SOME PROPERTIES OF PARTIALLY ORDERED SETS

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CHAPTER I

PARTIALLY ORDERED SETS AND OPERATIONS ON PARTIALLY ORDERED SETS

It may be said of certain pairs of elements of a set that one element precedes the other. If the collection of all such pairs of elements in a given set exhibits certain properties, the set and the collection of pairs is said to constitute a partially ordered set. The purpose of this paper is to explore some of the properties of partially ordered sets. This chapter will discuss operations on partially ordered sets, Chapter II will treat properties of ordinals and weak ordinals, and Chapter III will demonstrate some properties of automorphisms on partially ordered sets.

The notion of a partially ordered set is formally defined by the following.

Definition 1.1. A relation is a set of ordered pairs of elements. The domain of a relation R, designated by $\underline{D}(R)$, is the set of all first elements of the ordered pairs of R. The range of R, designated by $\underline{R}(R)$, is the set of all second elements of the ordered pairs of R.

Definition 1.2. The statement that R is a relation in a set A means that R is a relation such that $\underline{R}(R) \cup \underline{D}(R) \subset A$.

Definition 1.3. The statement that the ordered pair $(A, \leq)$ is a partially ordered set means A is a set and $\leq$ is a relation in A such that

i) if $(a,b) \in \leq$, $(b,c) \in \leq$ then $(a,c) \in \leq$,

ii) if $(a,b) \in \leq$ and $(b,a) \in \leq$ then $a=b$,

and iii) $(a,a) \in \leq$ for all $a \in A$.

The above properties are called transitive, antisymmetric and reflexive, respectively.

Unless otherwise noted a set will have only one relation defined in it and a partially ordered set will be denoted by the set name. The name partially ordered set will be shortened to poset. If A is a set and R is a relation in A, the statement $(a,b) \in R$ may be written aRb. If $B \subset A$, $a,b \in B$ then $a \leq b$ in B if and only if $a \leq b$ in A. Any two posets in a discussion are understood to be disjoint unless otherwise noted.

<u>Definition 1.4.</u> Let $<$ be the relation in a poset A such that for $a,b \in A$, $a<b$ if and only if $a \leq b$ and $b \not\leq a$.

<u>Definition 1.5.</u> If A is a poset, $a,b \in A$, let $a>b$ mean $b<a$ and let $a \geq b$ mean $b \leq a$.

<u>Definition 1.6.</u> The cardinal sum of posets X and Y is $X+Y = C$ where $C = X \cup Y$ and $a \leq b$ in C if and only if $a \leq b$ in X or $a \leq b$ in Y.

<u>Definition 1.7.</u> The cardinal product of posets X and Y is $XY = D$ where $D = X \times Y$ and $d_1 \leq d_2$ in D if and only if $d_1 = (a_1,b_1)$, $d_2 = (a_2,b_2)$ where $a_1 \leq a_2$ in X and $b_1 \leq b_2$ in Y.

<u>Definition 1.8.</u> A function f is a relation such that if $(a,b) \in f$ and $(a,c) \in f$ then $b = c$. An alternate notation for $(a,b) \in f$ is $b = f(a)$.

Definition 1.9. Let A be a set with a relation R_1 defined in it. Let B be a set with a relation R_2 defined in it. The statement that f is an isotone function such that D(f) = A, R(f) $\subset$ B means $f(a_1)R_2f(a_2)$ in B for all $a_1R_1a_2$ in A.

Definition 1.10. The cardinal power of poset Y to the exponent poset X is Y^X = E where f ε E if and only if f is an isotone function such that D(f) = X, R(f) $\subset$ Y. $f \leq g$ in E means $f(a) \leq g(a)$ in Y for all a ε X.

Theorem 1.1. If A and B are posets, then A+B is a poset.

Proof: Let A+B = C.

Let $a \leq b$ in C and $b \leq c$ in C. Then $a \leq b$ in A or $a \leq b$ in B. If $a \leq b$ in A then b ε A and hence c ε A for $b \leq c$ in C for no b ε A, c ε B. Thus $a \leq b$, $b \leq c$ in A. From the transitive property $a \leq c$ in A and hence $a \leq c$ in C. By a similar argument $a \leq b$ in B implies $a \leq c$ in C.

Let $a \leq b$ in C and $b \leq a$ in C. Then either $a \leq b$ in A or $a \leq b$ in B. If $a \leq b$ in A then $b \leq a$ in A. Thus b = a. If $a \leq b$ in B then $b \leq a$ in B. Thus b = a.

Let a ε C. Then a ε A, $a \leq a$ in A thus $a \leq a$ in C or a ε B, $a \leq a$ in B thus $a \leq a$ in C.

Thus the relation defined on C by the definition of A+B = C is transitive, antisymmetric and reflexive. So C is a poset.

Theorem 1.2. If A and B are posets, then AB is a poset.

Proof: Let AB = D.

Let $d_1 \leq d_2$ in D and $d_2 \leq d_3$ in D. Then $d_1 = (a_1, b_1)$, $d_2 = (a_2, b_2)$, $d_3 = (a_3, b_3)$ where $a_1 \leq a_2$, $a_2 \leq a_3$ in A and $b_1 \leq b_2$, $b_2 \leq b_3$ in B. By the transitive property $a_1 \leq a_3$ and $b_1 \leq b_3$ in A and B respectively and $d_1 \leq d_3$ in D follows from the cardinal product definition.

Let $d_1 \leq d_2$ in D and $d_2 \leq d_1$ in D. Then $d_1 = (a_1, b_1)$, $d_2 = (a_2, b_2)$ where $a_1 \leq a_2$, $a_2 \leq a_1$ in A and $b_1 \leq b_2$, $b_2 \leq b_1$ in B. Thus $a_1 = a_2$, $b_1 = b_2$ from the antisymmetric property of posets. Hence $d_1 = d_2$.

Let $d_1 \in D$. Then $d_1 = (a_1, b_1)$ where $a_1 \leq a_1$ in A and $b_1 \leq b_1$ in B by the reflexive property. Hence $d_1 \leq d_1$.

Thus the relation defined on D by the definition of AB = D is transitive, antisymmetric and reflexive. Hence D is a poset.

<u>Theorem 1.3.</u> If A and B are posets then B^A is a poset.

Proof: Let $B^A = E$.

Let $f \leq g$ in E and $g \leq h$ in E. Then $f(a) \leq g(a)$ in B for all $a \in A$ and $g(a) \leq h(a)$ for all $a \in A$. By the transitive property $f(a) \leq h(a)$ for all $a \in A$, and $f \leq h$ in E follows from the definition of the relation in B^A.

