DIFFERENTIABLE FUNCTIONS

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DIFFERENTIABLE FUNCTIONS

THESIS

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CHAPTER I

INTRODUCTION

Preliminary Remarks

1.1. Perhaps the greatest of all branches of mathematics and certainly the dominant force in mathematics for almost 300 years is the calculus. And the central idea of the calculus as developed by Newton is the derivative of a function. Although calculus in its simplest form pertains to a function of one real variable, the applications of the subject more often than not involve functions of several variables and may even concern functions of an infinite number of variables.

The primary purpose of this thesis is to carefully develop and prove some of the fundamental, classical theorems of the differential calculus for functions of two real variables. The generalization of the ideas of derivative and differentiable functions to two variables gives the essence of the generalization to n-variables. Since courses in advanced calculus seldom give rigorous proofs in two-variable calculus, first-year graduate courses usually restrict the treatment to functions of one variable, and since advanced graduate courses in analysis often involve abstract spaces, it seems that the two-variable case is neglected.

Chapter I includes most of the relevant definitions on sets and functions that are needed in the thesis, and a statement of certain basic results on sets of real numbers. Chapter II will consist of proofs of fundamental theorems on continuous and differentiable functions of one real variable. Chapter III will include proofs of fundamental, classical theorems on differentiable functions of two real variables, some of which are extensions of the theorems on differentiable functions of one real variable in Chapter II.

Definitions

1.2. A function from A into B is a correspondence which mates with every element t of the set A a unique element f(t) of the set B. A is called the domain of the function and $R_f = \{f(t) | t \in A\}$ is called the range of f. The function f is said to be defined on A.

1.3. A real function of one real variable is a function from A into R, where A is a set of real numbers and R is the set of all real numbers.

1.4. A real function of two real variables is a function from A into R, where A is a set of ordered pairs of real numbers and R is the set of all real numbers.

1.5. All functions considered in this thesis are functions of the type defined in 1.3 and 1.4, and the domain of definition will clarify which type is being considered.

1.6. If $c < d$, then $[c,d]$ will denote the set of all real numbers t such that $c \leq t \leq d$ and (c,d) will denote the set of all real numbers t such that $c < t < d$; $[c,d]$ is called a <u>closed interval</u> and (c,d) is called an <u>open interval</u>.

1.7. If ζ is a real number and $\delta > 0$, then $I(\zeta;\delta)$ will denote the set of all real numbers t such that

$$|t-\zeta| < \delta;$$

$I(\zeta;\delta)$ is an open interval of center ζ and length 2δ.

1.8. The symbol ::: will mean "means" or "means that."

1.9. $I_1, I_2, \ldots, I_n, \ldots$ <u>is a descending infinitesimal sequence of closed intervals</u> ::: for each n, $I_{n+1} \subset I_n$, and $\lim_{n\to\infty} l(I_n) = 0$, <u>i.e.</u> if $\epsilon > 0$ is chosen, then there exists a positive integer N so that whenever n is chosen such that $n > N$, then $l(I_n) < \epsilon$, where $l(I_n)$ denotes the length of the closed interval I_n.

1.10. <u>A set</u> T <u>of open intervals covers a set</u> S <u>of real numbers</u> ::: each element of S belongs to at least one of the intervals of T.

1.11. ζ <u>is an interior point of</u> S ::: there exists a $\delta > 0$ so that $I(\zeta;\delta) \subset S$.

1.12. ζ <u>is a limit point of</u> S ::: if $\delta > 0$ is chosen, then there is a $t \in S$ so that $t \in I(\zeta;\delta)$ and $t \neq \zeta$.

1.13. U <u>is an upper bound of</u> S ::: for each $t \in S$, $t \leq U$.

1.14. L <u>is a lower bound of</u> S ::: for each $t \in S$, $t \geq L$.

1.15. S <u>is bounded</u> ::: there exists an $M > 0$ so that for each $t \in S$, $|t| < M$.

1.16. K <u>is a least upper bound of</u> S ::: K is an upper bound of S, and if $\epsilon > 0$ is chosen, then there is a $t \in S$ such that $t > K - \epsilon$.

1.17. k *is a greatest lower bound of* S ::: k is a lower bound of S, and if $\epsilon > 0$ is chosen, then there is a $t \in S$ such that $t < k + \epsilon$.

1.18. If f is defined on $I(\xi ; \delta_1)$, then f *is continuous at* ξ ::: if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever t is chosen such that $|t - \xi| < \delta$, then $|f(t) - f(\xi)| < \epsilon$.

1.19. If there exists a $\delta_1 > 0$ so that for each t such that $\xi \leq t \leq \xi + \delta_1$ f(t) is defined, then f *is right-continuous at* ξ ::: if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever t is chosen such that $\xi \leq t < \xi + \delta$, then $|f(t) - f(\xi)| < \epsilon$.

1.20. If there exists a $\delta_1 > 0$ so that for each t such that $\xi - \delta_1 < t \leq \xi$ f(t) is defined, then f *is left-continuous at* ξ ::: if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever t is chosen such that $\xi - \delta < t \leq \xi$, then $|f(t) - f(\xi)| < \epsilon$.

1.21. If f is defined on $[c,d]$, then f *is continuous on* $[c,d]$ ::: f is continuous at each ξ such that $c < \xi < d$, f is right-continuous at c, and f is left-continuous at d.

1.22. If f is defined on $I(\xi ; \delta_1)$, then f *is differentiable at* ξ ::: there exists a real number A so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever t is chosen such that $0 < |t - \xi| < \delta$, then

$$\left| \frac{f(t) - f(\xi)}{t - \xi} - A \right| < \epsilon .$$

If f is differentiable at ξ, then A is denoted by $f'(\xi)$ and $f'(\xi)$ is called <u>the derivative of</u> f <u>at</u> ξ.

1.23. If (a,b) is an ordered pair of real numbers and $\delta > 0$, then $C((a,b); \delta)$ will denote the set of all ordered pairs of real numbers $(x,y) = (a+\Delta x,\ b+\Delta y)$ such that

$$\sqrt{\Delta^2 x + \Delta^2 y} < \delta;$$

$C((a,b); \delta)$ is called an <u>open circle</u>.

