)$ and $f(x+)$ exist (2). Similarly, $f(a+)$ and $f(b-)$ exist.

In fact, if $x \in (a,b)$, $f(x-)$ is merely the max of $f(y)$ for all y in $[a,b]$ so that $y < x$. Likewise, $f(x+)$ is the min of $f(y)$ for all y in $[a,b]$ so that $y > x$. It then follows easily that $f(x-) \leq f(x) \leq f(x+)$. This fact together with the following lemma proves quite useful.

Lemma. *If the function f is defined and monotone non-decreasing on $[a,b]$, ξ_1 and ξ_2 are in $[a,b]$, and $\xi_1 < \xi_2$, then* $f(\xi_1+) \leq f(\xi_2-)$.

Proof. Let $E = \{x | x \in (\xi_1, \xi_2)$. Then

$$f(\xi_1+) = \min E \leq \max E = f(\xi_2-).$$

1.44. Theorem. *If the function f is defined and monotone non-decreasing (monotone non-increasing) on $[a,b]$, then the set of points of discontinuity of f on $[a,b]$ is countable.*

Proof. Suppose that f is monotone non-decreasing on $[a,b]$, and let D denote the set of points of discontinuity of f on $[a,b]$.

Let $\xi \in D$. Then by 1.36 and 1.43 $f(\xi-) \leq f(\xi) \leq f(\xi+)$ and $f(\xi-) < f(\xi+)$. Let r_ξ be a rational number contained in the open interval $(f(\xi-), f(\xi+))$.

Then suppose that ξ_1 and ξ_2 are distinct elements of D. Let $\xi_1 < \xi_2$. By 1.43, $f(\xi_1+) \leq f(\xi_2-)$. Hence

$$(f(\xi_1-), f(\xi_1+)) \cap (f(\xi_2-), f(\xi_2+)) = \emptyset .$$

Thus $r_{\xi_1} \neq r_{\xi_2}$. The countability of D follows from the fact that the set of rational numbers is countable.

1.45. *If f is defined and monotone non-decreasing (monotone non-increasing) on $[a,b]$, then*

$$S(f;\xi) = f(\xi+) - f(\xi-) (S(f;\xi) = f(\xi-) - f(\xi+)).$$

Proof. The proof is given for the case when f is monotone non-decreasing.

Suppose that $\xi \in (a,b)$ in order that both one-sided limits may be considered. Then suppose by way of contradiction that $S(\xi) \neq f(\xi+)-f(\xi-)$, and consider the case where $S(\xi) > f(\xi+)-f(\xi-)$. Therefore there exists an $\epsilon > 0$ so that $S(\xi) = f(\xi+)-f(\xi-)+\epsilon = f(\xi+)+\frac{\epsilon}{2} - [f(\xi-)-\frac{\epsilon}{2}]$. But then there exist points x_1 and x_2 in (a,b) so that $x_1 < \xi < x_2$, $f(x_1) > f(\xi-)-\frac{\epsilon}{2}$, and $f(x_2) < f(\xi+)+\frac{\epsilon}{2}$. Hence $S(\xi) > f(x_2)-f(x_1) = |f(x_2)-f(x_1)|$, a contradiction.

The remaining case leads to a similar contradiction. Therefore the theorem is proved.

1.46. <u>If</u> f <u>is defined and monotone non-decreasing on</u> $[a,b]$, <u>and</u> $\{x_1, x_2, \ldots, x_n\}$ <u>is a finite set of points so that</u> $a < x_1 < x_2 < \ldots < x_n < b$, <u>then</u>

$$f(a+)-f(a)+\sum_{i=1}^{n} S(x_i)+f(b)-f(b-) \leq f(b)-f(a).$$

<u>Proof</u>. Let $\{y_1, y_2, \ldots, y_{n+1}\}$ be a finite collection of points so that $a < y_1 < x_1 < y_2 < x_2 < y_3 < \ldots < y_n < x_n < y_{n+1} < b$. Then by 1.43, $f(y_1)-f(a) \geq f(a+)-f(a)$, $f(y_2)-f(y_1) \geq S(x_1)$, $f(y_3)-f(y_2) \geq S(x_2)$, ..., $f(y_{n+1})-f(y_n) \geq S(x_n)$, and

$$f(b)-f(y_{n+1}) \geq f(b)-f(b-).$$

Adding these terms, the conclusion of the theorem is obtained.

<u>Corollary</u>. $\sum_{i=1}^{n} S(x_i) \leq f(b-)-f(a+)$.

1.47. Let f be defined and monotone non-decreasing on $[a,b]$. Let D be the set of points of discontinuity of f on (a,b). Define the interval function F as follows:

$F[a,x] = \max\{A \mid A = f(a+)-f(a)+\sum_{i=1}^{n} S(x_i)+f(x)-f(x-)$, where $x_1, x_2, \ldots, x_n$ is any finite collection of points in $D \cap (a,x)\}$ if $x > a$, and $F[a,a] = 0$. It should be remarked that the finite collections of points in D could be ordered without altering the definition. As a result of 1.46 it is clear that $F[a,x]$ is well defined for any x in $[a,b]$.

The following lemma will be helpful.

<u>Lemma</u>. <u>If</u> f <u>is defined and monotone non-decreasing on</u> $[a,b]$ <u>and</u> $a < c < b$, <u>then</u> $F[a,c]+F[c,b] = F[a,b]$.

<u>Proof</u>. Suppose that $F[a,c]+F[c,b] < F[a,b]$. Therefore there exists an $\epsilon > 0$ so that $F[a,c]+F[c,b]+\epsilon = F[a,b]$. Then $F[a,c]+F[c,b] = F[a,b]-\epsilon$. But then there exists a finite collection of points $\{x_1, x_2, \ldots, x_n, c\}$ in $[a,b]$ so that

$$\begin{aligned} & f(a+)-f(a)+\sum_{i=1}^{n} S(x_i)+S(c)+f(b)-f(b-) \\ &= f(a+)-f(a)+\sum_{i=1}^{n} S(x_i)+(f(c+)-f(c)+f(c)-f(c-)) \\ &+ f(b)-f(b-) > F[a,b]-\epsilon . \end{aligned}$$

Hence, $F[a,c]+F[c,b] < f(a+)-f(a)+\sum_{i=1}^{n} S(x_i)+(f(c+)-f(c-))+f(b)-f(b-)$.

Then let Δ_1 denote those values of i so that $x_i < c$, and let Δ_2 denote those values of i so that $x_i > c$. Then

$$\begin{aligned} & \Big[f(a+)-f(a)+\sum_{i\in\Delta_1} S(x_i)+f(c)-f(c-)\Big] \\ &+ f(c+)-f(c)+\sum_{i\in\Delta_2} S(x_i)+f(b)-f(b-) \\ &\leq F[a,c]+F[c,b], \end{aligned}$$

a contradiction.

The assumption that $F[a,c]+F[c,b] > F[a,b]$ leads to a similar contradiction. Thus $F[a,c]+F[c,b] = F[a,b]$, as was to be proved.

Corollary. <u>If</u> x_1 <u>and</u> x_2 <u>are in</u> $[a,b]$ <u>and</u> $x_1 < x_2$, <u>then</u> $F[a,x_1] \leq F[a,x_2]$.

1.48. Theorem. If f is defined and monotone non-decreasing on $[a,b]$, then the function $h = f-F$ is monotone non-decreasing and continuous on $[a,b]$.

Proof. Choose x_1 and x_2 in $[a,b]$ so that $x_1 < x_2$. Then

$$\begin{aligned} h(x_2)-h(x_1) &= f(x_2)-F[a,x_2]-(f(x_1)-F[a,x_1]) \\ &= f(x_2)-f(x_1)-F[x_1,x_2] \\ &\geq 0 \end{aligned}$$

by 1.46.

Then, without loss of generality, suppose that $\xi \in (a,b)$. Then choose $\epsilon > 0$. Therefore there exists a $\delta > 0$ so that if $x > \xi$ and $x-\xi < \delta$, then $|f(x)-f(\xi+)| < \epsilon$. Hence

$$\begin{aligned} |f(x)-F[a,x]-(f(\xi)-F[a,\xi])| &= |f(x)-f(\xi)-F[\xi,x]| \\ &= f(x)-f(\xi)-F[\xi,x] \end{aligned}$$

by 1.46. If $\{x_1, \ldots, x_n\}$ is any finite collection of points in $D \cap [a,x]$ (notation as in 1.47), then

$$-[f(\xi+)-f(\xi)+\sum_{i=1}^{n} S(x_i)+f(x)-f(x-)] \geq -F[\xi,x].$$

Hence

$$f(x)-f(\xi)-F[\xi,x] \leq f(x-)-f(\xi+)-\sum_{i=1}^{n} S(x_i) \leq f(x-)-f(\xi+).$$

But by 1.43, $f(x-)-f(\xi+) = |f(x-)-f(\xi+)| < \epsilon$.

The case where $x < \xi$ leads to a similar argument. Hence h is continuous at $x = \xi$.

5. The Riemann Integral

1.49. If the function f is monotone non-decreasing on $[a,b]$, then f is Riemann integrable (R-integrable) on $[a,b]$.

Proof. If $f(a) = f(b)$, f is constant on $[a,b]$ and the theorem follows. Therefore suppose that $f(b) > f(a)$.

Choose $\epsilon > 0$. Then let σ be any subdivision of $[a,b]$ so that

$$\| \sigma \| < \frac{\epsilon}{f(b)-f(a)}.$$

Then consider $\overline{\sum_\sigma} f - \underline{\sum_\sigma} f$. $\overline{\sum_\sigma} f - \underline{\sum_\sigma} f = \sum_{i=1}^{n} (M_i - m_i) l(I_i)$, where $l(I_i)$ denotes $(x_i - x_{i-1})$. But

$$\sum_{i=1}^{n} (M_i - m_i) l(I_i) < \frac{\epsilon}{f(b)-f(a)} \sum_{i=1}^{n} (M_i - m_i)$$

$$= \frac{\epsilon}{f(b)-f(a)} \cdot (f(b)-f(a))$$

$$= \epsilon.$$

R-integrability follows from 1.40.

1.50. Theorem. A necessary and sufficient condition for a bounded function f defined on $[a,b]$ to be R-integrable on $[a,b]$ is that for each $K > 0$, $E_K = \{x \in [a,b] \mid S(x) \geq K\}$ be of Jordan-Content-0.

Proof. Necessity. Suppose that f is R-integrable on $[a,b]$. Then suppose by way of contradiction that there exists a $K > 0$ so that E_K is of exterior Jordan-Content greater than ϵ, for some positive number ϵ. Therefore, if $I = \{I_n\}$ is any finite collection of open intervals covering E_K, then the length sum of the elements of I is greater than ϵ. Then if σ is any subdivision of $[a,b]$, the points of E_K not occurring as subdivision points of σ will have exterior Jordan-Content greater than ϵ. Hence $\overline{\sum_\sigma} f - \underline{\sum_\sigma} f > K \cdot \epsilon$, a contradiction of 1.40. Hence, for each $K > 0$, E_K must have Jordan-Content-0.

Sufficiency. Since f is bounded on $[a,b]$, there exists a positive number M so that if $x \in [a,b]$, then $|f(x)| < M$. Thus, $S(f;[a,b]) \leq 2M$.

Choose $\epsilon > 0$. Let k' be a positive number so that

$$k' < \frac{\epsilon}{2(b-a)}.$$

Then, by hypothesis, $E_{k'}$ is of Jordan-Content-0. Now $E_{k'}$ can be covered by a finite number of non-overlapping open intervals relative to $[a,b]$ with length sum as small as is desired. Furthermore, this collection can be chosen so that no element of $E_{k'}$ other than a or b could possibly be an end point of an interval in this collection. Choose one such collection $I' = \{I'_n\} = \{(a_n, b_n)\}$ with length sum less than

$$\frac{\epsilon}{4M}.$$

Then let $I = \{I_n\} = \{[a_n, b_n]\}$ denote the collection of closures of elements of I'. Then the elements of I have length sum less than

$$\frac{\epsilon}{4M}.$$

Let $\{x_n\}$ be the collection of end points of elements of I. Then $\{x_n\} \cup \{a,b\}$ forms a subdivision σ' of $[a,b]$.

Let H denote the collection of closed subintervals of $[a,b]$ introduced by σ' which do not belong to I. Let $H_1 \in H$. Then $H_1 \cap E_{k'} = \emptyset$. Therefore, around each element $x \in H_1$ there exists an open interval (relative to H_1) $I_x = (c_x, d_x)$ so that $S(f;[c_x, d_x]) < k'$. The collection of all such I_x for $x \in H_1$ forms an open covering of H_1. Then by 1.38 there exists a finite subcovering $I' = \{I_{x_1}, I_{x_2}, \ldots, I_{x_n}\}$. The set of end points of elements of I' together with the end points of H_i forms a subdivision σ_{H_i} of H_i so that if $I_i \in \sigma_{H_i}$, then $S(f; I_i) < k'$.

Then let $\sigma = \sigma' \cup \left[\bigcup_{H_i \in H} \sigma_{H_i}\right]$. Hence, σ forms a subdivision of $[a,b]$. Then

$$\overline{\sum_{\sigma}} f - \underline{\sum_{\sigma}} f = \sum_{I_i \in I} (M_i - m_i) l(I_i) + \sum_{H_i \in H} (M_i - m_i) l(I_i),$$

where

$$\sum_{H_i \in H} (M_i - m_i) l(I_i)$$

denotes the appropriate sum taken over σ_{H_i} for all $H_i \in H$.

But

$$\begin{aligned} &\sum_{I_i \in I} (M_i - m_i) l(I_i) + \sum_{H_i \in H} (M_i - m_i) l(I_i) \\ &< 2M \sum_{I_i \in I} l(I_i) + \frac{\epsilon}{2(b-a)} \sum_{H_i \in H} l(I_i) \\ &< 2M\left(\frac{\epsilon}{4M}\right) + \frac{\epsilon}{2(b-a)}(b-a) \\ &= \epsilon. \end{aligned}$$

R-integrability follows by 1.40.

1.51. <u>Theorem</u>. <u>The bounded function f defined on $[a,b]$ is R-integrable on $[a,b]$ if, and only if, the set D of points of discontinuity of f on $[a,b]$ is of Lebesgue-Measure-0.</u>

<u>Proof</u>. Suppose that f is R-integrable on $[a,b]$. Let $K = \{K_1, K_2, \ldots\}$ be the set of all positive rational numbers. Then, by 1.50, for every positive integer n,

$$E_{K_n} = \{x \mid S(x) \geq K_n\}$$

has Jordan-Content-0. Then, by 1.42, $E_{K_n} \subset D$.

Then cover E_{K_1} with a set of open intervals of length sum less than $\frac{\epsilon}{2}$, where ϵ is an arbitrary positive number. Cover E_{K_2} with a set of open intervals of length sum less than $\frac{\epsilon}{2^2}$. In general, cover E_{K_n} with a set of open intervals of length sum less than $\frac{\epsilon}{2^n}$. Clearly, as a result of 1.42, $D \subset \bigcup_{K_i \in K} E_{K_i}$, and the above process gives an open covering of

$$\bigcup_{K_i \in K} E_{K_i}$$

of length sum less than ϵ. Therefore, D is of Lebesgue-Measure-0.

Now suppose that D has Lebesgue-Measure-0. Choose $K > 0$. By previous remarks, $E_K = \{x \mid S(x) \geq K\}$ is a subset of D and has Lebesgue-Measure-0. It follows that E_K is closed, and obviously E_K is bounded. Then 1.38 implies that E_K is of Jordan-Content-0. Hence, f is R-integrable by 1.50, and the proof is finished.

CHAPTER BIBLIOGRAPHY

1. Saks, Stanislaw, *Theory of the Integral*, New York, G. E. Stechert and Company, 1937.

2. Rudin, Walter, *Principles of Mathematical Analysis*, McGraw-Hill Book Company, Inc., 1953.

CHAPTER II

ABSOLUTE CONTINUITY AND BOUNDED VARIATION

6. Definitions

2.1. If the function f is defined on $[a,b]$ $((a,b))$, the statement that f is __absolutely continuous__ on $[a,b]$ $((a,b))$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ so that if $\{(x_1,x_1'),(x_2,x_2'), \ldots, (x_n,x_n')\}$ is any finite collection of non-overlapping open intervals in $[a,b]$ $((a,b))$ so that

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta$$

then $\sum_{i=1}^{n} |f(x_i')-f(x_i)| < \epsilon$.

2.2. If f is defined on $[a,b]$ and

$$\sigma : a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

is a subdivision of $[a,b]$, then the __variation of__ f __with respect to__ σ, denoted by $V_\sigma(f;[a,b])$ or $V_\sigma(f)$, is defined to be $\sum_{i=1}^{n} |f(x_i)-f(x_{i-1})|$. The notation $\sum_\sigma f$ or

$$\sum_\sigma |f(x_i)-f(x_{i-1})|$$

is sometimes used to indicate this sum.

2.3. If the function f is defined on $[a,b]$, the __total variation__ of f on $[a,b]$, denoted by $V(f;[a,b])$, is defined to be the max of $V_\sigma(f)$ for all σ of $[a,b]$. The total variation of f on $[a,b]$ is sometimes denoted by $V(f)$.

2.4. If the function f is defined on $[a,b]$, then f is of bounded variation on $[a,b]$ if, and only if, the total variation of f on $[a,b]$ is finite.

2.5. If $I = (a,b)$ is an open interval, a subdivision σ of (a,b) will be a finite set of points $\{x_0, x_1, x_2, \ldots, x_n\}$ contained in (a,b) so that $x_0 < x_1 < x_2 < \ldots < x_n$.

2.6. If the function f is defined on the open interval $I = (a,b)$ and σ is a subdivision of I, then the variation of f with respect to σ, denoted by $V(f;(a,b))$, and the total variation of f on I are defined as in 2.2 and 2.3. Similarly, f is of bounded variation on I if, and only if, the total variation of f on I is finite.

2.7. If the function f is defined on some open interval I containing ξ, then the statement that f is of bounded variation at $x = \xi$ means that there exists an open interval $J = (c,d) \subset I$ so that f is of bounded variation on J.

2.8. If the function f is defined on the interval I (open or closed), then the statement that f is of unbounded variation on I means that f is not of bounded variation on I.

2.9. The statement that f is of unbounded variation at $x = \xi$ means that if I is an open interval containing ξ so that f is defined on I, then f is of unbounded variation on I.

2.10. If the function f is defined on $[a,b]$, then

$$\lim_{\|\sigma\| \to 0} V_\sigma(f) = K$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ so that if σ

is a subdivision of $[a,b]$ so that $\|\sigma\| < \delta$, then

$$|V_\sigma(f)-K| < \epsilon .$$

2.11. If the function f is defined on $[a,b]$ and σ_1 and σ_2 are two subdivisions of $[a,b]$, then $V_{\sigma_1/\sigma_2}(f)$ will denote only those terms of $V_{\sigma_1}(f)$ which are taken over intervals of σ_1 which lie entirely within some interval of σ_2.

2.12. Let a and b be two real numbers so that $a \leq b$. Let f_1 and f_2 be bounded functions defined for every real number t, $a \leq t \leq b$, so that $f_1(t) = x$ and $f_2(t) = y$. The functions f_1 and f_2 are termed <u>parameter functions</u>, and interval $[a,b]$ is called a <u>parameter interval for the parameter</u> t. The ordered set $A = \{(x,y) | x = f_1(t) \text{ and } y = f_2(t)\}$ is said to be a simple arc and is denoted by arc$[a,b]$.

2.13. Let $\sigma: a = t_0 < t_1 < t_2 < \dots < t_n = b$ be a subdivision of the parameter interval $[a,b]$. Then the <u>length</u> of arc$[a,b]$, denoted by L(arc$[a,b]$), is defined to be the max of

$$L_\sigma = \sum_{i=1,\sigma}^{n} \left[(f_1(t_i)-f_1(t_{i-1}))^2+(f_2(t_i)-f_2(t_{i-1}))^2\right]^{\frac{1}{2}}$$

for all σ of $[a,b]$. Arc$[a,b]$ is said to be <u>rectifiable</u> if, and only if, L(arc$[a,b]$) is finite.

