DIRECT SUMS OF RINGS

APPROVED:

Robert H. Cranford
Major Professor

Russell G. Bilyeu
Minor Professor

John T. Mahat
Director of the Department of Mathematics

Robert B. Toulouse
Dean of the Graduate School

DIRECT SUMS OF RINGS

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Dolin F. Hughes, B. S.
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PREFACE

This paper consists of a study of the direct sum U of two rings S and T. Such a direct sum is defined as the set of all ordered pairs (s_1, t_1), where s_1 is an arbitrary element in S and t_1 is an arbitrary element in T.

In the first two chapters, binary operations are defined on the set of ordered pairs so that this set is a ring. Also included are theorems concerning homomorphic mappings between the direct sum of S and T, subrings of this direct sum, and rings S and T.

The last two chapters contain a study of the relationship between direct sum rings or ideals and their components. Necessary and sufficient conditions are given in order that ideals in the direct sum of rings S and T with units be maximal, prime, or primary. Other theorems on topics including existence of zero divisors, irredundant primary representations of ideals, and the characteristic of a ring are stated and proved.

Examples are provided throughout the thesis in order to clarify definitions, to show that some theorems do not have converses, and to show the necessity of the hypothesis in some theorems.

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CHAPTER I

INTRODUCTORY CONCEPTS

Definition 1.1.--Let A be a non-empty set. A binary operation "x" defined on A is a correspondence which associates with each ordered pair (a,b) of elements of A, a uniquely determined element a x b of A.

Definition 1.2.--Consider a non-empty set R on which there are defined two binary operations which may be called addition "+" and multiplication " · ". If a, b $\in$ R then a + b and a · b are uniquely determined elements in R. Such a set is said to be a ring if the set has the following properties with respect to these binary operations. Let a, b, and c denote arbitrary elements in R.

P_1. $a + (b + c) = (a + b) + c$. This is called the associative law of addition.

P_2. There exists an element z of R such that $a + z = a$ for every element a of R. This z is called the zero element or additive identity. It can be shown that z is unique.

P_3. If a $\in$ R then there exists an x $\in$ R which depends on a such that $a + x = z$ where z is the additive identity. "x" is usually denoted by "-a" and is called an additive inverse. It can be shown that each a $\in$ R has a unique additive inverse.

P_4. $a + b = b + a$. This is called the commutative law of addition.

P_5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ is the associative law of multiplication.

P_6. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$. These are the left and right hand distributive laws respectively.

It may be observed that the words addition and multiplication do not necessarily refer to the familiar definitions given in the case of real numbers.

<u>Notation</u>.--Suppose a and b are arbitrary elements in a ring R such that "-b" is the additive inverse of b. For purposes of notation, denote $a + (-b)$ by $a - b$. It follows that $a - (-b) = a + b$.

<u>Definition 1.3</u>.--A ring R with the following additional property is called a commutative ring:

P_7. If a and b are arbitrary elements in R then $a \cdot b = b \cdot a$.

<u>Definition 1.4</u>.--A ring R is said to have a unity r_e if $r_e \cdot a = a \cdot r_e = a$ for every element a in R. This r_e is also called a multiplicative identity. It is unique for any ring.

<u>Definition 1.5</u>.--Let R be a ring, and let W be a subset of R. If W is also a ring, call W a subring of R. Since W is a subset of R, the binary operations defined on W are those operations defined on R restricted to elements of W.

Theorem 1.6.--A non-empty subset W of a ring R is a subring of R if and only if for every w_1, $w_2 \in W$, it is true that $w_1 \cdot w_2 \in W$ and $w_1 - w_2 \in W$.

Proof.--Suppose that for every w_1, $w_2 \in W$, it is true that $w_1 \cdot w_2 \in W$ and $w_1 - w_2 \in W$. Let $w_1 \in S$, then $w_1 - w_1 = w_z \in W$. Thus W contains an additive identity w_z.

Suppose w_1, $w_2 \in W$ then $w_z - w_2 = -w_2 \in W$. Thus $w_1 - (-w_2) = w_1 + w_2 \in W$ and it follows that W is closed under addition. W is also closed under multiplication since $w_1 \cdot w_2 \in W$. Observe that for any w_2 in W, it follows that $-w_2 \in W$ from the above argument. Thus each element in W has an additive inverse.

W inherits the associative laws of addition and multiplication, the commutative law of addition, and the distributive laws from the ring R.

Because W satisfies the above properties, it follows that W is a subring of R.

Now suppose that W is a subring of R. If w_1 and w_2 denote arbitrary elements of W then $w_1 \cdot w_2 \in W$ since W is closed under multiplication, and since $w_2 \in W$ implies $-w_2 \in W$, it follows by the closure of addition in W that $w_1 + (-w_2) = w_1 - w_2 \in W$.

Definition 1.7.--Let S and T denote arbitrary rings. Consider the ordered pairs (s_1, t_1) and (s_2, t_2) where s_1, s_2 and t_1, t_2 are arbitrary elements of S and T respectively. Define $(s_1, t_1) \bullet (s_2, t_2)$ if and only if $s_1 = s_2$ and

$t_1 = t_2$. The same symbol will be used to denote equality in rings S and T, but the notation will indicate the rings in which the equality refers. Now define addition, " $\oplus$ ", and multiplication, " $\odot$ ", of these ordered pairs by $(s_1, t_1) \oplus (s_2, t_2) \circleddash (s_1 +_s s_2, t_1 +_t t_2)$ and $(s_1, t_1) \odot (s_2, t_2) \circleddash (s_1 \cdot_s s_2, t_1 \cdot_t t_2)$ where $+_s$, $\cdot_s$ and $+_t$, $\cdot_t$ are the binary operations defined on S and T respectively.

Suppose $(s_1, t_1) \circleddash (s_2, t_2)$ and $(s_3, t_3) \circleddash (s_4, t_4)$, then $s_1 = s_2$, $t_1 = t_2$, $s_3 = s_4$, and $t_3 = t_4$. Observe that $(s_1, t_1) \oplus (s_3, t_3) \circleddash (s_1 +_s s_3, t_1 +_t t_3) \circleddash (s_2 +_s s_4, t_2 +_t t_4) \circleddash (s_2, t_2) \oplus (s_4, t_4)$. This means that the operation $\oplus$ is well defined.

Again suppose $(s_1, t_1) \circleddash (s_2, t_2)$ and $(s_3, t_3) \circleddash (s_4, t_4)$. It then follows that $(s_1, t_1) \odot (s_3, t_3) \circleddash (s_1 \cdot_s s_3, t_1 \cdot_t t_3) \circleddash (s_2 \cdot_s s_4, t_2 \cdot_t t_4) \circleddash (s_2, t_2) \odot (s_4, t_4)$; therefore the operation $\odot$ is well defined.

Notation.--The notation $+_s$, $\cdot_s$, and $+_t$, $\cdot_t$ for binary operations in the rings S and T respectively will be shortened to $+$ and $\cdot$ for usage in both rings S and T. The symbols used for elements in S or T will make it clear which binary operations are indicated by $+$ and $\cdot$. For examplé, $s_1 \cdot_s s_3$ will be written as $s_1 \cdot s_3$ and $t_1 \cdot_t t_4$ will be written as $t_1 \cdot t_4$.

Definition 1.8.--The set of all ordered pairs (s_1, t_1), where s_1 is a arbitrary element in S and t_1 is an arbitrary element in T, is called the direct sum of S and T and is denoted by $S \dotplus T$.

Notation.--U will be used to denote the direct sum of S and T hereafter. Also, for an ordered pair in U, an "s" with a subscript refers to an element in S and a "t" with a subscript refers to an element in T. These elements are arbitrary unless otherwise specified. For example, s_z, which always refers to the additive identity in the ring S and t_z, which always refers to the additive identity in the ring T are not arbitrary.

Theorem 1.9.--The direct sum U is a ring.

Proof.--Suppose (s_1, t_1) and (s_2, t_2) are arbitrary elements of U. Then $(s_1, t_1) \oplus (s_2, t_2) \ominus (s_1 + s_2, t_1 + t_2) \ominus (s_3, t_3)$ where $s_1 + s_2 = s_3 \in S$ and $t_1 + t_2 = t_3 \in T$ since S and T are closed under addition. So U is closed under the operation $\oplus$.

Observe that $[(s_1, t_1) \oplus (s_2, t_2)] \oplus (s_3, t_3) \ominus (s_1 + s_2, t_1 + t_2) \oplus (s_3, t_3) \ominus ([s_1 + s_2] + s_3, [t_1 + t_2] + t_3) \ominus (s_1 + [s_2 + s_3], t_1 + [t_2 + t_3]) \ominus (s_1, t_1) \oplus (s_2 + s_3, t_2 + t_3) \ominus (s_1, t_1) \oplus [(s_2, t_2) \oplus (s_3, t_3)]$ since the respective additive operations in S and T are associative. So the operation $\oplus$ defined in U is associative.

Also note that $(s_1, t_1) \oplus (s_z, t_z) \ominus (s_1 + s_z, t_1 + t_z) \ominus (s_1, t_1)$, where s_z is the additive identity of S and t_z is the additive identity of T. Thus U has an additive identity (s_z, t_z).

Let (s_1, t_1) be an arbitrary element in U. It follows that $(s_1, t_1) \oplus (-s_1, -t_1) \ominus (s_1 - s_1, t_1 - t_1) \ominus (s_z, t_z)$

where $-s_1$ is the additive inverse of s_1 and $-t_1$ is the additive inverse of t_1. Thus each element in U has an additive inverse.

Observe that $(s_1, t_1) \oplus (s_2, t_2) \ominus (s_1 + s_2, t_1 + t_2) \ominus (s_2 + s_1, t_2 + t_1) \ominus (s_2, t_2) \oplus (s_1, t_1)$ since the respective additive operations in S and T are commutative. Thus $\oplus$ is commutative in U.

Note $(s_1, t_1) \odot (s_2, t_2) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \ominus (s_3, t_3)$ where $s_1 \cdot s_2 = s_3 \in S$ and $t_1 \cdot t_2 = t_3 \in T$ since S and T are closed under their respective multiplicative operations. Thus U is closed under the operation $\odot$.

Also $(s_1, t_1) \odot [(s_2, t_2) \odot (s_3, t_3)] \ominus (s_1, t_1) \odot (s_2 \cdot s_3, t_2 \cdot t_3) \ominus (s_1 \cdot [s_2 \cdot s_3], t_1 \cdot [t_2 \cdot t_3]) \ominus ([s_1 \cdot s_2] \cdot s_3, [t_1 \cdot t_2] \cdot t_3) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \odot (s_3, t_3) \ominus [(s_1, t_1) \odot (s_2, t_2)] \odot (s_3, t_3)$ since the multiplicative operations of both S and T are associative. Thus U is associative under the operation $\odot$.

In addition, $(s_1, t_1) \odot [(s_2, t_2) \oplus (s_3, t_3)] \ominus (s_1, t_1) \odot (s_2 + s_3, t_2 + t_3) \ominus (s_1 \cdot [s_2 + s_3], t_1 \cdot [t_2 + t_3]) \ominus (s_1 \cdot s_2 + s_1 \cdot s_3, t_1 \cdot t_2 + t_1 \cdot t_3) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \oplus (s_1 \cdot s_3, t_1 \cdot t_3) \ominus [(s_1, t_1) \odot (s_2, t_2)] \oplus [(s_1, t_1) \odot (s_3, t_3)]$ since S and T are rings which satisfy the distributive property. This shows that U satisfies the distributive property from the left. Similarly, it follows that the property is satisfied from the right. So U is said to satisfy the distributive property.

Since U satisfies the above properties, it follows that U is a ring.

Notation.--By Theorem 1.9, the inverse of any ordered pair (s_1, t_1) is $(-s_1, -t_1)$ which hereafter will be denoted by $-(s_1, t_1)$.

Theorem 1.10.--The direct sum U is a commutative ring with unity if and only if both S and T are commutative rings with unities.