1.24. $N((a,b); \delta_1; \delta_2)$ will denote the set of all ordered pairs of real numbers $(x,y) = (a+\Delta x, b+\Delta y)$ such that $|\Delta x| < \delta_1$ and $|\Delta y| < \delta_2$; $N((a,b); \delta_1; \delta_2)$ is called an <u>open oriented rectangle</u>.

1.25. If P is a set of ordered pairs of real numbers, then (a,b) <u>is an interior point of</u> P ::: there exists a $\delta > 0$ so that $C((a,b); \delta) \subset P$.

1.26. If for each $x \in I(a; \delta_1)$ $f(x,b)$ is defined, then f <u>is continuous at</u> (a,b) <u>with respect to the first variable</u> ::: if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δx is chosen such that $|\Delta x| < \delta$, then $|f(a+\Delta x,b)-f(a,b)| < \epsilon$.

1.27. If for each $y \in I(b; \delta_1)$ $f(a,y)$ is defined, then f <u>is continuous at</u> (a,b) <u>with respect to the second variable</u> ::: if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δy is chosen such that $|\Delta y| < \delta$, then $|f(a,b+\Delta y)-f(a,b)| < \epsilon$.

1.28. If f is defined on $C((a,b); \delta_1)$, then f <u>is continuous at</u> (a,b) ::: if $\epsilon > 0$ is chosen, then there exists a

$\delta > 0$, $\delta \le \delta_1$, so that whenever Δx and Δy are chosen such that

$$\sqrt{\Delta^2 x + \Delta^2 y} < \delta,$$

then $|f(a+\Delta x, b+\Delta y) - f(a,b)| < \epsilon$. The function f <u>is</u> <u>continuous</u> <u>on</u> $C((a,b);\delta_1)$ ::: for each $(x,y) \in C((a,b);\delta_1)$, f is continuous at (x,y).

1.29. If for each $x \in I(a;\delta_1)$ $f(x,b)$ is defined, then f <u>is</u> <u>differentiable</u> <u>at</u> (a,b) <u>with</u> <u>respect</u> <u>to</u> <u>the</u> <u>first</u> <u>variable</u> ::: there exists a real number A so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \le \delta_1$, so that whenever Δx is chosen such that $0 < |\Delta x| < \delta$, then

$$\left| \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x} - A \right| < \epsilon .$$

If f is differentiable at (a,b) with respect to the first variable, then A is denoted by $f_1(a,b)$ and $f_1(a,b)$ is called <u>the</u> <u>first-order</u> <u>partial</u> <u>derivative</u> <u>of</u> f <u>at</u> (a,b) <u>with</u> <u>respect</u> <u>to</u> <u>the</u> <u>first</u> <u>variable</u>. The function f_1 <u>exists</u> <u>on</u> $C((a,b);\delta_1)$ ::: for each $(x,y) \in C((a,b);\delta_1)$, $f_1(x,y)$ exists.

1.30. If for each $y \in I(b;\delta_1)$ $f(a,y)$ is defined, then f <u>is</u> <u>differentiable</u> <u>at</u> (a,b) <u>with</u> <u>respect</u> <u>to</u> <u>the</u> <u>second</u> <u>variable</u> ::: there exists a real number A so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \le \delta_1$, so that whenever Δy is chosen such that $0 < |\Delta y| < \delta$, then

$$\left| \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} - A \right| < \epsilon .$$

If f is differentiable at (a,b) with respect to the second variable, then A is denoted by $f_2(a,b)$ and $f_2(a,b)$ is called <u>the first-order partial derivative of</u> f <u>at</u> (a,b) <u>with respect to the second variable</u>. The function f_2 <u>exists on</u> $C((a,b);\delta_1)$ ::: for each $(x,y) \in C((a,b);\delta_1)$, $f_2(x,y)$ exists.

1.31. If f is defined on $C((a,b);\delta_1)$, and if for each Δy such that $|\Delta y| < \delta_1$ $f_1(a,b+\Delta y)$ exists, then f <u>is differentiable at</u> (a,b) <u>with respect to the first variable and then with respect to the second variable</u> ::: there exists a real number A so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δy is chosen such that $0 < |\Delta y| < \delta$, then

$$\left| \frac{f_1(a,b+\Delta y)-f_1(a,b)}{\Delta y} - A \right| < \epsilon .$$

If f is differentiable at (a,b) with respect to the first variable and then with respect to the second variable, then A is denoted by $f_{12}(a,b)$ and $f_{12}(a,b)$ is called <u>the second-order partial derivative of</u> f <u>at</u> (a,b) <u>with respect to the first variable and then with respect to the second variable</u>. If f_1 exists on $C((a,b);\delta_1)$, then f_{12} <u>exists on</u> $C((a,b);\delta_1)$ ::: for each $(x,y) \in C((a,b);\delta_1)$, $f_{12}(x,y)$ exists.

1.32. If f is defined on $C((a,b);\delta_1)$, and if for each Δx such that $|\Delta x| < \delta_1$ $f_2(a+\Delta x,b)$ exists, then f <u>is differentiable at</u> (a,b) <u>with respect to the second variable and then with respect to the first variable</u> ::: there exists a real number A so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δx is chosen such that $0 < |\Delta x| < \delta$, then

$$\left| \frac{f_2(a+\Delta x,b)-f_2(a,b)}{\Delta x} - A \right| < \epsilon .$$

If f is differentaible at (a,b) with respect to the second variable and then with respect to the first variable, then A is denoted by $f_{21}(a,b)$ and $f_{21}(a,b)$ is called <u>the second-order partial derivative of</u> f <u>at</u> (a,b) <u>with respect to the second variable and then with respect to the first variable</u>. If f_2 exists on $C((a,b);\ \delta_1)$, then f_{21} <u>exists on</u> $C((a,b);\ \delta_1)$ :: for each $(x,y) \in C((a,b);\ \delta_1)$, $f_{21}(x,y)$ exists.

1.33. It should be clear that $f_{11}(a,b)$, $f_{22}(a,b)$, and "higher order" partial derivatives can be defined in a similar manner.