2.14. Let f be a function defined on $[a,b]$, and let $\sigma: a = x_0 < x_1 < \dots < x_n = b$ be a subdivision of $[a,b]$.

$$S(f; [x_{i-1}, x_i])$$

will denote the saltus of the function f on $[x_{i-1}, x_i]$. The S-<u>variation of</u> f <u>with respect to</u> σ, denoted by $\sum_\sigma S(f)$, is defined to be $\sum_{i=1,\sigma}^{n} S(f; [x_{i-1}, x_i])$. The __total__ S-__variation__ of f on $[a,b]$ is defined to be the max of $\sum_\sigma S(f)$ for all σ of $[a,b]$. Then f is said to have __bounded__ S-__variation__ on $[a,b]$ if, and only if, the total S-variation of f on $[a,b]$ is finite.

2.15. Let f and g be bounded functions defined on the closed interval $I = [a,b]$. Let

$$\sigma : a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

be a subdivision of $[a,b]$. Choose a point $\xi_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then define $\sum_\sigma fdg$ to be

$$\sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1})).$$

Then, if there exists a real number I so that $\lim_{\|\sigma\| \to 0} \sum_\sigma fdg = I$, independently of the choice of subdivision and the choice of ξ_i, I is called the __Riemann__-__Stieltjes__ __integral__ (or merely the Stieltjes integral) of f with respect to g on $[a,b]$. And f is said to be S-__integrable__ __with__ __respect__ __to__ g, or f is merely said to be g-__integrable__. I is denoted by

$$\int_a^b fdg.$$

The following properties of the Stieltjes integral follow from the definition and will be stated here without proofs. If each of f_1, f_2, g_1, and g_2 is a function defined on $[a,b]$, k and l are real numbers, and

$$\int_a^b f_1 dg_1, \quad \int_a^b f_2 dg_1, \text{ and } \int_a^b f_1 dg_2$$

exist, then

i) $$\int_a^b (f_1 + f_2) dg_1 = \int_a^b f_1 dg_1 + \int_a^b f_2 dg_1;$$

ii) $\int_a^b f_1(dg_1+dg_2) = \int_a^b f_1 dg_1 + \int_a^b f_1 dg_2;$

iii) $\int_a^b kf_1 d(lg_1) = kl \int_a^b f_1 d(g_1);$

iv) $\int_a^b d(g_1) = g_1(b)-g_1(a);$

v) if $a \le c \le b$, $\int_a^c f_1 dg_1 + \int_c^b f_1 dg_1 = \int_a^b f_1 dg_1.$

2.16. Let the function f be defined and be of bounded variation on $[a,b]$. If $x \in [a,b]$ and

$$\sigma : a = x_0 < x_1 < \dots < x_n = x$$

is a subdivision of $[a,x]$, define p(x) to be the max of

$$\sum_{A_\sigma} (f(x_i)-f(x_{i-1})),$$

where A_σ denotes the intervals of σ so that $f(x_i)-f(x_{i-1}) \ge 0$ and the max is taken with respect to all subdivisions σ of $[a,x]$. Define n(x) to be the max of $\sum_{B_\sigma} |f(x_i)-f(x_{i-1})|$, where B_σ denotes the intervals of σ so that $(f(x_i)-f(x_{i-1})) < 0$ and the max is taken with respect to all subdivisions σ of $[a,x]$. Define v(x) to be the total variation of f on $[a,x]$. It should be noted that $p(a) = n(a) = v(a) = 0$ and

$$v(b) = V(f;[a,b]).$$

The interval function v is called the <u>variation</u> <u>function</u> of f.

2.17. Let f be a function defined on $[a,b]$, and suppose that $\xi \in [a,b]$. The statement that f is differentiable at $x = \xi$ means that there exists a real number K so that

$$\lim_{x \to \xi} \frac{f(x)-f(\xi)}{x-\xi} = K.$$

7. Absolute Continuity and Bounded Variation

2.18. Remark. Because of the close connection between the notions of absolute continuity and bounded variation, the two concepts will be handled in the same section. However, the first part of §7 will be devoted primarily to a study of absolute continuity, while the latter part will be concerned mainly with a discussion of bounded variation.

2.19. Remark. It should be mentioned here that a considerable portion of the motivation and arrangement for the remainder of this chapter is due to Natanson (1).

2.20. *If the function f is defined and absolutely continuous on $[a,b]$, then for each $\epsilon > 0$ there exists a $\delta > 0$ so that if $\{(x_i,x_i')\}_{i=1}^{\infty}$ is a countably infinite collection of pairwise disjoint open intervals so that $\sum_{i=1}^{\infty}|x_i'-x_i| < \delta$, then $\sum_{i=1}^{\infty}|f(x_i')-f(x_i)| < \epsilon$.*

Proof. Suppose that the function f defined on $[a,b]$ is absolutely continuous by 2.1. Thus, if ϵ is a positive number, there exists a $\delta > 0$ so that if $\{(x_i,x_i')\}_{i=1}^{n}$ is a collection of non-overlapping open intervals so that

$$\sum_{i=1}^{n}(x_i'-x_i) < \delta,$$

then $\sum_{i=1}^{n}|f(x_i')-f(x_i)| < \epsilon$. Then let $I = \{I_n\} = \{(x_n,x_n')\}$ be a sequence of pairwise disjoint open intervals so that

$$\sum_{i=1}^{\infty}(x_i'-x_i) < \delta.$$

Then consider the following monotone sequence of non-negative numbers: $F_1 = |f(x_1')-f(x_1)|$, $F_2 = \sum_{i=1}^{2}|f(x_i')-f(x_i)|$, ...,

$F_n = \sum_{i=1}^{n} |f(x_i')-f(x_i)|$, $F_{n+1} = \sum_{i=1}^{n+1} |f(x_i')-f(x_i)|$, ...
Since f is absolutely continuous, $F_n < \epsilon$, and $F_n \leq F_{n+1}$ for every positive integer n. Hence the sequence $F = \{F_n\}$ converges to a positive number $\epsilon' \leq \epsilon$. Thus

$$\sum_{i=1}^{\infty} |f(x_i')-f(x_i)| \leq \epsilon .$$

But ϵ is arbitrary, and the assertion follows.

2.21. <u>If the function</u> f <u>is defined on</u> [a,b] <u>and has a continuous derivative</u> f' <u>on</u> [a,b], <u>then</u> f <u>satisfies a Lipschitz Condition on</u> [a,b].

<u>Proof</u>. Since f' is continuous on [a,b], f' is bounded on [a,b]. Let K be a bound for f'. Then choose x_1 and x_2 in [a,b] so that $x_1 < x_2$. By the Mean Value Theorem of differential calculus, there exists a point $x' \in (x_1,x_2)$ so that

$$f'(x') = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

But $|f'(x')| < K$. Hence

$$\left| \frac{f(x_2)-f(x_1)}{x_2-x_1} \right| < K$$

and the theorem is proved.

2.22. <u>If the function</u> f <u>satisfies a Lipschitz Condition on</u> [a,b], <u>then</u> f <u>is absolutely continuous on</u> [a,b].

<u>Proof</u>. Let $K > 0$ be a bound for the difference quotient of f on [a,b]. Let ϵ be a positive number. Then choose $\delta > 0$ so that $\delta < \frac{\epsilon}{K}$. Let $\{(x_i,x_i')\}_{i=1}^{n}$ be a finite collection of non-overlapping open intervals so that $\sum_{i=1}^{n}(x_i'-x_i) < \delta$. Then,

for $i = 1, 2, \ldots, n$,

$$\left| \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}} \right| < K.$$

Hence

$$\sum_{i=1}^{n} f(x_i')-f(x_i) < K \cdot \sum_{i=1}^{n} |x_i'-x_i| < K \cdot \frac{\epsilon}{K} = \epsilon .$$

Thus the theorem is proved.

2.23. <u>If the function</u> f <u>is absolutely continuous on</u> $[a,b]$, <u>then</u> f <u>is bounded on</u> $[a,b]$.

<u>Proof.</u> Let $n = 1$ in the definition of absolute continuity. Thus, f is seen to be continuous on $[a,b]$. Then f is bounded by 1.32.

2.24. <u>If each of</u> f <u>and</u> g <u>is an absolutely continuous function defined on</u> $[a,b]$, <u>then</u>

i) $f+g$ is absolutely continuous on $[a,b]$;

ii) $f-g$ is absolutely continuous on $[a,b]$;

iii) $f\cdot g$ is absolutely continuous on $[a,b]$;

iv) if $\frac{1}{g}$ is defined and bounded on $[a,b]$, then $\frac{f}{g}$ is absolutely continuous on $[a,b]$.

<u>Proof.</u> Proofs of (i) and (iii) will be given. The proofs of the other parts are similar to the ones given.

i) Choose $\epsilon > 0$. There exist positive numbers δ_1 and δ_2 so that if each of $\{(x_i,x_i')\}_{i=1}^{n}$ and $\{(y_i,y_i')\}_{i=1}^{m}$ is a finite collection of non-overlapping open intervals so that $\sum_{i=1}^{n}(x_i'-x_i) < \delta_1$ and $\sum_{i=1}^{m}(y_i'-y_i) < \delta_2$, then

$$\sum_{i=1}^{n} |f(x_i')-f(x_i)| < \frac{\epsilon}{2}$$

and $\sum_{i=1}^{m} f(y_i')-f(y_i) < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1, \delta_2)$, and let $\{(x_i,x_i')\}_{i=1}^{k}$ be a finite collection of pairwise disjoint open intervals of length sum less than δ. Then

$$\sum_{i=1}^{k} |f(x_i')+g(x_i')-(f(x_i)+g(x_i))|$$
$$\leq \sum_{i=1}^{k} |f(x_i')-f(x_i)| + \sum_{i=1}^{k} |g(x_i')-g(x_i)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon.$$

iii) By 2.23 both f and g are bounded on $[a,b]$. Let M_1 be a bound for f, and let M_2 be a bound for g.

Choose $\epsilon > 0$. There exists a $\delta_1 > 0$ so that if

$$\{(x_i,x_i')\}_{i=1}^{n}$$

is any finite collection of non-overlapping open intervals so that $\sum_{i=1}^{n} (x_i'-x_i) < \delta_1$, then

$$\sum_{i=1}^{n} |f(x_i')-f(x_i)| < \frac{\epsilon}{2M_2}.$$

Similarly, there exists a $\delta_2 > 0$ so that if $\{(x_i,x_i')\}_{i=1}^{m}$ is any finite collection of non-overlapping open intervals of length sum less than δ_2, then

$$\sum_{i=1}^{m} |g(x_i')-g(x_i)| < \frac{\epsilon}{2M_1}.$$

Let $\delta = \min(\delta_1, \delta_2)$, and let $\{(x_i,x_i')\}_{i=1}^{k}$ be a finite collection of non-overlapping open intervals of length sum less than δ. Then

$$\sum_{i=1}^{k} |f(x_i')g(x_i')-f(x_i)g(x_i)|$$
$$= \sum_{i=1}^{k} |f(x_i')g(x_i')-f(x_i)g(x_i)+f(x_i)g(x_i')-f(x_i)g(x_i')|$$

$$= \sum_{i=1}^{k} |f(x_i)(g(x_i')-g(x_i))+g(x_i')(f(x_i')-f(x_i))|$$

$$\leq \sum_{i=1}^{k} |f(x_i)|\,|g(x_i')-g(x_i)| + \sum_{i=1}^{k} |g(x_i')|\,|f(x_i')-f(x_i)|$$

$$< M_1 \sum_{i=1}^{k} |g(x_i')-g(x_i)| + M_2 \sum_{i=1}^{k} |f(x_i')-f(x_i)|$$

$$< M_1\left(\frac{\epsilon}{2M_1}\right)+M_2\left(\frac{\epsilon}{2M_2}\right)$$

$$= \epsilon .$$

Hence, $f\cdot g$ is absolutely continuous on $[a,b]$, and the proof is completed.

2.25. __Theorem__. __If the function__ f __is defined and absolutely continuous on__ $[a,b]$, __then__ f __maps sets of exterior Lebesgue-Measure-0 into sets of exterior Lebesgue-Measure-0__.

__Proof__. Without loss of generality, suppose that $E\subset(a,b)$ and that E has exterior measure 0. Denote the exterior measure of E by $m^*(E)$.

Choose $\epsilon > 0$. By 2.20 and 1.34 there exists a $\delta > 0$ so that if $I = \{I_n\}$ is a countably infinite collection (sequence) of non-overlapping open intervals so that $\sum_{n=1}^{\infty} l(I_n) < \delta$, then $\sum_{n=1}^{\infty}(M_n-m_n) < \epsilon$, where M_n and m_n denote the max and min respectively of f on I_n. Let $J = \{J_n\} = \{(a_n,b_n)\}$ be a sequence of pairwise disjoint open intervals covering E so that the length sum of J is less than δ. Clearly, $f(E)\subset \bigcup_{n=1}^{\infty} f([a_n,b_n])$ and $\bigcup_{n=1}^{\infty} f[a_n,b_n]\subset \bigcup_{n=1}^{\infty}[m_n,M_n]$. But $\sum_{n=1}^{\infty}(M_n-m_n) < \epsilon$. Hence $m^*f(E) < \epsilon$, and the theorem is proved.

2.26. __If the function__ f __is defined and absolutely continuous on__ $[a,b]((a,b))$, __then__ f __is of bounded variation on__ $[a,b]((a,b))$.

<u>Proof</u>. Suppose by way of contradiction that f is not of bounded variation on $[a,b]$. Choose $\epsilon = 1$. There exists a $\delta > 0$ so that if $\{(x_n,x_n')\}_{n=1}^{k}$ is any finite collection of pairwise disjoint open intervals so that $\sum_{n=1}^{k}(x_n'-x_n) < \delta$, then $\sum_{n=1}^{k}|f(x_n')-f(x_n)| < \epsilon$. Then form a subdivision σ of $[a,b]$ so that $\|\sigma\| < \delta$. It follows that there exists an interval $[x_{i-1},x_i] \in \sigma$ so that f is not of bounded variation on $[x_{i-1},x_i]$. Hence there exists a subdivision

$$\sigma': x_{i-1} = y_0 < y_1 < y_2 < \cdots < y_n = x_i$$

so that $V_{\sigma'}(f;[x_{i-1},x_i]) > 1$. But $\sum_{i=1}^{n}(y_i-y_{i-1}) < \delta$. Hence, from above, $\sum_{i=1}^{n}|f(y_i)-f(y_{i-1})| < 1$, a contradiction. Thus, f is of bounded variation on $[a,b]$.

The proof goes through similarly if (a,b) is taken to be open.

2.27. <u>If the function f is defined and absolutely continuous on its domain, the function g is absolute continuous and monotone non-decreasing on $[a,b]$, and the range of g is a subset of the domain of f, then the composition function $F(x) = f(g(x))$ is absolutely continuous on $[a,b]$.</u>

<u>Proof</u>. Choose $\epsilon > 0$. There exists a $\delta_1 > 0$ so that if $\{(y_i,y_i')\}_{i=1}^{n}$ is a finite collection of pairwise disjoint open intervals contained in the domain of f so that $\sum_{i=1}^{n}(y_i'-y_i) < \delta_1$, then $\sum_{i=1}^{n}|f(y_i')-f(y_i)| < \epsilon$. Now g is absolutely continuous on $[a,b]$. Therefore there exists a $\delta_2 > 0$ so that if

$$\{(x_i,x_i')\}_{i=1}^{k}$$

is a finite collection of non-overlapping open intervals of length sum less than δ_2, then $\sum_{i=1}^{k} |g(x_i')-g(x_i)| < \delta_1$. But g is monotone non-decreasing on [a,b]. Therefore

$$\{(g(x_i),g(x_i'))\}_{i=1}^{k}$$

is a finite collection of non-overlapping open intervals in the domain of f with length sum less than δ_1. Hence

$$\sum_{i=1}^{k} |F(x_i)-F(x_{i-1})| = \sum_{i=1}^{k} |f(g(x_i'))-f(g(x_i))| < \epsilon,$$

and F is absolutely continuous on [a,b].

2.28. Remark. Clearly, if the function f is absolutely continuous on [a,b], then f is continuous on [a,b]. Furthermore, 2.25 shows that an absolutely continuous function maps sets of exterior Lebesgue-Measure-0 into sets of exterior Lebesgue-Measure-0; and 2.26 shows that an absolutely continuous function is of bounded variation. Actually, these three conditions form a complete characterization of the class of absolutely continuous functions. That is, Varberg (2) proves that any continuous function f defined on [a,b] which is of bounded variation and maps sets of measure zero into sets of measure zero is absolutely continuous on [a,b]. The proof of this theorem involves considerable material about Lebesgue Measure and the Lebesgue Integral which is not discussed in this thesis. For this reason the proof will be omitted.

2.29. Example. The purpose of this example is to show that the converse of 2.26 is not true. An example is given of a function defined on [0,1] which is monotone non-decreasing, continuous, and not absolutely continuous. Admittedly, the function discussed here is not the most obvious one which disproves the converse of 2.26.

Let $I = [0,1]$. Consider the points of $[0,1]$ to be white. Blacken the open middle third of I, and call this blackened interval I_{11}. Blacken the open middle third of the two remaining white intervals and call them I_{21} and I_{22}. I_{21} is understood to be the open interval with the smallest left end point. Continue in this fashion. Let

$$G_0 = I_{11} \cup (I_{21} \cup I_{22}) \cup (I_{31} \cup I_{32} \cup I_{33} \cup I_{34}) \cup \ldots$$
$$\cup (I_{n1} \cup I_{n2} \cup \ldots \cup I_{n2^{n-1}}) \cup \ldots,$$

where the parentheses indicate the black intervals formed at the various stages. The set of white points $P_0 = [0,1] - G_0$ is called the Cantor set. The blackened intervals of G_0 are called the complementary intervals of the Cantor set.

The complementary intervals are now used to define a function f. Define $f(x) = \frac{1}{2}$ if $x \in I_{11}$. Define $f(x) = \frac{1}{4}$ if $x \in I_{21}$, and define $f(x) = \frac{3}{4}$ if $x \in I_{22}$. In general, consider the set of black intervals formed at the n^{th} stage:

$$I_{n1},\ I_{n2},\ \ldots,\ I_{n2^{n-1}}.$$

Define $f(x) = \frac{1}{2^n}$ if $x \in I_{n1}$, define $f(x) = \frac{3}{2^n}$ if $x \in I_{n2}$, ..., and define

$$f(x) = \frac{2^n-1}{2^n}$$

if $x \in I_{n2^{n-1}}$. Let $f(0) = 0$ and $f(1) = 1$. Then let $x_0 \in (0,1)$ so that $x_0 \in P_0$. Define $f(x_0)$ to be the max of

$$\{f(x) \mid x \in G_0 \text{ and } x < x_0\}.$$

It is asserted that the function f is monotone non-decreasing on $[0,1]$. Choose x_1 and x_2 in $[0,1]$ so that $x_1 < x_2$. If x_1 and x_2 belong to the same complementary interval of G_0, then $f(x_1) = f(x_2)$ and the monotonicity follows. If x_1 and x_2 belong to complementary intervals formed at the same stage, then $f(x_1) < f(x_2)$ by definition. Then suppose that x_1 and x_2 belong to G_0 but that the interval containing x_1 was produced at the j^{th} stage and the interval containing x_2 was produced at the k^{th} stage. Let $m = \max(j,k)$. Then at the m^{th} stage there will be 2^m-1 total black intervals. Let

$$A = \{I_1, I_2, \ldots, I_{2^m-1}\}$$

denote this subset of G_0. The values that f assumes on these intervals will be

$$\frac{1}{2^m}, \frac{2}{2^m}, \frac{3}{2^m}, \ldots \frac{2^m-1}{2^m},$$

respectively. It is then apparent that $f(x_1) < f(x_2)$. Then suppose that $x_1 \notin G_0$ and $x_2 \in G_0$. But if $x \in G_0$ and $x < x_2$, $f(x) \leq f(x_2)$ by the preceding argument. It follows immediately that $f(x_1) \leq f(x_2)$ since $f(x_1)$ is the max of all $x \in G_0$ so that $x < x_1$. If $x_1 \in G_0$ and $x_2 \notin G_0$, then by definition $f(x_1) \leq f(x_2)$. Then suppose that $x_1 \notin G_0$ and $x_2 \notin G_0$. If $E_{x_1} = \{x \in G_0 | x < x_1\}$ and $E_{x_2} = \{x \in G_0 | x < x_2\}$, it is immediate that $E_{x_1} \subset E_{x_2}$. Hence, f is monotone non-decreasing on $[0,1]$.