Proof.--Suppose S and T are commutative rings with unities. Then by Theorem 1.9, U is a ring. Let s_1, s_2 and t_1, t_2 be arbitrary elements of S and T respectively, then $(s_1, t_1) \odot (s_2, t_2) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \ominus (s_2 \cdot s_1, t_2 \cdot t_1) \ominus (s_2, t_2) \odot (s_1, t_1)$ since S and T are commutative rings. Thus U is a commutative ring.

The following observation confirms that U has a unity: $(s_1, t_1) \odot (s_e, t_e) \ominus (s_1 \cdot s_e, t_1 \cdot t_e) \ominus (s_1, t_1)$ where s_e the multiplicative identity of S and t_e is the multiplicative identity of T. Thus (s_e, t_e) is the multiplicative identity of U.

Now suppose U is a commutative ring with unity. Furthermore suppose $(s_1, t_1), (s_2, t_2) \in U$. Then $(s_1, t_1) \odot (s_2, t_2) \ominus (s_2, t_2) \odot (s_1, t_1)$. This means that $(s_1 \cdot s_2, t_1 \cdot t_2) \ominus (s_2 \cdot s_1, t_2 \cdot t_1)$ so that $s_1 \cdot s_2 = s_2 \cdot s_1$ and $t_1 \cdot t_2 = t_2 \cdot t_1$. It follows that S and T are commutative rings.

Note $(s_1, t_1) \odot (s_e, t_e) \ominus (s_1, t_1)$, where (s_e, t_e) is the multiplicative identity of U. This means that $(s_1 \cdot s_e, t_1 \cdot t_e) = (s_1, t_1)$; thus $s_1 \cdot s_e = s_1$ and $t_1 \cdot t_e = t_1$. So S and T have s_e and t_e for multiplicative identities respectively.

Hereafter s_e and t_e will refer to the multiplicative identities in the rings S and T respectively.

Definition 1.11.--A non-empty subset Q of a ring R is called an ideal in R if and only if it has the following properties:

i. If a and b are elements in Q, then a - b is an element in Q.

ii. If a is an element in Q, then for every element r in R, $a \cdot r$ and $r \cdot a$ are elements in Q. Since both $a \cdot r$ and $r \cdot a$ are elements in Q, this is the definition of a two-sided ideal.

Theorem 1.12.--An ideal Q in R is necessarily a subring of R.

Proof.--By i. of Definition 1.11, if a and b are elements in Q then a - b is an element in Q. Using the facts that Q is a subset of R and ii. of Definition 1.11, it follows that if a is an element in Q, then for every c in Q, $a \cdot c$ and $c \cdot a$ are elements in Q. Now by applying Theorem 1.6, it follows that Q is a subring of R.

Example 1.13.--This example shows that the converse of Theorem 1.12 is not true. That is, there exists a subring Q in a ring R which is not an ideal. Let Q be the set of all real numbers of the form $x + y\sqrt{2}$ where x and y are integers. Define addition and multiplication in the usual way. Now Q is a subring in the ring of real numbers. Note $\sqrt{3}\,(x + y\sqrt{2}) = x\sqrt{3} + y\sqrt{3}\sqrt{2}$, but this is not an element in Q. Thus Q is not an ideal because ii. of Definition 1.11 is not satisfied.

CHAPTER II

HOMOMORPHISMS AND ISOMORPHISMS

Definition 2.1.--Let R and M be two rings such that for arbitrary elements a and b in R, there is associated in some determined way, unique image elements a^1 and b^1 in M such that $(a +_r b)^1 = a^1 +_m b^1$ and $(a \cdot_r b)^1 = a^1 \cdot_m b^1$, where $+_r$, $\cdot_r$ and $+_m$, $\cdot_m$ denote additive and multiplicative operations in R and M respectively. This mapping is called a homomorphism of R into M. If every element of M is the image of some element of R, the homomorphism is of R onto M, denoted by $R \sim M$.

Definition 2.2.--If in a homomorphism of a ring R onto a ring M, each element of M is the image of a unique element of R, the homomorphism is said to be an isomorphism, denoted by $R \cong M$. This correspondence between elements is said to be one-to-one.

Theorem 2.3.--The set of all elements in U of the form (s, t_z) where $s \in S$ and t_z is the additive identity of T is a subring U_s^1 of U which is isomorphic to S by the correspondence $s \leftrightarrow (s, t_z)$ or $s^1 = (s, t_z)$, where s^1 denotes the image of s under the proposed mapping.

Proof.--Let s_1, s_2 be arbitrary elements in S then $(s_1, t_z) \odot (s_2, t_z) \ominus (s_1 \cdot s_2, t_z \cdot t_z) \ominus (s_3, t_z)$, where $s_1 \cdot s_2 = s_3 \in S$ since S is closed under multiplication.

Thus it follows that $(s_1, t_z) \odot (s_2, t_z) \in U_s^1$.

Observe that $(s_1, t_z) \oplus (-s_2, -t_z) \circleddash (s_1, t_z) \ominus (s_2, t_z) \circleddash (s_1 - s_2, t_z - t_z) \circleddash (s_1 - s_2, t_z) \in U_s^1$ since the additive inverse of each element in S is contained in S and also because S is closed under addition. Thus it follows that $(s_1, t_z) \ominus (s_2, t_z) \in U_s^1$, and it can now be said that U_s^1 is a subring of U by applying Theorem 1.6.

An arbitrary element s in S has (s, t_z) for its image, so the mapping is into. Also, an arbitrary element (s, t_z) in U_s^1 has s for its preimage, so the mapping is onto.

Now let $s_1, s_2 \in S$ where $s_1 \neq s_2$. Assume that these two different elements have the same image. Then $s_1 \to (s_1, t_z)$ and $s_2 \to (s_1, t_z)$, but $s_2 \to (s_2, t_z)$ by the correspondence. Thus $(s_1, t_z) \circleddash (s_2, t_z)$, and it follows that $s_1 = s_2$, a contradiction to the assumption that the elements were different. Thus no two different elements in S have the same image in U_s^1 which makes the mapping one-to-one.

Note $(s_1 + s_2)^1 \circleddash (s_1 + s_2, t_z) \circleddash (s_1 + s_2, t_z + t_z) \circleddash (s_1, t_z) \oplus (s_2, t_z) \circleddash s_1^1 \oplus s_2^1$. Similarly, $(s_1 \cdot s_2)^1 \circleddash (s_1 \cdot s_2, t_z) \circleddash (s_1 \cdot s_2, t_z \cdot t_z) \circleddash (s_1, t_z) \odot (s_2, t_z) \circleddash s_1^1 \odot s_2^1$. The above equalities makes the mapping a homomorphism. So this one-to-one homomorphism means that U_s^1 is isomorphic to S.

<u>Theorem 2.4</u>.--The set of all elements of U of the form (s_z, t) where s_z is the additive identity of S and $t \in T$ is a subring U_t^1 of U which is isomorphic to T by the

correspondence $t \leftrightarrow (s_z, t)$ or $t^1 = (s_z, t)$ where t^1 denotes the image of t under the proposed mapping.

The proof of this theorem follows from the argument used in the proof of Theorem 2.3 upon replacing elements of the form (s, t_z) by elements of the form (s_z, t) and replacing U_s^1 by U_t^1.

<u>Theorem 2.5</u>.--The subring U_s^1 of U consisting of all elements in U of the form (s_1, t_z) where $s_1 \in S$ and t_z is the additive identity of T, is an ideal in U.

<u>Proof</u>.--By Theorem 2.3 U_s^1 is a subring of U, therefore if (s_1, t_z), (s_2, t_z) are arbitrary elements of U_s^1 then $(s_1, t_z) \ominus (s_2, t_z) \in U_s^1$ by Theorem 1.6. Thus property i. of Definition 1.11 is satisfied.

Suppose (s_1, t_z) is an arbitrary element of U_s^1 and (s_2, t_2) is an arbitrary element of U. Then $(s_1, t_z) \odot (s_2, t_2) \circleddash (s_1 \cdot s_2, t_z \cdot t_2) \circleddash (s_1 \cdot s_2, t_z) \in U_s^1$ since S is closed under multiplication. Similarly $(s_2, t_2) \odot (s_1, t_z) \circleddash (s_2 \cdot s_1, t_2 \cdot t_z) \circleddash (s_2 \cdot s_1, t_z) \in U_s^1$, which satisfies property ii. of Definition 1.11. It then follows that U_s^1 is an ideal.

<u>Theorem 2.6</u>.--The subring U_t^1 of U consisting of all elements in U of the form (s_z, t_1) where s_z is the additive identity of S and $t_1 \in T$, is an ideal in U.

The proof of this theorem follows in a manner similar to the proof of Theorem 2.5.

<u>Theorem 2.7</u>.--The correspondence $(s_1, t_1) \rightarrow (s_1, t_z)$ is

a homomorphism of U onto U_s^1. The elements of U which correspond to the zero element of U_s^1 are the elements of U_t^1.

Proof.--An arbitrary element (s_1, t_1) in U has (s_1, t_z) for its image, so the mapping is into. Also, an arbitrary element (s_1, t_z) in U_s^1 has (s_1, t_1) for its preimage where t_1 represents any element of T. So the mapping is onto.

Let (s_1, t_1) and (s_2, t_2) denote arbitrary elements in U, then $(s_1, t_1)^1 \circeq (s_1, t_z)$ and $(s_2, t_2)^1 \circeq (s_2, t_z)$. It follows that $(s_1, t_1)^1 \oplus (s_2, t_2)^1 \circeq (s_1, t_z) \oplus (s_2, t_z) \circeq (s_1 + s_2, t_z + t_z) \circeq (s_1 + s_2, t_z)$. Note also that $[(s_1, t_1) \oplus (s_2, t_2)]^1 \circeq (s_1 + s_2, t_1 + t_2)^1 \circeq (s_1 + s_2, t_z)$. Therefore it can be said that $[(s_1, t_1) \oplus (s_2, t_2)]^1 \circeq (s_1, t_1)^1 \oplus (s_2, t_2)^1$. Similarly $(s_1, t_1)^1 \odot (s_2, t_2)^1 \circeq (s_1, t_z) \odot (s_2, t_z) \circeq (s_1 \cdot s_2, t_z \cdot t_z) \circeq (s_1 \cdot s_2, t_z)$, and $[(s_1, t_1) \odot (s_2, t_2)]^1 \circeq (s_1 \cdot s_2, t_1 \cdot t_2)^1 \circeq (s_1 \cdot s_2, t_z)$. Thus the mapping is a homomorphism of U onto U_s^1.

The zero element of U_s^1 is (s_z, t_z). By the correspondence, its preimage is any element of the form (s_z, t_1) where $t_1 \in T$. But (s_z, t_1) represents any element of U_t^1. Thus the elements of U which correspond to the zero element of U_s^1 are the elements of U_t^1.

Theorem 2.8.--The correspondence $(s_1, t_1) \rightarrow s_1$ is a homomorphism of U onto S and the correspondence $(s_1, t_1) \rightarrow t_1$ is a homomorphism of U onto T.

Proof.--By Theorem 2.7 the correspondence $(s_1, t_1) \rightarrow$

(s_1, t_z) is a homomorphism of U onto U_s^1, and by Theorem 2.3 U_s^1 is isomorphic to S by the correspondence $(s_1, t_z) \rightarrow s_1$. Because of the relationship between homomorphic and isomorphic sets, it can be concluded that the correspondence $(s_1, t_1) \rightarrow s_1$ is a homomorphism of U onto S. That is, if A is homomorphic to B, and B is isomorphic to C then A is homomorphic to C. It may be similarly concluded that the correspondence $(s_1, t_1) \rightarrow t_1$ is a homomorphism of U onto T.