Let $f \leq g$ in E and $g \leq f$ in E. Then $f(a) \leq g(a)$ and $g(a) \leq f(a)$ in B for all $a \in A$. Since B is a poset $f(a) = g(a)$ for all $a \in A$ by the antisymmetric property. Hence $f = g$.

Let $f \in E$. $f(a) \leq f(a)$ in B for all $a \in A$ so $f \leq f$.

Thus the relation defined on E by the definition of B^A is transitive, reflexive and antisymmetric. So E is a poset.

The definitions of cardinal operations invite investigation of the commutative, associative and distributive properties of cardinal sums, products and powers. The relation "=" has been taken without formal definition to mean "is the same as". Isomorphism will be the relation used to compare two posets.

Definition 1.11. A reversible function is a function f such that f(a) = b and f(c) = b if and only if a = c.

Definition 1.12. If A is a set with a relation R_1 in it and B is a set with a relation R_2 in it, then $A \cong B$ (A is isomorphic to B) if and only if there exists a reversible function θ such that $\underline{D}(\theta) = A$, $\underline{R}(\theta) = B$ and aR_1b in A if and only if $\theta(a)R_2\theta(b)$ in B.

Theorem 1.4. If A and B are posets then $A+B \cong B+A$ and $AB \cong BA$.

Proof: i) Let C = A+B. Let D = B+A. Let θ be a function whose domain is C such that for all $a \in C$, $\theta(a) = a$ in D. Clearly $\underline{R}(\theta) = D$ and θ is reversible. Let $a \leq b$ in C. Then $a,b \in A$ or $a,b \in B$. If $a,b \in A$ then $a \leq b$ in A so $a \leq b$ in D or $\theta(a) \leq \theta(b)$ in D. Similarly if $a,b \in B$, $\theta(a) \leq \theta(b)$ in D. The proof that $\theta(a) \leq \theta(b)$ implies $a \leq b$ follows similarly. Thus $A+B \cong B+A$.

ii) Let C = AB. Let D = BA. Let θ be a function whose domain is C and for all $d \in C$, d = (a,b) where $a \in A$, $b \in B$, $\theta(d) = (b,a) \in D$. $\underline{R}(\theta) = D$ and θ is reversible for every $(b,a) \in D$ is the image of exactly one $(a,b) \in C$. Let $d_1 \leq d_2$ in C. Let $d_1 = (a_1,b_1)$ and $d_2 = (a_2,b_2)$. Then $a_1 \leq a_2$

in A and $b_1 \leq b_2$ in B. Thus $(b_1,a_1) \leq (b_2,a_2)$ in D or $\theta(d_1) \leq \theta(d_2)$ in D. Let $\theta(d_1) \leq \theta(d_2)$ in D. Then $\theta(d_1) = (b_1,a_1)$, $\theta(d_2) = (b_2,a_2)$ where $b_1 \leq b_2$ in B and $a_1 \leq a_2$ in A. $(a_1,b_1) \leq (a_2,b_2)$ in C or $d_1 \leq d_2$ in C. Thus AB $\cong$ BA.

Thus cardinal addition and multiplication are each commutative within isomorphism.

<u>Theorem 1.5.</u> If A, B and C are posets, then A+(B+C) $\cong$ (A+B)+C and A(BC) $\cong$ (AB)C.

Proof: Let A, B and C each be a poset.

i) Let D = A+(B+C). Let E = (A+B)+C. Let θ be a function whose domain is D such that if $a \in D$ then $\theta(a) = a \in E$. This is clearly a reversible function whose range is E.

If $a \leq b$ in D then 1) $a \leq b$ in A so $a \leq b$ in A+B, hence $a \leq b$ in E or $\theta(a) \leq \theta(b)$ in E; or 2) $a \leq b$ in B+C in which case 2.1) $a \leq b$ in B so $a \leq b$ in A+B and $a \leq b$ in E or 2.2) $a \leq b$ in C so $a \leq b$ in E. In any case $a \leq b$ in D implies $\theta(a) \leq \theta(b)$ in E. Similarly $\theta(a) \leq \theta(b)$ in E implies $a \leq b$ in D and hence A+(B+C) $\cong$ (A+B)+C.

ii) Let D = A(BC). Let E = (AB)C. Let θ be a function whose domain is D. Let $d \in D$. Then $d = (a,t)$ where $a \in A$, $t \in BC$ and $t = (b,c)$ where $b \in B$, $c \in C$. Let $\theta(d) = ((a,b),c) \in E$. θ is a reversible function whose range is E.

Let $d_1 \leq d_2$ in D where $d_1 = (a_1,t_1) = (a_1,(b_1,c_1))$ and $d_2 = (a_2,t_2) = (a_2,(b_2,c_2))$. Then $a_1 \leq a_2$ and $t_1 \leq t_2$. Since $t_1 \leq t_2$ then $b_1 \leq b_2$, $c_1 \leq c_2$. $a_1 \leq a_2$, $b_1 \leq b_2$ implies $(a_1,b_1) \leq (a_2,b_2)$

which with $c_1 \leq c_2$ implies $((a_1,b_1),c_1) \leq ((a_2,b_2),c_2)$ or $\theta(d_1) \leq \theta(d_2)$. Similarly if $\theta(d_1) \leq \theta(d_2)$ then $d_1 \leq d_2$ and thus $A(BC) \cong (AB)C$.

Thus cardinal addition and multiplication are each associative within isomorphism.

<u>Theorem 1.6.</u> If A, B and C are posets, then $A(B+C) \cong AB+AC$.

Proof: Let A, B and C be posets. Let D = B+C. Let E = AD. Let F = AB. Let G = AC. Let H = F+G. Let θ be a function whose domain is E. Let $(a,b) \in E$. Then $a \in A$, $b \in D$. Since $b \in D$ then $b \in B$ or $b \in C$. Suppose $b \in B$. Then $(a,b) \in AB$ and thus $(a,b) \in H$. Suppose $b \in C$. Then $(a,b) \in AC$ and thus $(a,b) \in H$. For $(a,b) \in E$ let $\theta((a,b)) = (a,b)$ in H. Clearly since H consists of elements (a,b) as described above, θ is a reversible function whose range is H.