If $x \in G_0$, it is apparent that f is continuous at x. Suppose that $x \notin G_0$. By definition, $f(x) = f(x-)$. Furthermore, by the construction of G_0, there exists a black interval in G_0

with left end point greater than x but as close to x as is desired. Hence, $f(x) = f(x+)$, and continuity follows by 1.42 and 1.45.

In addition, it is easy to see that P_0 is of Jordan-content-0. Then choose $\epsilon = \frac{1}{2}$. Let δ be any positive number. Hence there exists a finite collection $I = \{I_i\} = \{(a_i, b_i)\}$ of non-overlapping open intervals (relative to $[0,1]$) so that $P_0 \subset \bigcup_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} l(I_i) < \epsilon$. Furthermore, let the elements of I be ordered in terms of their left end points, i.e. $a_i < a_{i+1}$. Clearly, $a_1 = 0$. Now $b_1 \in G_0$ since the intervals of I are non-overlapping. Thus, $a_2 \in G_0$. Furthermore, b_1 and a_2 must belong to the same complementary interval, for otherwise there exists a point of P_0 between b_1 and a_2, and I would not cover P_0. In general, $f(b_i) = f(a_{i+1})$ for

$$i = 1, 2, \ldots, n-1.$$

Thus

$$\begin{aligned} \sum_{i=1}^{n} |f(b_i)-f(a_i)| &= \sum_{i=1}^{n} (f(b_i)-f(a_i)) \\ &= f(b_n)-f(a_1) \\ &= f(b_n). \end{aligned}$$

But b_n must be 1. Thus, $\sum_{i=1}^{n} |f(b_i)-f(a_i)| = f(1) = 1 > \epsilon$. Therefore, f is not absolutely continuous on $[a,b]$.

2.30. <u>If the function f is defined and monotone non-decreasing (monotone non-increasing) on $[a,b]$, then f is of bounded variation on $[a,b]$.</u>

<u>Proof</u>. For definiteness, suppose that f is monotone non-decreasing on $[a,b]$. Let $\sigma : a = x_0 < x_1 < \ldots < x_n = b$ be a

subdivision of $[a,b]$. Then

$$\sum_{\sigma}|f(x_i)-f(x_{i-1})| = \sum_{\sigma}(f(x_i)-f(x_{i-1}))$$
$$= f(b)-f(a).$$

The theorem follows immediately.

2.31. <u>If the function</u> f <u>is defined and of bounded variation on</u> $[a,b]$, <u>then</u> $|f|$ <u>is of bounded variation on</u> $[a,b]$.

<u>Proof</u>. Let $\sigma: a = x_0 < x_1 < x_2 < \dots < x_n$ be a subdivision of $[a,b]$. Then

$$\sum_{\sigma}|f(x_i)-f(x_{i-1})| \geq \sum_{\sigma}\big||f(x_i)|-|f(x_{i-1})|\big|$$

by 1.35. Hence

$$\sum_{\sigma}\big||f(x_i)|-|f(x_{i-1})|\big| \leq V(f;[a,b]),$$

and the theorem follows.

2.32. <u>If the function</u> f <u>is defined and of bounded variation on</u> $[a,b]$, <u>then</u> f <u>is bounded on</u> $[a,b]$.

<u>Proof</u>. A proof of the contrapositive is given. Choose $K > 0$. Then there exists an $x' \in (a,b)$ so that

$$|f(x')| > K+|f(a)| .$$

Thus, $|f(x')|-|f(a)| > K$. But $|f(x')-f(a)| \geq |f(x')|-|f(a)|$. Then let $\sigma: a = x_0 < x_1 < x_2 = b$ be a subdivision of $[a,b]$ so that $x_1 = x'$. Thus

$$\begin{aligned} V_{\sigma}(f) &= |f(x_1)-f(x_0)|+|f(x_2)-f(x_1)| \\ &= |f(x')-f(a)|+|f(b)-f(x')| \\ &\geq |f(x')-f(a)| \\ &> K. \end{aligned}$$

Hence, f is not of bounded variation on $[a,b]$, and the proof is finished.

Corollary. If the function f is defined and absolutely continuous on (a,b), then f is bounded on (a,b).

Proof. The corollary is an immediate consequence of 2.27 and 2.32.

2.33. *Example.* Theorem 2.32 showed that bounded variation implies boundedness. The converse is not true, for consider the function $f(x) = x \sin \frac{\pi}{x}$ if $x \quad (0,2]$ and $f(0) = 0$. The function f is bounded and continuous on $[0,2]$. But f is not of bounded variation on $[0,2]$ (3, p.100).

2.34. *If the function f is defined on $[a,b]$, σ and σ_1 are two subdivisions of $[a,b]$, and $\sigma \subset \sigma_1$, then $V_\sigma(f) \leq V_{\sigma_1}(f)$.*

Proof. The proof follows immediately from the fact that $\sigma \subset \sigma_1$ and 1.35.

2.35. *If each of the functions f and g is defined and of bounded variation on $[a,b]$, then*

i) f+g is of bounded variation on $[a,b]$;

ii) f-g is of bounded variation on $[a,b]$;

iii) $f \cdot g$ is of bounded variation on $[a,b]$;

iv) if $\frac{1}{g}$ is defined and bounded on $[a,b]$, then $\frac{f}{g}$ is of bounded variation on $[a,b]$.

Proof. Proofs of (i) and (iii) will be given. The proofs of the remaining two parts are similar.

i) Let $\sigma : a = x_0 < x_1 < \dots < x_n = b$ be a subdivision of $[a,b]$. Then

$$\sum_\sigma \left| (f(x_i)+g(x_i)) - (f(x_{i-1})+g(x_{i-1})) \right|$$
$$= \sum_\sigma \left| (f(x_i)-f(x_{i-1})) + (g(x_i)-g(x_{i-1})) \right|$$

$$\leq \sum_{\sigma} |f(x_i)-f(x_{i-1})| + \sum_{\sigma} |g(x_i)-g(x_{i-1})|$$
$$\leq V(f;[a,b])+V(g;[a,b]).$$

Part (i) follows.

iii) By 2.32, f and g are bounded on $[a,b]$. Let M_1 be a bound for f, and let M_2 be a bound for g. Let

$$\sigma : a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

be a subdivision of $[a,b]$. Then

$$\sum_{\sigma} |f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})|$$
$$= \sum_{\sigma} |f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})+f(x_i)g(x_{i-1})-f(x_i)g(x_{i-1})|$$
$$= \sum_{\sigma} |f(x_i)(g(x_i)-g(x_{i-1}))+g(x_{i-1})(f(x_i)-f(x_{i-1}))|$$
$$\leq \sum_{\sigma} |f(x_i)|\,|g(x_i)-g(x_{i-1})| + |g(x_{i-1})|\,|f(x_i)-f(x_{i-1})|$$
$$\leq M_1 \sum_{\sigma} |g(x_i)-g(x_{i-1})| + M_2 \sum_{\sigma} |g(x_i)-g(x_{i-1})|$$
$$\leq M_1(V(g;[a,b]))+M_2(V(f;[a,b])).$$

Thus the theorem follows.

2.36. <u>Corollary</u>. *If f and g are defined and of bounded variation on $[a,b]$, and c and d are real numbers, then $c\cdot f(x)+d\cdot g(x)$ is of bounded variation on $[a,b]$.*

2.37. <u>Example</u>. In 2.35 the function g was restricted so that its reciprocal was defined and bounded. This example indicates that the boundedness restriction cannot be removed.

Let $f(x) = 1$ for all $x \in [0,1]$. Clearly, f is of bounded variation on $[0,1]$. Then consider the following function h: $h(x) = x$ if $x \in (0,1]$, and $h(0) = 1$. Let

$$\sigma : a = x_0 < x_1 < \ldots < x_n = b$$

be a subdivision of $[a,b]$. Then

$$V_\sigma(h) = \sum_{i=1}^{n} |h(x_i)-h(x_{i-1})|$$
$$= |1-h(x_1)|+|h(x_2)-h(x_1)|$$
$$+|h(x_3)-h(x_2)|+\ldots+|h(1)-h(x_{n-1})|$$
$$= 2\left[(1-h(x_1))\right] < 2.$$

In fact, it follows easily that $V(h;[0,1]) = 2$.

Then let g be a function defined on $[a,b]$ so that

$$g(x) = \frac{1}{x}$$

if $x \in (0,1]$, and $g(0) = 1$. Clearly, g is not bounded on $[0,1]$. Then g is not of bounded variation on $[0,1]$ by 2.32. But

$$g(x) = \frac{1}{h(x)} = \frac{f(x)}{h(x)},$$

where both f and g have previously been proved to be of bounded variation on $[0,1]$.

2.38. _If the function_ f _is defined on_ $[a,b]$, _then a necessary and sufficient condition that_ f _be of bounded variation on_ $[a,b]$ _is that if_ $c \in [a,b]$, _then_ f _be of bounded variation on_ $[a,c]$ _and_ $[c,b]$.

Proof. Suppose that f is of bounded variation on $[a,b]$. Clearly, if $c = a$ or $c = b$, then f is of bounded variation on $[a,c]$ and $[c,b]$. Obviously, $V(f;[c,c]) = 0$, _i.e._ the variation of f _on_ c is 0.

Then suppose that $c \in (a,b)$. Let σ_1 be a subdivision of $[a,c]$, and let σ_2 be a subdivision of $[c,b]$. Then

$$\sigma = \sigma_1 \cup \sigma_2$$

forms a subdivision of $[a,b]$. But

$$V_{\sigma}(f) = V_{\sigma_1}(f;[a,c])+V_{\sigma_2}(f;[c,b]) \leq V(f;[a,b]).$$

Thus, f is of bounded variation on $[a,c]$ and $[c,b]$.

Now suppose that f is of bounded variation on $[a,c]$ and $[c,b]$. Let σ be a subdivision of $[a,b]$. Let

$$\sigma_2 : a = x_0 < c < x_2 = b$$

be a subdivision of $[a,b]$. Then, setting $\sigma_1 = \sigma \cup \sigma_2$, by 2.34,

$$\begin{aligned} V_{\sigma}(f) &\leq V_{\sigma_1}(f) \\ &= \sum_{i=1}^{x_m=c} |f(x_i)-f(x_{i-1})| + \sum_{x_m=c}^{n} |f(x_i)-f(x_{i-1})| \\ &\leq V(f;[a,c])+V(f;[c,b]). \end{aligned}$$

Thus, f is of bounded variation on $[a,b]$.

2.39. *If f is of bounded variation on $[a,b]$ and $a < c < b$, then* $V(f;[a,c])+V(f;[c,b]) = V(f;[a,b])$.

Proof. By 2.38, f is of bounded variation on $[a,c]$ and $[c,b]$. Let σ_1 be a subdivision of $[a,c]$, and let σ_2 be a subdivision of $[c,b]$. Then let $\sigma = \sigma_1 \cup \sigma_2$. Therefore, $V_{\sigma}(f;[a,b]) = V_{\sigma_1}(f;[a,c])+V_{\sigma_2}(f;[c,b])$. Hence

$$V_{\sigma}(f;[a,b]) \geq V_{\sigma_1}(f;[a,c])+V_{\sigma_2}(f;[a,c]).$$

Thus

$$V(f;[a,b]) \geq V(f;[a,c])+V(f;[c,b]).$$

In a similar fashion it can be shown that

$$V(f;[a,b]) \leq V(f;[a,c])+V(f;[c,b]).$$

Hence, $V(f;[a,b]) = V(f;[a,c])+V(f;[c,b])$, and the proof is finished.

__Note__. In 2.40, 2.42, 2.43, 2.44, 2.45, and 2.46, complete characterizations are given of the class of functions which are of bounded variation. These theorems are especially important, and reference will be made to them periodically throughout the remainder of this paper.

2.40. __Theorem__. __A function f defined on [a,b] is of bounded variation on [a,b] if, and only if, f can be written as the difference of two monotone non-decreasing functions__.

__Proof__. Suppose that f is of bounded variation on [a,b]. Let v(x) be the variation function of f which is defined in 2.16. As a result of 2.38 and 2.39, it is clear that v is well-defined and monotone non-decreasing on [a,b]. Then consider the function h = v-f defined for all $x \in [a,b]$. Choose x_1 and x_2 in [a,b] so that $x_1 < x_2$. Then

$$h(x_2)-h(x_1) = v(x_2)-f(x_2)-[v(x_1)-f(x_1)]$$
$$= v(x_2)-v(x_1)-(f(x_2)-f(x_1)).$$

But $v(x_2) = V(f;[a,x_1])+V(f;[x_1,x_2])$ and $v(x_1) = V(f;[a,x_1])$. Therefore

$$h(x_2)-h(x_1)$$
$$= V(f;[a,x_1])+V(f;[x_1,x_2])-V(f;[a,x_1])-(f(x_2)-f(x_1))$$
$$= V(f;[x_1,x_2])-(f(x_2)-f(x_1)).$$

Obviously, $V(f;[x_1,x_2]) \geq f(x_2)-f(x_1)$. Hence, h is monotone non-decreasing on [a,b]. Noticing that f(x) = v(x)-h(x), the first part of the proof is completed.

If f is expressible as the difference of two monotone non-decreasing functions, the conclusion follows immediately from 2.30 and 2.35.

2.41. Theorem. *If the function f is defined and of bounded variation on $[a,b]$, then the set D of points of discontinuity of f on $[a,b]$ is countable.*

Proof. By 2.40 there exist monotone functions g and h so that $f = g-h$. Clearly, if f is discontinuous at

$$x = \xi \in [a,b],$$

then either g or h must be discontinuous at $x = \xi$. The conclusion of the theorem then follows from 1.44 and the fact that the union of two countable sets is countable.

2.42. Theorem. *A function f defined on $[a,b]$ is of bounded variation on $[a,b]$ if, and only if, there exists a monotone non-decreasing function g defined on $[a,b]$ so that if x_1 and x_2 are in $[a,b]$ and $x_1 < x_2$, then*

$$f(x_2)-f(x_1) \leq g(x_2)-g(x_1).$$

Proof. Suppose that $V(f;[a,b])$ is finite. Therefore, by the theorem of 2.40, there exist monotone non-decreasing functions h_1 and h_2 defined on $[a,b]$ so that

$$f(x) = h_1(x)-h_2(x)$$

for all $x \in [a,b]$. Then choose x_1 and x_2 in $[a,b]$ so that $x_1 < x_2$. Then

$$\begin{aligned} f(x_2)-f(x_1) &= h_1(x_2)-h_2(x_2)-[h_1(x_1)-h_2(x_1)] \\ &= h_1(x_2)-h_1(x_1)-[h_2(x_2)-h_2(x_1)]. \end{aligned}$$

But h_2 is monotone non-decreasing on $[a,b]$. Thus

$$f(x_2)-f(x_1) \leq h_1(x_2)-h_1(x_1).$$

Setting $h_1 = g$, the first part of 2.42 is completed.

Now suppose there exists such a function g as is described in the hypothesis. Choose x_1 and x_2 in $[a,b]$ so that $x_1 < x_2$. Hence, $g(x_2)-g(x_1)-[f(x_2)-f(x_1)] \geq 0$. But

$$g(x_2)-g(x_1)-[f(x_2)-f(x_1)] = g(x_2)-f(x_2)-[g(x_1)-f(x_1)].$$

Hence the function $h = g-f$ is monotone non-decreasing on $[a,b]$. But $f = g-h$, and f is of bounded variation by 2.40.

2.43. __Theorem__. __Suppose that__ $a \leq t \leq b$ __is a parameter interval and that__ $f_1(t) = x$ __and__ $f_2(t) = y$ __are functions defined for each__ $t \in [a,b]$. __Then__ arc $[a,b]$ __is rectifiable if, and only if,__ f_1 __and__ f_2 __are of bounded variation on__ $[a,b]$.

__Proof__. Suppose that f_1 and f_2 are of bounded variation on $[a,b]$. Therefore there exist real numbers $V(f_1;[a,b])$ and $V(f_2;[a,b])$ so that if σ is any subdivision of $[a,b]$, then $V_\sigma(f_1) \leq V(f_1;[a,b])$ and $V_\sigma(f_2) \leq V(f_2;[a,b])$.

Let t_i and t_{i-1} be real numbers in $[a,b]$ so that $t_{i-1} < t_i$. Then, by the Minkowski inequality,

$$\left[(f_1(t_i)-f_1(t_{i-1}))^2+(f_2(t_i)-f_2(t_{i-1}))^2\right]^{\frac{1}{2}}$$
$$\leq \left[(f_1(t_i)-f_1(t_{i-1}))^2\right]^{\frac{1}{2}} + \left[(f_2(t_i)-f_2(t_{i-1}))^2\right]^{\frac{1}{2}}$$
$$= |f_1(t_i)-f_1(t_{i-1})| + |f_2(t_i)-f_2(t_{i-1})|.$$

Hence

$$L_\sigma = \sum_{i=1}^{n} \left[(f_1(t_i)-f_1(t_{i-1}))^2+(f_2(t_i)-f_2(t_{i-1}))^2\right]^{\frac{1}{2}}$$
$$\leq \sum_{i=1}^{n} |f_1(t_i)-f_1(t_{i-1})| + \sum_{i=1}^{n} |f_2(t_i)-f_2(t_{i-1})|$$
$$\leq V(f_1;[a,b])+V(f_2;[a,b]).$$

Thus, arc $[a,b]$ is rectifiable.

Now suppose that arc $[a,b]$ is rectifiable and that either f_1 or f_2 is of unbounded variation on $[a,b]$. Without loss of generality, suppose that f_1 is not of bounded variation on $[a,b]$. Choose $K > 0$. Hence there exists a subdivision σ of $[a,b]$ so that

$$\sum_{\sigma} |f(t_i)-f(t_{i-1})| = \sum_{\sigma} \left[(f(t_i)-f(t_{i-1}))^2\right]^{\frac{1}{2}} > K.$$

But

$$\sum_{\sigma} |f_1(t_i)-f_1(t_{i-1})|^2 \leq \sum_{\sigma} |f_1(t_i)-f_1(t_{i-1})|^2 + \sum_{\sigma} |f_2(t_i)-f_2(t_{i-1})|^2.$$

Thus

$$\sum_{\sigma} \left[(f_1(t_i)-f_1(t_{i-1}))^2+(f_2(t_i)-f_2(t_{i-1}))^2\right]^{\frac{1}{2}} \geq \sum_{\sigma} \left[(f(t_i)-f(t_{i-1}))^2\right]^{\frac{1}{2}} > K.$$

Therefore a contradiction is achieved since K was arbitrarily chosen, and arc$[a,b]$ is rectifiable. The theorem follows.

2.44. <u>Corollary</u>. <u>In the special case where</u> $t \equiv x$, <u>the function</u> $y = f(x)$ <u>is of bounded variation if, and only if, the function</u> f <u>is rectifiable</u>.

2.45. <u>Theorem</u>. <u>The function</u> f <u>defined on</u> $[a,b]$ <u>is of bounded variation on</u> $[a,b]$ <u>if, and only if,</u> f <u>is of bounded</u> S-<u>variation on</u> $[a,b]$. <u>Furthermore, if</u> f <u>is of bounded</u> S-<u>variation on</u> $[a,b]$, <u>then</u> $S_V(f;[a,b]) = V(f;[a,b])$.