Definition 2.9.--The ideal U_s^1 in the ring U defines a partition of U into sets which are called residue classes modulo U_s^1. Two elements (s_1, t_1), (s_2, t_2) in U are in the same residue class modulo U_s^1 if $(s_1, t_1) \ominus (s_2, t_2) \in U_s^1$. In this case, (s_1, t_1) is said to be congruent to (s_2, t_2) modulo U_s^1, and this is written $(s_1, t_1) \equiv (s_2, t_2) \bmod U_s^1$. Express this set of classes by U/U_s^1.

Notation.--If $(s_1, t_1) \ominus (s_2, t_2) \in U_s^1$ then $(s_1, t_1) \ominus (s_2, t_2) \ominus (s_1 - s_2, t_z)$. Since $t_1 - t_2 = t_z$, it follows that $t_1 = t_2$. This means that any residue class modulo U_s^1 can be expressed as $\{(s_1, t_1) \mid s_1$ is an arbitrary element in S, and t_1 is a fixed element in $T\}$. Since each $t \in T$ determines a unique residue class modulo U_t^1, denote this residue class by $\{S, t\}$.

Definition 2.10.--For the residue classes $\{S, t_1\}$ and $\{S, t_2\}$ modulo U_s^1, define $\{S, t_1\} \ominus \{S, t_2\}$ if and only if $t_1 = t_2$. Now define addition, $\oplus$, and multiplication, $\odot$, of these residue classes modulo U_s^1 by $\{S, t_1\} \oplus \{S, t_2\} \ominus$

$\{S, t_1 + t_2\}$ and $\{S, t_1\} \diamond \{S, t_2\} \oplus \{S, t_1 \cdot t_2\}$ respectively where + and · are binary operations defined on T. Here $\oplus$ and $\diamond$ represent well-defined binary operations.

Theorem 2.11.--The residue classes of U modulo U_s^1 form a ring which has $\{S, t_z\}$ for its additive identity.

Proof.--By using the fact that T is a ring, it follows that U/U_s^1 is a ring.

Definition 2.12.--In a manner similar to Definition 2.9 the set of residue classes U/U_t^1 may be defined. Here each s_1 in S determines a unique residue class of U modulo U_t^1 denoted by $\{s_1, T\}$.

Notation.--The symbols $\oplus$ and $\diamond$ will be used to denote the binary operations of U/U_t^1 as well as those of U/U_s^1.

Theorem 2.13.--There exists an isomorphism between U_s^1 and the residue class ring U/U_t^1, that is $U_s^1 \cong U/U_t^1$. The correspondence to be considered is $(s_1, t_z) \leftrightarrow \{s_1, T\}$.

Proof.--An arbitrary element (s_1, t_z) in U_s^1 has $\{s_1, T\}$ for its image, thus the mapping is into. Similarly, an arbitrary residue class $\{s_1, T\}$ in U has (s_1, t_z) as a pre-image. Thus the mapping is onto.

Arbitrarily choose two different elements (s_1, t_z) and (s_2, t_z) in U_s^1. Assume that they have the same image, then $(s_1, t_z) \rightarrow \{s_1, T\}$ and $(s_2, t_z) \rightarrow \{s_1, T\}$. But $(s_2, t_z) \rightarrow \{s_2, T\}$ by the correspondence, so $\{s_1, T\} \oplus \{s_2, T\}$ and it follows that $s_1 = s_2$. Therefore $(s_1, t_z) \ominus (s_2, t_z)$ is a contradiction that the elements were different. So no two different elements have the same image which makes the mapping one-to-one.

Observe that $[(s_1, t_z) \oplus (s_2, t_z)]^1 \Leftrightarrow (s_1 + s_2, t_z + t_z)^1 \Leftrightarrow (s_1 + s_2, t_z)^1 \Leftrightarrow \{s_1 + s_2, T\} \Leftrightarrow \{s_1, T\} \Leftrightarrow \{s_2, T\} \Leftrightarrow (s_1, t_z)^1 \oplus (s_2, t_z)^1$.

Also, $[(s_1, t_z) \odot (s_2, t_z)]^1 \Leftrightarrow (s_1 \cdot s_2, t_z \cdot t_z)^1 \Leftrightarrow (s_1 \cdot s_2, t_z)^1 \Leftrightarrow \{s_1 \cdot s_2, T\} \Leftrightarrow \{s_1, T\} \odot \{s_2, T\} \Leftrightarrow (s_1, t_z)^1 \odot (s_2, t_z)^1$.

Thus there exists a one-to-one onto homomorphism, therefore it can be said that U_s^1 is isomorphic to U/U_t^1.

<u>Theorem 2.14</u>.--The correspondence $(s_1, t_1) \rightarrow (s_z, t_1)$ is a homomorphism of U onto U_t^1, and U_s^1 is an ideal in U which maps onto the zero of U_t^1. Furthermore, there exists an isomorphism between U_t^1 and the residue class U/U_s^1, that is $U_t^1 \cong U/U_s^1$.

The proof of this theorem is similar to the proof of Theorems 2.7 and 2.13.

<u>Example 2.15</u>.--Consider the ring I_6 of integers modulo 6 whose elements are 0_6^1, 1_6^1, 2_6^1, 3_6^1, 4_6^1, and 5_6^1. Consider also rings I_2 and I_3 whose elements are 0_2^1, 1_2^1 and 0_3^1, 1_3^1, 2_3^1 respectively. It follows that $I_2 \dotplus I_3 = \{(0_2^1, 0_3^1), (1_2^1, 0_3^1), (0_2^1, 1_3^1), (1_2^1, 1_3^1), (0_2^1, 2_3^1), (1_2^1, 2_3^1)\}$. Note that the following correspondence between elements of I_6 and those of $I_2 \dotplus I_3$ is an isomorphism:

$$0_6^1 \leftrightarrow (0_2^1, 0_3^1)\ ,\ 1_6^1 \leftrightarrow (1_2^1, 1_3^1)\ ,\ 2_6^1 \leftrightarrow (0_2^1, 2_3^1)$$
$$3_6^1 \leftrightarrow (1_2^1, 0_3^1)\ ,\ 4_6^1 \leftrightarrow (0_2^1, 1_3^1)\ ,\ 5_6^1 \leftrightarrow (1_2^1, 2_3^1).$$

<u>Example 2.16</u>.--Consider the direct sum of the two rings I_2 and I_4. Denote the elements of I_2 by 0_2^1, 1_2^1 and those of I_4 by 0_4^1, 1_4^1, 2_4^1, 3_4^1. It follows that $I_2 \dotplus I_4 = \{(0_2^1, 0_4^1), (0_2^1, 1_4^1), (0_2^1, 2_4^1), (0_2^1, 3_4^1), (1_2^1, 0_4^1), (1_2^1, 1_4^1), (1_2^1, 2_4^1), (1_2^1, 3_4^1)\}$. The set consisting of the elements $(0_2^1, 0_4^1)$, $(1_2^1, 1_4^1)$, $(0_2^1, 2_4^1)$, and $(1_2^1, 3_4^1)$ is a subring of the direct sum. It may be noted that the following set is a different subring of the direct sum: $\{(0_2^1, 0_4^1), (0_2^1, 2_4^1), (1_2^1, 0_4^1), (1_2^1, 2_4^1)\}$.

CHAPTER III

PROPERTIES OF RINGS

Theorem 3.1.--The correspondence $(s_1, t_1) \leftrightarrow (t_1, s_1)$ between the elements of $S \dot{+} T$ and $T \dot{+} S$ is an isomorphism.

Proof.--An arbitrary element (s_1, t_1) in $S \dot{+} T$ has (t_1, s_1) in $T \dot{+} S$ for its image, so the mapping is into. Also an arbitrary element (t_1, s_1) in $T \dot{+} S$ has (s_1, t_1) in $S \dot{+} T$ for its preimage. Thus the mapping is onto.

Arbitrarily choose two different elements (s_1, t_1) and (s_2, t_2) in $S \dot{+} T$. Assume that these elements have the same image. Then $(s_1, t_1) \rightarrow (t_1, s_1)$ and $(s_2, t_2) \rightarrow (t_1, s_1)$, but $(s_2, t_2) \rightarrow (t_2, s_2)$ by the correspondence. Thus $(t_1, s_1) \circeq (t_2, s_2)$, and it follows that $t_1 = t_2$ and $s_1 = s_2$. This means that $(s_1, t_1) \circeq (s_2, t_2)$, a contradiction to the assumption that the elements were different. Thus no two different elements have the same image which makes the mapping one-to-one.

Note $[(s_1, t_1) \oplus (s_2, t_2)]^1 \circeq [(s_1 + s_2, t_1 + t_2)]^1 \circeq (t_1 + t_2, s_1 + s_2) \circeq (t_1, s_1) \oplus (t_2, s_2) \circeq (s_1, t_1)^1 \oplus (s_2, t_2)^1$. Similarly $[(s_1, t_1) \odot (s_2, t_2)]^1 \circeq [(s_1 \cdot s_2, t_1 \cdot t_2)]^1 \circeq (t_1 \cdot t_2, s_1 \cdot s_2) \circeq (t_1, s_1) \odot (t_2, s_2) \circeq (s_1, t_1)^1 \odot (s_2, t_2)^1$. Thus there exists a one-to-one onto homomorphism between $S \dot{+} T$ and $T \dot{+} S$. This completes the proof that $S \dot{+} T \cong T \dot{+} S$.

Remark.--Since $S \dotplus T \cong T \dotplus S$, any theory concerning the ring $S \dotplus T$ will correspond to the theory concerning the ring $T \dotplus S$. Therefore one may speak of the direct sum of two rings without regard to the order of the direct sum.

Definition 3.2.--A commutative ring F with more than one element and having a unity is called a field if it has the additional property:

1. For every non-zero element "a" in F, there exists an "x" in F such that the multiplication of "a" by "x" yields the unity. This element "x" is called the multiplicative inverse of "a".

Theorem 3.3.--Assume U is the direct sum of rings S and T where U is a commutative ring with unity. Also assume that both of S and T has more than one element. It then follows that U cannot be a field.

Proof.--Suppose S has a non-zero element s_1. Then (s_1, t_z) is a non-zero element in U. The multiplicative inverse of (s_1, t_z) is the ordered pair with the multiplicative inverse of s_1 in the first position and the multiplicative inverse of t_z in the second position. Hence (s_1, t_z) does not have a multiplicative inverse because t_z, the additive identity of T, does not have a multiplicative inverse in T.

Definition 3.4.--Suppose a, b, and c are elements in a ring R whose additive identity is represented by r_z. If there exists a non-zero element b such that $a \cdot b = r_z$ or a non-zero element c such that $c \cdot a = r_z$, then a is said to be a divisor of zero. A non-zero divisor of zero is called a proper divisor of zero.

Theorem 3.5.--If both S and T have more than one element, their direct sum U has proper divisors of zero.

Proof.--Suppose S and T are rings such that s_1 represents a non-zero element of S and t_1 represents a non-zero element of T. Observe that $(s_1, t_z) \odot (s_z, t_1) \ominus (s_1 \cdot s_z, t_1 \cdot t_z) \ominus (s_z, t_z)$. Both (s_1, t_z) and (s_z, t_1) are proper zero divisors.

Definition 3.6.--A commutative ring R with more than one element and having a unity is called an integral domain if it has the following additional property:

i. If $r_1, r_2 \in R$ such that $r_1 \cdot r_2 = r_z$ then $r_1 = r_z$ or $r_2 = r_z$, where r_z is the additive identity of R.

Theorem 3.7.--Suppose U is the direct sum of rings S and T where U is a commutative ring with unity. Also suppose each of S and T has more than one element. Then U is not an integral domain.

This proof is contained in the proof of Theorem 3.5.

Example 3.8.--This example shows that if one of S and T has only one element and the other is an integral domain, then their direct sum U has no proper divisors of zero. It then follows that U is an integral domain in this example.