1.34. If f is defined on $C((a,b);\ \delta_1)$, then f <u>is differentiable at</u> (a,b) :: there exists an ordered pair of real numbers (A,B) so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta ,$$

then

$$\left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x + \Delta^2 y}} - \frac{A\Delta x + B\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right| < \epsilon .$$

1.35. If f is defined on $C((a,b);\ \delta_1)$, and if the angle α, $0 \leq \alpha \leq 2\pi$, is measured in radians by a counter-clockwise rotation from the positive x-axis, then f <u>is differentiable at</u> (a,b) <u>in the direction</u> α :: there exists a real number A so that if $\epsilon > 0$ is chosen, then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δs is chosen such that $0 < \Delta s < \delta$, then

$$\left| \frac{f(a+\Delta s \cos\alpha, b+\Delta s \sin\alpha)-f(a,b)}{\Delta s} - A \right| < \epsilon .$$

If f is differentiable at (a,b) in the direction α, then A is denoted by $D(f;(a,b);\alpha)$ and $D(f;(a,b);\alpha)$ is called <u>the directional derivative of f at (a,b) in the direction</u> α.

Assumed Theorems

1.36. <u>Borel Covering Theorem. If T is a set of open intervals which covers</u> $[c,d]$, <u>then there exists a finite subset</u> T_1 <u>of T which also covers</u> $[c,d]$.

1.37. <u>Bolzano-Weierstrass Theorem. If a set S of real numbers is infinite and bounded, then S has at least one limit point.</u>

1.38. <u>If S is non-empty and S has an upper bound, then S has a unique least upper bound.</u>

1.39. <u>If S is non-empty and S has a lower bound, then S has a unique greatest lower bound.</u>

1.40. <u>If</u> $I_1, I_2, \ldots, I_n, \ldots$ <u>is a descending infinitesimal sequence of closed intervals, then there exists a unique real number</u> ξ <u>such that for each</u> n, $\xi \in I_n$.

1.41. <u>A closed interval contains all of its limit points.</u>

1.42. <u>If c and d are real numbers, then</u>

$$\big|\,|c|-|d|\,\big| \leq |c+d| \leq |c|+|d|$$

and $|cd| = |c|\,|d|$.

CHAPTER II

THEOREMS ON REAL FUNCTIONS OF ONE REAL VARIABLE

2.1. Theorem. *If f is defined on* $[c,d]$ *and f is continuous on* $[c,d]$, *and if* $f(c)$ *and* $f(d)$ *differ in sign, then there exists a point* ξ, $c < \xi < d$, *such that* $f(\xi) = 0$.

Proof. Assume $f(c) < 0$ and $f(d) > 0$. Divide $[c,d]$ in half, *i.e.* let

$$[c,d] = \left[c,\frac{c+d}{2}\right] \cup \left[\frac{c+d}{2},d\right].$$

If

$$f\left(\frac{c+d}{2}\right) = 0$$

the theorem is proved. Assume

$$f\left(\frac{c+d}{2}\right) \neq 0.$$

Then if

$$f\left(\frac{c+d}{2}\right) > 0,$$

choose

$$\left[c,\frac{c+d}{2}\right];$$

if

$$f\left(\frac{c+d}{2}\right) < 0,$$

choose

$$\left[\frac{c+d}{2},d\right].$$

Denote the chosen half by $I_1 = [c_1,d_1]$; notice that

$$f(c_1) < 0, f(d_1) > 0,$$

and

$$l(I_1) = \frac{d-c}{2}.$$

Continue the process, assuming that each time a closed interval is divided in half, the mid-point obtained, say p, is such that $f(p) \neq 0$; for if at any time $f(p) = 0$, the theorem follows. In general, $I_n = [c_n,d_n]$, $f(c_n) < 0$, $f(d_n) > 0$, and

$$l(I_n) = \frac{d-c}{2^n}.$$

Clearly, I_1, I_2, ..., I_n, ... is a descending infinitesimal sequence of closed intervals. By 1.40 there exists a unique real number ξ such that for each n, $\xi \in I_n$. Certainly, $\xi \in [c,d]$.

Assume $f(\xi) > 0$. By 1.21, f is either continuous at ξ, right-continuous at ξ, or left-continuous at ξ; then clearly, there exists a $\delta > 0$ so that whenever $t \in [c,d]$ is chosen such that $|t-\xi| < \delta$, then $|f(t)-f(\xi)| < f(\xi)$. Since $\lim_{n\to\infty} l(I_n) = 0$, there exists a positive integer N so that whenever n is chosen such that $n > N$, then $l(I_n) < \delta$, <u>i</u>.<u>e</u>.

$$d_n-c_n < \delta .$$

Choose $n > N$ and consider c_n. Clearly, $|c_n - \xi| < \delta$. Therefore, $|f(c_n) - f(\xi)| < f(\xi)$, and hence, $f(c_n) > 0$. But this contradicts the fact that $f(c_n) < 0$. Therefore, $f(\xi) \not> 0$, and in a similar manner it can be shown that $f(\xi) \not< 0$. Thus, $f(\xi) = 0$, and clearly $c < \xi < d$.

Assume $f(c) > 0$ and $f(d) < 0$. Then the theorem follows by a similar argument.

2.2. <u>Theorem</u>. <u>If</u> f <u>is defined on</u> $I(\xi;\delta_1)$, <u>and if</u> f <u>is continuous at</u> ξ, <u>then there exists a</u> $\delta > 0$, $\delta \le \delta_1$, <u>so that</u> $E = \{f(t) \mid t \in I(\xi;\delta)\}$ <u>is bounded</u>.

<u>Proof</u>. Since f is continuous at ξ, there exists a $\delta > 0$, $\delta \le \delta_1$, so that whenever t is chosen such that

$$|t - \xi| < \delta ,$$

then $|f(t) - f(\xi)| < 1$. Choose $M = 1 + f(\xi)$ and let

$$t \in I(\xi;\delta).$$

Therefore, $|f(t) - f(\xi)| < 1$, and, by 1.42, $|f(t)| - |f(\xi)| < 1$, <u>i.e.</u> $|f(t)| < 1 + |f(\xi)| = M$. Therefore, E is bounded.