<u>Proof</u>. Clearly, if f is of bounded S-variation on $[a,b]$, then f is of bounded variation on $[a,b]$. Therefore, suppose that f is of bounded variation on $[a,b]$. Let

$$\sigma : a = x_0 < x_1 < \ldots < x_n = b.$$

Consider $\sum_{\sigma} S(I_i)$. Choose $\epsilon > 0$. There exist numbers y_1 and y_1' in $[x_0,x_1]$ so that $|f(y_1')-f(y_1)| > S(f;[x_0,x_1])- \frac{\epsilon}{n}$.

Repeating this process with respect to each $I_i \in \sigma$, a subdivision σ^* of $[a,b]$ is formed so that $V_{\sigma^*}(f) > \sum_\sigma S(I_i) - \epsilon$. Since ϵ was arbitrary, it follows that

$$S_V(f;[a,b]) \leq V(f;[a,b]).$$

Hence, f is of bounded S-variation on $[a,b]$. The preceding remarks make the equality of $V(f;[a,b])$ and $S_V(f;[a,b])$ obvious. Therefore the theorem has been proved.

2.46. Corollary. *A function f defined on $[a,b]$ is of bounded S-variation on $[a,b]$ if, and only if, there exist monotone non-decreasing functions g and h defined on $[a,b]$ so that $f = g-h$.*

2.47. Several characterizations have been given of functions of bounded variation. The next theorem gives some indication of the behavior of a function which is not of bounded variation on the closed interval $[a,b]$.

Theorem. *If the function f is defined and of unbounded variation on $[a,b]$, then there exists a point $\delta \in [a,b]$ so that f is of unbounded variation at $x = \delta$. Furthermore, if $E = \{x \in [a,b] \mid f(x)$ is of unbounded variation at $x\}$, then E is closed.*

Proof. Divide $I = [a,b]$ in half. f must be of unbounded variation on at least one of the halves by 2.38. If f is of unbounded variation on both closed halves, choose the right closed half and call it $I_1 = [a_1,b_1]$. Otherwise, choose the left closed half and call it I_1. Then divide I_1 in half and choose that closed half I_2 of I_1 having the largest right end point so that f is of unbounded variation on I_2. Continue this process. By induction, a descending, infinitesimal sequence $\{I_n\}$ of closed intervals is formed so that f is of unbounded variation on I_n for each positive integer n. Then by 1.31 there exists a unique point ξ which belongs to I_n for every n.

Then let (c,d) be an open interval (relative to $I = [a,b]$) containing ξ. Since $\{I_n\}$ is infinitesimal, there exists a positive integer N so that $I_N \subset (c,d)$. Hence, f is of unbounded variation on (c,d). Therefore, by definition, f is of unbounded variation at ξ.

Now let y be a limit point of $E = \{x \in [a,b] \mid f$ is of unbounded variation at $x\}$. Let J be an open interval containing y. Thus, $J \cap E \neq \emptyset$, and f is of unbounded variation on J. Therefore, $y \in E$, and the theorem is proved.

2.48. Theorem. If the function f is defined and of bounded variation on $[a,b]$, and $x \in [a,b]$, then f is continuous at x if, and only if, the function $v(x) = V(f;[a,x])$ is continuous at x.

Proof. Rather than appeal directly to the definition of continuity, an attempt has been made to indicate an application of some of the material of Chapter I. Furthermore, in order that both one-sided limits may be considered, suppose that $\xi \in (a,b)$. The body of the proof makes it clear how the end points could be handled.

Suppose that f is continuous. Choose $\epsilon > 0$. By definition, there exists a subdivision σ of $[a, \xi]$ so that

$$V_\sigma(f) > V(f;[a,\xi]) - \epsilon .$$

Also, there exists a $\delta > 0$ so that if $x \in [a,b]$ and $|x-\xi| < \delta$, then $|f(x)-f(\xi)| < \epsilon$. Consider the refinement σ^1 of σ which is formed by adding a point x' to σ so that

$$x_{n-1} < x' < \xi$$

and $|x'-\xi| < \delta$. Then $V_{\sigma^1}(f) \geq V_\sigma(f) > V(f;[a,\xi]) - \epsilon$. Thus, $V_{\sigma^1}(f) > V(f;[a,\xi]) - \epsilon$. Now

$$V_{\sigma^1}(f) = \sum_{i=1}^{n-1} |f(x_i)-f(x_{i-1})| + |f(x')-f(x_{n-1})| + |f(\xi)-f(x')| .$$

Therefore

$$\sum_{i=1}^{n-1} |f(x_i)-f(x_{i-1})| + |f(x')-f(x_{n-1})| > V(f;[a,\xi]) - 2\epsilon .$$

Thus, if x'' is any number so that $x' \leq x'' < \xi$,

$$V(f;[a,x'']) > V(f;[a,\xi]) - 2\epsilon .$$

Therefore, by 1.43, $v(\xi-) \geq v(\xi)$.

The proof of the fact that $v(\xi+) \leq v(\xi)$ differs somewhat from the work immediately above. There exists a subdivision σ of $[\xi,b]$ so that $V_\sigma(f) > V(f;[\xi,b]) - \epsilon$, where ϵ is an arbitrary positive number. There exists a positive number δ so that if $\xi < x < \xi + \delta$, then $|f(x)-f(\xi)| < \epsilon$. Let σ' be a refinement of σ formed by adding a point x' so that $\xi < x' < x_1$ and $x' - \xi < \delta$. Then

$$\begin{aligned} V_{\sigma'}(f) &= |f(x')-f(\xi)| + |f(x_1)-f(x')| \\ &\quad + \sum_{i=2,\sigma}^{n} |f(x_i)-f(x_{i-1})| + \epsilon > V(f;[\xi,b]) \\ &= V(f;[\xi,x']) + V(f;[x',b]). \end{aligned}$$

But $|f(x')-f(\xi)| < \epsilon$ and

$$|f(x_1)-f(x')| + \sum_{i=2,\sigma}^{n} |f(x_i)-f(x_{i-1})| \leq V(f;[x',b]).$$

Hence, $2\epsilon > V(f;[\xi, x'])$. Thus

$$V(f;[a,\xi])+2\epsilon > V(f;[a,x']) \geq v(\xi+).$$

Therefore, $v(\xi) \geq v(\xi+)$. The continuity of v follows from 1.36.

Now suppose that v is continuous at $x = \xi \in [a,b]$. Choose $\epsilon > 0$. There exists a $\delta > 0$ so that if $x \in [a,b]$ and $|x-\xi| < \delta$, then $|v(x)-v(\xi)| < \epsilon$. For definiteness, suppose that $x > \xi$. Then

$$|v(x)-v(\xi)| = |V(f;[\xi,x])| = V(f;[\xi,x]).$$

But obviously $|f(\xi)-f(x)| \leq V(f;[\xi,x])$, i.e. $f(x)-f(\xi) < \epsilon$.

2.49. Example. Theorem 2.48 demonstrated that a function f of bounded variation has exactly the same points of discontinuity as its variation function v. This example shows that a function of bounded variation and its variation function need not be differentiable at the same points. Consider the following function f defined on $[0,2]$: $f(x) = x$ if $0 \leq x \leq 1$, and $f(x) = 2-x$ if $1 < x \leq 2$. Clearly, f is of bounded variation on $[0,2]$ $(V(f;[0,2]) = 2)$ but not differentiable at $x = 1$. However, $v(x) = x$ for every $x \in [0,2]$. Thus, v is differentiable at all points of $[0,2]$.

2.50. If f is a function defined and of bounded variation on $[a,b]$, then, by 2.40, f can be written as the difference of two monotone non-decreasing functions, say g and h, i.e. $f = g-h$. Let $G(x) = G[a,x]$ be the interval function of 1.47 that is obtained from g, and let $H(x) = H[a,x]$ be the interval function that is obtained from h. If $x \in [a,b]$, define

$f^*[a,x] = f^*(x)$ to be $G[a,x]-H[a,x]$. The interval function f^* then leads to another decomposition of functions of bounded variation.

Theorem. If the function f is defined and of bounded variation on $[a,b]$, then f can be written as the sum of a continuous function of bounded variation and a pre-determined interval function f^*.

Proof. By 2.40, $f = g-h$, where both g and h are monotone non-decreasing on $[a,b]$. Let G and H be the interval functions of 1.47 discussed above. Then, by 1.48, both g-G and h-H are monotone non-decreasing and continuous on $[a,b]$. Hence both functions are of bounded variation on $[a,b]$. Furthermore, the function $f-f^* = (g-G)-(h-H)$ is continuous and of bounded variation on $[a,b]$. Thus, $f = (f-f^*)+f^*$, and the proof is through.

2.51. Corollary. If the function f is of bounded variation on $[a,b]$, then the set of points of discontinuity of f on $[a,b]$ is the set of points of discontinuity of f^* on $[a,b]$.

2.52. The following two theorems indicate some conditions under which the total variation of a function may be defined by a limiting process.

Theorem. If the function f is continuous and of bounded variation on $[a,b]$, then $\lim_{\|\sigma\| \to 0} V(f;[a,b])$ exists and is the total variation of f on $[a,b]$.

<u>Proof</u>. Choose $\epsilon > 0$. By definition, there exists a subdivision σ^*: $a = x_0 < x_1 < \dots < x_n = b$ of a,b so that $V_{\sigma^*}(f) > V(f;[a,b]) - \frac{\epsilon}{2}$. σ^* has a finite number of subdivision points, say N. Also, by 2.48, the variation function v is continuous on $[a,b]$. Therefore there exists a $\delta_1 > 0$ so that if y_1 and y_2 belong to $[a,b]$ and

$$|y_1 - y_2| < \delta_1,$$

then

$$v(y_1) - v(y_2) < \frac{\epsilon}{4N}.$$

Choose a positive number $\delta < \min(\delta_1, \frac{\|\sigma\|^*}{2})$. Then let σ: $a = x_0' < x_1' < \dots < x_n' = b$ be any subdivision so that

$$\|\sigma\| < \delta.$$

Then let i be the largest positive integer so that $x_i' \le x_1$. If $x_i' = x_1$, then $V(f;[a,x_i']) \ge |f(x_1) - f(x_0)|$. But suppose that $x_i' < x_1$. Then

$$V(f;[a,x_i']) + \frac{\epsilon}{4N} > V_\sigma(f;[a,x_i']) + |f(x_1) - f(x_i')|$$
$$\ge |f(x_1) - f(x_0)|.$$

Then let j be the largest positive integer so that $x_j' \le x_2$. It follows that

$$2\left(\frac{\epsilon}{4N}\right) + V_\sigma(f;[x_i', x_j']) > |f(x_1) - f(x_{i+1}')|$$
$$+ V_\sigma(f;[x_{i+1}', x_j']) + |f(x_2) - f(x_j')|$$
$$\ge f(x_2) - f(x_1) \ .$$

But since there exist N-1 subdivision points of σ in (a,b), it follows that

$$\frac{V_\sigma}{\sigma^*} + 2N\frac{\epsilon}{4N} > V_{\sigma^*}(f).$$

Obviously, $V_\sigma(f) \geq V_{\sigma^*}(f)$. Hence, $V_\sigma(f)+\frac{\epsilon}{2} > V_{\sigma^*}(f) > V(f)-\frac{\epsilon}{2}$. Therefore, $|V(f)-V_\sigma(f)| < \epsilon$, and the proof is finished.

2.53. __Theorem.__ __If the function__ f __is defined, continuous, and of bounded variation on__ $[a,b]$, __then__ $\lim_{\|\sigma\|\to 0}\sum_\sigma S(f;[x_{i-1},x_i])$ __exists and is the total variation of__ f __on__ $[a,b]$.

__Proof.__ Choose $\epsilon > 0$. By 2.52 there exists a $\delta > 0$ so that if σ is a subdivision of $[a,b]$ so that $\|\sigma\| < \delta$, then $V_\sigma(f) > V(f;[a,b]) - \epsilon$. Let σ_1: $a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a subdivision of $[a,b]$ so that $\|\sigma_1\| < \delta$. Since f is continuous on $[a,b]$, f assumes its max and min on $[x_{i-1},x_i]$, say at ξ_{1_i} and ξ_{2_i}. Let $x_{i-1} \leq \xi_{1_i} \leq \xi_{2_i} \leq x_i$, i.e. the extreme points are labeled according to the natural ordering of the real numbers. To reiterate, no mention has been made of whether $f(\xi_{1_i})$ or $f(\xi_{2_i})$ is the max or min. This convention is maintained in the remainder of the proof.

Then let σ_2 be the refinement of σ_1 formed by adding the points ξ_{1_i} and ξ_{2_i} to $[x_{i-1},x_i]$ for $i = 1, 2, \dots, n$. Clearly, $\|\sigma_2\| < \delta$. Thus, $V_{\sigma_2}(f) > V(f;[a,b]) - \epsilon$. But

$$\begin{aligned}&|f(\xi_{1_i})-f(x_{i-1})|+|f(\xi_{2_i})-f(\xi_{1_i})|+|f(x_i)-f(\xi_{2_i})|\\ \geq\ &|f(\xi_{2_i})-f(\xi_{1_i})|\\ =\ &S(f;[x_{i-1},x_i]).\end{aligned}$$

Then $\sum_{\sigma_1} S(f;[x_{i-1},x_i]) \leq V_{\sigma_2}(f) \leq V(f;[a,b])$. But

$$\sum_{\sigma_1} S(f;[x_{i-1},x_i]) \geq V_\sigma(f) > V(f;[a,b]) - \epsilon .$$

Therefore, $V(f;[a,b]) - \epsilon < \sum_\sigma S(f;[x_{i-1},x_i]) < V(f;[a,b]) + \epsilon$, and the theorem is proved.

2.54. The functions p and n defined in 2.16 are called the positive and negative variations, respectively, of f on [a,b].

It is easily seen that if σ is a subdivision of [a,b] and x' is a point in [a,b], then

$$\sum_{A_\sigma \cup \{x'\}} (f(x_i)-f(x_{i-1})) \geq \sum_{A_\sigma} (f(x_i)-f(x_{i-1})).$$

Similarly

$$\sum_{B_\sigma \cup \{x'\}} |f(x_i)-f(x_{i-1})| \geq \sum_{B_\sigma} |f(x_i)-f(x_{i-1})|.$$

Hence, if σ' is a refinement of σ, then

$$\sum_{A_{\sigma'}} (f(x_i)-f(x_{i-1})) \geq \sum_{A_\sigma} (f(x_i)-f(x_{i-1}))$$

and

$$\sum_{B_{\sigma'}} |f(x_i)-f(x_{i-1})| \geq \sum_{B_\sigma} |f(x_i)-f(x_{i-1})|.$$

These observations simplify the proof of the following theorem.

<u>Theorem</u>. <u>If the function</u> f <u>is defined and of bounded variation on</u> [a,b], <u>then</u>

i) $f(x)-f(a) = p(x)-n(x)$; and

ii) the total variation of the function f on [a,x] is the sum of the positive and negative variations of f on [a,x], <u>i.e.</u> $v(x) = p(x)+n(x)$.

<u>Proof</u>. (i) Suppose by way of contradiction that there exists an $x \in [a,b]$ so that $f(x)-f(a) > p(x)-n(x)$. Hence there exists an $\epsilon > 0$ so that $f(x)-f(a) = p(x)-n(x)+\epsilon = p(x)-(n(x)-\epsilon)$. There exists a subdivision σ of [a,x] so that

$$\sum_{B_\sigma} |f(x_i)-f(x_{i-1})| > n(x)-\epsilon .$$

Certainly, $\sum_{A_\sigma} (f(x_i)-f(x_{i-1})) \leq p(x)$. Hence

$$f(x)-f(a) > \sum_{A_\sigma}(f(x_i)-f(x_{i-1})) - \sum_{B_\sigma}|f(x_i)-f(x_{i-1})|$$
$$= \sum_{A_\sigma}(f(x_i)-f(x_{i-1})) + \sum_{B_\sigma}(f(x_i)-f(x_{i-1}))$$
$$= f(x)-f(a),$$

and a contradiction is achieved.

The assumption that $f(x)-f(a) < p(x)-n(x)$ leads to a similar contradiction. Whence, $f(x)-f(a) = p(x)-n(x)$.

(ii) Let $\sigma: a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a subdivision of $[a,b]$. Then

$$V_\sigma(f) = \sum_{A_\sigma}(f(x_i)-f(x_{i-1})) + \sum_{B_\sigma}|f(x_i)-f(x_{i-1})|$$
$$\leq p(x)+n(x).$$

Hence, $V(f;[a,x]) = v(x) \leq p(x)+n(x)$.

Then let σ_1 and σ_2 be subdivisions of $[a,x]$. Hence

$$\sum_{A_{\sigma_1}}(f(x_i)-f(x_{i-1})) + \sum_{B_{\sigma_2}}|f(x_i)-f(x_{i-1})|$$
$$\leq \sum_{A_{\sigma_1\cup\sigma_2}}(f(x_i)-f(x_{i-1})) + \sum_{B_{\sigma_1\cup\sigma_2}}|f(x_i)-f(x_{i-1})|$$
$$= V_{\sigma_1\cup\sigma_2}(f)$$
$$\leq V(f;[a,x]).$$

Thus, $p(x)+n(x) \leq v(x)$, and the theorem is proved.

2.55. Theorem. If $A = \{f_n\}$ is a sequence of functions defined on $[a,b]$, each function of A is of bounded variation on $[a,b]$, the total variation of f_n is less than a fixed positive number K for each positive integer n, and $\{f_n\}$ converges to a function f defined on $[a,b]$, then f is of bounded variation on $[a,b]$. Furthermore, the total variation of f on $[a,b]$ is not greater than K.

Proof. Choose $\epsilon > 0$, let $\sigma: a = x_0 < x_1 < x_2 < \dots < x_k = b$ be a subdivision of $[a,b]$. There exists a positive integer N_1

so that if $n > N_1$, then

$$|f_n(x_0)-f(x_0)| < \frac{\epsilon}{2(k+1)};$$

there exists a positive integer N_2 so that if $n > N_2$, then

$$|f_n(x_1)-f(x_1)| < \frac{\epsilon}{2(k+1)}; \ \ldots;$$

there exists a positive integer N_{k+1} so that if $n > N_{k+1}$, then

$$|f_n(x_k)-f(x_k)| < \frac{\epsilon}{2(k+1)}.$$

Let $N = \max(N_1, N_2, \ldots, N_k)$, and choose $n > N$. Therefore

$$\begin{aligned} &|f(x_i)-f(x_{i-1})| - |f_n(x_i)-f_n(x_{i-1})| \\ \leq\ &|f_n(x_i)-f(x_i)+f(x_{i-1})-f_n(x_{i-1})| \\ \leq\ &|f_n(x_i)-f(x_i)|+|f(x_{i-1})-f_n(x_{i-1})| \\ <\ &\frac{\epsilon}{k+1}. \end{aligned}$$

Therefore, $|f(x_i)-f(x_{i-1})| < |f_n(x_i)-f_n(x_{i-1})| + \frac{\epsilon}{k+1}$. Summing over σ, $V_\sigma(f) < V_\sigma(f_n)+\epsilon \leq V(f_n;[a,b])+\epsilon < K+\epsilon$. Hence, $V(f;[a,b]) \leq K$, and the proof is finished.

2.56. Example. In Theorem 2.55 the variations of the functions in the sequence $A = \{f_n\}$ were bounded in a uniform fashion, i.e. there existed a positive number K so that $V(f_n;[a,b]) < K$ for every positive integer n. This theorem cannot be strengthened to the extent of omitting the uniform bound of the variations, for consider the following sequence of functions defined on $[0,1]$:

$$f_1\left(\frac{1}{2}\right) = 1, \ f_1(x) = 0 \text{ if } x \neq \frac{1}{2};$$

$$f_2(\frac{1}{4}) = f_2(\frac{1}{2}) = f_2(\frac{3}{4}) = 1,$$

$$f(x) = 0 \text{ if } x \notin \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}; \ldots;$$

$$f_n(\frac{m}{2^n}) = 1 \text{ if } m = 1, 2, \ldots, 2^n-1, \; f_n(x) = 0$$

otherwise; ... . Clearly, $\lim \{f_n\} = f$, where $f(x) = 1$ if $x \in [0,1]$ and $x = \frac{m}{n}$, where m is an integer and n is a positive integral power of 2; and $f(x) = 0$ otherwise. Obviously, f is not of bounded variation on $[0,1]$.