Suppose S is the zero ring which contains only the element 0. Let T be the ring of integers. Then if $(0, t_1)$,

$(0, t_2) \in U$ such that $(0, t_1) \odot (0, t_2) \ominus (0, 0)$ where t_1 and t_2 are arbitrary integers, then $(0, t_1) \ominus (0, 0)$ or $(0, t_2) \ominus (0, 0)$ because $t_1 \cdot t_2 = 0$ implies $t_1 = 0$ or $t_2 = 0$.

Definition 3.9.--If for an arbitrary ring R, there exists a positive integer n such that "a" added to itself n times equals the additive identity r_z in R, denoted by $na = r_z$, for every element "a" in R, the least such n is called the characteristic of R, and R is said to have positive characteristic. If no such integer exists, R is said to have characteristic zero.

Definition 3.10.--Suppose m and n are positive integers such that $km = pn = r$ where k and p are the smallest such positive integers such that the equation is true. The positive integer r is called the least common multiple of m and n, and this is denoted by l. c. m. $\{m,n\} = r$.

Theorem 3.11.--If S has characteristic $m > 0$ and T has characteristic $n > 0$, then U has characteristic l. c. m. $\{m,n\} = r$.

Proof.--Since l. c. m. $\{m,n\} = r$, there exists positive integers k and p such that $km = pn = r$. Since S has characteristic m and T has characteristic n, then for any element s_1 in S, $ms_1 = s_z$, and for any element t_1 in T, $nt_1 = t_z$. Thus for any element (s_1, t_1) in U, it follows that $r(s_1, t_1) \ominus (rs_1, rt_1) \ominus (kms_1, pnt_1) \ominus (ks_z, pt_z) \ominus (s_z, t_z)$. Thus the characteristic of U is either less than or equal to r.

Suppose that the characteristic of U is less than r. Then there exists an integer $0 < j < r$ such that for every element (s_1, t_1) in U, $j(s_1, t_1) \circledcirc (js_1, jt_1) \circledcirc (s_z, t_z)$. Thus $js_1 = s_z$ for all elements s_1 in S and $jt_1 = t_z$ for all elements t_1 in T. Since j is not the l. c. m. $\{m, n\}$, it follows that m does not divide j or n does not divide j. Suppose that m does not divide j, then $m \neq j$, and since m is the characteristic of S, it follows that j is not less than m. This means that $m < j$, thus $j = mz + w$ where z is an integer and $0 < w < m$. It follows that $js_1 = (mz + w) s_1 = mzs_1 + ws_1 = zms_1 + ws_1 = zs_z + ws_1 = s_z + ws_1 = ws_1$ and $js_1 = s_z$, thus $ws_1 = s_z$. But since $w < m$, $ws_1 = s_z$ for all s_1 in S contradicts that m is the characteristic of S, therefore m must divide j. In a similar manner, it follows that n must divide j; hence the least common multiple of m and n divides j. This contradicts $j < r$, hence the characteristic of U is not less than r. The characteristic of U is therefore equal to the l. c. m. $\{m, n\} = r$.

<u>Theorem 3.12</u>.--If S has characteristic zero, then U has characteristic zero.

<u>Proof</u>.--Since S has characteristic zero, it follows that for each arbitrary integer $m > 0$, there exists an element s_1 in S such that $ms_1 \neq s_z$, (s_1 may depend on m). Thus for any integer $m > 0$, there exists an element (s_1, t_1) in U such that $m(s_1, t_1) \circledcirc (ms_1, mt_1) \not\circledcirc (s_z, t_z)$, which means that U has characteristic zero.

Theorem 3.13.--If U has characteristic $r > 0$, then S has characteristic $m > 0$ and T has characteristic $n > 0$ such that l. c. m. $\{m, n\} = r$.

Proof.--Since U has characteristic $r > 0$, it follows that for any element (s_1, t_1) in U, $r(s_1, t_1) \ominus (rs_1, rt_1) \ominus (s_z, t_z)$. This means that for any element s_1 in S, $rs_1 = s_z$ and for any element t_1 in T, $rt_1 = t_z$. It follows that $0 < m \leq r$ and $0 < n \leq r$.

Suppose that r is not a multiple of m, then $r = am + b$ where a and b are positive integers and $0 < b < m$. For each s_1 in S, it follows that $rs_1 = [am + b]\ s_1 = ams_1 + bs_1 = as_z + bs_1 = bs_1$ and $rs_1 = s_z$. Thus $bs_1 = s_z$ for any element s_1 in S which contradicts that m is the characteristic of S. This means that r is a multiple of m, and in a similar manner it follows that r is a multiple of n.

Now suppose that there exists a common multiple j of m and n less than r. Then for integers a^1 and b^1, $a^1m = b^1n = j < r$. Observe that $j(s_1, t_1) \ominus (js_1, jt_1) \ominus (a^1ms_1, b^1nt_1) \ominus (a^1s_z, b^1t_z) \ominus (s_z, t_z)$ for any element (s_1, t_1) in U. But this contradicts that the characteristic of U is r. Thus since r is a common multiple of m and n and since there is no common multiple of m and n less than r, it follows that $r =$ l. c. m. $\{m, n\}$.

Theorem 3.14.--If U has characteristic zero, then either S has characteristic zero or T has characteristic zero.

Proof.--If S has characteristic $m > 0$ and T has characteristic n > 0 then U has characteristic l. c. m. $\{m, n\}$ by Theorem 3.11. This contradicts the zero characteristic of U; hence either S or T has zero characteristic.

CHAPTER IV

PROPERTIES OF IDEALS

<u>Theorem 4.1</u>.--Suppose U' is an ideal in U. If S has a unity s_e and T has a unity t_e then there exist ideals S' and T' in S and T respectively such that $U' = S' \dotplus T'$.

<u>Proof</u>.--By a previous theorem, the correspondence $(s_1, t_1) \to s_1$ is a homomorphism of U onto S and $(s_1, t_1) \to t_1$ is a homomorphism of U onto T. Let S' be the image of U' in the first homomorphism and T' be the image of U' in the second homomorphism. The following argument for the first homomorphism shows that S' is an ideal.

Choose two arbitrary elements s_1, s_2 in S'. Then there exists $t_1, t_2 \in T$ such that $(s_1, t_1), (s_2, t_2) \in U'$ with $(s_1, t_1) \to s_1$ and $(s_2, t_2) \to s_2$. Observe that a preimage of $s_1 - s_2$ is $(s_1 - s_2, t_1 - t_2) \circledequal (s_1, t_1) \ominus (s_2, t_2) \in U'$ since U' is an ideal, so $s_1 - s_2 \in S'$. Now arbitrarily choose $s_4 \in S$ and $s_5 \in S'$. Then there exists $(s_4, t_4) \in U$ and $(s_5, t_5) \in U'$ with $t_4, t_5 \in T$ such that $(s_4, t_4) \to s_4$ and $(s_5, t_5) \to s_5$. Note that a preimage of $s_4 \cdot s_5$ is $(s_4 \cdot s_5, t_4 \cdot t_5) \circledequal (s_4, t_4) \odot (s_5, t_5) \in U'$ because U' is an ideal. Similarly $(s_5 \cdot s_4, t_5 \cdot t_4) \circledequal (s_5, t_5) \odot (s_4, t_4) \in U'$. Thus $s_4 \cdot s_5 \in S'$ and $s_5 \cdot s_4 \in S'$. It can now be said that S' is an ideal.

A similar argument for the second homomorphism yields that T' is also an ideal.

Once again consider the ideal S' in S. If s_1 is any element of S', then there exists an element in U' with s_1 in the first position, say $(s_1, t_1) \in U'$. It follows that $(s_1, t_1) \odot (s_e, t_z) \circleddash (s_1, t_z) \in U'$. Thus U' contains all elements in U of the form (s_1, t_z). In a similar manner, U' contains all elements of the form (s_z, t_1) where $t_1 \in T'$. Thus U' contains all sums of these elements since U' is an ideal. That is, U' contains $S' \dotplus T'$.

Now let (s_1, t_1) be an arbitrary element in U' then $s_1 \in S'$ and $t_1 \in T'$ by the construction of S' and T'. Hence U' is contained in $S' \dotplus T'$. Thus it can be concluded that $U' = S' \dotplus T'$.

<u>Remark</u>.--In all of the following theorems where S' and T' are ideals in S and T respectively, both S and T are assumed to have unities. This will insure that the ideal U' is the direct sum of S' and T', and the symbol U' will be reserved for this direct sum.

<u>Theorem 4.2</u>--S' is an ideal in S and T' is an ideal in T if and only if U' is an ideal in U.

<u>Proof</u>.--Suppose S' is an ideal in S and T' is an ideal in T. Let $(s_1, t_1), (s_2, t_2) \in U'$, then $(s_1, t_1) \ominus (s_2, t_2) \circleddash (s_1 - s_2, t_1 - t_2) \in U'$ because S' and T' are ideals closed under their respective subtractions.

Now let $(s_1, t_1) \in U$ and $(s_2, t_2) \in U'$, then $(s_1, t_1) \odot$

$(s_2, t_2) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \in U'$ because S' and T' are ideals. Similarly $(s_2, t_2) \odot (s_1, t_1) \in U'$, so U' is an ideal in U. The converse follows from Theorem 4.1.

Definition 4.3.--If A and B are sets with the property that every element of A is also an element of B, then A is called a subset of B and the relationship is denoted by $A \subseteq B$. If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B, and the notation $A \subset B$ is used.

Lemma 4.4.--Suppose that both S' and S" are subrings in S and that both T' and T" are subrings in T. If $S' \subset S''$ and $T' \subseteq T''$ then $S' \dotplus T' \subset S'' \dotplus T''$.

Proof.--Let $S' = \{s' \mid s' \in S'\}$ and $T' = \{t' \mid t' \in T'\}$ then $S' \dotplus T' = \{(s', t') \mid s' \in S', t' \in T'\}$. Now let $T'' = \{t' \mid t' \in T'\} \cup \{t'' \mid t'' \in T'', t'' \notin T'\}$ then $S' \dotplus T'' = \{(s', t') \mid s' \in S', t' \in T'\} \cup \{(s', t'') \mid s' \in S', t'' \in T'', t'' \notin T'\}$. Any arbitrary element (s_1', t_1') in $S' \dotplus T'$ is also in $S' \dotplus T''$ since $s_1' \in S'$ and $t_1' \in T''$. Thus it follows that $S' \dotplus T' \subseteq S' \dotplus T''$.

In a similar manner it follows that $S' \dotplus T'' \subseteq S'' \dotplus T''$. But there exists an element s_1'' in S" such that $s_1'' \notin S'$ since $S' \subset S''$. Thus for any element t_1'' in T", $(s_1'', t_1'') \in S'' \dotplus T''$ but $(s_1'', t_1'') \notin S' \dotplus T''$. It follows that $S' \dotplus T'' \subset S'' \dotplus T''$. Thus since $S' \dotplus T' \subseteq S' \dotplus T''$ and $S' \dotplus T'' \subset S'' \dotplus T''$, it can be concluded that $S' \dotplus T' \subset S'' \dotplus T''$.

Lemma 4.5.--Suppose that both S' and S" are subrings in S and that T' is a subring in T. If $S' \dotplus T' \subset S'' \dotplus T'$, then

$S' \subset S''$.

Proof.--Let $S' \dot{+} T' = \{(s', t') \mid s' \in S', t' \in T'\}$ and $S'' \dot{+} T' = \{(s'', t') \mid s'' \in S'', t' \in T'\}$. Since $S' \dot{+} T' \subset S'' \dot{+} T'$, it follows that for any element (s_1', t_1') in $S' \dot{+} T'$, (s_1', t_1') is also in $S'' \dot{+} T'$. This implies that for any element s_1' in S', $s_1' \in S''$; hence $S' \subseteq S''$. Also since $S' \dot{+} T' \subset S'' \dot{+} T'$, there exists an element (s_1'', t_1') in $S'' \dot{+} T'$ such that $(s_1'', t_1') \notin S' \dot{+} T'$. But since $t_1' \in T'$ it follows that $s_1'' \notin S'$, thus $S' \subset S''$.