Let $(a_1,b_1) \leq (a_2,b_2)$ in E. Then $a_1 \leq a_2$ in A, $b_1 \leq b_2$ in D. Thus $b_1 \leq b_2$ in B or $b_1 \leq b_2$ in C. Suppose $b_1 \leq b_2$ in B. Then $(a_1,b_1) \leq (a_2,b_2)$ in F and hence $(a_1,b_1) \leq (a_2,b_2)$ in H. Suppose $b_1 \leq b_2$ in C. Then $(a_1,b_1) \leq (a_2,b_2)$ in G and hence $(a_1,b_1) \leq (a_2,b_2)$ in H. In both cases $(a_1,b_1) \leq (a_2,b_2)$ in E implies $\theta(a_1,b_1) \leq \theta(a_2,b_2)$ in H.

A similar argument shows $\theta(a_1,b_1) \leq \theta(a_2,b_2)$ in H implies $(a_1,b_1) \leq (a_2,b_2)$ in E. Thus $A(B+C) \cong AB+AC$.

Thus cardinal multiplication is distributive to the right over cardinal addition within automorphism.

<u>Corollary 1.1.</u> If A, B and C are posets, then $(A+B)C \cong AC+BC$.

Proof: From Theorem 1.5 $(A+B)C \cong C(A+B)$.

From Theorem 1.6 $C(A+B) \cong CA+CB$.

From Theorem 1.5 $CA+CB \cong AC+BC$.

Hence $(A+B)C \cong AC+BC$.

<u>Definition 1.13.</u> The ordinal sum of two posets A and B is $A \oplus B = C$ where $C = A \cup B$ and $a \leq b$ in C if and only if 1) $a \leq b$ in A, 2) $a \leq b$ in B or 3) $a \in A$, $b \in B$.

<u>Definition 1.14.</u> The ordinal product of two posets A and B is $A \circ B = D$ where $D = A \times B$ and $d_1 \leq d_2$ in D means $d_1 = (a_1, b_1)$, $d_2 = (a_2, b_2)$ for $a_1, a_2 \in A$, $b_1, b_2 \in B$ and

i) $a_1 < a_2$ in A

or ii) $a_1 = a_2$ in A and $b_1 \leq b_2$ in B.

<u>Definition 1.15.</u> The ordinal power of poset B to the exponent poset A is $E = {}^{A}B$ where $f \in E$ if and only if f is a function whose domain is A and whose range is a subset of B. $f \leq g$ in E if and only if $f(a) \leq g(a)$ or $g(a) \leq f(a)$ in B for all $a \in A$, and for every $a \in A$ such that $g(a) < f(a)$ in B there is an $a_1 < a$ in A for which $f(a_1) \leq g(a_1)$ in B.

<u>Theorem 1.7.</u> If A and B are posets, then $A \oplus B$ is a poset.

Proof: Let $C = A \oplus B$. Let $a \leq b$ and $b \leq c$ in C. Either 1) $c \in A$ or 2) $c \in B$. If 1) $c \in A$, then $a, b \in A$ so $a \leq b$, $b \leq c$, $a \leq c$ in A. Hence $a \leq c$ in C. If 2) $c \in B$ then either 2.1) $a \in A$ or 2.2) $a \in B$. If 2.1) $a \in A$ then $a \leq c$ by the addition definition. If 2.2) $a \in B$ then $b \in B$ for $a \leq x$ for no $x \in A$. Thus $a \leq b$, $b \leq c$ in B so $a \leq c$ in B and $a \leq c$ in C. The relation on C is transitive.

Let $a \leq b$ and $b \leq a$ in C. If $a \in A$ then $b \in A$ for $x \leq a$ for no $x \in B$. Thus $a \leq b$, $b \leq a$ in A so $a = b$. If $a \in B$ then $b \in B$ for $a \leq x$ for no $x \in A$. Thus $a \leq b$, $b \leq a$ in B so $a = b$. The relation on C is antisymmetric.

Let $a \in C$. If $a \in A$ then $a \leq a$ in A so $a \leq a$ in C. If $a \in B$ then $a \leq a$ in B so $a \leq a$ in C. In either case the relation on C is reflexive.

The relation on C is transitive, reflexive and antisymmetric. Thus C is a poset.

<u>Theorem 1.8.</u> The ordinal product of two posets is a poset.

Proof: Let $D = A \circ B$ be the ordinal product of posets A and B.

Let $d_1 \leq d_2$, $d_2 \leq d_3$ in D. Then $d_1 = (a_1, b_1)$, $d_2 = (a_2, b_2)$ and $d_3 = (a_3, b_3)$ where $a_1 < a_2$ in A or $a_1 = a_2$ in A, $b_1 \leq b_2$ in B and $a_2 < a_3$ in A or $a_2 = a_3$ in A, $b_2 \leq b_3$ in B. Suppose $a_1 < a_2$ in A. Then $a_1 < a_3$ in A and hence $d_1 \leq d_3$ in D. Suppose $a_1 = a_2$ in A. Then either $a_2 < a_3$ in A, in which case $a_1 < a_3$ in A so $d_1 \leq d_3$ in D, or $a_2 = a_3$ in A. If $a_1 = a_2 = a_3$ in A then $b_1 \leq b_2$, $b_2 \leq b_3$ so $b_1 \leq b_3$ in B and hence $d_1 \leq d_3$ in D. In every case $d_1 \leq d_2$, $d_2 \leq d_3$ in D implies $d_1 \leq d_3$ in D.

Let $d_1 \leq d_2$, $d_2 \leq d_1$ in D. $a_1 < a_2$ in A, $a_2 < a_1$ in A is not possible. Hence $a_1 = a_2$. Then $b_1 \leq b_2$ and $b_2 \leq b_1$ in B, so $b_1 = b_2$. Thus $d_1 \leq d_2$, $d_2 \leq d_1$ in D implies $d_1 = d_2$.

Let $d \in D$. Then $a \in A$ and $a = a$ in A, $b \in B$ and $b \leq b$ in B. Hence $d \leq d$ in D.

The relation imposed on $A \circ B$ by the definition of ordinal multiplication is transitive, antisymmetric and reflexive. Thus D is a poset.

It is not generally true that the ordinal power of one poset to another poset is a poset.

<u>Example 1.1.</u> Let B be the ordered set of two elements a and b where $a \leq a$, $a \leq b$, $b \leq b$. Let J be the set of integers with their usual ordering. Clearly B and J are posets. Let $E = {}^{J}B$.

f, g, h ϵ E exist such that $f = \{(x,y) \mid x \in J$ and if x is odd $y = b$, if x is even $y = a\}$,

$g = \{(x,y) \mid x \in J$ and if x is even $y = b$, if x is odd $y = a\}$,

and $h = \{(x,y) \mid x \in J$ and if x is odd $y = a$, if x is even and $x/2$ is even $y = a$, if x is even and $x/2$ is odd $y = b\}$.