2.57. <u>Theorem</u>. <u>If</u> E <u>is an infinite collection of functions defined on</u> $I = [a,b]$, <u>and if there exists a positive number</u> K <u>so that</u> $f \in E$ <u>implies that</u> f <u>is bounded by</u> K, <u>then for any countably infinite collection</u> I' <u>of points of</u> I <u>and for each countably infinite collection</u> E^* <u>of</u> E, <u>there exists a sequence</u> E' <u>of functions from</u> E^* <u>which converges at every point of</u> I'.

<u>Proof</u>. Let $I' = \{x_1, x_2, \ldots, x_n, \ldots\}$, and let E^* be a countably infinite subcollection of E'. Then list the elements of E^*, <u>i.e.</u> $E^* = \{f_1, f_2, \ldots, f_n, \ldots\}$. Let

$$B_1 = \{f_1(x_1), f_2(x_1), \ldots, f_n(x_1), \ldots\}.$$

To eliminate the trivial case, suppose that B_1 is infinite. Then by the Bolzano-Weierstrass Theorem there exists a convergent sequence of points in B_1. Denote this sequence by $A_1 = \{f_{11}(x_1), f_{12}(x_1), \ldots, f_{1n}(x_1), \ldots\}$. Let

$$E_1 = \{f_{11}, f_{12}, \ldots, f_{1n}, \ldots\},$$

and let $B_2 = \{f_{11}(x_2), f_{12}(x_2), \ldots, f_{1n}(x_2), \ldots\}$. Again, there exists a convergent sequence of points from B_2, say $A_2 = \{f_{21}(x_2), f_{22}(x_2), \ldots, f_{2n}(x_2), \ldots\}$. Let

$$E_2 = \{f_{21}, f_{22}, \ldots, f_{2n}, \ldots\}.$$

Continue in this fashion.

Thus, a nested sequence $\{E_n\}$ of collections of functions has been formed. Let $f_{1_{n1}} = f_{11}$; let n_2 be the next smallest positive integer so that $f_{1_{n2}} \in E_2$; let n_3 be the next smallest positive integer so that $f_{1_{n3}} \in E_3$; ... . Consider the sequence $F = \{f_{1_{n1}}, f_{1_{n2}}, \ldots f_{1_{ni}}, \ldots\}$. Obviously, $F \subset E^*$.

Then let j be a positive integer. There exists only a finite number of elements of F not in E_j. And that subsequence of F in E_j converges at x_j since any rearrangement of E_j will converge at x_j. Putting $F = E'$, the proof is finished.

2.58. __Theorem.__ __If__ E __is an infinite collection of functions, all defined and monotone non-decreasing on__ $[a,b]$, __and if there exists a positive number__ K __so that__ $f \in E$ __implies that__ f __is bounded by__ K, __then there exists a sequence of functions of__ E __which converges to a monotone non-decreasing function__ g __at every point of__ $[a,b]$. __Furthermore, if__ $x \in [a,b]$, __then__ $|g(x)| \leq K$.

__Proof.__ Let $R = \{r_1, r_2, \ldots\}$ be the set of rational numbers in $[a,b]$ together with the number a. By 2.57 there exists a sequence $F = \{f_n\}$ of functions of E which converges at every point of R. Let x_1 and x_2 be two elements of R so that $x_1 < x_2$. Let $A_{x_1} = \lim_{n\to\infty} \{f_n(x_1)\}$, and let

$$A_{x_2} = \lim_{n\to\infty} \{f_n(x_2)\}.$$

Suppose that $A_{x_2} < A_{x_1}$. Hence there exists an $\epsilon > 0$ so that $A_{x_2} + \epsilon = A_{x_1}$. But there exists a positive integer N so that if $n > N$, then $|f_n(x_1) - A_{x_1}| < \frac{\epsilon}{2}$ and $|f_n(x_2) - A_{x_2}| < \frac{\epsilon}{2}$. Therefore, $|f_n(x_2) - f_n(x_1) + A_{x_1} - A_{x_2}| = f_n(x_2) - f_n(x_1) + A_{x_1} - A_{x_2} < \epsilon$. Therefore, $f_n(x_2) - f_n(x_1) < \epsilon - [A_{x_1} - A_{x_2}] = 0$, a contradiction of the monotonicity of f_n. Thus, putting $g'(r_i) = \lim_{n\to\infty} \{f_n(r_i)\}$, g' is seen to be monotone non-decreasing on $[a,b]$. If $x \notin R$, define $g'(x)$ to be the max of $g'(r_i)$ for all $r_i \in R$ so that $r_i < x$. It follows easily that g' is monotone non-decreasing on $[a,b]$.

Then by 2.41 the set of points of discontinuity of g' on $[a,b]$ is countable. Let D_1 denote this set. Let D_2 denote the set of points of $[a,b]$ on which g' is continuous, i.e. $D_2 = [a,b] - D_1$. Suppose that $x_1 \in D_2$ and $x_1 \notin R$. Choose $\epsilon > 0$. There exists a $\delta > 0$ so that if $x \in [a,b]$ and $|x - x_1| < \delta$, then $|g'(x) - g'(x_1)| < \frac{\epsilon}{2}$. Choose x_2 and x_3 in R so that

$$x_1 - \delta < x_2 < x_1 < x_3 < x_1 + \delta.$$

Therefore, $|g'(x_1) - g'(x_2)| < \frac{\epsilon}{2}$ and $|g'(x_1) - g'(x_3)| < \frac{\epsilon}{2}$. But $\lim_{n\to\infty} \{f_n(x_2)\} = g'(x_2)$ and $\lim_{n\to\infty} \{f_n(x_3)\} = g'(x_3)$. Hence there exists a positive integer N so that if n N, then

$$|f_n(x_2) - g'(x_2)| < \frac{\epsilon}{2}$$

and $|f_n(x_3)-g'(x_3)| < \frac{\epsilon}{2}$. Therefore, $|f_n(x_2)-g'(x_1)| < \epsilon$ and $|f_n(x_3)-g'(x_1)| < \epsilon$. But $f_n(x_2) \le f_n(x_1) \le f_n(x_3)$. Therefore, $|f_n(x_1)-g'(x_1)| < \epsilon$, and $\{f_n\}$ converges at every point of D_2.

Now, by 2.57, there exists a sequence $\{f_n'\}$ of elements from F which converges at every point of D_1. It follows easily that the sequence $\{f_n'\}$ converges at every point of $[a,b]$. In fact, if $\{f_n\}$ converges at $x \in [a,b]$, then

$$\lim_{n\to\infty} \{f_n(x)\} = \lim_{n\to\infty} \{f_n'(x)\}.$$

Define the function g to be the limit function of the sequence $\{f_n'\}$. The proof of the monotonicity of g is identical to that of g'.

Then consider g(b). Since $g(b) = \lim_{n\to\infty} \{f_n'(b)\}$, and

$$\{f_n'(b)\} < K$$

for every positive integer n, $g(b) \le K$. Thus, the proof is completed.

2.59. __Theorem__. __If__ E __is an infinite collection of functions defined on__ $[a,b]$ __so that each function and the total variation of each function are bounded by the positive number__ K, __then there exists a sequence of functions in__ E __which converges to a function__ F __at each point of__ $[a,b]$. __Furthermore__, F __is of bounded variation on__ $[a,b]$.

__Proof__. Let $f \in E$ and define $g_f(x)$ to be $V(f;[a,x])$ and $h_f(x)$ to be $V(f;[a,x])-f(x)$. The functions g_f and h_f are monotone non-decreasing by 2.37. Furthermore, for each $f \in E$, g_f and h_f are bounded by 2K.

Then let $E_1 = \{g_f \mid f \in E\}$ and let $E_2 = \{h_f \mid f \in E\}$. By 2.58 there exists an infinite sequence $E_3 = \{(g_f)_n\}$ of elements of E_1 so that E_3 converges at every point of $[a,b]$. Denote $(g_f)_n$ by g_{f_n}. Let $E_4 = \{f_n\}$ be the sequence of elements of E associated with E_3, i.e. the function f_i is the function from which g_{f_i} was obtained. Let $E_5 = \{h_{f_n} \mid f_n \in E_4\}$ Applying 2.58 again, there exists a sequence $E_6 = \{h_{f_n}\}$ of elements of E_5 so that the sequence E_6 converges at every point of $[a,b]$. Furthermore, if E_6 is ordered so that the associated sequence of f's forms a subsequence of E_4, the convergence of E_6 will not be disturbed. Let E_6 be ordered in this fashion. Let E_7 be the subsequence $\{f_n'\}$ of E_4 which is associated with E_6.

Then let g be the limit function of the sequence E_3, and let h be the limit function of the sequence E_6. Then choose $\epsilon > 0$ and let $x \in [a,b]$. There exists a positive integer N so that if $n > N$, then $|g_{f_n}(x)-g(x)| < \frac{\epsilon}{2}$ and $|h_{f_n}(x)-h(x)| < \frac{\epsilon}{2}$. Consider h_{f_n}. The subscript of f_n in the sequence E_4 is certainly as large as n. Let i denote the subscript of f_n in E_4. Of course, $f_i \in E_7$. Since $i > N$, it follows that

$$|g_{f_i}(x)-g(x)| + |h_{f_i}(x)-h(x)| < \epsilon .$$

Therefore, $|g_{f_i}(x)-h_{f_i}(x)-(g(x)-h(x))| < \epsilon$. But

$$g_{f_i}(x)-h_{f_i}(x) = f_i(x).$$

Hence, $|f_i(x)-(g(x)-h(x))| < \epsilon$. Thus the sequence $E_7 = \{f_n'\}$ converges to $F = g-h$ on $[a,b]$. By 2.37, F is of bounded variation on $[a,b]$, and the proof is finished.

8. The Riemann-Stieltjes Integral

2.60. ***If each of f and g is a function defined on [a,b], f is continuous and g is of bounded variation on [a,b], then*** $\int_a^b f\,dg$ ***exists.***

Proof. Without loss of generality, suppose that $g \equiv 0$ and that g is monotone non-decreasing on [a,b]. Furthermore, since f is continuous on [a,b], then f is uniformly continuous on [a,b]; and f assumes its max and min as values.

Let σ: $a = x_0 < x_1 < \ldots < x_n = b$ be a subdivision of [a,b]. S_σ will denote $\sum_\sigma M_i(g(x_i)-g(x_{i-1}))$, where M_i is the max of f on $[x_{i-1},x_i]$. Let I be the min of S_σ for all σ of [a,b]. Certainly, I is well defined since f is bounded from below.

Then choose $\epsilon > 0$, and let σ^*: $a = x_0 < x_1 < \ldots < x_n = b$ be a subdivision of [a,b] so that $|I-S_{\sigma^*}| < \epsilon$. Let δ_1 be a positive number so that if x_1 and x_2 are in [a,b] and

$$|x_1-x_2| < \delta_1,$$

then

$$f(x_1)-f(x_2) < \frac{\epsilon}{2(g(b)-g(a))}.$$

Let δ_2 be the min of $\{(x_1-x_0),(x_2-x_1), \ldots, (x_n-x_{n-1})\}$. Then let δ be a positive number so that

$$\delta < \min(\delta_1, \frac{\delta_2}{2}).$$

Let σ be a subdivision of [a,b] so that $\|\sigma\| < \delta$.

Then consider the subdivision $\sigma \cup \sigma^*$ and the sum $S_{\sigma \cup \sigma^*}$. Certainly, $S_{\sigma \cup \sigma^*} \le S_{\sigma^*}$. Therefore,

$$|S_{\sigma \cup \sigma^*} - I| < \epsilon .$$

Let $I = \{I_k\}$ denote the subintervals of σ which contain subdivision points of σ^*, and let $J = \{J_k\}$ denote the subintervals of σ which lie entirely within some subinterval of σ^*. Choose $J_i \in J$. Then the term of $\sum_{\sigma} fdg$ involving J_i is of the form $f(\xi_i)(g(x_i)-g(x_i))$. If m_i and M_i are the min and max, respectively, of f on J_i, then obviously

$$m_i \le f(\xi_i) \le M_i .$$

But $\|\sigma\| < \delta$. Hence

$$M_i - f(\xi_i) < \frac{\epsilon}{2(g(b)-g(a))}.$$

Therefore, $|S_{J}{}_{\sigma \cup \sigma^*} - \sum_{J\sigma} f(\xi_i)(g(x_i)-g(x_{i-1}))| < \frac{\epsilon}{2}$, where $S_{J}{}_{\sigma \cup \sigma^*}$ and $\sum_{J\sigma}$ denote the appropriate sums taken only over the collection J of closed intervals which occurs in both subdivisions σ and $\sigma \cup \sigma^*$. Then choose $I_n \in I$. There exists a point $x' \in \sigma^*$ so that if y_{n-1} and y_n are the end points of I_n, then $y_{n-1} < x' < y_n$. A comparison is made of the terms $f(\xi_n)g(y_n)-g(y_{n-1})$ and $M_{n-1}(g(x')-g(y_{n-1}))+M_n(g(y_n)-g(x'))$, where M_{n-1} and M_n denote the max of f on $[y_{n-1}, x']$ and $[x', y_n]$, respectively.

$$|f(\xi_n)(g(y_n)-g(y_{n-1})) - M_{n-1}(g(x')-g(y_{n-1}))$$
$$+ M_n(g(y_n)-g(x'))|$$
$$= |f(\xi_n)(g(y_n)-g(y_{n-1})+g(x')-g(x'))-M_{n-1}(g(x')-g(y_{n-1}))$$
$$- M_n(g(y_n)-g(x'))|$$

$$= |(f(\xi_n)-M_n)(g(y_n)-g(x'))+(f(\xi_n)-M_{n-1})(g(x')-g(y_{n-1}))|$$
$$\leq |(f(\xi_n)-M_n)||g(y_n)-g(x')|+|f(\xi_n)-M_{n-1}||g(x')-g(y_{n-1})|.$$

But $|y_n-y_{n-1}|<\delta_1$. Therefore

$$|(f(\xi_n)-M_n)||g(y_n)-g(x')|+|f(\xi_n)-M_{n-1}||g(x')-g(y_{n-1})|$$
$$< \frac{\epsilon}{2(g(b)-g(a))}(g(y_n)-g(y_{n-1})).$$

Thus, $|s\sigma_1\cup\sigma^* - \sum_{I\sigma} fdg| < \frac{\epsilon}{2}$. Hence, $|\sum_\sigma fdg - s_\sigma \cup_\sigma * | < \epsilon$. But $|s_\sigma \cup_\sigma * - I| < \epsilon$. Therefore, $|\sum_\sigma fdg - I| < 2\epsilon$, and the theorem is proved.

2.61. Remark. It should be noted that if s_σ is defined to be $\sum_\sigma m_i(g(x_i)-g(x_{i-1}))$, where m_i is the min of f on

$$[x_{i-1}, x_i],$$

the I of 2.60 can be defined to be the max of s_σ for all σ of $[a,b]$.

2.62. <u>If each of f and g is a function defined on $[a,b]$, f is bounded by the positive number K, g is of bounded variation, and $\int_a^b fdg$ exists, then $|\int_a^b fdg| \leq K\cdot V(g;[a,b])$.</u>

Proof. Let $\sigma: a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a subdivision of $[a,b]$. Then

$$\left|\sum_{i=1}^{n} f(\xi_i)(g(x_i)-g(x_{i-1}))\right|$$
$$\leq \sum_{i=1}^{n} |f(\xi_i)(g(x_i)-g(x_{i-1}))|$$
$$< K\sum_{i=1}^{n} |g(x_i)-g(x_{i-1})|$$
$$\leq K\cdot V(g;[a,b]).$$

The theorem follows.

2.63. Two proofs will be given of the following theorem. In particular, the second proof indicates how the properties of functions of bounded variation applies to certain theorems in integration theory.

To facilitate the proof, let

$$\sum_{\sigma} fdg = \sum_{i=1}^{n} m_i(g(x_i)-g(x_{i-1})),$$

where m_i is the min of f on $[x_{i-1}, x_i]$. That is, $\sum_{\sigma} fdg$ is merely the lower sum s_{σ} of 2.61 defined for the particular functions f and g.

<u>Theorem</u>. <u>If</u> g <u>is a function of bounded variation defined on</u> $[a,b]$, <u>and</u> $\{f_n\}$ <u>is a sequence of continuous functions which converges uniformly to the function</u> f <u>defined on</u> $[a,b]$, <u>then</u> $\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b fdg$.

<u>Proof</u>. (1) The integral $\int_a^b fdg$ obviously exists since f is necessarily continuous. Since g is of bounded variation on $[a,b]$, it may be assumed that g is monotone non-decreasing on $[a,b]$. Furthermore, suppose that $g \not\equiv 0$ in order to avoid the trivial case.

Choose $\epsilon > 0$. There exists a positive integer N so that if $n > N$ and $x \in [a,b]$, then $|f_n(x)-f(x)| < \epsilon$. Choose $n > N$. There exists a subdivision σ_1 of $[a,b]$ so that

$$\left| \int_a^b f_n dg - \sum_{\sigma_1} f_n dg \right| < \epsilon,$$

and there exists a subdivision σ_2 of $[a,b]$ so that

$$\left| \int_a^b fdg - \sum_{\sigma_2} fdg \right| < \epsilon.$$

Also, there exists a $\delta > 0$ so that if x_1 and x_2 are in $[a,b]$ so that $|x_1-x_2| < \delta$, then $|f(x_1)-f(x_2)| < \epsilon$ and

$$|f_n(x_1)-f_n(x_2)| < \epsilon.$$

Let σ_3 be any subdivision of $[a,b]$ so that $\|\sigma_3\| < \delta$. Then let $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$. Clearly, $\left| \int_a^b f dg - \sum_\sigma f dg \right| < \epsilon$ and $\left| \int_a^b f_n dg - \sum_\sigma f dg \right| < \epsilon$. Choose x and x' in $[x_{i-1}, x_i]$. Thus, $|x-x'| < \delta$, and $|f(x)-f(x')| < \epsilon$. Similarly,

$$|f_n(x)-f_n(x')| < \epsilon .$$

Then let y_1 be an element of $[x_{i-1}, x_i]$ so that $f(y_1)$ is the min of f on $[x_{i-1}, x_i]$. Similarly, let $y_2 \in [x_{i-1}, x_i]$ so that $f_n(y_2)$ is the min of f_n on $[x_{i-1}, x_i]$. Then let y_3 be an arbitrary point in $[x_{i-1}, x_i]$. Thus

$$|f(y_3)-f(y_1)| + |f_n(y_2)-f(y_3)| < 2\,\epsilon .$$

Hence, $|f_n(y_2)-f(y_1)| - |f_n(y_3)-f(y_3)| < 2\,\epsilon$. Therefore,

$$|f_n(y_2)-f(y_1)| < 3\,\epsilon .$$

It follows that $\left| \sum_\sigma f dg - \sum_\sigma f_n dg \right| < 3\,\epsilon\,(g(b)-g(a))$. But

$$\left| \sum_\sigma f dg - \int_a^b f dg \right| + \left| \sum_\sigma f_n dg - \int_a^b f_n dg \right| < 2\,\epsilon .$$

Therefore

$$\left| \int_a^b f_n dg - \int_a^b f dg \right|$$

$$< 2\epsilon + \left| \sum_\sigma f dg - \sum_\sigma f_n dg \right| < 2\,\epsilon + 3\epsilon \cdot (g(b)-g(a)).$$

Since ϵ may be chosen arbitrarily small, the conclusion follows.