Definition 4.6.--Suppose R is a ring which contains a set of ideals A_i, for $i = 1, 2, \ldots$, such that $A_1 \subset A_2 \subset \ldots$. These ideals A_i are said to form a strictly increasing sequence.

Definition 4.7.--If for a ring R, every strictly increasing sequence of ideals contains only a finite number of ideals, then the ascending chain condition is said to hold in R.

Theorem 4.8.--Suppose S and T are rings which have unities. The ascending chain condition holds for S and T if and only if it holds for $U = S \dot{+} T$.

Proof.--Suppose the ascending chain condition holds for U. Also suppose that the ascending chain condition does not hold for S. Then there exists a strictly increasing sequence of ideals in S denoted by $S_1 \subset S_2 \subset \ldots$ which is not finite. Observe that $S_i \dot{+} T$ is an ideal in U by Theorem 4.2 for $i = 1, 2, \ldots$, and that $S_1 \dot{+} T \subset S_2 \dot{+} T \subset \ldots$ is an infinitely strictly increasing sequence in U, where the containment follows from Lemma 4.4. But this contradicts the assumption that the ascending chain condition holds for U. An argument similar to the above could be made if T were assumed to not satisfy the ascending chain condition instead of S. Thus it follows that the ascending chain condition holds for both S and T if it holds for U.

Now assume that the ascending chain condition holds for both S and T. Also suppose that it does not hold for U. Then there exists a strictly increasing sequence of ideals in U denoted by $U_1 \subset U_2 \subset \ldots$, which is not finite. By Theorem 4.1, there exist ideals S_i and T_i such that $U_i = S_i \dotplus T_i$ for i = 1, 2, ... This means that either $S_1, S_2, \ldots$ such that $S_1 \subset S_2 \subset \ldots$ is infinite or $T_1, T_2, \ldots$ such that $T_1 \subset T_2 \subset \ldots$ is infinite, where these containments follow from Lemma 4.5. But this contradicts the assumption that both of S and T satisfy the ascending chain condition. Hence it can be concluded that U satisfies the ascending chain condition if S and T satisfy the ascending chain condition.

<u>Definition 4.9</u>.--Suppose R is a ring which contains a set of ideals A_i for i = 1, 2,..., such that each subsequent ideal is properly contained in the preceding one, denoted by $A_1 \supset A_2 \supset \ldots$. These ideals A_i are said to form a strictly decreasing sequence.

<u>Definition 4.10</u>.--If for a ring R, every strictly decreasing sequence of ideals contains only a finite number of ideals, then the descending chain condition is said to

hold in R.

Theorem 4.11.--Suppose S and T are rings which have a unity. The descending chain condition holds for S and T if and only if it holds for $U = S \dotplus T$.

Proof.--Exchange the words descending for ascending and decreasing for increasing, and the proof of Theorem 4.8 can be used here.

Definition 4.12.--Suppose R is a ring. An ideal A in R is said to be maximal if $A \neq R$ and there exists no ideals between A and R. Thus if A is a maximal ideal and K is an ideal such that $A \subseteq K \subseteq R$, then either $K = A$ or $K = R$.

Theorem 4.13.--An ideal $U' = S' \dotplus T'$ is maximal if and only if either $S' = S$ and T' is maximal in T or $T' = T$ and S' is maximal in S.

Proof.--Suppose U' is a maximal ideal in U. Also suppose that $T' = T$ and S' is not maximal in S. Then there exists an ideal S'' such that $S' \subset S'' \subset S$, and by Theorem 4.2, $S'' \dotplus T'$ is an ideal in U. Furthermore by Lemma 4.4, $S' \dotplus T' \subset S'' \dotplus T' \subset U$. But this contradicts the assumption that $S' \dotplus T' = U'$ is a maximal ideal in U.

Secondly suppose U' is a maximal ideal in U. Also assume that $T' \neq T$ and S' is maximal in S. Hence $T' \subset T$ and it follows that $S' \dotplus T' \subset S' \dotplus T \subset U$ by Lemma 4.4. But this contradicts the assumption that $S' \dotplus T' = U'$ is a maximal ideal in U.

Thirdly suppose U' is a maximal ideal in U, and also assume that $T' \neq T$ and S' is not maximal in S. Then there exists an ideal S'' such that $S' \subset S'' \subset S$, and it again follows by Lemma 4.4 that $S' \dotplus T' \subset S'' \dotplus T' \subset U$. But this contradicts the assumption that $S' \dotplus T' = U'$ is a maximal ideal in U.

In a similar manner, it can be shown that U' is not maximal in each of the following three cases: (1) $S' = S$, T' is not maximal in T; (2) $S' \neq S$, T' is maximal in T; and (3) $S' \neq S$, T' is not maximal in T. The only other cases are the conclusions desired. Thus $S' = S$ and T' is maximal in T or $T' = T$ and S' is maximal in S if U' is a maximal ideal in U.

Now suppose $T' = T$ and S' is a maximal ideal in S. Also suppose U' is not a maximal ideal in U. Then there exists an ideal U'' such that $U' \subset U'' \subset U$. By Theorem 4.1, U'' may be expressed as a direct sum, say $U'' = S'' \dotplus T''$. Thus it follows that $S' \dotplus T' \subset S'' \dotplus T'' \subset S \dotplus T$. Since $T' = T$, it may be concluded that $T'' = T$. This means $S' \subset S'' \subset S$ by Lemma 4.5. But this contradicts the assumption that S' is a maximal ideal in S.

In a similar way, it can be shown that U' is maximal whenever $S' = S$ and T' is maximal in T. Thus U' is a maximal ideal in U if $S' = S$ and T' is maximal in T or $T' = T$ and S' is maximal in S.

Example 4.14.--Let $S = T$ be the set of integers and $S' = T'$ be the even integers. Observe that $U'' = S \dotplus T'$ is an ideal such that $U' \subset U'' \subset U$. It follows that S' and T' are maximal ideals in S and T respectively but U' is not maximal in U.

Definition 4.15.--Suppose R is a commutative ring. An ideal A in R is said to be prime if whenever a product $b \cdot c \in A$ with $b, c \in R$ then $b \in A$ or $c \in A$.

Theorem 4.16.--An ideal $U' = S' \dotplus T'$ is prime if and only if either $S' = S$ and T' is prime in the commutative ring T or $T' = T$ and S' is prime in the commutative ring S.

Proof.--Assume U' is a prime ideal in the ring U. Then one and only one of the following four cases is possible:
(1) $S' = S$, $T' = T$; (2) $S' = S$, $T' \neq T$; (3) $S' \neq S$, $T' = T$;
(4) $S' \neq S$, $T' \neq T$.

Suppose case (1) is true; then both parts of the conclusion of Theorem 4.16 are implied.

Suppose case (2) is true when T' is not prime. Then there exists a product $t_1 \cdot t_2 \in T'$ where t_1 and t_2 are elements in T such that $t_1 \notin T'$ and $t_2 \notin T'$. If $s_1, s_2 \in S'$ then $(s_1 \cdot s_2, t_1 \cdot t_2) = (s_1, t_1) \odot (s_2, t_2) \in U'$ but $(s_1, t_1) \notin U'$ and $(s_2, t_2) \notin U'$ because $t_1 \notin T'$ and $t_2 \notin T'$. This then contradicts the assumption that U' is a prime ideal in U. This situation for (2) is thus impossible.

Now suppose (2) is true when T' is prime; then this case for (2) is one of the conclusions of the theorem.

Now suppose (3) is true when S' is not prime. Then there exists a product $s_1 \cdot s_2 \in S'$ where s_1 and s_2 are arbitrary elements of S such that $s_1 \notin S'$ and $s_2 \notin S'$. Consider (s_1, t_1), $(s_2, t_2) \in U$ where s_1 and s_2 represent the above mentioned elements and $t_1, t_2 \in T'$. Now $(s_1, t_1) \odot (s_2, t_2) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \in U'$ with $(s_1, t_1) \notin U'$ and $(s_2, t_2) \notin U'$ because $s_1 \notin S'$ and $s_2 \notin S'$. But this contradicts the assumption that U' is a prime ideal in U; so this situation for (3) is impossible.

Consider when (3) is true where S' is prime, then this case for (3) is one of the conclusions of the theorem.

Now for case (4), since $S \neq S'$ there exists an element s_1 in S such that $s_1 \notin S'$. There is also an element s_2 in S' such that $s_1 \cdot s_2 \in S'$ since S' is an ideal. Since $T \neq T'$ there exists an element t_1 in T', and there is an element $t_2 \in T$ such that $t_2 \notin T'$. It follows that $t_1 \cdot t_2 \in T'$ because T' is an ideal. Thus $(s_1 \cdot s_2, t_1 \cdot t_2) \ominus (s_1, t_1) \odot (s_2, t_2) \in U'$ but $(s_1, t_1) \notin U'$ since $s_1 \notin S'$ and $(s_2, t_2) \notin U'$ because $t_2 \notin T'$. This means that U' is not a prime ideal, a contradiction which means that case (4) is impossible.

Thus cases (1), (2), and (3) imply the conclusion and case (4) is impossible. This means that the hypothesis of the theorem implies the conclusion; that is, if U' is a prime ideal in U then $S' = S$ and T' is prime in T or $T' = T$ and S' is prime in S.

Now assume that $T' = T$ and S' is a prime ideal in the ring S. Assume also that U' is not prime. Then there exists (s_1, t_1), $(s_2, t_2) \in U$ and $(s_1, t_1) \odot (s_2, t_2) \in U'$ such that

$(s_1, t_1) \notin U'$ and $(s_2, t_2) \notin U'$. It follows that $s_1 \cdot s_2 \in S'$ but $s_1 \notin S'$ and $s_2 \notin S'$ because $T' = T$, thus making $t_1 \in T'$ and $t_2 \in T'$. This contradicts the assumption that S' is a prime ideal, and it can be concluded that U' is a prime ideal, in U.

In a similar way, it can be shown that U' is prime whenever S' = S and T' is prime in T. Thus U' is a prime ideal in U if S' = S and T' is prime in T or T' = T and S' is prime in S.

Example 4.17.--Let S and T be the set of all integers. Then let S' be the set consisting of all multiples of 3 and T' be the set consisting of all multiples of 7. Multiplication to be used is the ordinary multiplication defined for integers. Here both S' and T' are prime ideals. Observe $6 \cdot 4 = 24 \in S'$ with $6 \in S'$, but $4 \notin S'$ and $8 \cdot 7 = 56 \in T'$ with $8 \notin T'$, but $7 \in T'$. Thus $(6 \cdot 4, 8 \cdot 7) \ominus (6, 8) \odot (4, 7) \in U'$ since $6 \cdot 4 \in S'$ and $8 \cdot 7 \in T'$, but $(6, 8) \notin U'$ and $(4, 7) \notin U'$. This means that U' is not a prime ideal, but both S' and T' are prime ideals.

Definition 4.18.--Suppose R is a commutative ring. An ideal A in R is said to be primary if the conditions $a, b \in R$ with $a \cdot b \in A$ and $a \notin A$ implies the existence of an integer $n > 0$ such that $b^n \in A$.

Theorem 4.19.--An ideal $U' = S' \dotplus T'$ is primary if and only if either S' = S and T' is primary in the commutative ring T or T' = T and S' is primary in the commutative ring S.

Proof.--The following two contradictions will be useful later in the proof. First let U' be a primary ideal in the ring U. Suppose S' is not a primary ideal in the ring S and T' is an ideal in the ring T. Then there exists $s_1, s_2 \in S$ such that $s_1 \cdot s_2 \in S'$ with $s_1 \notin S'$ and such that for every integer $n > 0$, $s_2^n \notin S'$. Let (s_1, t_1) and (s_2, t_2) be elements of U, where s_1 and s_2 represent the above mentioned elements and where t_1 and t_2 are arbitrary elements in T'. Now $(s_1, t_1) \odot (s_2, t_2) \ominus (s_1 \cdot s_2, t_1 \cdot t_2) \in U'$ with $(s_1, t_1) \notin U'$ because $s_1 \notin S'$. And for every integer $n > 0$ $(s_2, t_2)^n \ominus (s_2^n, t_2^n) \notin U'$ because $s_2^n \notin S'$ which contradicts the assumption that U' is a primary ideal.