The following illustrates these sets:

$$f = \begin{array}{c} \vdots \\ (4, a) \\ (3, b) \\ (2, a) \\ (1, b) \\ (0, a) \\ (-1, b) \\ (-2, a) \\ (-3, b) \\ (-4, a) \\ (-5, b) \\ (-6, a) \\ \vdots \end{array} \qquad g = \begin{array}{c} \vdots \\ (4, b) \\ (3, a) \\ (2, b) \\ (1, a) \\ (0, b) \\ (-1, a) \\ (-2, b) \\ (-3, a) \\ (-4, b) \\ (-5, a) \\ (-6, b) \\ \vdots \end{array} \qquad h = \begin{array}{c} \vdots \\ (4, a) \\ (3, a) \\ (2, b) \\ (1, a) \\ (0, a) \\ (-1, a) \\ (-2, b) \\ (-3, a) \\ (-4, a) \\ (-5, a) \\ (-6, b) \\ \vdots \end{array}$$

Now $g \leqq f$ in E for let $x \in J$ such that $f(x) < g(x)$ in B. Then x is even. Now $x-1 < x$ in J and $g(x-1) < f(x-1)$ in B.

$f \leqq h$ in E for let $x \in J$ such that $h(x) < f(x)$ in B. Then x is odd. Now since x is odd either $(x-1)/2$ is odd or $(x-3)/2$ is odd. If $(x-1)/2$ is odd, let $x_1 = x-1$. If not, let $x_1 = x-3$. Now $x_1 < x$ in J and $f(x_1) < h(x_1)$ in B.

By the transitive property if E is a poset $g \leqq h$.

Consider $g(4) = b$ and $h(4) = a$. Clearly $h(4) < g(4)$ in B. But for all x in J either $g(x) = h(x)$ (when x is odd or x/2 is odd) or $h(x) < g(x)$ in B (when x/2 is even). Thus no $x_1 < 4$ exists for which $g(x_1) < h(x_1)$ and hence $g \nless h$. Hence E is not a poset.

CHAPTER II

WEAK ORDINALS, CHAINS AND ORDINALS

Definition 2.1. The element a of poset A is a minimal element of A if and only if $x<a$ for no $x \in A$.

Definition 2.2. Let $O[A]$ be the set of all minimal elements of poset A.

Definition 2.3. A partly ordered set X is weakly well ordered, or a weak ordinal, if and only if every subset of X contains at least one minimal element.

Definition 2.4. A weak ordinal A is an ordinal if and only if every subset S of A has exactly one minimal element. The minimal element of an ordinal is called the least element.

Theorem 2.1. If B is a subset of a weak ordinal A, then B is a weak ordinal.

Proof: Let $S \subset B$. Then $S \subset A$. Hence S has a minimal element. Every subset of B has a minimal element; so B is a weak ordinal.

Theorem 2.2. The ordinal sum of two weak ordinals is a weak ordinal.

Proof: Let $C = A \oplus B$ where A and B are each weak ordinals. Let $S \subset C$. Suppose $S \subset B$. Then S has a minimal element. Suppose $S \not\subset B$. Then $S_1 = S \cap A$ is a non void subset of A and hence there is an $x \in S_1$ such that $x \in O[S_1]$. Now $x \in O[S]$ for let $b \in S$. Either $b \in A$ or $b \in B$. If $b \in A$, then b ε S_1 so $b \nless x$. If b ε B then $x<b$. In every case S has a minimal element. Hence C is a weak ordinal.

Theorem 2.3. The ordinal product of two weak ordinals is a weak ordinal.

Proof: Let $C = A \circ B$ where A and B are both weak ordinals. Let $S \subset C$. The following exhibits a minimal element in S.

Recall from the definition of ordinal multiplication that $C = A \times B$. Thus $\underline{D}(S) \subset \underline{D}(C) = A$. Hence $O[\underline{D}(S)]$ is not empty. Let a ε $O[\underline{D}(S)]$. Let $S_1 \subset S$ such that $\underline{D}(S_1) = \{a\}$. Since $\underline{R}(S_1) \subset B$ then $O[\underline{R}(S_1)]$ is not empty. Let b ε $O[\underline{R}(S_1)]$ be chosen. The following argument shows that (a,b) ε O[S].

Let (c,d) ε S where $(c,d) \neq (a,b)$. Then $c \nless a$ for a ε $O[\underline{D}(S)]$. Suppose $c \neq a$. Then $(c,d) \nless (a,b)$ for $(c,d)<(a,b)$ only if $c \leq a$. Suppose $c = a$. Then $d \neq b$ for $c = a$, $d = b$ implies $(c,d) = (a,b)$. Also $d \nless b$ for d ε $\underline{R}(S_1)$ and b ε $O[\underline{R}(S_1)]$. Thus $d \nless b$ so $(c,d) \nless (a,b)$. Thus (a,b) ε O[S]. Hence C is a weak ordinal.

Theorem 2.4. The cardinal sum of two weak ordinals is a weak ordinal.

Proof: Let $C = A+B$ where A and B are weak ordinals. Let $S \subset C$. Suppose $S_1 = S \cap A$ is not empty. Then $O[S_1]$ is not empty since $S_1 \subset A$. Let a ε $O[S_1]$ be chosen. a ε O[S] for if x ε S then x ε A or x ε B. If x ε A then $x \nless a$. If x ε B then $x \nless a$. Suppose $S_1 = S \cap A$ is empty. Then $S \subset B$ so O[S] is not empty. Hence C is a weak ordinal.

Theorem 2.5. The cardinal product of two weak ordinals is a weak ordinal.

Proof: Let $C = AB$ where A and B are both weak ordinals. Let $S \subset C$. Then $\underline{D}(S) \subset A$ and hence has a minimal element. Let $a \in O[\underline{D}(S)]$ be chosen. Let $b \in \underline{R}(S) \subset B$ be chosen. $(a,b) \in O[S]$ for if $(c,d) \in S$ then $c \nless a$ so $(c,d) \nless (a,b)$. Hence C is a weak ordinal.

It is not generally true that the cardinal power of one weak ordinal to another is a weak ordinal.