<u>Proof</u>. (2) Since f is continuous on $[a,b]$, the integral $\int_a^b f dg$ exists by 2.60. Choose $\epsilon > 0$. There exists a positive integer N so that if $n > N$ and $x \in [a,b]$, then

$$|f_n(x)-f(x)| < \frac{\epsilon}{V(g;[a,b])+1}.$$

Now $\left| \int_a^b f_n dg - \int_a^b f dg \right| = \left| \int_a^b (f_n-f) dg \right|$. But, by 2.62,

$$\left| \int_a^b (f_n-f) dg \right| \leq K \cdot V(g;[a,b]),$$

where K is the bound for f_n-f. Clearly

$$K \leq \frac{\epsilon}{V(g;[a,b])+1}.$$

Hence

$$\left| \int_a^b (f_n - f)dg \right| \leq K \cdot V(g;[a,b])$$
$$\leq \frac{\epsilon}{V(g;[a,b])+1} \cdot V(g;[a,b]) < \epsilon .$$

Thus, the proof is completed.

2.64. <u>If</u> $\int_a^b fdg$ <u>and</u> $\int_a^b |f|dg$ <u>exist and</u> g <u>is monotone non-decreasing on</u> $[a,b]$, <u>then</u> $\left| \int_a^b fdg \right| \leq \int_a^b |f|dg$.

<u>Proof</u>. Choose $\epsilon > 0$. There exists a $\delta > 0$ so that if σ is a subdivision of $[a,b]$ so that $\|\sigma\| < \delta$, then

$$\left| \int_a^b fdg - \sum_\sigma fdg \right| < \frac{\epsilon}{2}$$

and $\left|\sum_\sigma |f|dg - \int_a^b |f|dg\right| < \frac{\epsilon}{2}$. Therefore

$$\left| \int_a^b fdg \right| < \left| \sum_\sigma fdg \right| + \frac{\epsilon}{2} \leq \sum_\sigma |f|dg + \frac{\epsilon}{2}.$$

But $\sum_\sigma |f|dg < \left| \int_a^b |f|dg \right| + \frac{\epsilon}{2} = \int_a^b |f|dg + \frac{\epsilon}{2}$. Thus

$$\left| \int_a^b fdg \right| < \int_a^b |f|dg + \epsilon ,$$

and the proof is through.

2.65. <u>If the function</u> f <u>is continuous and the function</u> g <u>is of bounded variation on</u> $[a,b]$, <u>then</u> $F(x) = \int_a^x fdg$ <u>is of bounded variation on</u> $[a,b]$; <u>and</u> F <u>is continuous at all points of continuity of</u> g <u>on</u> $[a,b]$.

<u>Proof</u>. The function g is assumed to be monotone non-decreasing on $[a,b]$ without loss of generality.

Let σ : $a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a subdivision of $[a,b]$. Then

$$\sum_{\sigma} |F(x_i)-F(x_{i-1})| = \sum_{\sigma} \left| \int_a^{x_i} f dg - \int_a^{x_{i-1}} f dg \right|$$
$$= \sum_{\sigma} \left| \int_a^{x_{i-1}} f dg + \int_{x_{i-1}}^{x_i} f dg - \int_a^{x_{i-1}} f dg \right|$$
$$= \sum_{\sigma} \left| \int_{x_{i-1}}^{x_i} f dg \right|$$
$$\leq \sum_{\sigma} K_i V(g;[x_{i-1},x_i]),$$

where K_i is the bound of f on $[x_{i-1},x_i]$. Letting $K > 0$ be the bound for f on $[a,b]$,

$$\sum_{\sigma} K_i V(g;[x_{i-1},x_i]) \leq K \cdot \sum_{\sigma} V(g;[x_{i-1},x_i])$$
$$= K(g(b)-g(a)).$$

Hence, F is of bounded variation on $[a,b]$.

Now suppose that $\xi \in [a,b]$ is a point of continuity of g. By 2.48, the variation function $v(x) = V(f;[a,x])$ is continuous at $x = \xi$. Choose $\epsilon > 0$. There exists a $\delta > 0$ so that if $x \in [a,b]$ and $|x-\xi| < \delta$, then

$$|V(g;[a,x])-V(g;[a,\xi])| < \frac{\epsilon}{K}$$

(where K is the aforementioned bound for f). For definiteness, suppose that $x > \xi$ and $x-\xi < \delta$. Then

$$|F(x)-F(\xi)| = \left| \int_{\xi}^{x} f dg \right| < K \cdot V(g;[\xi,x]) < K \cdot \frac{\epsilon}{K}.$$

The continuity of F is now apparent.

2.66. <u>Theorem</u>. <u>Let f be a continuous function defined on $[a,b]$. If $G = \{g_n\}$ is a sequence of functions of bounded variation defined on $[a,b]$, the bound for the variation of elements of G is uniform, and the sequence $\{g_n\}$ converges to the function g at each point of $[a,b]$, then $\int_a^b f dg$ exists and</u> $\lim_{n \to \infty} \int_a^b f dg_n = \int_a^b f dg.$

__Proof.__ Let K be the uniform bound for the variation of the functions in G. Then, by 2.55, g is of bounded variation on $[a,b]$, and $V(g;[a,b]) \leq K$. By 2.60, $\int_a^b f dg$ exists.

Now, by 1.33, f is uniformly continuous on $[a,b]$. Choose $\epsilon > 0$, and let δ be a positive number so that if x_1 and x_2 are in $[a,b]$ so that $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. Let $\sigma: a = x_0 < x_1 < \dots < x_k = b$ be a subdivision of $[a,b]$ so that $\|\sigma\| < \delta$. Now there exists a positive integer N_1 so that if $n > N_1$, then

$$|f(x_1)(g(x_1)-g(x_0))-f(x_1)(g_n(x_1)-g_n(x_0))| < \frac{\epsilon}{k};$$

there exists a positive integer N_2 so that if $n > N_2$, then

$$|f(x_2)(g(x_2)-g(x_1))-f(x_2)(g_n(x_2)-g_n(x_1))| < \frac{\epsilon}{k}; \quad \dots;$$

there exists a positive integer N_k so that if $n > N_k$, then

$$|f(x_k)(g(x_k)-g(x_{k-1}))-f(x_k)(g_n(x_k)-g_n(x_{k-1}))| < \frac{\epsilon}{k}.$$

Let $N = \max(N_1, N_2, \dots, N_k)$, and choose $n^* > N$. Therefore

$$\left|\sum_{i=1}^{k} f(x_i)(g(x_i)-g(x_{i-1})) - \sum_{i}^{k} f(x_i)(g(x_i)-g(x_{i-1}))\right| < \epsilon .$$

Then consider

$$\left| \int_a^b f dg - \int_a^b f d(g_{n^*}) \right|.$$

$$\begin{aligned}
\int_a^b f dg &= \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} f dg \\
&= \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i)) dg + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} f(x_i) dg \\
&= \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i)) dg + \sum_{i=1}^{k} f(x_i) \int_{x_{i-1}}^{x_i} f dg \\
&= \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i)) dg + \sum_{i=1}^{k} f(x_i)(g(x_i)-g(x_{i-1})).
\end{aligned}$$

And

$$\int_a^b f\,dg_{n*} = \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i))\,dg_{n*} + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} f(x_i)\,dg_{n*}$$
$$= \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i))\,dg_{n*} + \sum_{i=1}^{k} f(x_i)(g_{n*}(x_i)-g_{n*}(x_{i-1})).$$

Therefore

$$\left| \int_a^b f\,dg - \int_a^b f\,dg_{n*} \right| = \Bigg| \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i)\,dg$$
$$+ \sum_{i=1}^{k} f(x_i)(g(x_{i-1})) - \Bigg[\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i))\,dg_{n*}$$
$$+ \sum_{i=1}^{k} f(x_i)(g_{n*}(x_i)-g_{n*}(x_{i-1}))\Bigg] \Bigg|$$
$$\leq \left| \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i))\,dg - \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i))\,dg_{n*} \right.$$
$$+ \sum_{i=1}^{k} f(x_i)(g(x_i)-g(x_{i-1}))$$
$$\left. - \sum_{i=1}^{k} f(x_i)(g_{n*}(x_i)-g_{n*}(x_{i-1})) \right|$$
$$< \left| \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (f-f(x_i))\,d(g-g_{n*}) \right| + \epsilon$$
$$\leq \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} (f-f(x_i))\,d(g-g_{n*}) \right| + \epsilon .$$

But if $\xi_i \in [x_{i-1}, x_i]$, then $|x_i - \xi_i| < \delta$, and $|f(x_i)-f(\xi_i)| < \epsilon$. Furthermore, for every positive integer n, $V(g-g_n; [a,b]) < 2K$.

Hence

$$\left| \int_a^b f\,dg - \int_a^b f\,dg_{n*} \right|$$
$$< \sum_{i=1,\sigma}^{k} \left| \int_{x_{i-1}}^{x_i} (f-f_{x_i})\,d(g-g_{n*}) \right| + \epsilon$$
$$< \epsilon \cdot V(g-g_{n*}; [a,b]) + \epsilon$$
$$< \epsilon(2K) + \epsilon .$$

Since ϵ may be chosen arbitrarily small, the proof is finished.

CHAPTER BIBLIOGRAPHY

1. Natanson, I. P., *Theory of Functions of a Real Variable*, translated by Leo F. Boron, edited by Edwin Hewitt, New York, Frederick Ungar Publishing Company, 1961.

2. Varberg, Dale E., "On Absolutely Continuous Functions," *American Mathematical Monthly*, Vol. 72 (October, 1965), 831-841.

3. Rudin, Walter, *Principles of Mathematical Analysis*, New York, McGraw-Hill Book Company, 1964.

CHAPTER III

SOME GENERALIZATIONS OF BOUNDED VARIATION

9. Introductory Remarks

3.1. Remark. In Chapter II, bounded variation was discussed almost entirely in terms of functions defined on closed intervals. The notion of the variation of a function defined on an open interval was defined in 2.6, but very little material was presented in this context.

This chapter seeks to indicate how the concept of bounded variation can be generalized. Several methods of generalization are explored. A special type of subdivision or partition of a closed interval is defined. This type of subdivision, called a B-partition, leads to a discussion of a class of functions which are termed S-measurable. In addition, bounded variation is defined for functions defined on a broader class of point sets than closed intervals on the real line. In particular, interval functions in Euclidean n-space are discussed. This discussion prompts the short development of the Burkill integral which was mentioned in the introduction to the thesis.

10. Definitions

3.2. If E is a set of real numbers, then a subdivision of E is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ so that

$x_i \in E$ for $i = 0, 1, 2, \ldots, n$, and

$$x_0 < x_1 < x_2 < \cdots < x_n.$$

3.3. If f is a function defined on the point set E and $\sigma : x_0 < x_1 < \cdots < x_n$ is a subdivision of E, then the <u>variation of</u> f <u>with respect to</u> σ is defined to be $\sum_{\sigma} |f(x_i) - f(x_{i-1})|$. The notation $V_{\sigma}(f;E)$, $V_{\sigma}(f)$, or $\sum_{\sigma} f$ is sometimes used. The <u>total variation</u> of f on E, denoted by $V(f;E)$, is defined to be the max of $V_{\sigma}(f)$ for all σ of E. Then f is said to be of <u>bounded variation</u> on E if, and only if, $V(f;E)$ is finite.

3.4. If $J = [a,b]$ is a closed interval, then the statement that the countable collection $T = \{I_n\}$ of closed subintervals of J is a B-partition of J means that

i) $[a,b] = \bigcup_{I_n \in T} I_n$, and

ii) if $i \neq k$ and $I_i \cap I_k \neq \emptyset$, then I_i and I_k are abutting intervals. In addition, a degenerate interval of the form $[\xi, \xi]$ is allowed provided that ξ does not belong to any other interval in the collection T. A similar definition is made for J being open or half open.

3.5. If $\beta_1 = \{I_n\}$ and $\beta_2 = \{J_n\}$ are B-partitions of $[a,b]$, then the <u>product</u> of the two B-partitions, denoted by $\beta_1 \cdot \beta_2$, is defined to be the set

$$A = \{I_n \cap J_m\}_{I_n \in \beta_1,\, J_m \in \beta_2},$$

where the following restrictions are placed on A:

i) if $I_n \cap J_m \in A$, then $I_n \cap J_m \neq \emptyset$, and

ii) if $I_n \cap J_m = \{x\}$, then $I_n \cap J_m \in A$ if, and only if, there do not exist positive integers n' and m' so that $I_{n'} \cap J_{m'}$ is non-degenerate and contains x.

3.6. If the function f is defined on $[a,b]$, then the statement that f is <u>generalized</u> <u>monotone</u> <u>non-decreasing</u> on $[a,b]$ means that there exists a B-partition $\beta = \{I_n\}$ of $[a,b]$ so that if $I_n \in \beta$, then f is monotone non-decreasing on I_n.

3.7. If the function f is defined on $[a,b]$, then the statement that the function f is <u>absolutely</u> <u>continuous</u> <u>in</u> <u>the</u> <u>generalized</u> <u>sense</u> on $[a,b]$(GAC on $[a,b]$) means that there exists a B -partition $\beta = \{I_n\}$ so that f is absolutely continuous on each $I_n \in \beta$.

3.8. The function f defined on $[a,b]$ is of <u>generalized</u> <u>bounded</u> <u>variation</u> (GBV) on $[a,b]$ if, and only if, there exists a B-partition $\beta = \{I_n\}$ of $[a,b]$ so that f is of bounded variation on I_n for every $I_n \in \beta$.

3.9. Let each of f and g be a function defined on $[a,b]$. The statement that f is S-<u>measurable</u> <u>with</u> <u>respect</u> <u>to</u> g on $[a,b]$ means that there exists a B-partition $\beta = \{I_n\}$of $[a,b]$ so that f is Stieltjes integrable with respect to g on I_n for each $I_n \in \beta$. It follows that both f and g are bounded on I_n for every $I_n \in \beta$.

3.10. Let $A = \{a_n\}$ be a sequence of real numbers. The statement that the <u>limit</u> <u>superior</u> $\{a_n\} = K$ means that

i) for each $\epsilon > 0$ there exists only a finite number of values of n so that $a_n > K + \epsilon$, and

ii) for each $\epsilon > 0$ there exists an infinite number of values of n so that $a_n > K - \epsilon$. The statement that the <u>limit</u> <u>inferior</u> $\{a_n\} = K$ means that

i) for each $\epsilon > 0$ there exists only a finite number of values of n so that $a_n < K - \epsilon$, and

ii) for each $\epsilon > 0$ there exists an infinite number of values of n so that $a_n < K + \epsilon$. The notation <u>lim</u> <u>sup</u> and <u>lim</u> <u>inf</u> is used to denote limit superior and limit inferior, respectively. It follows readily from the definitions that if $A = \{a_n\}$ is any sequence of real numbers, then

$$\lim\inf\{a_n\} \leq \lim\sup\{a_n\}.$$

3.11. Let E_n denote Euclidean n-space, <u>i</u>.<u>e</u>. the space of all n-tuples $(x_1, x_2, \ldots, x_n)$, where x_i is a real number for $i = 1, 2, 3, \ldots, n$. An n-tuple of this form is called a point in E_n, and a point set in E_n is merely a collection of points in E_n. An <u>interval</u> I in E_n is the collection of all n-tuples $(x_1, x_2, \ldots, x_n)$ so that $a_i \leq x_i \leq b_i$, and a_i and b_i are real numbers so that $a_i < b_i$ for $i = 1, 2, \ldots, n$. It should be remarked that the interval I is actually determined by the two points $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots b_n)$. The terms <u>oriented</u> <u>rectangle</u> and <u>interval</u> will be used interchangeably when referring to E_n. The notation

$$I = [a_1, b_1; a_2, b_2; \ldots; a_n, b_n]$$

is sometimes used to denote the interval I.

3.12. The statement that I is an <u>open</u> <u>interval</u> in E_n means that I is the set of all points $(x_1, x_2, \ldots, x_n)$

determined by inequalities of the form

$$a_1 < x_1 < b_1, a_2 < x_2 < b_2, \ldots, a_n < x_n < b_n,$$

where a_i and b_i are real numbers and $a_i \quad b_i$ for $i = 1, 2, \ldots, n$.

3.13. Let R_0 be an oriented rectangle in E_n. $C(R_0)$ will denote the class of all oriented rectangles in E_n which are subsets of R_0.

3.14. Let C be a class of oriented rectangles I in E_n. If ϕ is a real-valued function defined for each $I \in C$, then ϕ is called a <u>rectangle function</u> (<u>interval function</u>) defined on C. The term interval function will be used almost exclusively.

3.15. If $I = [a_1, b_1; a_2, b_2; \ldots; a_n, b_n]$ is an oriented rectangle in E_n, then the <u>volume</u> of I, denoted by $|I|$, is defined to be $\prod_{i=1}^{n}(b_i - a_i)$.

3.16. If $I = [a_1, b_1; a_2, b_2; \ldots; a_n, b_n]$ is an interval in E_n, then the <u>interior</u> of I, denoted by I^0, is defined to be the open interval defined by the following inequalities:

$$a_1 < x_1 < b_1, a_2 < x_2 < b_2, \ldots, a_n < x_n < b_n.$$

3.17. Let R_0 be a fixed oriented rectangle in E_n. A <u>partial subdivision</u> $p(\sigma)$ of R_0 is a finite collection

$$R_1, R_2, \ldots, R_n$$

of oriented rectangles in $C(R_0)$ so that $R_i^0 \cap R_j^0 = \emptyset$ if $i \neq j$. The notation $p(\sigma)$: $R_1, R_2, \ldots, R_n$ is used to indicate a partial subdivision.

3.18. A <u>subdivision</u> σ of R_0 is a partial subdivision $p(\sigma)$: $R_1, R_2, \ldots, R_n$ of R_0 so that $\bigcup_{i=1}^{n} R_i = R_0$. The notation σ: $R_1, R_2, \ldots, R_n$ is used to indicate a subdivision.

3.19. Let R_0 be an oriented rectangle in E_n, and let ϕ be an interval function defined on $C(R_0)$. The function ϕ is said to be absolutely continuous on R_0 if, and only if, for each $\epsilon > 0$ there exists a $\delta > 0$ so that if

$$p(\sigma): R_1, R_2, \ldots, R_n$$

is any partial subdivision of R_0 so that $\sum_{i=1}^{n} |R_i| < \delta$, then $\sum_{i=1}^{n} |\phi(R_i)| < \epsilon$.

3.20. Let R_0 be a fixed oriented rectangle in E_n, and let ϕ be an interval function defined on $C(R_0)$. Let

$$p(\sigma): R_1, R_2, \ldots, R_n$$

be a partial subdivision of R_0. Define $V_{p(\sigma)}(\phi; R_0)$ to be $\sum_{i=1}^{n} |\phi(R_i)|$. $V_{p(\sigma)}(\phi; R_0)$ is called the variation of ϕ on R_0 with respect to $p(\sigma)$ and is sometimes denoted by $V_{p(\sigma)}(\phi)$. The total variation of ϕ on R_0, denoted by $V(\phi; R_0)$, is defined to be the max of $V_{p(\sigma)}(\phi; R_0)$ for all partial subdivisions of R_0. ϕ is said to be of bounded variation on R_0 if, and only if, $V(\phi; R_0)$ is finite.

3.21. Let R_0 be an oriented rectangle in E_n, and let ϕ be an interval function defined on $C(R_0)$. Then ϕ is said to be of restricted bounded variation (RBV) on R_0 provided that there exists two positive numbers M and δ so that if $p(\sigma)$ is a partial subdivision of R_0 so that $\|p(\sigma)\| < \delta$, then $\sum_{p(\sigma)} |\phi(R_i)| < M$. The function ϕ is said to be of upper restricted bounded variation (lower restricted bounded variation) on $C(R_0)$ provided that there exists two positive numbers δ and M so that if $p(\sigma)$ is any partial subdivision of R_0 so that $\|p(\sigma)\| < \delta$, then $\sum_{p(\sigma)} \phi(R_i) < M(\sum_{p(\sigma)} \phi(R_i) > -M)$.