Again let U' be a primary ideal in the ring U. Suppose also that S' is an ideal in the ring S and T' is not a primary ideal in the ring T. In a manner similar to the one used in the first contradiction, it follows that U' cannot be primary, a contradiction to the assumption that U' is primary.

Once again assume U' is a primary ideal in U. Then one and only one of the following four cases is possible: (1) $S' = S$, $T' = T$; (2) $S' = S$, $T' \neq T$; (3) $S' \neq S$, $T' = T$; (4) $S' \neq S$, $T' \neq T$.

Suppose (1) is true; then both parts of the conclusion of the theorem are satisfied.

Suppose (2) is true when T' is not primary. Then a contradiction to the assumption that U' is primary is reached as was shown earlier.

Now suppose (2) is true when T' is primary. Then this case for (2) is one of the conclusions of the theorem.

Now suppose (3) is true when S' is not primary. Then by the result at the beginning of the proof, a contradiction to the assumption that U' is primary is reached.

Consider when (3) is true where S' is primary. Then this case for (3) is one of the conclusions of the theorem.

Now for case (4) there exist four possibilities: (A) S' is not primary, T' is not primary; (B) S' is not primary, T' is primary; (C) S' is primary, T' is not primary; (D) S' is primary, T' is primary. The possibilities (A), (B), and (C) lead to a contradiction that U' is primary by the results at the beginning of the proof.

Consider possibility (D) when $S' \neq S$ where S' is primary and $T' \neq T$ where T' is primary. First note that s_e is not an element of S'. For suppose $s_e \in S'$; then for every $s_1 \in S$ it follows that $s_1 \cdot s_e = s_1 \in S'$. This means that $S' = S$, which contradicts the assumption that $S' \neq S$. Now let s_1 be an arbitrary element of S'. It then follows that $s_1 + s_e \notin S'$. For suppose $s_1 + s_e \in S'$ then $(s_1 + s_e) - s_1 = s_e \in S'$, a contradiction to the fact that $s_e \notin S'$. Nor is any power of $s_1 + s_e$ an element of S'. For suppose $(s_1 + s_e)^n = s_1^n + ns_1^{n-1} + \ldots + ns_1 + s_e^n \in S'$; then since all of the terms preceding the last term in the expression contain s_1, it follows that the sum of these terms may be expressed as $s_2 \in S'$. Thus $(s_1 + s_e)^n = s_2 + s_e \in S'$ which contradicts the above fact that $s_1 + s_e \notin S'$ for every $s_1 \in S'$.

Since $T' \neq T$ for possibility (D), there exists $t_1 \in T$ such that $t_1 \notin T'$. Observe that $(s_z, t_1) \odot ([s_1 + s_e], t_z) \circleddash (s_z \cdot [s_1 + s_e], t_1 \cdot t_z) \circleddash (s_z, t_z) \in U'$ where t_1 is defined above and s_1 is an arbitrary element of S'. Note $(s_z, t_1) \notin U'$ since $t_1 \notin T'$, and for every integer $n > 0$ $([s_1 + s_e], t_z)^n \circleddash ([s_1 + s_e]^n, t_z^n) \notin U'$ since $[s_1 + s_e]^n \notin S'$. It then follows that U' is not a primary ideal which is a contradiction to the assumption. Thus case (4) is impossible, and cases (1), (2), and (3) imply the conclusion of the theorem.

Conversely, suppose $T' = T$ and S' is primary in S. Since S' is primary in S then if $s_1, s_2 \in S$ with $s_1 \cdot s_2 \in S'$ and $s_1 \notin S'$ it follows that there exists an integer $n > 0$ such that $s_2^n \in S'$. Observe also that for every element t_1 in T' and for every integer $m > 0$, $t_1^m \in T'$. Thus if (s_1, t_1), $(s_2, t_2) \in U$ with $(s_1, t_1) \odot (s_2, t_2) \in U'$ and $(s_1, t_1) \notin U'$, it follows that $s_1 \notin S'$ and thus there exists an integer $n > 0$ such that $(s_2, t_2)^n \circleddash (s_2^n, t_2^n) \in U'$. This means that U' is a primary ideal for this case.

In a similar way, it can be shown that U' is a primary ideal whenever $S' = S$ and T' is a primary ideal in T.

<u>Example 4.20</u>.--In this example, the ring $S = \{0, \pm 2, \pm 4, \ldots\}$ has no unity and the ring $T = \{0, \pm 2, \pm 4, \ldots\}$ has no unity. It is then shown that there exists proper primary

ideals $S' = \{0, \pm 4, \pm 8, \ldots\}$ and $T' = \{0, \pm 4, \pm 8, \ldots\}$ in S and T respectively such that their direct sum is a primary ideal.

Observe that S' is a primary ideal, for suppose that $s_1 \cdot s_2 \in S'$ and $s_1 \notin S'$ where $s_1, s_2 \in S$. Since $s_2 \in S$ is an even integer, it is of the form 2p for some integer p. It follows that $s_2^2 = (2p)^2 = 4p^2 \in S'$, so s_2 raised to the power two is in S' thus making S' a primary ideal. In a similar fashion, it can be shown that T' is a primary ideal.

It follows that U', the direct sum of S' and T' is also a primary ideal. For suppose that $(s_1, t_1) \odot (s_2, t_2) \in U'$ and $(s_1, t_1) \notin U'$ where $(s_1, t_1), (s_2, t_2) \in U$. Since $(s_2, t_2) \in U$, it follows that $s_2 \in S$ and $t_2 \in T$. Thus $s_2 =$ 2m and $t_2 = 2n$ for some integers m and n. Thus $(s_2, t_2)^2 \ominus (2m, 2n)^2 \ominus (4n^2, 4m^2) \in U'$, so (s_2, t_2) raised to the power two is in U'. This means that U' is a primary ideal.

<u>Definition 4.21</u>.--Let R be a commutative ring. Denote any element "x" in R added to itself n times by nx where n is a positive integer. If n is a negative integer, nx represents the additive inverse of x added to itself n times. Let $A = \{r \cdot a + na \mid r$ is an arbitrary element in R, a is a fixed element in R, and n is any integer$\}$. Here A is said to be a principal ideal in R generated by a.

In particular suppose $U = S \dotplus T$ is a direct sum. Also suppose $U' = S' \dotplus T'$ is an ideal such that $U' = \{(p, q) \odot (s_1, t_1) \oplus r(s_1, t_1) \mid$ p and q are arbitrary elements in S and T respectively, s_1 and t_1 are fixed elements in S and T respectively, and r is any integer . Here U' is said to be a principal ideal in the ring U where U' is generated by (s_1, t_1).

Theorem 4.22.--Let S' and T' be ideals in the commutative rings S and T respectively. $U' = S' \dotplus T'$ is a principal ideal if and only if S' and T' are principal ideals.

Proof.--Suppose that $S' = \{s_2 \mid s_2 = p \cdot s_1 + ns_1$, where p is an arbitrary element in S, s_1 is a fixed element in S, and n is any integer$\}$ and $T' = \{t_2 \mid t_2 = q \cdot t_1 + mt_1$, where q is an arbitrary element in T, t_1 is a fixed element in T, and m is any integer$\}$ are principal ideals in the rings S and T respectively. Thus an arbitrary element in U' may be expressed as $(s_2, t_2) \ominus (p \cdot s_1 + ns_1, q \cdot t_1 + mt_1)$. There exists an integer k such that n = k + m, thus $(s_2, t_2) \ominus (p \cdot s_1 + [k + m] s_e \cdot s_1, q \cdot t_1 + mt_1) \ominus (p \cdot s_1 + [ks_e + ms_e] \cdot s_1, q \cdot t_1 + mt_1) \ominus ([p + ks_e] \cdot s_1 + ms_1, q \cdot t_1 + mt_1) \ominus ([p + ks_e] \cdot s_1, q \cdot t_1) \oplus (ms_1, mt_1) \ominus (p + ks_e, q) \odot (s_1, t_1) \oplus m(s_1, t_1)$. Thus $U' = \{(p + ks_e, q) \odot (s_1, t_1) + m(s_1, t_1)\}$ where all of the symbols are defined above which means that U' is a principal ideal generated by (s_1, t_1).

Now suppose that U' is a principal ideal. Then U' can be expressed as $\{(p, q) \odot (s_1, t_1) \oplus r(s_1, t_1) \ominus (p \cdot s_1, q \cdot t_1) \oplus (rs_1, rt_1) \ominus (p \cdot s_1 + rs_1, q \cdot t_1 + rt_1) \mid$ p and q are arbitrary elements in S and T respectively, s_1 and t_1 are fixed elements in S and T respectively, and r is any integer}. This means that s_1 is the principal generator of S' and that t_1 is the principal generator of T'. Thus S' and T' are principal ideals whenever U' is a principal ideal.

Example 4.23.--Let both S and T be the ring of integers. Also let the ideal S' in S be the even integers and the ideal T' be T. It then follows that the direct sum of S' and T' is a maximal ideal in the direct sum of S and T. U' is also a prime ideal and a primary ideal. It also follows that (2, 1) generates U', so U' is principal in U.

Lemma 4.24.--Suppose that the ideal S' is the intersection of m ideals denoted by $S' = S_1' \cap S_2' \cap \ldots \cap S_m'$, and also suppose the ideal T' is the intersection of m ideals denoted by $T' = T_1' \cap T_2' \cap \ldots \cap T_m'$. The following equation then holds: $(S_1' \cap S_2' \cap \ldots \cap S_m') \dot{+} (T_1' \cap T_2' \cap \ldots \cap T_m') = (S_1' \dot{+} T_1') \cap (S_2' \dot{+} T_2') \cap \ldots \cap (S_m' \dot{+} T_m')$.

Proof.--Choose an arbitrary element (s_1, t_1) in $(S_1' \cap S_2' \cap \ldots \cap S_m') \dot{+} (T_1' \cap T_2' \cap \ldots \cap T_m')$, and it then follows that $s_1 \in S_i'$ for $i = 1, 2, \ldots, m$ and $t_1 \in T_i'$ for $i = 1, 2, \ldots, m$. This means that $(s_1, t_1) \in S_i' \dot{+} T_i'$ for $i = 1, 2, \ldots, m$; therefore $(s_1, t_1) \in (S_1' \dot{+} T_1') \cap (S_2' \dot{+} T_2') \cap \ldots \cap (S_m' \dot{+} T_m')$. Thus it can be concluded that $(S_1' \cap S_2' \cap \ldots \cap S_m') \dot{+} (T_1' \cap T_2' \cap \ldots \cap T_m') \subseteq (S_1' \dot{+} T_1') \cap (S_2' \dot{+} T_2') \cap \ldots \cap (S_m' \dot{+} T_m')$.

Now choose an arbitrary element (s_2, t_2) in $(S_1' \dot{+} T_1') \cap (S_2' \dot{+} T_2') \cap \ldots \cap (S_m' \dot{+} T_m')$. It then follows that $(s_2, t_2) \in S_i' \dot{+} T_i'$, for $i = 1, 2, \ldots, m$. This means that $s_2 \in S_i'$ for

$i = 1, 2, \ldots, m$ and $t_2 \in T'_i$ for $i = 1, 2, \ldots, m$. Therefore $s_2 \in S'_1 \cap S'_2 \cap \ldots \cap S'_m$ and $t_2 \in T'_1 \cap T'_2 \cap \ldots \cap T'_m$, so $(s_2, t_2) \in (S'_1 \cap S'_2 \cap \ldots \cap S'_m) \dot{+} (T'_1 \cap T'_2 \cap \ldots \cap T'_m)$, and in conclusion $(S'_1 \dot{+} T'_1) \cap (S'_2 \dot{+} T'_2) \cap \ldots \cap (S'_m \dot{+} T'_m) \subseteq (S'_1 \cap S'_2 \cap \ldots \cap S'_m) \dot{+} (T'_1 \cap T'_2 \cap \ldots \cap T'_m)$. These two containments yield the desired equality.