Example 2.1. Let $C = B^{\omega}$ where $B = \{0,1\}$ and ω is the set of non negative integers with both B and ω ordered in the usual manner. Let $S \subset C$ such that $g \in S$ if and only if $1 \in \underline{R}(g)$. Let $f \in S$ be chosen. Since $\underline{D}(f) = \omega$ and ω is an ordinal, let n be the least element in $\underline{D}(f)$ such that $f(n) = 1$. Then $f(x) = 0$ for all $x<n$. Since f is isotone, $f(x) = 1$ for all $x>n$. Let $g \in S$ such that $g(x) = 1$ for all $x>n$ and $g(x) = 0$ for all $x \leq n$. Then $g(x) \leq f(x)$ for all $x \in \omega$ and $f(n) \nleq g(n)$ so $g<f$. Since for $f \in S$ there exists $g \in S$ such that $g<f$, C is not a weak ordinal.

It is not generally true that the ordinal power of one weak ordinal to another is a weak ordinal.

Example 2.2. Let $C = {}^{\omega}B$ for B and ω as defined in Example 2.1. Let $f \in C$ such that $f(n) = 0$ for all $n \in \omega$. Let $C_1 = C - \{f\}$.

Suppose $g \in C_1$ is a minimal element. Let M be the set of all $m \in \omega$ such that $g(m) = 1$. Since ω is an ordinal, there is a least element in M. Let m_1 be the least element in M. Let $h \in C_1$ such that $h(n) = g(n)$ for all $n \in \omega$ except m_1. Let $h(m_1) = 0$. Then $h(m_1)<g(m_1)$. Now $g(m) \nleq h(m)$ for all $m \in \omega$. Hence $h<g$. This contradicts the supposition that g is a minimal element in C_1. Hence C_1 has no minimal element.

Since C has a subset with no minimal element, then C is not a weak ordinal.

Definition 2.5. A chain A is a poset such that $a,b \in A$ implies $a \leq b$ or $b \leq a$.

Definition 2.6. If A is a poset the statement that C is a chain in A means that $C \subset A$ and C is a chain.

Definition 2.7. The statement that C is a maximal chain in a poset A means that C is a chain in A; and if N is a chain in A, then $C \not\subset N$.

Definition 2.8. Let A be a poset with relation $\leq$. A relation $\prec$ on A such that A is a chain under $\prec$ and $a \prec b$ if $a \leq b$ is called a strengthening of A.

The set A is clearly a chain under the relation $\prec$.

Definition 2.9. Let A and B be two chains which are not necessarily disjoint. Let R_α be the chain relation on A and let R_β be the chain relation on B. Furthermore let $R_\alpha = R_\beta$ for the set $A \cap B$. Let $C = A \cup B$. Let R_γ be the relation on C such that

1) $aR_\gamma b$ if $aR_\alpha b$;

2) $aR_\gamma b$ if $aR_\beta b$;

3) $aR_\gamma b$ if there exists a c such that $aR_\alpha c$, $cR_\beta b$ or $aR_\beta c$, $cR_\alpha b$;

and 4) $aR_\gamma b$ if $a \in A$, $b \in B$ and $bR_\gamma a$ is not implied by 1), 2) or 3).

Then C is a merger of A into B.

R_γ is obviously a chain relation on C. Also note that in the case of disjoint chains a merger is an ordinal sum.

Axiom 2.1. Every chain C in a poset A is a subset of a maximal chain M in A.

Birkhoff shows that Axiom 2.1 is equivalent to the axiom of choice.[1]

Theorem 2.6. A strengthening exists for every poset.

Proof: Let A be a poset. Then by Axiom 2.1 each chain in A is contained in a maximal chain in A. Let S be the set of maximal chains in A. Let S be well ordered.

Let s_1, s_2 be the first two elements of S. Let t_2 be the merger of s_1 into s_2. Let $s_\beta \in S$. Let $t_\beta' = \bigcup_{\alpha<\beta} t_\alpha$. If $a,b \in t_\beta'$ let $a \leq b$ if and only if $a \leq b$ in t_γ for some $\gamma < \beta$. Let t_β be t_β' merged into s_β. Observe that t_β' is a well defined chain which preserves the order of elements in all previous chains. For suppose there exists a first β such that t_β' is not such a chain. Let $a,b \in t_\beta'$. Then $a,b \in \bigcup_{\alpha<\beta} t_\alpha$. Let s_υ and s_ξ be the first maximal chains in which a and b appear,

[1]Garrett Birkhoff, Lattice Theory, Vol. XXV of American Mathematical Society Colloquim Publications, (Rhode Island, 1961), pp. 42-44.

respectively. Suppose $\upsilon = \xi$ and $a \leq b$ in s_υ. Then $a \leq b$ in all t_η for $\upsilon \leq \eta < \beta$. Hence $a \leq b \in t_\beta'$. Suppose $\upsilon < \xi$. Then $a \in t_\xi'$ and hence $a \leq b$ or $b \leq a$ in t_ξ and in all t_η for $\xi \leq \eta < \beta$. Thus $a \leq b$ or $b \leq a$ in t_β'. Hence, by contradiction, t_β is well defined for all β.

Let $\prec$ be the relation resulting from the merger of all elements of S as described above. $\prec$ is a strengthening of A.

Obviously a strengthening on A is not unique unless A is a chain, for the ordering of the elements of S is arbitrary.

Theorem 2.7. A poset A may be strengthened to an ordinal if and only if A is a weak ordinal.

Proof: 1) If A is a weak ordinal the strengthening described in Theorem 2.6 is an ordinal. For let $A_1 \subset A$. Each element of $0[A_1]$ is in a different $s \in S$. Let S_0 be the set of all such s. Let $x_1 \in 0[A_1]$ such that x_1 is an element of the least $s \in S_0$. Then x_1 is the least element in $0[A_1]$ by the construction of $\prec$. Suppose $a \in A_1$ such that $a \notin 0[A_1]$. Then $a \geq x$ for some $x \in 0[A_1]$. Hence $x_1 \prec x \prec a$ so $x_1 \prec a$. So x_1 is the least element of A_1 under $\prec$. Hence A is an ordinal.

2) If A is not a weak ordinal some subset of A has no minimal element. Let $A_1 \subset A$ be such a set. Let $x_0 \in A_1$. Since $x_0 \notin 0[a_1]$ there exists $x_1 < x_0$ in A_1. Suppose $x_0 > x_1 > \ldots x_n$ in A_1. Since $x_n \notin 0[A_1]$ there exists $x_{n+1} < x_n$ in A_1. Let $C = \{x_0, x_1, \ldots\}$. Now $x_0 \succ x_1 \succ \ldots x_n \succ x_{n+1} \ldots$ in C. Hence $C \subset A$ with no least element under $\prec$.

CHAPTER III

AUTOMORPHISMS ON POSETS

Definition 3.1. An automorphism on a partly ordered set X is an isomorphism whose domain is X and whose range is X.