3.22. If R_0 is an oriented rectangle, and ϕ is an interval function defined on $C(R_0)$, then ϕ is said to be <u>bounded</u> on $R_0(C(R_0))$ provided that there exists a positive number M so that if $R \in C(R_0)$, then $|\phi(R)| < M$.

3.23. Let R_0 be a fixed oriented rectangle in E_n, and let $R \in C(R_0)$. R is defined by a set of inequalities of the form $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2$, ..., $a_n \le x_n \le b_n$. The <u>norm</u> of R, denoted by $\|R\|$, is defined to be the max of $(b_i - a_i)$ for $i = 1, 2, \ldots, n$.

3.24. Let R_0 be an oriented rectangle in E_n, and let : $R_1, R_2, \ldots, R_n$ be a subdivision of R_0. The <u>norm</u> of σ, denoted by $\|\sigma\|$, is defined to be the max of $\|R_i\|$ for $i = 1, 2, \ldots, n$.

3.25. Let R_0 be an oriented rectangle in E_n, and let $\sigma: R_1, R_2, \ldots, R_n$ be a subdivision of R_0. If ϕ is an interval function defined on $C(R_0)$, define $\phi(\sigma; R_0)$ to be $\sum_{i=1}^{n} \phi(R_i)$.

3.26. Let R_0 be an oriented rectangle in E_n, and let ϕ be an interval function defined on $C(R_0)$. Then ϕ is said to be <u>monotone non-decreasing</u> (<u>monotone non-increasing</u>) on R_0 if, and only if, for each $R \in C(R_0)$ and for each subdivision σ of R, $\phi(R) \le \phi(\sigma; R)(\phi(\sigma; R) \le \phi(R))$.

3.27. If R_0 is an interval in E_n and ϕ is an interval function defined on $C(R_0)$, then the statement that ϕ is (*finitely*) *additive* on R_0 means that ϕ is both monotone non-decreasing and monotone non-increasing on R_0.

3.28. Let R_0 be an oriented rectangle in E_n, and let ϕ be an interval function defined on $C(R_0)$. Let $R \in C(R_0)$. Then define the *upper Burkill integral* of ϕ over R to be the $\max(\limsup \phi(\sigma_n;R))$, where the max is taken with respect to all sequences $\{\sigma_n\}$ of subdivisions of R so that

$$\lim_{n\to\infty} \|\sigma_n\| = 0;$$

and define the *lower Burkill integral* of ϕ over R to be the $\min(\liminf \phi(\sigma_n;R))$, where the min is taken with respect to all sequences $\{\sigma_n\}$ of subdivisions of R so that

$$\lim_{n\to\infty} \|\sigma_n\| = 0.$$

The upper and lower integrals are denoted by $\overline{\int_R} \phi$ and $\underline{\int_R} \phi$, respectively. The function ϕ is said to be *Burkill integrable* (*B-integrable*) provided that both $\overline{\int_R} \phi$ and $\underline{\int_R} \phi$ are finite and $\overline{\int_R} \phi = \underline{\int_R} \phi$. This common value is taken to be the *Burkill integral* of ϕ on R.

11. Bounded Variation on an Arbitrary Point Set E and the Open Intervals (a,b) and $(-\infty,+\infty)$

3.29. *If the function f is defined and of bounded variation on the point set E, then f is bounded on E.*

Proof. Let $\xi \in E$. Then, if $x \in E$, $|f(x)-f(\xi)| \leq V(f;E)$. Hence, $|f(x)| \leq V(f;E)+|f(\xi)|$, and the conclusion follows.

3.30. *If the functions f and g are defined on the point set E, and f and g are of bounded variation on E, then*

i) $f+g$ is of bounded variation on E;

ii) $f-g$ is of bounded variation on E;

iii) $f\cdot g$ is of bounded variation on E; and

iv) if $\frac{1}{g}$ is defined and bounded on E, then $\frac{f}{g}$ is of bounded variation on E.

Proof. The proof follows in a similar fashion to that of 2.35.

3.31. *If the function f is defined and continuous on* $[a,b]$, E *is everywhere dense in* $[a,b]$, *and f is of bounded variation on* E, *then f is of bounded variation on* a,b . *Furthermore*, $V(f;[a,b]) = V(f;E)$.

Proof. Let $\sigma: a = x_0 < x_1 < \dots < x_n = b$ be a subdivision of $[a,b]$, and choose $\epsilon > 0$. Then let $\sigma': y_0 < y_1 < \dots < y_n$ be a subdivision of E so that

$$|f(x_i)-f(y_i)| < \frac{\epsilon}{2n},$$

for $i = 0, 1, 2, \dots, n$. Then

$$\sum_{i=1}^{n} |f(x_i)-f(x_{i-1})|$$

$$\le \sum_{i=1}^{n} |f(x_i)-f(y_{i-1})|+|f(y_{i-1})-f(x_{i-1})|$$

$$< \sum_{i=1}^{n} |f(y_i)-f(y_{i-1})|+|f(x_i)-f(y_i)|+\frac{\epsilon}{2n}$$

$$< \sum_{i=1}^{n} |f(y_i)+f(y_{i-1})|+\frac{\epsilon}{n}$$

$$\le \quad V(f;E)+\epsilon .$$

Therefore, $V(f;[a,b]) \le V(f;E)$. But, obviously, $V(f;E) \le V(f;[a,b])$. Thus, the theorem is proved.

3.32. Remark. As has been indicated in 3.29 and 3.30, several of the theorems of Chapter II may be generalized for functions having an arbitrary set of real numbers E as their domain. However, such generalization can prove to be quite mechanical. Therefore, rather than presenting a reiteration of the theorems of the preceding chapter, attention will now be focused briefly on the notion of bounded variation for functions defined on open intervals. An extensive cataloging of theorems for such functions will not be attempted. Rather, a few selected properties are stated and verified. These serve to indicate in general how the concept of bounded variation might be handled for such functions. In particular, a close connection between the development of the theory for functions defined on a "finite" open interval (a,b) and the theory for functions defined on $(-\infty,+\infty)$ is demonstrated.

3.33. *If the function f is defined and of bounded variation on (a,b) and $(a,b)\supset(a_1,b_1)$, then f is of bounded variation on (a_1,b_1). Furthermore,* $V(f;(a_1,b_1))\leq V(f;(a,b))$.

Proof. The proof is an immediate consequence of Definitions 3.2 and 3.3.

3.34. *If the function f is defined and of bounded variation on* $(a,b)((-\infty,+\infty))$, *then*

i) if $\delta = \frac{b+a}{2}$, $D = \frac{b-a}{2}$, and $0 < x < D$, then $\lim_{n\to 0} V(f;(\delta-x,\delta+x)) = V(f;(a,b))$;

ii) if $x > 0$, $\lim_{x\to\infty} V(f;(-x,x)) = V(f;(-\infty,+\infty))$;

iii) if $a < x < b$,

$$\lim_{x \to b} V(f;(a,x))$$
$$= V(f;(a,b))(\lim_{x \to \infty} V(f;(-\infty,x))$$
$$= V(f;(-\infty,+\infty)));$$

iv) (for the case where f is defined and of bounded variation on $(-\infty,+\infty)$) f can be written as the difference of two monotone non-decreasing functions g and h so that

$$\lim_{x \to -\infty} g(x) = 0.$$

Proof. (i) Choose $\epsilon > 0$. There exists a subdivision $\sigma: x_0 < x_1 < \cdots < x_n$ of (a,b) so that

$$V_\sigma(f;(a,b)) > V(f;(a,b)) - \epsilon .$$

Let $\delta = \min(b-x_n, x_0-a)$, and choose x so that $0 < |D-x| = D-x < \delta$. Therefore, $a < \xi - x < x_0 < x_n < \xi + x < b$. Obviously,

$$V(f;(\xi - x, \xi + x)) \geq V_\sigma(f;(a,b)),$$

but $V(f;(\xi - x, \xi + x)) \leq V(f;(a,b))$. Hence

$$|V(f;(a,b)) - V(f;(\xi - x, \xi + x))| < \epsilon ,$$

and the proof of (i) is finished.

(ii) Choose $\epsilon > 0$. There exists a subdivision

$$\sigma: x_0 < x_1 < \cdots < x_n$$

of $(-\infty,+\infty)$ so that $V_\sigma(f;(-\infty,+\infty)) > V(f;(-\infty,+\infty)) - \epsilon$. Let $\delta = \max(|x_0|, |x_n|)$, and choose $x > \delta$. It follows easily that $|V(f;(-\infty,+\infty)) - V(f;(-x,x))| < \epsilon$.

(iii) The proof of (iii) is similar to the proofs of the preceding two parts.

(iv) Let $g(x) = V(f;(-\infty,x])$. It is to be understood that any subdivision σ of $(-\infty,x]$ must include the point x.

Let $h(x) = g(x)-f(x)$. Clearly, g is monotone non-decreasing on $(-\infty,+\infty)$. Then let x_1 and x_2 be two real numbers so that $x_1 < x_2$. It follows that

$$\begin{aligned} g(x_2) &= V(f;(-\infty,x_2]) \\ &= V(f;(-\infty,x_1])+V(f;[x_1,x_2]). \end{aligned}$$

Hence, $h(x_2)-h(x_1) = V(f;[x_1,x_2])-(f(x_2)-f(x_1)) \geq 0$. Thus, h is monotone non-decreasing. Therefore, $f = g-h$, and the first assertion in (iv) is proved.

Now choose $\epsilon > 0$. There exists a subdivision

$$\sigma : x_0 < x_1 < \cdots < x_n$$

of $(-\infty,+\infty)$ so that $V_\sigma(f) > V(f;(-\infty,+\infty))-\epsilon$. Let N be the largest integer so that $N \leq x_0$. Choose x so that $x < N$, and suppose that $V(f;(-\infty,x]) > \epsilon$. But now, $V(f;(x,+\infty)) \geq V\ (f)$. Therefore, $V(f;(-\infty,x])+V(f;(x,+\infty)) > V(f;(-\infty,+\infty))$, a contradiction. Hence, $g(x) = V(f;(-\infty,x)) \leq \epsilon$, and the proof is finished.

12. Generalized Bounded Variation and the B-partition

3.35. __Theorem__. __If each of__ $\beta_1 = \{I_n\}$ __and__ $\beta_2 = \{J_n\}$ __is a__ B-__partition of__ $[a,b]$, __then__ $\beta_1 \cdot \beta_2$ __is a__ B-__partition of__ $[a,b]$.

__Proof__. Let $I_n \in \beta_1$ and $J_m \in \beta_2$ so that $I_n \cap J_m \neq \emptyset$. Clearly, this intersection forms a closed interval. Furthermore, by definition, $\beta_1 \cdot \beta_2$ is countable. Therefore, represent $\beta_1 \cdot \beta_2$ by a collection $\{T_n\}$ of closed intervals. Let i and i' be two positive integers so that $T_i \cap T_{i'} \neq \emptyset$.

Now, by definition, there exist positive integers n and n' so that $T_i = I_n \cap J_{n'}$, and there exist positive integers m and m' so that $T_{i'} = I_m \cap J_{m'}$. By the definition of B-partition, $I_m = I_n$, I_m and I_n are abutting, or $I_m \cap I_n = \emptyset$. If $I_m = I_n$, it follows immediately that $J_{m'}$ and $J_{n'}$ must be abutting intervals. If this is not the case, then I_m and I_n must be abutting closed intervals. In either case, there exists a unique point x which belongs to T_i and $T_{i'}$.

Furthermore, neither T_i nor $T_{i'}$ is degenerate. Suppose that T_i were degenerate. Then either I_n or $J_{n'}$ is degenerate, or I_n and $J_{n'}$ are abutting intervals. If $I_n = [\xi, \xi]$, then either I_m is I_n or $\xi \notin I_m$. If $I_m = I_n = [\xi, \xi]$, then $T_i = T_{i'}$, a contradiction since $\beta_1 \cdot \beta_2$ is a set and an element is named in a set only one time. If $\xi \notin I_m$, then $T_i \cap T_{i'} = \emptyset$, a contradiction. Then suppose that I_n and $J_{n'}$ are abutting. Then $J_{n'}$ is degenerate or not. If it is degenerate, similar contradictions to the ones above are reached. If I_n and $J_{n'}$ are abutting and neither is degenerate, it follows that $T_{i'}$ is non-degenerate. Thus, $T_i \notin \beta_1 \cdot \beta_2$, a contradiction. Obviously, $\bigcup_{T_i \in \{T_n\}} T_i = [a,b]$, and the theorem is proved.

3.36. <u>If each of the functions</u> f <u>and</u> g <u>is defined and</u> GAC <u>on</u> [a,b], <u>then</u>

i) f+g is GAC on [a,b];

ii) f-g is GAC on [a,b];

iii) f·g is GAC on [a,b];

iv) if $\frac{1}{g}$ is defined on $[a,b]$ and there exists a B-partition $\beta = \{I_n\}$ so that $\frac{1}{g}$ is bounded on each $I_n \in \beta$, then $\frac{f}{g}$ is GAC on $[a,b]$.

Proof. (i) There exists a B-partition $\beta_1 = \{I_n\}$ of $[a,b]$ so that f is absolutely continuous on each $I_n \in \beta_1$. Similarly, there exists a B-partition $\beta_2 = \{J_n\}$ of $[a,b]$ so that g is absolutely continuous on each $J_n \in \beta_2$. Let $\beta = \beta_1 \cdot \beta_2$. The conclusion then follows from 2.24.

The proofs of (ii), (iii), and (iv) follow in a similar fashion, with the aid of 2.24.

3.37. If the function f is defined and GAC on $[a,b]$, then f is GBV on $[a,b]$.

Proof. The proof follows from the definition of generalized bounded variation and 2.27.

3.38. If the function f is defined and generalized monotone non-decreasing on $[a,b]$, then f is GBV on $[a,b]$.

Proof. The proof follows from 3.6 and 2.30.

3.39. If each of the functions f and g is GBV on $[a,b]$, then

i) f+g is GBV on $[a,b]$;

ii) f-g is GBV on $[a,b]$;

iii) f·g is GBV on $[a,b]$;

iv) if there exists a B-partition $\beta_1 = \{I_n\}$ so that $\frac{1}{g}$ is defined and bounded on each $I_n \in \beta_1$, then $\frac{f}{g}$ is GBV on $[a,b]$.

Proof. With the aid of 2.35, the proof follows in a similar fashion to that of 3.36.

3.40. Example. Consider the following function f defined on $[0,1]$: $f(x) = \frac{1}{x}$ if $x \ (0,1]$ and $f(0) = 0$.

$$[0,1] = [0,0] \cup [\tfrac{1}{2},1] \cup [\tfrac{1}{4},\tfrac{1}{2}] \cup \cdots \cup [\frac{1}{2^n},\frac{1}{2^{n-1}}] \cup \cdots .$$

Let $I_0 = [0,0]$ and

$$I_n = [\frac{1}{2^n},\frac{1}{2^{n-1}}]$$

for $n = 1, 2, \ldots$. Obviously, f is of bounded variation on I_0 and I_n for every positive integer n. However, f is not bounded on $[0,1]$; and, therefore, f is not of bounded variation on $[0,1]$.

3.41. A necessary and sufficient condition that the function f defined on $[a,b]$ be of generalized bounded variation on $[a,b]$ is that there exist two generalized non-decreasing functions g and h defined on $[a,b]$ so that $f = g-h$.

Proof. Suppose that f is GBV on $[a,b]$. Hence, there exists a B-partition $\beta = \{I_n\}$ so that f is of bounded variation on each $I_n \in \beta$. It follows then from 2.40 that f can be written as the difference of two monotone non-decreasing functions on each $I_n \in \beta$.

Now suppose that f is the difference of two generalized monotone non-decreasing functions g and h on $[a,b]$. Therefore, there exist B-partitions $\beta_1 = \{I_n\}$ and $\beta_2 = \{J_n\}$ so that g is monotone non-decreasing on each $I_n \in \beta_1$ and h is monotone non-decreasing on each $J_n \in \beta_2$. Let $\beta = \beta_1 \cdot \beta_2$. Let $T_i \in \beta$. Then there exist positive integers n and n' so that $T_i \subset I_n$ and $T_i \subset J_{n'}$. Hence, both g and h are monotone non-decreasing on T_i. Thus, f is of bounded variation on each $T_i \in \beta$, and the proof is finished.

3.42. <u>If the function</u> f <u>is</u> S-<u>measurable with respect to the functions</u> g_1 <u>and</u> g_2 <u>on</u> $[a,b]$, <u>and if the functions</u> f_1 <u>and</u> f_2 <u>are</u> S-<u>measurable with respect to the function</u> g <u>on</u> $[a,b]$, <u>then</u>

i) f is S-measurable with respect to g_1+g_2 on $[a,b]$;

ii) f_1+f_2 is S-measurable with respect to g on $[a,b]$;

iii) if α and γ are real numbers, then $\alpha \cdot f$ is S-measurable with respect to $\gamma \cdot g_1$ on $[a,b]$.

<u>Proof</u>. (i) Let $\beta_1 = \{I_n\}$ be a B-partition so that f is S-integrable with respect to g_1 on each $I_n \in \beta_1$, and let $\beta_2 = \{J_n\}$ be a B-partition of $[a,b]$ so that f is S-integrable with respect to g_2 on each $J_n \in \beta_2$. Let

$$\beta = \beta_1 \cdot \beta_2 = \{T_n\}.$$

It follows from 2.15 that f is S-integrable with respect to g_1+g_2 on each $T_n \in \beta$. Hence, f is S-measurable with respect to g_1+g_2 on $[a,b]$.

The proofs of (ii) and (iii) follow in a similar fashion.

3.43. <u>If there exists a</u> B-<u>partition</u> $\beta = \{I_n\}$ <u>of</u> $[a,b]$ <u>so that the function</u> f <u>is continuous on</u> I_n <u>for each</u> $I_n \in \beta$,

<u>and the function</u> g <u>is defined and</u> GBV <u>on</u> $[a,b]$, <u>then</u> f <u>is</u> S-<u>measurable with respect to</u> g <u>on</u> $[a,b]$.

<u>Proof</u>. The proof follows from 2.15, 2.60, and 3.35.

3.44. Clearly, the class of S-measurable functions contains functions which are not S-integrable in the usual sense. Let f be the function defined in 3.40, and let $g(x) = x$ for $0 \leq x \leq 1$. Then, obviously, f is S-measurable with respect to g on $[0,1]$. But the function f is not bounded on $[a,b]$. Thus, under Definition 2.15, f is not S-integrable with respect to any function g defined on $[0,1]$.

13. Interval Functions and the Burkill Integral

3.45. <u>Theorem</u>. (1) <u>If</u> R_0 <u>is an oriented rectangle in</u> E_n <u>and</u> $\emptyset$ <u>is a bounded, absolutely continuous interval function defined on</u> $C(R_0)$, <u>then</u> $\emptyset$ <u>is of bounded variation on</u> $C(R_0)$.

<u>Proof</u>. Let $B > 0$ be a bound for $\emptyset$ on R_0. Choose $\epsilon > 0$ and let $p(\sigma)$: $R_1, R_2, \ldots, R_n$ be a partial subdivision on R_0. There exists a $\delta > 0$ so that if $R_1', R_2', \ldots, R_m'$ is any finite collection of oriented rectangles in $C(R_0)$ so that $(R_i')^0 \cap (R_j')^0 = \emptyset$ when $i \neq j$ and $\sum_{i=1}^{m} |R_i'| < 2\delta$, then

$$\sum_{i=1}^{m} |\emptyset(R_i')| < \epsilon .$$

Let $\Delta_1 = \{R_i \in p(\sigma) \mid |R_i| < 2\delta\}$ and let $\Delta_2 = \{R_i \in p(\sigma) \mid R_i \notin \Delta_1\}$. Then let $G = \{G_1, G_2, \ldots, G_l\}$ be a grouping of the elements of $\Delta_1 \ni G_i \cap G_j = \emptyset$ if $i \neq j$ and $\sum_{R_i \in G_j} |R_i| < 2\delta$ for $j = 1, 2, \ldots, l$.