2.3. <u>Theorem</u>. <u>If</u> f <u>is defined on</u> $[c,d]$ <u>and</u> f <u>is continuous on</u> $[c,d]$, <u>then</u>

i) $E = \{f(t) \mid t \in [c,d]\}$ <u>is bounded</u>,

ii) E <u>has a unique least upper</u> K <u>and a unique greatest lower bound</u> k, <u>and</u>

iii) <u>there exists an</u> $\alpha_1 \in [c,d]$ <u>such that</u> $f(\alpha_1) = K$ <u>and there exists an</u> $\alpha_2 \in [c,d]$ <u>such that</u> $f(\alpha_2) = k$.

<u>Proof of</u> (i). For each ξ such that $c < \xi < d$, it is clear that there exists a $\delta_1 > 0$ so that $I(\xi; \delta_1) \subset [c,d]$. By 2.2 there exists a $\delta > 0$, $\delta \leq \delta_1$, so that

$$E_\xi = \{f(t) \mid t \in I(\xi; \delta)\}$$

is bounded. Since f is right-continuous at c and left-continuous at d, an argument similar to the one used in the proof of 2.2 can be employed to find δ_2, $\delta_3 > 0$ so that each of $E_c = \{f(t) \mid t \in I(c; \delta_2) \cap [c,d]\}$ and

$$E_d = \{f(t) \mid t \in I(d; \delta_2) \cap [c,d]\}$$

is bounded.

Let T denote the set of open intervals obtained in the preceding paragraph. Clearly, T covers $[c,d]$. By 1.36 there exists a finite subset T_1 of T which also covers $[c,d]$. For each interval J of T_1, there exists a positive number M_J so that for each $t \in J \cap [c,d]$, $|f(t)| < M_J$. Let M be the largest such M_J. Choose $t \in [c,d]$. Then $t \in J$ for some $J \in T_1$, and $|f(t)| < M_J \leq M$. Therefore, E is bounded.

<u>Proof of</u> (ii). Since E is bounded, it is clear that E has an upper bound and a lower bound. By 1.38 and 1.39, E has a unique least upper bound K and a unique greatest lower bound k.

<u>Proof of</u> (iii). Assume there does not exist an

$$\alpha_1 \in [c,d]$$

such that $f(\alpha_1) = K$. By 1.16 there is a $t_1 \in [c,d]$ such that $K > f(t_1) > K-1$. Similarly, there is a $t_2 \in [c,d]$ such that

$k > f(t_2) > \text{maximum}(f(t_1), K-\frac{1}{2})$. The process is continued, thereby constructing an infinite set S of points

$$t_1, t_2, \ldots, t_n, \ldots$$

such that for each n,

$$K > f(t_n) > K - \frac{1}{n}.$$

Clearly, S is bounded and the elements t_n are distinct. By 1.37, S has at least one limit point, say ξ. By 1.41 it follows that $\xi \in [o,d]$.

From the assumption and the definition of K, it follows that $f(\xi) < K$. Since f is either continuous at , right-continuous at , or left-continuous at ξ, then there exists a $\delta > 0$ so that whenever $t \in [o,d]$ is chosen such that $|t-\xi| < \delta$, then

$$|f(t)-f(\xi)| < \frac{K-f(\xi)}{2}.$$

Choose n such that

$$n > \frac{2}{K-f(\xi)}$$

and $|t_n - \xi| < \delta$. Then

$$|f(t_n)-f(\xi)| < \frac{K-f(\xi)}{2}$$

and therefore

$$f(t_n) < f(\xi) + \frac{K-f(\xi)}{2} = K - \frac{K-f(\xi)}{2}.$$

But this contradicts the fact that

$$f(t_n) > K - \frac{1}{n} > K - \frac{K - f(\xi)}{2}.$$

A similar argument will determine an $\alpha_2 \in [c,d]$ such that $f(\alpha_2) = k$.

2.4. <u>Theorem</u>. <u>If</u> f <u>is defined on</u> $[c,d]$, <u>and if there exists a point</u> ξ, $c < \xi < d$, <u>such that</u> $f(\xi)$ <u>is an upper bound</u> (<u>lower bound</u>) <u>of</u> $E = \{f(t) \mid t \in [c,d]\}$, <u>and if</u> $f'(\xi)$ <u>exists</u>, <u>then</u> $f'(\xi) = 0$.

<u>Proof</u>. Assume $f(\xi)$ is an upper bound of E. Since ξ is such that $c < \xi < d$, there exists a $\delta_1 > 0$ so that

$$I(\xi ; \delta_1) \subset [c,d] .$$

Assume $f'(\xi) > 0$. Then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever t is chosen such that $0 < |t - \xi| < \delta$, then

$$\left| \frac{f(t) - f(\xi)}{t - \xi} - f'(\xi) \right| < f'(\xi).$$

Choose t such that $0 < t - \xi < \delta$. Since $t - \xi > 0$, $f'(\xi) > 0$, and

$$\left| \frac{f(t) - f(\xi)}{t - \xi} - f'(\xi) \right| < f'(\xi),$$

it follows that $f(t) - f(\xi) > 0$. This contradicts the assumption that $f(\xi)$ is an upper bound of E. Thus, $f'(\xi) \not> 0$, and in a similar manner it can be shown that $f'(\xi) \not< 0$. Hence,

$$f'(\xi) = 0.$$

Assume $f(\xi)$ is a lower bound of E. Then the theorem follows by a similar argument.

2.5. <u>Theorem</u>. <u>If</u> f <u>is defined on</u> $I(\xi, \delta_1)$, <u>and if</u> $f'(\xi)$ <u>exists, then</u> f <u>is continuous at</u> ξ.

<u>Proof</u>. Choose $\epsilon > 0$. Since $f'(\xi)$ exists, there exists a $\delta_2 > 0$, $\delta_2 \leq \delta_1$, so that whenever t is chosen such that $0 < |t-\xi| < \delta_2$, then

$$\left| \frac{f(t)-f(\xi)}{t-\xi} - f'(\xi) \right| < 1;$$

it follows that

$$\left| \frac{f(t)-f(\xi)}{t-\xi} \right| - |f'(\xi)| < 1,$$

<u>i.e.</u>

$$\left| \frac{f(t)-f(\xi)}{t-\xi} \right| < 1 + |f'(\xi)|.$$

Choose

$$\delta = \min\left(\delta_2, \frac{\epsilon}{1+|f'(\xi)|}\right).$$

Choose t such that $|t-\xi| < \delta$. Assume $t = \xi$. Then

$$f(t) = f(\xi)$$

since f is a function, and therefore, $|f(t)-f(\xi)| = 0 < \epsilon$. Assume $t \neq \xi$. Thus, $0 < |t-\xi| < \delta$, and therefore,

$$\left| \frac{f(t)-f(\xi)}{t-\xi} \right| < 1 + |f'(\xi)|;$$

it follows that

$$|f(t)-f(\xi)| < |t-\xi|\,(1+|f'(\xi)|) < \delta(1+|f'(\xi)|)$$

$$\leq \frac{\epsilon}{1+|f'(\xi)|}(1+|f'(\xi)|) = \epsilon.$$

Thus, $|f(t)-f(\xi)| < \epsilon$, and therefore, f is continuous at ξ.