Remark.--The above result may be expanded by observing that for any two sets A and B, $A \cap B = B \cap A$. Thus the lemma is still true even after the order of elements is arbitrarily interchanged in either $\{S'_1, S'_2, \ldots, S'_m\}$ or in $\{T'_1, T'_2, \ldots, T'_m\}$. As a consequence for example, if $m = 3$ then the following equation is true: $(S'_1 \dot{+} T'_1) \cap (S'_2 \dot{+} T'_2) \cap (S'_3 \dot{+} T'_3) = (S'_2 \dot{+} T'_3) \cap (S'_1 \dot{+} T'_2) \cap (S'_3 \dot{+} T'_1)$.

Definition 4.25.--An ideal E is said to be the irredundant intersection of a finite sequence $E_1, E_2, \ldots, E_m$ of ideals if E is the intersection of the sequence $E_1, E_2, \ldots, E_m$ and E is not the intersection of some proper subcollection of the sequence $E_1, E_2, \ldots, E_m$.

Theorem 4.26.--Suppose $U' = S' \dot{+} T'$ where S' and T' are defined as the intersection of the ideals in the sequence $S'_1, S'_2, \ldots, S'_m$ and as the intersection of the ideals in the sequence $T'_1, T'_2, \ldots, T'_n$ respectively. If S' is the irredundant intersection of the sequence $S'_1, S'_2, \ldots, S'_m$ of ideals then U' is the irredundant intersection of the sequence $S'_1 \dot{+} T''_1, S'_2 \dot{+} T''_2, \ldots, S'_m \dot{+} T''_m$ of ideals where the sequence $T''_1, T''_2, \ldots, T''_m$ of ideals may be constructed from $\{T'_1, T'_2,$

..., $T'_n\}$ such that $T' = \bigcap_{i=1}^{m} T''_i$.

Proof.--Since $\{S'_1, S'_2, \ldots, S'_m\}$ has m ideals and $\{T'_1, T'_2, \ldots, T'_n\}$ has n ideals it follows that either $m < n$, $m = n$, or $m > n$. If $m \leq n$ define $T''_i = T'_i$ for $1 \leq i \leq m - 1$ and $T''_m = \bigcap_{i=m}^{n} T'_i$. Observe that $\bigcap_{i=m}^{n} T'_i$ is also an ideal since it may be shown that the intersection of any finite number of ideals is an ideal. Now if $m > n$, define $T''_i = T'_i$ for $1 \leq i \leq n$ and $T''_i = T'$ for $n < i \leq m$. Thus it is possible to express T' as the intersection of m ideals.

Now suppose that U' is not the irredundant intersection of the sequence $S'_1 \dotplus T''_1, S'_2 \dotplus T''_2, \ldots, S'_m \dotplus T''_m$ of ideals where $U' = S' \dotplus T' = (S'_1 \cap S'_2 \cap \ldots \cap S'_m) \dotplus (T''_1 \cap T''_2 \cap \ldots \cap T''_m) = (S'_1 \dotplus T''_1) \cap (S'_2 \dotplus T''_2) \cap \ldots \cap (S'_m \dotplus T''_m)$ by Lemma 4.24. Then U' may be expressed as the intersection of a proper subcollection of the sequence $S'_1 \dotplus T''_1, S'_2 \dotplus T''_2, \ldots, S'_m \dotplus T''_m$ of ideals. Thus upon omission of a particular direct sum, say $S'_r \dotplus T''_r$, and upon renumbering the remaining direct sums, it follows that $U' = (S'_1 \dotplus T''_1) \cap (S'_2 \dotplus T''_2) \cap \ldots \cap (S'_{m-1} \dotplus T''_{m-1}) = (S'_1 \cap S'_2 \cap \ldots \cap S'_{m-1}) \dotplus (T''_1 \cap T''_2 \cap \ldots \cap T''_{m-1}) = S' \dotplus T'$. This means that S' is not the irredundant intersection of the sequence $S'_1, S'_2, \ldots, S'_m$ of ideals because S' may be written as a proper subcollection of this sequence. This is a contradiction to the assumption that S' is the irredundant intersection of the sequence $S'_1, S'_2, \ldots, S'_m$ of ideals. Thus U' is the irredundant intersection of the sequence $S'_1 \dotplus T''_1, S'_2 \dotplus T''_2, \ldots, S'_m \dotplus T''_m$ of ideals.

- <u>Example 4.27</u>.--In this example, let both S and T be the ring of integers with the usual binary operations. The following is an irredundant representation of U' such that the representations of S' and T' are not irredundant.

Let $U' = (S_1' \dotplus T_1') \cap (S_2' \dotplus T_1') \cap (S_3' \dotplus T_2') \cap (S_3' \dotplus T_3')$ where $S_1' = T_3' = \{0, \pm 3, \pm 6, \ldots\}$, $S_2' = T_2' = \{0, \pm 4, \pm 8, \ldots\}$, and $S_3' = T_1' = \{0, \pm 2, \pm 4, \ldots\}$. This means that $U' = (S_1' \cap S_2' \cap S_3' \cap S_3') \dotplus (T_1' \cap T_1' \cap T_2' \cap T_3') = \{(12k, 12j) \mid k$ and j are integers$\}$. Observe that $(S_1' \dotplus T_1') \cap (S_2' \dotplus T_1') \cap (S_3' \dotplus T_2') = (S_1' \cap S_2' \cap S_3') \dotplus (T_1' \cap T_2') \neq U'$ since $(0, 4) \in (S_1' \cap S_2' \cap S_3') \dotplus (T_1' \cap T_2')$ but $(0, 4) \notin U'$. Also $(S_1' \dotplus T_1') \cap (S_2' \dotplus T_1') \cap (S_3' \dotplus T_3') = (S_1' \cap S_2' \cap S_3') \dotplus (T_1' \cap T_3') \neq U'$ since $(0, 6) \in (S_1' \cap S_2' \cap S_3') \dotplus (T_1' \cap T_3')$ but $(0, 6) \notin U'$. And $(S_1' \dotplus T_1') \cap (S_3' \dotplus T_2') \cap (S_3' \dotplus T_3') = (S_1' \cap S_3') \dotplus (T_1' \cap T_2' \cap T_3') \neq U'$ since $(6, 0) \in (S_1' \cap S_3') \dotplus (T_1' \cap T_2' \cap T_3')$ but $(6, 0) \notin U'$. Observe also that $(S_2' \dotplus T_1') \cap (S_3' \dotplus T_2') \cap (S_3' \dotplus T_3') = (S_2' \cap S_3') \dotplus (T_1' \cap T_2' \cap T_3') \neq U'$ since $(4,0) \in (S_2' \cap S_3') \dotplus (T_1' \cap T_2' \cap T_3')$ but $(4, 0) \notin U'$. Therefore U' has an irredundant intersection. But $S_1' \cap S_2' \cap S_3' \cap S_3' = S'$ and $T_1' \cap T_1' \cap T_2' \cap T_3' = T'$ are not irredundant intersections since $S' = S_1' \cap S_2'$ and $T' = T_2' \cap T_3'$.

<u>Theorem 4.28</u>.--Suppose $U' = S' \dotplus T'$ where S' and T' are proper ideals in S and T respectively such that $S' = \bigcap_{i=1}^{m} S_i'$ is an irredundant intersection of primary ideals. Then for every set of ideals $T_1', T_2', \ldots, T_m'$ in T such that $T' = \bigcap_{i=1}^{m} T_i'$, there exists $S_i' \dotplus T_i'$ for some $i = 1, 2, \ldots, m$ which is

not a primary ideal. Also, $U' = \bigcap_{i=1}^{m} (S_i' \dotplus T_i')$ is not a primary representation of U'.

Proof.--Since S' is a proper ideal in S expressed as an irredundant intersection of $\{S_1', S_2', \ldots, S_m'\}$ for every S_i', $i = 1, 2, \ldots, m$, it follows that $S_i' \neq S$. Since T' is a proper ideal in T, it follows that for every set of ideals T_1', T_2', ..., T_m' in T' such that $T' = \bigcap_{i=1}^{m} T_i'$, there exists an ideal $T_p' \neq T$ for some integer p where $1 \leq p \leq m$. Contained in the proof of Theorem 4.19 is the result that if $S_i' \neq S$ and $T_p' \neq T$ then $S_i' \dotplus T_p'$ is not a primary ideal. Since $S_i' \dotplus T_p'$ is not a primary ideal, it follows that $U' = \bigcap_{i=1}^{m} (S_i' \dotplus T_i')$ is not a primary representation of U'.

Theorem 4.29.--Suppose $S' = \bigcap_{i=1}^{m} S_i'$, where S' is the irredundant intersection of primary ideals $S_1', S_2', \ldots, S_m'$, and $T' = T$. Then $S_i' \dotplus T$ for every $i = 1, 2, \ldots, m$ is a primary ideal, and $U' = S' \dotplus T = \bigcap_{i=1}^{m} (S_i' \dotplus T)$ is an irredundant intersection of primary ideals.

Proof.--Since S_i' is a primary ideal in S and $T' = T$, it follows from Theorem 4.19 that $S_i' \dotplus T'$ is a primary ideal in U. Since S_i' is a primary ideal for all $i = 1, 2, \ldots, m$, it follows that $U' = S' \dotplus T = \bigcap_{i=1}^{m} (S_i' \dotplus T)$ is an intersection of primary ideals. From Theorem 4.26, it may be concluded that the intersection of $S_i' \dotplus T$ for $i = 1, 2, \ldots, m$ is irredundant.

Theorem 4.30.--Suppose U' is a proper ideal in U where $U' = \bigcap_{i=1}^{m} (S_i' \dotplus T_i')$ is an irredundant intersection of primary ideals. Then $A = \{U_i' \mid U_i' = S_i' \dot{+} T_i', i = 1, 2, \ldots, m\}$ may be expressed as $B \cup C$, where $B = \{U_i' \mid U_i' \in A, U_i' = S_i' \dot{+} T\}$ and $C = \{U_i' \mid U_i' \in A, U_i' = S \dot{+} T_i'\}$. Furthermore, if $U' = S' \dot{+} T'$ then the intersection of all S_i' such that $S_i' \dot{+} T \in B$ and the intersection of all T_i' such that $S \dot{+} T_i' \in C$ are irredundant primary representations of S' and T' respectively.

Proof.--Each ideal $S_p' \dot{+} T_p'$ in A for $p = 1, 2, \ldots, m$, is primary in U', hence S_p' is primary in S and $T_p' = T$ or $S_p' = S$ and T_p' is primary in T by Theorem 4.19. Therefore every ideal in A is in B or C. The ring $U = S \dot{+} T \neq S_p' \dot{+} T_p'$ for all $p = 1, 2, \ldots, m$ since the representation of U' is irredundant and since U' is a proper subset in U. Thus the sets B and C are disjoint and $A = B \cup C$.

Now U' is an irredundant primary intersection of those ideals in $B \cup C$. Since each $S_i' \dot{+} T$ in B is a primary ideal in U, it follows from the above argument that each such S_i' is primary in S. Let $U' = S' \dot{+} T'$ then $S' = \bigcap_{i=1}^{m} S_i'$ and $T' = \bigcap_{i=1}^{m} T_i'$ by Lemma 4.24. However $S_j' = S$ if $S_j' \dot{+} T_j' \notin B$, thus S' is the intersection of S_i' such that $S_i' \dot{+} T \in B$. Furthermore, the intersection is irredundant, for if some S_k' for $S_k' \dot{+} T \in B$ can be eliminated from the representation of S', then $S_k' \dot{+} T$ can be eliminated from the representation of U'. But this cannot happen since this representation of U' is irredundant. Hence the intersection of elements S_i' such that $S_i' \dot{+} T \in B$ is an irredundant primary representation of S'. Using a similar argument, it follows that the intersection of elements T_1' such that $S \dotplus T_1' \in C$ is an irredundant primary representation of T'.