Definition 3.2. A group is a set with a closed binary operation defined on it which is associative, has an identity and has an inverse for each element of the set.

Definition 3.3. Let f and g be two automorphisms on a poset A. Let $f \cdot g$ denote the relation whose domain is A and such that $f \cdot g(a) = f(g(a))$ for all $a \in A$.

Theorem 3.1. The set of automorphisms on a poset forms a group under the product operation defined in Definition 3.3.

Proof: Let f and g be automorphisms on a poset A. Then $f \cdot g$ is a reversible function. Let $a \leq b$ in A. Then $g(a) \leq g(b)$ so $f \cdot g(a) \leq f \cdot g(b)$. Hence $f \cdot g$ is an automorphism on A.

Let f_1 be the automorphism on A such that $f_1(a) = a$ for all $a \in A$. Clearly $f_1 \cdot g = g \cdot f_1 = g$ for every automorphism g on A. f_1 is the identity element.

Let f be an automorphism on A. Let f^{-1} be an automorphism on A such that $f^{-1}(a) = b$ if and only if $f(b) = a$. Then $f \cdot f^{-1} = f^{-1} \cdot f = f_1$. f^{-1} is the inverse of f.

Let f, g and h be automorphisms on A. Let $a \in A$. $(f\cdot(g\cdot h))(a) = f(g(h(a)))$. Also $((f\cdot g)\cdot h)(a) = f(g(h(a)))$. Hence $f\cdot(g\cdot h) = (f\cdot g)\cdot h$ and the product operation is associative.

Lemma 3.1. If a chain C has an automorphism f on it, $f \neq f_1$, then f^i is an automorphism on C for positive integer i and $f^i \neq f^j$ for positive integers i, j, $i \neq j$.

Proof: f^i is an automorphism on C by the closure property of groups.

Without loss of generality suppose $i<j$. $f^i \neq f^{i+1} = f\cdot f^i$ for, since f is not the identity, there is an $x \in C$ such that $f(f^i(x)) \neq f^i(x)$. Suppose $f^i(x)<f(f^i(x))$. Then $f(f^i(x))<f(f(f^i(x))$ so $f^i(x)<f^{i+1}(x)<f^{i+2}(x)$ hence $f^i \neq f^{i+2}$.

Suppose $f^i \neq f^{i+n}$ for n, a positive integer, because there is an $x \in C$ such that $f^i(x)<f^{i+n}(x)$. Then $f(f^i(x))<f(f^{i+n}(x))$ so $f^i \neq f^{i+n+1}$. A similar argument follows the supposition that $f^i(x)>f(f^i(x))$. Hence by induction $f^i \neq f^j$.

Theorem 3.2. If a chain C has an automorphism f on it, $f \neq f_1$, then there are infinitely many automorphisms on C.

Proof: Let f be an automorphism on a chain C where $f \neq f_1$, the identity element. Define $A = \{g \mid g_n = f^n, n = 1, 2, 3 \ldots.\}$. Since by Lemma 3.1, $g_i \neq g_j$ for $i \neq j$, there are infinitely many elements of A and hence infinitely many automorphisms on C.

Corollary 3.1. The set of integers J has infinitely many automorphisms on it.

Proof: The function f such that $f(i) = i+1$ for all $i \in J$ is a non identity automorphism on J.

Lemma 3.2. If $n \in J$ then $f(1) = n$ for exactly one automorphism f on J.

Proof: Let $n \in J$. Let f be a function such that for $i \in J$, $f(i) = i+n-1$. Then $f(1) = 1+n-1 = n$. Let $j \in J$. $f(j-n+1) = (j-n+1)+n-1 = j$. Thus f is a reversible function whose domain is J and whose range is J. Let $k,m \in J$ such that $k \leq m$. Then $k+n-1 \leq m+n-1$ so $f(k) \leq f(m)$. Also if $f(k) \leq f(m)$ then $k+n-1 \leq m+n-1$ so $k \leq m$. Thus f is an automorphism on J.

Suppose g is an automorphism on J such that $g(1) = n$. Furthermore suppose $g \neq f$. Then $g(i) \neq f(i)$ for some $i \in J$. Let J_1 be the set of all such i. Further let J_2 be the set of all such $i > 1$. This is a well ordered set. Let i_1 be the first element in J_2. Now $g(i_1-1) = i_1-1+n-1 = i_1+n-2$ so $g(i_1) > i_1+n-2$. But $g(i_1) \neq i_1+n-1$ for $f(i_1) = i_1+n-1$. Say $g(i_1) = k$. Then $i_1+n-2 < i_1+n-1 < k$. Thus i_1+n-1 is the image of an element m such that $i_1-1 < m < i_1$. No such m exists in J. J_2 has no first element and hence has no elements. A similar discussion for J_3, the set of all $i \in J_1$ such that $i < 1$, shows that J_1 has no elements. No i exists such that $g(i) \neq f(i)$. This contradicts the supposition $g \neq f$. Hence $g = f$.

Definition 3.4. Let f and g be automorphisms on J. Then fRg if and only if $f(1) \leq g(1)$.

Theorem 3.3. The set A_J of automorphisms on J is isomorphic to J.

Proof: Let θ be a function whose domain is A_J such that $\theta(f) = f(1)$. Then by Lemma 3.2 θ is a reversible function. Let fRg in A_J. Then $\theta(f) = f(1) \leq g(1) = \theta(g)$. If $\theta(f) \leq \theta(g)$ then f(1)Rg(1) so fRg. Hence $A_J \cong J$.

Definition 3.5. Let A and B be sets. The set of all functions whose domain is B and whose range is a subset of A is denoted by A^{*B}.

Theorem 3.4. There are uncountably many automorphisms on the set R^+ of non negative rational numbers.

Proof: Let $f = 2^{*\omega}$ where 2 is the cardinal set $\{0,1\}$ and ω is the cardinal set of non negative integers.

Let ' be a function whose domain is R^+ and whose range is ω such that if $r \in R^+$ then $r' \leq r < r'+1$.