2.6. __Theorem.__ __If__ f __is defined on__ $[c,d]$, $f(c) = f(d)$, __and if__ f __is right-continuous at__ c __and left-continuous at__ d, __and if for each__ t __such that__ $c < t < d$ $f'(t)$ __exists, then there exists a point__ ξ, $c < \xi < d$, __such that__ $f'(\xi) = 0$.

__Proof.__ By 2.5 and 1.21, f is continuous on $[c,d]$. Let $E = \{f(t) | t \in [c,d]\}$. By 2.3, E has a unique least upper bound K and a unique greatest lower bound k, and there is an $\alpha_1 \in [c,d]$ such that $f(\alpha_1) = K$ and an $\alpha_2 \in [c,d]$ such that $f(\alpha_2) = k$. If $K = k$, then clearly $f'(t) = 0$ for each t such that $c < t < d$ and the theorem follows.

Assume $K \neq k$. Then at least one of K and k is different from f(c). Suppose $K \neq f(c)$. Then $K = f(\alpha_1) > f(c)$, and since $f(c) = f(d)$, $K = f(\alpha_1) > f(d)$. Since f is a function, it is clear that $\alpha_1 \neq c$ and $\alpha_1 \neq d$; thus, $c < \alpha_1 < d$. Denote α_1 by ξ. Then by 2.4, $f'(\xi) = 0$. The argument is similar if it is assumed that $k \neq f(c)$.

2.7. __Theorem.__ __If__ f __and__ g __are defined on__ $I(\xi; \delta_1)$, __and if__ $f'(\xi)$ __and__ $g'(\xi)$ __exist, then__

i) $f \pm g$ __are defined on__ $I(\xi; \delta_1)$, $(f \pm g)'(\xi)$ __exist and__ $(f \pm g)'(\xi) = f'(\xi) \pm g'(\xi)$,

ii) fg __is defined on__ $I(\xi; \delta_1)$, $(fg)'(\xi)$ __exists and__ $(fg)'(\xi) = f(\xi)g'(\xi) + g(\xi)f'(\xi)$, and

iii) __if__ $g(\xi) \neq 0$, __then there exists a__ $\delta' > 0$ __so that__

$\frac{f}{g}$ <u>is defined on</u> $I(\xi ; \delta')$, $(\frac{f}{g})'(\xi)$ <u>exists and</u>

$$\left(\frac{f}{g}\right)'(\xi) = \frac{g(\xi)f'(\xi)-f(\xi)g'(\xi)}{[g(\xi)]^2}.$$

<u>Proof of</u> (i). Consider the function f+g. Clearly, f+g is defined on $I(\xi ; \delta_1)$.

Choose $\epsilon > 0$. Since $f'(\xi)$ and $g'(\xi)$ exist, there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever t is chosen such that $0 < |t-\xi| < \delta$, then

$$\left|\frac{f(t)-f(\xi)}{t-\xi} - f'(\xi)\right| < \frac{\epsilon}{2}$$

and

$$\left|\frac{g(t)-g(\xi)}{t-\xi} - g'(\xi)\right| < \frac{\epsilon}{2}.$$

Choose t such that $0 < |t-\xi| < \delta$. Then

$$\left|\frac{(f+g)(t)-(f+g)(\xi)}{t-\xi} - [f'(\xi)+g'(\xi)\right|$$

$$= \left|\frac{f(t)+g(t)-f(\xi)-g(\xi)}{t-\xi} - f'(\xi)-g'(\xi)\right|$$

$$\leq \left|\frac{f(t)-f(\xi)}{t-\xi} - f'(\xi)\right| + \left|\frac{g(t)-g(\xi)}{t-\xi} - g'(\xi)\right|$$

$$< \epsilon .$$

Therefore, $(f+g)'(\xi)$ exists and $(f+g)'(\xi) = f'(\xi)+g'(\xi)$. The proof for the function f-g is similar.

<u>Proof of</u> (ii). Clearly, fg is defined on $I(\xi ; \delta_1)$. Choose $\epsilon > 0$. Since $f'(\xi)$ exists, there exists a $\delta_2 > 0$, $\delta_2 \leq \delta_1$, so that whenever t is chosen such that

$$0 < |t-\xi| < \delta_2,$$

then

$$\left| \frac{f(t)-f(\zeta)}{t-\zeta} - f'(\zeta) \right| < \frac{\epsilon}{3(|g(\zeta)|+1)}.$$

By 2.5, f is continuous at ζ; therefore, there exists a $\delta_3 > 0$, $\delta_3 \leq \delta_1$, so that whenever t is chosen such that $|t-\zeta| < \delta_3$, then

$$|f(t)-f(\zeta)| < \frac{\epsilon}{3(|g'(\zeta)|+1)}.$$

By 2.2 and 1.15 there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_1$, and an $M > 0$ so that whenever t is chosen such that $|t-\zeta| < \delta_4$, then $|f(t)| < M$. Since $g'(\zeta)$ exists, there exists a $\delta_5 > 0$, $_5$ $_1$, so that whenever t is chosen such that

$$0 < |t-\zeta| < \delta_5,$$

then

$$\left| \frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta) \right| < \frac{\epsilon}{3M}.$$