Furthermore, the grouping G is such that the integer l is a minimum; that is, any other grouping satisfying the above two conditions would have as many as l elements. Thus,

$$(l-1)\delta \leq \sum_{\substack{R_i \in G_i, \\ G_i \in G}} |R_i| \leq |R_0|.$$

Hence, $(l-1)\delta \leq R_0$. Therefore,

$$l \leq \frac{R_0}{\delta} + 1.$$

Now suppose that Δ_2 consists of j elements. Clearly, $\sum_{R_i \in \Delta_2} |R_i| \leq |R_0|$. Therefore, $j(2\delta) \leq |R_0|$. Hence

$$j \leq \frac{|R_0|}{2\delta}.$$

But then

$$\begin{aligned}
\sum_{p(\sigma)} |\phi(R_i)| &= \sum_{\Delta_1} |\phi(R_i)| + \sum_{\Delta_2} |\phi(R_i)| \\
&= \sum_{\substack{R_i \in G_i \\ G_i \in G}} |\phi(R_i)| + \sum_{\Delta_2} |\phi(R_i)| \\
&< l \cdot \epsilon + j \cdot B \\
&\leq \left(\frac{R_0}{\delta} + 1\right) \cdot \epsilon + \left(\frac{R_0}{2\delta}\right) B.
\end{aligned}$$

Since $|R_0|$, ϵ, δ, and B are in no way functions of $p(\sigma)$, the theorem follows.

3.46. <u>If R_0 is an oriented rectangle in E_n and each of the interval functions ϕ_1 and ϕ_2 is defined and absolutely continuous on $C(R_0)$, then</u>

i) $\phi_1 + \phi_2$ is absolutely continuous on $C(R_0)$;

ii) $\phi_1 - \phi_2$ is absolutely continuous on $C(R_0)$;

iii) if either ϕ_1 or ϕ_2 is bounded on $C(R_0)$, then $\phi_1 \cdot \phi_2$ is absolutely continuous on $C(R_0)$;

iv) if $\frac{1}{\phi_2}$ is defined and bounded on $C(R_0)$, then $\frac{\phi_1}{\phi_2}$ is absolutely continuous on $C(R_0)$.

Proof. The proof is similar to that of 2.24.

3.47. *If R_0 is an oriented rectangle in E_n and the interval function ϕ is defined and of bounded variation on $C(R_0)$, then ϕ is bounded on $C(R_0)$.*

Proof. Let $R \in C(R_0)$. Then $p(\sigma): R$ forms a partial subdivision of R_0, and $|\phi(R)| \leq V(\phi; R_0)$.

3.48. *If R_0 is an oriented rectangle in E_n and each of the interval functions ϕ_1 and ϕ_2 is defined and of bounded variation on $C(R_0)$, then*

i) $\phi_1 + \phi_2$ is of bounded variation on $C(R_0)$;

ii) $\phi_1 - \phi_2$ is of bounded variation on $C(R_0)$;

iii) $\phi_1 \cdot \phi_2$ is of bounded variation on $C(R_0)$;

iv) if $\frac{1}{\phi_2}$ is defined and bounded on $C(R_0)$, then $\frac{\phi_1}{\phi_2}$ is of bounded variation on $C(R_0)$.

Proof. The proof is similar to that of 2.35.

3.49. *If R_0 is an oriented rectangle in E_n and ϕ is an interval function defined on $C(R_0)$, then ϕ is of restricted bounded variation* (RBV) *on $C(R_0)$ if, and only if, ϕ is of upper restricted bounded variation* (URBV) *and lower restricted bounded variation* (LRBV) *on $C(R_0)$.*

Proof. Clearly, if ϕ is of RBV on $C(R_0)$, then it is of LRBV and URBV on $C(R_0)$, for there exist positive numbers δ and M so that if $p(\sigma)$ is any partial subdivision of R_0 so that $\|p(\sigma)\| < \delta$, then $V_{p(\sigma)}(\emptyset) < M$. Hence,

$$-M < \sum_{p(\sigma)} \emptyset(R_i) < M,$$

and the first part of the theorem is proved.

Now suppose that $\emptyset$ is of LRBV and URBV on $C(R_0)$. Therefore, there exist positive numbers δ_1, M_1, δ_2, and M_2 so that if each of $p(\sigma_1)$ and $p(\sigma_2)$ is a partial subdivision of R_0 so that $\|p(\sigma_1)\| < \delta_1$ and $\|p(\sigma_2)\| < \delta_2$, then

$$\sum_{p(\sigma_1)} \emptyset(R_i) < M_1$$

and $\sum_{p(\sigma_2)} \emptyset(R_i) > -M_2$. Let $\delta = \min(\delta_1, \delta_2)$, and let

$$M = \max(m_1, m_2).$$

Then let $p(\sigma)$: $R_1, R_2, \ldots, R_n$ be a partial subdivision of R_0 so that $\|p(\sigma)\| < \delta$. Let $\Delta_1 = \{R_i \in p(\sigma) | \emptyset(R_i) \geq 0\}$ and let $\Delta_2 = p(\sigma) - \Delta_1$. Hence $|\sum_{R_i \in \Delta_1} \emptyset(R_i)| < M$ and

$$|\sum_{R_i \in \Delta_2} \emptyset(R_i)| < M.$$

But $|\sum_{R_i \in \Delta_1} \emptyset(R_i)| = \sum_{R_i \in \Delta_1} |\emptyset(R_i)| < M$. Similarly,

$$|\sum_{R_i \in \Delta_2} \emptyset(R_i)| = |\sum_{R_i \in \Delta_2} -\emptyset(R_i)| = \sum_{R_i \in \Delta_2} |-\emptyset(R_i)| < M.$$

Hence, $\sum_{\Delta_1} |\emptyset(R_i)| + \sum_{\Delta_2} |\emptyset(R_i)| = \sum_{p\sigma} |\emptyset(R_i)| < 2M$, and the theorem is proved.

3.50. <u>Let</u> R_0 <u>be an oriented rectangle in</u> E_n, <u>and let</u> $\emptyset$ <u>be an interval function defined on</u> $C(R_0)$. <u>If</u> $R \in C(R_0)$, <u>then</u> $\underline{\int_R} \emptyset \leq \overline{\int_R} \emptyset$.

<u>Proof</u>. The proof follows from remarks made in 3.10.

3.51. <u>Theorem</u>. <u>If</u> R_0 <u>is an oriented rectangle in</u> E_n <u>and</u> $\emptyset$ <u>is</u> R-<u>integrable on</u> R_0, <u>then for each</u> $\epsilon > 0$ <u>there exists a</u> $\delta > 0$ <u>so that if</u> σ <u>is any subdivision of</u> R_0 <u>so that</u> $\|\sigma\| < \delta$, <u>then</u> $|\int_{R_0} \emptyset - \emptyset(\sigma; R_0)| < \epsilon$.

Proof. Suppose by way of contradiction that the theorem is false. Hence, there exists an $\epsilon > 0$ so that if δ is any positive number, then there exists a subdivision σ of R_0 so that $\|\sigma\| < \delta$, and $\left|\int_{R_0}\phi - \phi(\sigma; R_0)\right| \geq \epsilon$. Then let $\{\delta_n\}$ be a sequence of positive numbers so that $\lim_{n\to\infty} \{\delta_n\} = 0$. Let $\{\sigma_n\}$ be an associated sequence of subdivisions so that $\|\sigma_n\| < \delta_n$ and so that $\left|\int_{R_0}\phi - \phi(\sigma_n; R_0)\right| \geq \epsilon$ for each positive integer n. Clearly, $\lim_{n\to\infty} \|\sigma_n\| = 0$.

Then consider $\limsup \phi(\sigma_n; R_0)$ and $\liminf \phi(\sigma_n; R_0)$. Since $\underline{\int}_{R_0}\phi = \overline{\int}_{R_0}\phi$, it follows that

$$\limsup \phi(\sigma_n; R_0) = \liminf \phi(\sigma_n; R_0) = \int_{R_0}\phi.$$

Hence, there exists only a finite number of values of n so that $\phi(\sigma_n; R_0) < \int_{R_0}\phi - \frac{\epsilon}{2}$ and there exists only a finite number of values of n so that $\phi(\sigma_n; R_0) > \int_{R_0}\phi + \frac{\epsilon}{2}$. Let N be a positive integer so that

$$\int_{R_0}\phi - \frac{\epsilon}{2} \leq \phi(\sigma_N; R_0) \leq \int_{R_0}\phi + \frac{\epsilon}{2}.$$

Hence, $\left|\int_{R_0}\phi - \phi(\sigma_N; R_0)\right| \leq \frac{\epsilon}{2} < \epsilon$, a contradiction of the original assumption. Thus, the theorem is proved.

3.52. Theorem. Let R_0 be an oriented rectangle in E_n, and let ϕ be an interval function defined on $C(R_0)$. If $R \in C(R_0)$, then f is B-integrable on R if, and only if, for each $\epsilon > 0$ there exists a $\delta > 0$ so that if σ_1 and σ_2 are subdivisions of R so that $\|\sigma_1\| < \delta$ and $\|\sigma_2\| < \delta$, then $|\phi(\sigma_1; R) - \phi(\sigma_2; R)| < \epsilon$.

Proof. Suppose that ϕ is B-integrable on R. Choose $\epsilon > 0$. By 3.51 there exists a $\delta > 0$ so that if σ is a subdivision of R so that $\|\sigma\| < \delta$, then

$$\left|\int_{R_0} \phi - \phi(\sigma;R)\right| < \frac{\epsilon}{2}.$$

Let σ_1 and σ_2 be two subdivisions of R so that $\|\sigma_1\| < \delta$ and $\|\sigma_2\| < \delta$. Hence

$$\left|\int_R \phi - \phi(\sigma_1;R)\right| + \left|\int_R \phi - \phi(\sigma_2;R)\right| < \epsilon.$$

Therefore, $|\phi(\sigma_1;R) - \phi(\sigma_2;R)| < \epsilon$, and the first part of the proof is complete.

Now choose $\epsilon > 0$. Let $\{\sigma_n\}$ and $\{\sigma_{n'}\}$ be sequences of subdivisions with norms going to 0 so that

$$\limsup \phi(\sigma_n;R) \geq \overline{\int_R} \phi - \frac{\epsilon}{5}$$

and $\liminf \phi(\sigma_{n'};R) \leq \underline{\int_R} \phi + \frac{\epsilon}{5}$. Now there exists a positive number δ so that if σ_1 and σ_2 are subdivisions of R so that $\|\sigma_1\| < \delta$ and $\|\sigma_2\| < \delta$, then

$$|\phi(\sigma_1;R) - \phi(\sigma_2;R)| < \frac{\epsilon}{5}.$$

Let j and k be positive integers so that $\|\sigma_j\| < \delta$, $\|\sigma_{k'}\| < \delta$, $\phi(\sigma_j;R) \geq \limsup \phi(\sigma_n;R) - \frac{\epsilon}{5}$, and

$$\phi(\sigma_{k'};R) \geq \liminf \phi(\sigma_n;R) + \frac{\epsilon}{5}.$$

Hence, $\phi(\sigma_j;R) \geq \overline{\int_R} \phi - \frac{2\epsilon}{5}$ and $\phi(\sigma_{k'};R) \leq \underline{\int_R} \phi + \frac{2\epsilon}{5}$. But $|\phi(\sigma_j;R) - \phi(\sigma_{k'};R)| < \frac{\epsilon}{5}$. Therefore,

$$\overline{\int_R} \phi \leq \underline{\int_R} \phi + \epsilon.$$

But ϵ is arbitrary. Hence,

$$\overline{\int_R} \phi \leq \underline{\int_R} \phi,$$

and the B-integrability follows from 3.50.

3.53. *If R_0 is an oriented rectangle in E_n and the interval function ϕ is B-integrable on R_0, then ϕ is B-integrable on any $R \in C(R_0)$. Furthermore, the B-integral is an additive interval function defined on $C(R_0)$.*

Proof. A definition will be introduced first. If σ and σ' are subdivisions of the oriented rectangle R, then σ' will be said to form a refinement of σ provided that each subrectangle of σ' is a subset of some subrectangle of σ.

Let $R' \in C(R_0)$, and choose $\epsilon > 0$. There exists a $\delta > 0$ so that if each of σ and σ' is a subdivision of R_0 so that $\|\sigma\| < \delta$ and $\|\sigma'\| < \delta$, then $|\phi(\sigma;R_0)-\phi(\sigma';R_0)| < \epsilon$. Let σ_1 be a subdivision of R_0 so that no subrectangle of σ_1 contains an interior point of both R' and R_0-R'. Furthermore, choose σ_1 so that $\|\sigma_1\| < \delta$. Then let σ_2 be a subdivision of R_0 so that $\|\sigma_2\| < \delta$, σ_2 coincides with σ_1 on R_0-R', and σ_2 differs from σ_1 on R'. Then

$$
\begin{aligned}
&|\phi(\sigma_1;R_0)-\phi(\sigma_2;R_0)| \\
&= |\phi(\sigma_1;R_0-R')+\phi(\sigma_1;R)-[\phi(\sigma_2;R_0-R')+\phi(\sigma_2;R')]| \\
&= |\phi(\sigma_1;R')-\phi(\sigma_2;R')| < \epsilon .
\end{aligned}
$$

Hence, ϕ is B-integrable on R' by 3.52.

Then let $R \in C(R_0)$. Let σ : R_1, R_2, ..., R_k be an arbitrary subdivision of R_1 and choose $\epsilon > 0$. By the first part of this theorem, ϕ is B-integrable on R_1. Therefore, by 3.51, there exists a $\delta_1 > 0$ so that if σ is a subdivision of R_1 so that $||\sigma|| < \delta_1$, then $|\int_{R_1}\phi - \phi(\sigma;R_1)| < \epsilon$. Let

$$\sigma': R_1', R_2', \ldots, R_n'$$

be one such subdivision of R_1. Similarly, there exists a $\delta_i' > 0$ so that if σ_i' is a subdivision of R_i' so that

$$||\sigma_i'|| < \delta_i',$$

then

$$|\int_{R_i'}\phi - \phi(\sigma_i';R_i')| < \frac{\epsilon}{2^i},\ i = 1, 2, \ldots, n.$$

Let $\delta_i'' = \min(\delta_1, \delta_i')$, $i = 1, 2, \ldots, n$. Then let σ_i'' be a refinement of $\bigcup_{i=1}^{n}\sigma_i'$ so that $||\sigma_i''|| < \delta_i''$. Hence, $|\sum_{i=1}^{n}\int_{R_i'}\phi - \sum_{i=1}^{n}\phi(\sigma_i'';R_i')| < \epsilon$, and $|\int_{R_1}\phi - \phi(\sigma_1'';R_1)| < \epsilon$.

But

$$\sum_{i=1}^{n}\phi(\sigma_1'';R_i') = \phi(\sigma_1'';R_1).$$

Hence, $|\int_{R_1}\phi - \sum_{i=1}^{n}\int_{R_i'}\phi| < 2\epsilon$.

By repeating this process with respect to each $R_i \in \sigma$, a positive number δ_2 and a subdivision σ_2 of R are obtained so that if σ_3 is a refinement of σ_2 so that $||\sigma_3|| < \delta_2$, then $|\sum_{i=1}^{k}\int_{R_i}\phi - \sum_{\sigma_3}\int_{R_i}\phi| < k\cdot 2\epsilon$. Repeating this process for a third time, a refinement of σ_4 of σ_3 is formed so that $||\sigma_4|| < \delta_2$ and $|\int_R\phi - \sum_{\sigma_4}\int_{R_i}\phi| < 2\epsilon$. It follows that σ_4 satisfies the conditions placed on σ_3, i.e.

$$|\sum_{i=1}^{k}\int_{R_i}\phi - \sum_{\sigma_4}\int_{R_i}\phi| < k\cdot 2\epsilon.$$

Hence,

$$|\int_R\phi - \sum_{i=1}^{k}\int_{R_i}\phi| < (k+1)(2\epsilon).$$

Since ϵ is arbitrary, $\int_R\phi = \sum_{i=1}^{k}\int_{R_i}\phi$, and the additivity of the integral follows.

3.54. <u>If R_0 is an oriented rectangle in E_n, and the interval function ϕ is absolutely continuous and B-integrable on R_0, then the interval function $\phi(R) = \int_R \phi$ is absolutely continuous on R_0.</u>

<u>Proof</u>. Let $R \in C(R_0)$, and choose $\epsilon > 0$. There exists a $\delta > 0$ so that if $p(\sigma)$ is any partial subdivision of R so that $\sum_{p(\sigma)} |R_i| < \delta$, then $\sum_{p(\sigma)} |\phi(R_i)| < \epsilon$. Let

$$p(\sigma): R_1, R_2, \ldots, R_n$$

be one such partial subdivision of R. Now there exists a $\delta_1 > 0$ so that if σ_1 is a subdivision of $R_1 \in p(\sigma)$ so that $\|\sigma_1\| < \delta_1$, then $\left| \int_{R_1} \phi - \phi(\sigma_1; R_1) \right| < \frac{\epsilon}{2}$; there exists a positive number δ_2 so that if σ_2 is a subdivision of R_2 so that $\|\sigma_2\| < \delta_2$, then $\left| \int_{R_2} \phi - \phi(\sigma_2; R_2) \right| < \frac{\epsilon}{2^2}$; ...; there exists a $\delta_n > 0$ so that if σ_n is a subdivision of R_n so that $\|\sigma_n\| < \delta_n$, then $\left| \int_{R_n} \phi - \phi(\sigma_n; R_n) \right| < \frac{\epsilon}{2^n}$. Let

$$\delta^* = \min(\delta_1, \delta_2, \ldots, \delta_n)$$

and let $\sigma_1', \sigma_2', \ldots, \sigma_n'$ be subdivisions of $R_1, R_2, \ldots, R_n$, respectively, so that $\|\sigma_i'\| < \delta^*$ for $i = 1, 2, \ldots, n$. Hence, $\sum_{i=1}^{n} \left| \int_{R_i} \phi - \phi(\sigma_i'; R_i) \right| < \epsilon$. Therefore

$$\sum_{i=1}^{n} \left| \int_{R_i} \phi \right| < \epsilon + \sum_{i=1}^{n} |\phi(\sigma_i'; R_i)|.$$

But $p(\sigma^*) = \bigcup_{i=1}^{n} \sigma_i'$ forms a partial subdivision of R so that $\sum_{p(\sigma^*)} |R_i| = \sum_{p(\sigma)} |R_i| < \delta$. Hence, $\sum_{i=1}^{n} |\phi(\sigma_i'; R_i)| < \epsilon$, and $\sum_{i=1}^{n} \left| \int_{R_i} \phi \right| = \sum_{p(\sigma)} \left| \int_{R_i} \phi \right| < 2\epsilon$. Thus, the theorem is proved.

CHAPTER BIBLIOGRAPHY

1. Rado, Tibor, Length and Area, New York, American Mathematical Society, Colloquium Publications, 1948.

BIBLIOGRAPHY

Books

Natanson, I. P., _Theory of Functions of a Real Variable_, translated by Leo F. Boron, edited by Edwin Hewitt, New York, Frederick Ungar Publishing Company, 1961.

Rado, Tibor, _Length and Area_, New York, American Mathematical Society, Colloquium Publications, 1948.

Rudin, Walter, _Principles of Mathematical Analysis_, McGraw-Hill Book Company, Inc., 1953.

Saks, Stanislaw, _Theory of the Integral_, New York, G. E. Stechert and Company, 1937.

Articles

Ringenberg, Lawrence A., "The Theory of the Burkill Integral," _Duke Mathematical Journal_, Vol. 15, No. 1 (March, 1948), 239-270.

Varberg, Dale E., "On Absolutely Continuous Functions," _American Mathematical Monthly_, Vol. 72 (October, 1965), 831-841.