Let θ be a function whose domain is R^+ and such that for $r \in R^+$

$$\theta(r) = r'+a(r-r')+b(r-r')^2$$

where $b = f(r')$ and $a = \mathrm{mod}_2(b+1)$. The range of θ is R^+ for let $x \in R^+$. If $f(x') = 0$ then x is the θ image of x for

$$\theta(x) = x'+1(x-x')+0(x-x')^2 = x.$$

If $f(x') = 1$ then x is the θ image of $\sqrt{x-x'}+x'$ for first note that $(\sqrt{x-x'}+x')' = (x')' = x'$ since $\sqrt{x-x'}<1$ and $x' \in \omega \subset R^+$. Then

$$\theta(\sqrt{x-x'}+x') = x'+0(\sqrt{x-x'}+x'-x')+1(\sqrt{x-x'}+x'-x')^2$$
$$= x'+(\sqrt{x-x'})^2$$
$$= x'+x-x'$$
$$= x.$$

θ is reversible for let $x \in R^+$ such that $\theta(r_1) = x$ and $\theta(r_2) = x$. Then

$$1)\quad x = r_1'+a_1(r_1-r_1')+b_1(r_1-r_1')^2$$

and $$2)\quad x = r_2'+a_2(r_2-r_2')+b_2(r_2-r_2')^2.$$

But since $0\leq s_1 = a_1(r_1-r_1')+b_1(r_1-r_1')^2<1$ and $0\leq s_2 = a_2(r_2-r_2')+b_2(r_2-r_2')^2<1$, then $-1<s_2-s_1<1$. Since $r_1' = r_2'+s_2-s_1$, then $r_2'-1<r_1'<r_2'+1$. Now $r_1', r_2' \in \omega$. Hence $r_1' = r_2'$. It follows from the definition that $a_1 = a_2$, $b_1 = b_2$; so 1) and 2) above become

$$r_1'+a_1(r_1-r_1')+b_1(r_1-r_1')^2 = r_1'+a_1(r_2-r_1')+b_1(r_2-r_1')^2.$$

This simplifies to

$$a_1(r_1-r_2)+b_1((r_1-r_1')^2-(r_2-r_1')^2) = 0.$$

If $b_1 = 0$, then $a_1 = 1$ and $r_1 = r_2$. If $b_1 = 1$, then $a_1 = 0$ and $(r_1-r_1')^2 = (r_2-r_1')^2$. Since $r_1-r_1'\geq 0$, and $r_2-r_1'\geq 0$, then $r_1-r_1' = r_2-r_1'$ and $r_1 = r_2$. In every case $\theta(r_1) = x$, $\theta(r_2) = x$ implies $r_1 = r_2$.

θ preserves order, for let $p\leq q$ in R^+.

$$\theta(p) = p'+a_1(p-p')+b_1(p-p')^2$$

and $$\theta(q) = q'+a_2(q-q')+b_2(q-q')^2.$$

Either $p' = q'$ or $p'<q'$. Suppose $p'<q'$. Then $p'+1\leq q'$. Also $a_1(p-p')+a_2(p-p')^2<1$. Then

$$\begin{aligned}\theta(p) &= p'+a_1(p-p')+b_1(p-p')^2\\ &< p'+1\\ &< q'\\ &\leq q'+a_2(q-q')+b_2(q-q')^2\\ &= \theta(q).\end{aligned}$$

Suppose $p'=q'$. Then $a_1 = a_2$, $b_1 = b_2$ and $p-q' \leq q-q'$. Hence

$$\begin{aligned}\theta(p) &= p'+a_1(p-p')+b_1(p-p')^2\\ &= q'+a_2(p-q')+b_2(p-q')^2\\ &\leq q'+a_2(q-q')+b_2(q-q')^2\\ &= \theta(q).\end{aligned}$$

In every case $p \leq q$ in R^+ implies $\theta(p) \leq \theta(q)$ in R^+.

Let $\theta(p) \leq \theta(q)$ in R^+. Suppose $q<p$. Then $\theta(q) \leq \theta(p)$. Thus $\theta(q) = \theta(p)$. Thus either $q = p$ or θ is not reversible. Either is a contradiction. Hence $\theta(p) \leq \theta(q)$ implies $p \leq q$.

Thus for $f \in 2^{*\omega}$ an automorphism θ has been defined. No two such automorphisms are equal for let $f,g \in 2^{*\omega}$, $f \neq g$. Let θ_f be the automorphism defined on R^+ using f to define a and b. Let θ_g be the automorphism on R^+ using g to define a and b. For some $n \in \omega$, $f(n) \neq g(n)$. Say $f(n) = 1$, $g(n) = 0$. Consider the rational $n+1/2$.

$$\begin{aligned}\theta_f(n+1/2) &= n+1(n+1/2-n)+0(n+1/2-n)^2\\ &= n+1/2.\\ \theta_g(n+1/2) &= n+0(n+1/2-n)+1(n+1/2-n)^2\\ &= n+1/4.\end{aligned}$$

Hence $f \neq g$ implies $\theta_f \neq \theta_g$.

There are uncountably many elements of $2^{*\omega}$ and for each $f \in 2^{*\omega}$ there exists a distinct automorphism θ_f on R^+. Hence R^+ has uncountably many automorphisms on it.

Definition 3.6. The statement that the cardinal of a set A is less than or equal to the cardinal of a set B means that there is a reversible function whose domain is A and whose range is a subset of B.

Definition 3.7. The cardinal of a set A is equal to the cardinal of a set B if and only if the cardinal of each is respectively less than or equal to the cardinal of the other.

Definition 3.8. The cardinal of A is less than the cardinal of B if and only if the cardinal of A is less than or equal to that of B and their cardinals are not equal.

Theorem 3.5. Let S be a set with cardinal greater than or equal to the cardinal of $R^\#$, the real numbers. There exists a chain D whose cardinal is equal that of S such that the cardinal of all automorphisms on D is greater than the cardinal of S.

Proof: Let E be a well ordered set whose cardinal is equal to S. Let (0,1) be the open unit interval in $R^\#$. Let $D = E \circ (0,1)$.

D is a chain, for let $(a_1,b_1),(a_2,b_2) \in D$. Then $a_1,a_2 \in E$ so

1) $a_1<a_2$, hence $(a_1,b_1)<(a_2,b_2)$;

2) $a_2<a_1$, hence $(a_2,b_2)<(a_1,b_1)$;

or 3) $a_1 = a_2$, in which case since $b_1,b_2 \in (0,1)$ then

3.1) $b_1 \leq b_2$, hence $(a_1,b_1) \leq (a_2,b_2)$;

or 3.2) $b_2 \leq b_1$, hence $(a_2,b_2) \leq (a_1,b_1)$.

The cardinal of D is equal to the greatest of the cardinals of E and of (0,1). Hence the cardinal of D is equal to the cardinal of S.

Let $f \in 2^{*E}$. Let θ be a function on D such that

1) if $(x,b) \in D$ and $f(x) = 1$, $\theta((x,b)) = (x,b)$,

and 2) if $(x,b) \in D$ and $f(x) = 0$, $\theta((x,b)) = (x,b^2)$.