Choose $\delta = \min(\delta_2, \delta_3, \delta_4, \delta_5)$. Choose t such that

$$0 < |t-\zeta| < \delta .$$

Then

$$\left| \frac{(fg)(t)-(fg)(\zeta)}{t-\zeta} - [f(\zeta)g'(\zeta)+g(\zeta)f'(\zeta)] \right|$$

$$= \left| \frac{f(t)g(t)-t(\zeta)g(\zeta)}{t-\zeta} - f(\zeta)g'(\zeta) - g(\zeta)f'(\zeta) \right|$$

$$= \left| \frac{f(t)[g(t)-g(\zeta)]+g(\zeta)[f(t)-f(\zeta)]}{t-\zeta} - f(\zeta)g'(\zeta) - g(\zeta)f'(\zeta) \right|$$

$$\le \left| \frac{g(\zeta)[f(t)-f(\zeta)]}{t-\zeta} - g(\zeta)f'(\zeta) \right| + \left| \frac{f(t)[g(t)-g(\zeta)]}{t-\zeta} - f(\zeta)g'(\zeta) \right|$$

$$= |g(\zeta)| \left| \frac{f(t)-f(\zeta)}{t-\zeta} - f'(\zeta) \right|$$

$$+ \left| \frac{f(t)[g(t)-g(\zeta)]}{t-\zeta} - f(t)g'(\zeta) + f(t)g'(\zeta) - f(\zeta)g'(\zeta) \right|$$

$$\le |g(\zeta)| \left| \frac{f(t)-f(\zeta)}{t-\zeta} - f'(\zeta) \right|$$

$$+ |f(t)| \left| \frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta) \right| + |g'(\zeta)| \, |f(t)-f(\zeta)|$$

$$\le |g(\zeta)| \frac{\epsilon}{3(|g(\zeta)|+1)} + M\frac{\epsilon}{3M} + |g'(\zeta)| \frac{\epsilon}{3(|g'(\zeta)|+1)}$$

.

Therefore, $(fg)'(\zeta)$ exists and

$$(fg)'(\zeta) = f(\zeta)g'(\zeta) + g(\zeta)f'(\zeta).$$

<u>Proof</u> <u>of</u> (iii). Since $g'(\zeta)$ exists, then, by 2.5, g is continuous at ζ. Therefore, there exists a $\delta_2 > 0$, $\delta_2 \le \delta_1$, so that whenever t is chosen such that $|t-\zeta| < \delta_2$, then $|g(t)-g(\zeta)| < |g(\zeta)|$, <u>i.e.</u> $g(t) \ne 0$. Then $\frac{1}{g}$ is defined on $I(\zeta ; \delta_2)$.

Choose $\epsilon > 0$. Again, since g is continuous at ζ, there exists a $\delta_3 > 0$, $\delta_3 \le \delta_1$, so that whenever t is chosen such that $|t-\zeta| < \delta_3$, then

$$|g(t)-g(\zeta)| < \min\left(\frac{|g(\zeta)|}{2}, \frac{\epsilon\left|[g(\zeta)]^3\right|}{4(|g'(\zeta)|+1)}\right).$$

For each such t,

$$\left|g(\zeta)\right| - \left|g(t)\right| \leq \left|g(\zeta)-g(t)\right| = \left|g(t)-g(\zeta)\right| < \frac{|g(\zeta)|}{2};$$

in other words,

$$-\left|g(t)\right| < -\frac{|g(\zeta)|}{2},$$

and therefore,

$$\left|g(t)\right|\left|g(\zeta)\right| > \frac{[g(\zeta)]^2}{2}.$$

Since $g'(\zeta)$ exists, there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_1$, so that whenever t is chosen such that $0 < |t-\zeta| < \delta_4$, then

$$\left|\frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta)\right| < \frac{\epsilon[g(\zeta)]^2}{4}.$$

Choose $\delta = \min(\delta_2, \delta_3, \delta_4)$. Choose t such that

$$0 < |t-\zeta| < \delta .$$

Then

$$\left|\frac{\frac{1}{g}(t)-\frac{1}{g}(\zeta)}{t-\zeta} - \frac{-g'(\zeta)}{[g(\zeta)]^2}\right| = \left|\frac{\frac{1}{g(t)}-\frac{1}{g(\zeta)}}{t-\zeta} + \frac{g'(\zeta)}{[g(\zeta)]^2}\right|$$

$$= \left|\frac{g(\zeta)-g(t)}{g(t)g(\zeta)(t-\zeta)} + \frac{g'(\zeta)}{g(t)g(\zeta)} - \frac{g'(\zeta)}{g(t)g(\zeta)} + \frac{g'(\zeta)}{[g(\zeta)]^2}\right|$$

$$\leq \left|\frac{1}{g(t)g(\zeta)}\right|\left|\frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta)\right| + \left|\frac{g'(\zeta)}{g(\zeta)}\right|\left|\frac{1}{g(\zeta)} - \frac{1}{g(t)}\right|$$

$$= \frac{1}{|g(t)||g(\zeta)|}\left|\frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta)\right| + \left|\frac{|g'(\zeta)|}{|g(t)||g(\zeta)||g(\zeta)|}\right|\left|g(t)-g(\zeta)\right|$$

$$< \frac{2}{[g(\xi)]^2}\,\frac{\epsilon[g(\xi)]^2}{4} + \frac{2|g'(\xi)|}{[g(\xi)]^2|g(\xi)|}\,\frac{\epsilon|[g(\xi)]^3|}{4(|g'(\xi)|+1)}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon .$$

Therefore, $(\frac{1}{g})'(\xi)$ exists and

$$(\frac{1}{g})'(\xi) = \frac{-g'(\xi)}{[g(\xi)]^2}.$$

Since

$$\frac{f}{g} = f\frac{1}{g},$$

and since it is clear that there exists a $\delta' > 0$ so that f and $\frac{1}{g}$ are defined on $I(\xi; \delta')$, then by (ii), $\frac{f}{g}$ is defined on $I(\xi; \delta')$, $(\frac{f}{g})'(\xi)$ exists and

$$(\frac{f}{g})'(\xi) = (f\frac{1}{g})'(\xi)$$

$$= f(\xi)(\frac{1}{g})'(\xi) + (\frac{1}{g})(\xi)f'(\xi)$$

$$= f(\xi)\frac{-g'(\xi)}{[g(\xi)]^2} + \frac{1}{g(\xi)}f'(\xi)$$

$$= \frac{g(\xi)f'(\xi) - f(\xi)g'(\xi)}{[g(\xi)]^2}.$$

2.8. <u>Theorem</u>. <u>If</u> f <u>is defined on</u> [c,d], <u>and if</u> f <u>is right-continuous at</u> c <u>and left-continuous at</u> d, <u>and if for</u>