<u>Definition 4.31</u>.--Suppose R is a ring whose additive identity is denoted by r_z. Then $a \in R$ is said to be nilpotent if there exists an integer $n > 1$ such that $a^n = r_z$.

<u>Theorem 4.32</u>.--The ideal $U' = S' \dotplus T'$ has a non-zero nilpotent element if and only if either of the ideals S' or T' have non-zero nilpotent elements.

<u>Proof</u>.--Suppose S' has a non-zero nilpotent element s_1. Then there exists an integer $n > 1$ such that $s_1^n = s_z$. Thus $(s_1, t_z) \in U'$ and it follows that $(s_1, t_z)^n \circledcirc (s_1^n, t_z^n) \circledcirc (s_z, t_z)$ which makes (s_1, t_z) a non-zero nilpotent element of U'. A similar argument could be used to show that U' has a non-zero nilpotent element if T' has a non-zero nilpotent element.

Now suppose U' has a non-zero nilpotent element (s_1, t_1). Then one and only one of the following three cases is possible: (1) $s_1 = s_z$, $t_1 \neq t_z$; (2) $s_1 \neq s_z$, $t_1 = t_z$; (3) $s_1 \neq s_z$, $t_1 \neq t_z$. For case one there exists an integer $n > 1$ such that $(s_z, t_1)^n \circledcirc (s_z^n, t_1^n) \circledcirc (s_z, t_z)$. Thus t_1 is a non-zero nilpotent element in T since $t_1^n = t_z$. For case two there exists an integer $m > 1$ such that $(s_1, t_z)^m \circledcirc (s_1^m, t_z^m) \circledcirc (s_z, t_z)$. Thus s_1 is a non-zero nilpotent element in S since $s_1^m = s_z$. For case three there exists an integer $p > 1$ such that $(s_1, t_1)^p \circledcirc (s_1^p, t_1^p) \circledcirc (s_z, t_z)$. Thus s_1 and t_1 are non-zero nilpotent elements in S and T respectively since

$s_1^p = s_z$ and $t_1^p = t_z$. This means if U' has a non-zero nilpotent element then at least one of S' and T' has a non-zero nilpotent element.

Example 4.33.--This example shows that U' may have a non-zero nilpotent element and yet not both S' and T' have non-zero nilpotent elements. Let S' be the ring of integers with the zero nilpotent element 0. Note S' does not have a non-zero nilpotent element. Let T' be the residue class ring I/(4). Note that in T', [2] [2] = [0], so [2] is a non-zero nilpotent element in T'. It follows that (0, [2]) is a non-zero nilpotent element in U'.

Definition 4.34.--If A is an ideal in the ring R, then the radical of A, denoted by $\sqrt{A}$, consists of all elements b of R such that some power of b is contained in A.

Theorem 4.35.--If $U' = S' \dotplus T'$ where S' and T' are ideals in the rings S and T respectively then $\sqrt{S'} \dotplus \sqrt{T'} = \sqrt{U'} = \sqrt{S' \dotplus T'}$.

Proof.--Suppose $s_1 \in \sqrt{S'}$ and $t_1 \in \sqrt{T'}$; then there exist integers m and n such that $s_1^m \in S'$ and $t_1^n \in T'$. Since S' and T' are closed under multiplication, it follows that $(s_1^m)^n \in S'$ and $(t_1^n)^m \in T'$. Thus $(s_1^{mn}, t_1^{nm}) \circledeq (s_1, t_1)^{mn} \in U'$, and it can be concluded that $(s_1, t_1) \in \sqrt{U'}$. This means that $\sqrt{S'} \dotplus \sqrt{T'} \subseteq \sqrt{U'}$.

Now suppose $(s_1, t_1) \in \sqrt{U'}$, then there exists an integer r such that $(s_1, t_1)^r = (s_1^r, t_1^r) \in U'$. It then follows that $s_1^r \in S'$ and $t_1^r \in T'$ so that $s_1 \in \sqrt{S'}$ and $t_1 \in \sqrt{T'}$. Thus

$\sqrt{U'} \subseteq \sqrt{S'} \dotplus \sqrt{T'}$, and it can now be concluded that $\sqrt{S'} \dotplus \sqrt{T'} = \sqrt{U'}$.

- <u>Definition 4.36</u>.--Let A and B denote ideals in the commutative ring R, then $A \hat{+} B = \{a + b \mid a \in A, b \in B\}$.

<u>Theorem 4.37</u>.--If $U' = S' \dotplus T'$ and $U'' = S'' \dotplus T''$ are ideals in the commutative ring U, then $U' \hat{+} U'' = (S' \hat{+} S'') \dotplus (T' \hat{+} T'')$.

<u>Proof</u>.--Suppose $(s_1, t_1) \in U'$ and $(s_1'', t_1'') \in U''$ where $s_1' \in S'$, $s_1'' \in S''$, $t_1' \in T'$, and $t_1'' \in T''$ then $(s_1', t_1') \oplus (s_1'', t_1'') \in U' \hat{+} U''$. It follows that $(s_1', t_1') \oplus (s_1'', t_1'') \ominus (s_1' + s_1'', t_1' + t_1'') \in (S' \hat{+} S'') \dotplus (T' \hat{+} T'')$; thus $U' \hat{+} U'' \subseteq (S' \hat{+} S'') \dotplus (T' \hat{+} T'')$.

An arbitrary y in $(S' \hat{+} S'') \dotplus (T' \hat{+} T'')$ is of the form $(s_1' + s_1'', t_1' + t_1'')$ so let $(s_1' + s_1'', t_1' + t_1'') \in (S' \hat{+} S'') \dotplus (T' \hat{+} T'')$. It follows that $(s_1' + s_1'', t_1' + t_1'') \ominus (s_1', t_1') \oplus (s_1'', t_1'') \in U' \hat{+} U''$; thus $(S' \hat{+} S'') \dotplus (T' \hat{+} T'') \subseteq U' \hat{+} U''$. These two containments imply that $U' \hat{+} U'' = (S' \hat{+} S'') \dotplus (T' \hat{+} T'')$.

<u>Remark</u>.--It may be shown that in a commutative ring, the sum of any two ideals, that is $A \hat{+} B$, is also an ideal.

<u>Definition 4.38</u>.--Let A and B denote ideals in the commutative ring R, then $A \hat{\cdot} B = \sum_{i=1}^{p} a_i \cdot b_i \mid a_i \in A, b_i \in B$, p is a positive integer$\}$.

<u>Theorem 4.39</u>.--If $U' = S' \dotplus T'$ and $U'' = S'' \dotplus T''$ are ideals in the commutative ring U then $U' \hat{\cdot} U'' = (S' \hat{\cdot} S'') \dotplus (T' \hat{\cdot} T'')$.

Proof.--An arbitrary element x in $U' \hat{} U''$ is of the form $\sum_{i=1}^{n}[(s_i', t_i') \odot (s_i'', t_i'')]$, where $s_i' \in S'$, $t_i' \in T'$, $s_i'' \in S''$, and $t_i'' \in T''$. It then follows that $x \ominus \sum_{i=1}^{n} [(s_i', t_i') \odot (s_i'', t_i'')] \ominus \sum_{i=1}^{n} [(s_i' \cdot s_i'', t_i' \cdot t_i'')] \ominus (\sum_{i=1}^{n} s_i' \cdot s_i'', \sum_{i=1}^{n} t_i' \cdot t_i'') \in (S' \hat{} S'') \dot{+} (T' \hat{} T'')$. Thus $U' \hat{} U'' \subseteq (S' \hat{} S'') \dot{+} (T' \hat{} T'')$.

An arbitrary element y in $(S' \hat{} S'') \dot{+} (T' \hat{} T'')$ is of the form $(\sum_{i=1}^{k}[s_i' \cdot s_i''], \sum_{i=1}^{m}[t_i' \cdot t_i''])$. Now assume that $k \geq m$; then $y = (\sum_{i=1}^{k}[s_i' \cdot s_i''], \sum_{i=1}^{k}[t_i' \cdot t_i''])$, where $t_i' \cdot t_i'' = t_z$ for $i = m + 1, m + 2, \ldots, k$. It follows that y may be expressed as $\sum_{i=1}^{k}[(s_i' \cdot s_i'', t_i' \cdot t_i'')] \ominus \sum_{i=1}^{k}[(s_i' \cdot t_i') \odot (s_i'' \cdot t_i'')] \in U' \hat{} U''$. Thus $(S' \hat{} S'') \dot{+} (T' \hat{} T'') \subseteq U' \hat{} U''$. These two containments imply that $U' \hat{} U'' = (S' \hat{} S'') \dot{+} (T' \hat{} T'')$.

Remark.--It may be shown that in a commutative ring, the product of any two ideals, that is $A \hat{} B$, is also an ideal.

Definition 4.40.--Let A and B denote ideals in the commutative ring R; then $A : B$, called the quotient of A and B, consists of all elements c in R such that $c \cdot b \in A$ for every $b \in B$.

Theorem 4.41.--If $U' = S' \dot{+} T'$ and $U'' = S'' \dot{+} T''$ are ideals in the commutative ring U, then $U' : U'' = (S' : S'') \dot{+} (T' : T'')$.

Proof.--Let $(s_1, t_1) \in U' : U''$; then $(s_1, t_1) \odot (s_1'', t_1'') \in U'$ for every $(s_1'', t_1'') \in U''$. Since $(s_1, t_1) \odot (s_1'', t_1'') \ominus (s_1 \cdot s_1'', t_1 \cdot t_1'') \in U'$ for every $(s_1'', t_1'') \in U''$ it follows that $s_1 \cdot s_1'' \in S'$ for every $s_1'' \in S''$, and $t_1 \cdot t_1'' \in T'$ for every $t_1'' \in T''$. Thus $s_1 \in S' : S''$ and $t_1 \in T' : T''$, which means $(s_1, t_1) \in (S' : S'') \dot{+} (T' : T'')$. It follows that $U' : U'' \subseteq (S' : S'') \dot{+} (T' : T'')$.

Now let (s_1, t_1) be an arbitrary element of $(S' : S'') \dot{+} (T' : T'')$; then $s_1 \cdot s_1'' \in S'$ for every $s_1'' \in S''$ and $t_1 \cdot t_1'' \in T'$ for every $t_1'' \in T''$. Thus $(s_1 \cdot s_1'', t_1 \cdot t_1'') \ominus (s_1, t_1) \odot (s_1'', t_1'') \in S' \dot{+} T' = U'$ for every $(s_1'', t_1'') \in S'' \dot{+} T'' = U''$. This means $(s_1, t_1) \in U' : U''$. It follows that $(S' : S'') \dot{+} (T' : T'') \subseteq U' : U''$, and the above containments imply $U' : U'' = (S' : S'') \dot{+} (T' : T'')$.

Remark.--It may be shown that in a commutative ring, the quotient of any two ideals is also an ideal.

BIBLIOGRAPHY

Books

McCoy, Neal Henry, Introduction to Modern Algebra, Boston, Allyn and Bacon, Inc., 1960.

McCoy, Neal Henry, Rings and Ideals, Buffalo, The Mathematical Association of America, 1948.

McCoy, Neal Henry, The Theory of Rings, New York, The Macmillian Co., 1964.

MacDuffee, Cyrus Colton, An Introduction to Abstract Algebra, New York, John Wiley and Sons, Inc., 1940.

Moore, John T., Elements of Abstract Algebra, New York, The Macmillian Co., 1962.

Zariski, Oscar and Samuel, Pierce, Commutative Algebra, Princeton, D. Van Nostrand Co., Inc., 1959.