Clearly if $(x,b) \in D$ then (x,b) is the θ image of exactly one element in D, namely

1) (x,b) if $f(x) = 1$

or 2) $(x,\sqrt{b})$ if $f(x) = 0$.

Thus θ is a reversible function and $\underline{R}(\theta) = D$.

Let $(x_1,b_1) \leq (x_2,b_2)$ in D. Then $x_2 \nless x_1$. Suppose $x_1 < x_2$. Now $(x_1,a) < (x_2,b)$ for all $a,b \in (0,1)$ so

$$\theta((x_1,b_1)) \leq (x_1,b_1) \leq \theta((x_2,b_2)).$$

Suppose $x_1 = x_2$. Then $b_1 \leq b_2$. Suppose $f(x_1) = 1$. Then

$$\theta((x_1,b_1)) = (x_1,b_1) \leq (x_2,b_2) = \theta((x_2,b_2)).$$

Suppose $f(x_1) = 0$. Then

$$\theta((x_1,b_1)) = (x_1,b_1^2) \leq (x_2,b_2^2) = \theta((x_2,b_2)).$$

In every case $(x_1,b_1) \leq (x_2,b_2)$ in D implies $\theta((x_1,b_1)) \leq \theta((x_2,b_2))$ in D.

Suppose $\theta(x_1,b_1) \leq \theta(x_2,b_2)$ for some $(x_1,b_1),(x_2,b_2) \in D$. Suppose further that $(x_2,b_2) < (x_1,b_1)$. Then $x_2 \leq x_1$. But

$$\theta((x_1,b_1)) = (x_1,a_1) \leq (x_2,a_2) = \theta((x_2,b_2))$$

contradicts $x_2 < x_1$. Suppose $x_2 = x_1$. Then $b_2 < b_1$. But this leads to a contradiction. For if $f(x_1) = 1$, then $(x_2,b_2)<(x_1,b_1)$ implies $\theta((x_2,b_2))<\theta((x_1,b_1))$. And if $f(x_1) = 0$, then $(x_2,b_2)<(x_1,b_1)$ implies $b_2^2<b_1^2$; but $\theta(x_1,b_1)\leq\theta(x_2,b_2)$ implies $b_1^2\leq b_2^2$. Thus θ is an isomorphism on D.

For each $f \in 2^{*E}$ define θ_f as above. Let A be the set of all such θ.

If $g,f \in 2^{*E}$ and $g \neq f$, then $\theta_g \neq \theta_f$. For $g \neq f$ implies $x \in E$ such that $g(x) \neq f(x)$. Say $g(x) = 1$, $f(x) = 0$. Consider the element $(x,1/2) \in D$.

$$\begin{aligned}\theta_g((x,1/2)) &= (x,1/2)\\ &\neq (x,1/4)\\ &= \theta_f((x,1/2)).\end{aligned}$$

Hence for each element $f \in 2^{*E}$ there exists an element $\theta_f \in A$. Hence the cardinal of A is greater than or equal to the cardinal of 2^{*E} which is equal to the cardinal of 2^{*S}.

The cardinal of the set of all automorphisms on $R^{\#}$ is clearly greater than or equal to the cardinal of $R^{\#}$. It may be that it is exactly equal to the cardinal of $R^{\#}$. If the cardinal of the set of automorphisms on an interval (a,b) in $R^{\#}$ is equal to the cardinal of $R^{\#}$ then so is the cardinal of the set of automorphisms on $R^{\#}$. The interval $(-1,1)$ is chosen in the next theorem to simplify arithmetic. It can easily be generalized to any $(a,b) \subset R^{\#}$.

<u>Theorem 3.6.</u> The cardinal of the set of automorphisms on the open interval $(-1,1)$ in $R^{\#}$ is equal to the cardinal of the set of automorphisms on $R^{\#}$.

Proof: Let A be the set of automorphisms on $(-1,1)$ and let B be the set of automorphisms on $R^{\#}$.

1) Then the cardinal of A is less than or equal to the cardinal of B. For, let θ be a function whose domain is A and such that for $f \in A$, $\theta(f) = g$, a function defined as follows. $g(x) = x$ for all $x \in R^{\#}$, $x \notin (-1,1)$. $g(x) = f(x)$ for all $x \in (-1,1)$. Clearly $g \in B$. Let $f_1 \neq f_2$ in A. Then there is an $x \in (-1,1)$ such that $f_1(x) \neq f_2(x)$. Since for this x, $(\theta(f_1))(x) = f_1(x)$ and $(\theta(f_2))(x) = f_2(x)$ then $\theta(f_1) \neq \theta(f_2)$.

Thus θ is a reversible function whose domain is A and whose range is a subset of B. Hence the cardinal of A is less than or equal to the cardinal of B.

2) The cardinal of B is less than or equal to that of A.

Let ϕ be a function whose domain is B and such that if $f \in B$ then $\phi(f) = g$, a function defined as follows. Let $g(x) = \tanh(f(\tanh^{-1}(x)))$ for $x \in (-1,1)$. Since tanh, f and $\tanh^{-1}$ are each reversible, then g is clearly reversible. Let $x_1 \leq x_2$ in $(-1,1)$. Then $\tanh^{-1}(x_1) \leq \tanh^{-1}(x_2)$. Since f is an automorphism, $f(\tanh^{-1}(x_1)) \leq f(\tanh^{-1}(x_2))$. Finally $\tanh(f(\tanh^{-1}(x_1))) \leq \tanh(f(\tanh^{-1}(x_2)))$ or $g(x_1) \leq g(x_2)$. In a similar manner $g(x_1) \leq g(x_2)$ implies $x_1 \leq x_2$. Thus g is an automorphism on $(-1,1)$. So $g \in A$.

Suppose $f_1 \neq f_2$ in B. Then $f_1(x) \neq f_2(x)$ for some $x \in R^{\#}$. Now $y = \tanh(x)$ for some $y \in (-1,1)$. So

$$\begin{aligned} g_1(y) &= \tanh(f_1(\tanh^{-1}(\tanh(x)))) \\ &= \tanh(f_1(x)) \\ &\neq \tanh(f_2(x)) \\ &= \tanh(f_2(\tanh^{-1}(\tanh(x)))) \\ &= g_2(y). \end{aligned}$$

Thus $g_1 \neq g_2$.

Thus ϕ is a reversible function whose domain is B and whose range is a subset of A. Hence the cardinal of B is less than or equal to the cardinal of A.

3) From 1) and 2) the cardinal of B is equal to the cardinal of A.

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