<u>each</u> t <u>such</u> <u>that</u> $c < t < d$ $f'(t)$ <u>exists</u>, <u>then</u> <u>there</u> <u>exists</u> <u>a</u> <u>point</u> ξ, $c < \xi < d$, <u>such</u> <u>that</u>

$$f'(\xi) = \frac{f(d)-f(c)}{d-c}.$$

<u>Proof</u>. For each $t \in [c,d]$, let

$$g(t) = f(c)+\frac{f(d)-f(c)}{d-c}(t-c)$$

and $F(t) = f(t)-g(t)$; clearly, g is right-continuous at c and left-continuous at d, and although the details will not be given, it follows that F is right-continuous at c and left-continuous at d. It is also clear that for each t such that $c < t < d$ $g'(t)$ exists and

$$g'(t) = \frac{f(d)-f(c)}{d-c};$$

therefore, by 2.7 (i), $F'(t)$ exists and

$$F'(t) = f'(t)- \frac{f(d)-f(c)}{d-c}.$$

Now $F(c) = F(d)$; therefore, by 2.6, there exists a point ξ, $c < \xi < d$, such that $F'(\xi) = 0$. Thus,

$$0 = F'(\xi) = f'(\xi)- \frac{f(d)-f(c)}{d-c}.$$

In other words,

$$f'(\xi) = \frac{f(d)-f(c)}{d-c}$$

and the theorem is proved.

2.9. <u>Theorem</u>. <u>If</u> g <u>is defined on</u> $I(\xi;\delta_1)$ <u>and</u> $g'(\xi)$ <u>exists, and if</u> f <u>is defined on</u> $I(a;\delta_2)$, $a = g(\xi)$, <u>and</u> $f'(a)$ <u>exists, then there exists a</u> $\delta_3 > 0$ <u>so that</u> $F = f(g)$ <u>is defined on</u> $I(\xi;\delta_3)$, $F'(\xi)$ <u>exists, and</u> $F'(\xi) = f'(a)g'(\xi)$.

<u>Proof</u>. By 2.5, g is continuous at ξ, and with this fact it will follow that there exists a $\delta_3 > 0$ so that $F = f(g)$ is defined on $I(\xi;\delta_3)$. It shall now be shown that $F'(\xi)$ exists and $F'(\xi) = f'(a)g'(\xi)$.

<u>Case</u> I. Assume $g'(\xi) = 0$. Choose $\epsilon > 0$. Since $g'(\xi) = 0$, there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_1$, so that whenever t is chosen such that $0 < |t-\xi| < \delta_4$, then

$$\left| \frac{g(t)-g(\xi)}{t-\xi} \right| < \frac{\epsilon}{|f'(a)|+1}.$$

Since $f'(a)$ exists, there exists a $\delta_5 > 0$, $\delta_5 \leq \delta_2$, so that whenever x is chosen such that $0 < |x-a| < \delta_5$, then

$$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < 1,$$

and hence,

$$\left| \frac{f(x)-f(a)}{x-a} \right| < |f'(a)| + 1.$$

Since g is continuous at ξ, there exists a $\delta_6 > 0$, $\delta_6 \leq \delta_1$, so that whenever t is chosen such that $|t-\xi| < \delta_6$, then $|g(t)-g(\xi)| < \delta_5$.

Choose $\delta = \min(\delta_3, \delta_4, \delta_6)$. Choose t such that

$$0 < |t-\xi| < \delta .$$

Let $x = g(t)$. Assume $g(t) = g(\xi)$; then $f(g(t)) = f(g(\xi))$

since f is a function, and therefore,

$$\left| \frac{F(t)-F(\zeta)}{t-\zeta} - f'(a)g'(\zeta) \right| = \left| \frac{f(g(t))-f(g(\zeta))}{t-\zeta} \right| = 0 < \epsilon .$$

Assume $g(t) \neq g(\zeta)$, i.e. $x \neq a$; then

$$\left| \frac{F(t)-F(\zeta)}{t-\zeta} - f'(a)g'(\zeta) \right| = \left| \frac{f(g(t))-f(g(\zeta))}{t-\zeta} \right|$$

$$= \left| \frac{f(x)-f(a)}{t-\zeta} \right|$$

$$= \left| \frac{f(x)-f(a)}{x-a} \; \frac{x-a}{t-\zeta} \right|$$

$$= \left| \frac{f(x)-f(a)}{x-a} \right| \left| \frac{g(t)-g(\zeta)}{t-\zeta} \right|$$

$$< (|f'(a)|+1) \frac{\epsilon}{|f'(a)|+1}$$

$$= \epsilon .$$

Case II. Assume $g'(\zeta) \neq 0$. Choose $\epsilon > 0$. Since $g'(\zeta) \neq 0$, there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_1$, so that whenever t is chosen such that $0 < |t-\zeta| < \delta_4$, then

$$\left| \frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta) \right| < \min\left(\frac{\epsilon}{3(|f'(a)|+1)}, \frac{1}{3}, \frac{|g'(\zeta)|}{2} \right).$$

Since $f'(a)$ exists, there exists a $\delta_5 > 0$, $\delta_5 \leq \delta_2$, so that whenever x is chosen such that $0 < |x-a| < \delta_5$, then

$$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \min\left(\frac{\epsilon}{3|g'(\zeta)|}, \frac{\epsilon}{3} \right).$$

Since g is continuous at ζ, there exists a $\delta_6 > 0$, $\delta_6 \leq \delta_1$, so that whenever t is chosen such that $|t-\zeta| < \delta_6$, then $|g(t)-g(\zeta)| < \delta_5$.

Choose $\delta = \min(\delta_3, \delta_4, \delta_6)$. Choose t such that $0 < |t-\xi| < \delta$. Let $x = g(t)$. Now $g(t) \neq g(\xi)$ since $g(t) = g(\xi)$ contradicts the fact that

$$\left| \frac{g(t)-g(\xi)}{t-} - g'(\xi) \right| < \frac{|g'(\xi)|}{2}.$$

Thus, $x \neq a$.

Then

$$\left| \frac{F(t)-F(\xi)}{t-\xi} - f'(a)g'(\xi) \right| = \left| \frac{f(g(t))-f(g(\xi))}{t-\xi} - f'(a)g'(\xi) \right|$$

$$= \left| \frac{f(x)-f(a)}{x-a}\,\frac{x-a}{t-\xi} - f'(a)g'(\xi) \right|$$

$$= \left| \frac{f(x)-f(a)}{x-a}\,\frac{x-a}{t-\xi} - \frac{x-a}{t-\xi}f'(a) + \frac{x-a}{t-\xi}f'(a) - f'(a)g'(\xi) \right|$$

$$\leq \left| \frac{x-a}{t-\xi} \right| \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| + |f'(a)| \left| \frac{x-a}{t-\xi} - g'(\xi) \right|$$

$$= \left| \frac{g(t)-g(\xi)}{t-\xi} - g'(\xi) + g'(\xi) \right| \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right|$$

$$+ |f'(a)| \left| \frac{g(t)-g(\xi)}{t-\xi} - g'(\xi) \right|$$

$$\leq \left| \frac{g(t)-g(\xi)}{t-\xi} - g'(\xi) \right| \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| + |g'(\xi)| \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right|$$

$$+ |f'(a)| \left| \frac{g(t)-g(\xi)}{t-\xi} - g'(\xi) \right|$$

$$< \frac{1}{3}\,\frac{\epsilon}{3} + |g'(\xi)| \frac{\epsilon}{3|g'(\xi)|} + |f'(a)| \frac{\epsilon}{3(|f'(a)|+1)}$$

$$< \epsilon .$$

Cases I and II verify that $F'(\zeta)$ exists and

$$F'(\zeta) = f'(a)g'(\zeta).$$

CHAPTER III

DIFFERENTIABLE FUNCTIONS OF TWO REAL VARIABLES

3.1. Suppose f and g are functions of two real variables defined on $C((a,b); \delta)$. Then $f(x,b)$, $x \in I(a; \delta)$, and

$$f(a,y), \; y \in I(b; \delta),$$

are functions of one real variable; thus, $f(x,b)$ and $f(a,y)$ could be denoted by $F(x)$ and $G(y)$, for instance. It is evident, therefore, that the theorems of Chapter II have immediate analogues for functions of two variables when one of the variables is held constant. One such analogue is the following theorem on partial derivatives.

> *If* f *and* g *are defined on* $C((a,b); \delta)$, *and if* $f_1(a,b)$, $f_2(a,b)$, $g_1(a,b)$, *and* $g_2(a,b)$ *exist, then* f+g *is defined on* $C((a,b); \delta)$, $(f+g)_1(a,b)$ *and* $(f+g)_2(a,b)$ *exist, and* $(f+g)_1(a,b) = f_1(a,b)+g_1(a,b)$ *and* $(f+g)_2(a,b) = f_2(a,b)+g_2(a,b)$.

It is clear that similar theorems on partial derivatives can be stated for the functions $f-g$, fg, and $\frac{f}{g}$.

Theorem 3.2 will be an analogue of 2.9. The other analogues will not be stated, although they will be used from time to time by referring to the theorems in Chapter II.

3.2. *Theorem. If* r_0 *and* s_0 *are real numbers and for each* $r \in I(r_0; \delta_1)$ $g(r,s_0)$ *is defined, and* $g_1(r_0,s_0)$ *exists, and if* f *is defined on* $I(\zeta; \delta_2)$, $\zeta = g(r_0,s_0)$, *and* $f'(\zeta)$ *exists, then there exists a* $\delta_3 > 0$ *so that for each*

$$r \in I(r_0, \delta_3),$$

$F(r,s_0) = f(g(r,s_0))$ *is defined,* $F_1(r_0,s_0)$ *exists, and*

$$F_1(r_0,s_0) = f'(\zeta)g_1(r_0,s_0).$$

Proof. The theorem follows by 2.9.

3.3. It should be clear that an appropriate change in the hypothesis of 3.1 concerning g would be sufficient for $F_2(r_0,s_0)$ to exist and for $F_2(r_0,s_0)$ to be equal to

$$f'(\zeta)g_2(r_0,s_0).$$

3.4. It will be convenient to have 2.8 stated in the following, more general form. *If* a *and* Δx *are real numbers,* $\Delta x \neq 0$ *and if* f *is defined on*

$$[a,a+\Delta x] \ ([a+\Delta x,a], \text{ if } \Delta x < 0),$$

and if f *is right-continuous at* $a(a+\Delta x)$ *and left-continuous at* $a+\Delta x(a)$, *and if for each* t *such that*

$$a < t < a+\Delta x(a+\Delta x < t < a)$$

$f'(t)$ *exists, then there exists a* θ, $0 < \theta < 1$, *so that* $f(a+\Delta x)-f(a) = \Delta x f'(a+\theta \Delta x)$.

3.5. *Theorem. Let* $\Delta x, \Delta y > 0$. *If for each* x *such that* $a \leq x \leq a+\Delta x$ $f(x,b)$ *is defined, and* f *is right-continuous at* (a,b) *and left-continuous at* $(a+\Delta x,b)$ *with respect to the first variable, and for each* x *such that* $a < x < a+\Delta x$ $f_1(x,b)$ *exists, and if for each* y *such that* $b \leq y \leq b+\Delta y$ $f(a+\Delta x,y)$

is defined, and f is right-continuous at $(a+\Delta x, b)$ *and left-continuous at* $(a+\Delta x, b+\Delta y)$ *with respect to the second variable, and for each* y *such that* $b < y < b+\Delta y$ $f_2(a+\Delta x, y)$ *exists, then there exist* θ_1 *and* θ_2, $0 < \theta_1, \theta_2 < 1$, *such that*

$$f(a+\Delta x, b+\Delta y) - f(a,b)$$
$$= \Delta x f_1(a+\theta_1\Delta x, b) + \Delta y f_2(a+\Delta x, b+\theta_2\Delta y).$$

Proof. Let

$$f(a+\Delta x, b+\Delta y) - f(a,b)$$
$$= f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) + f(a+\Delta x, b) - f(a,b).$$

By 3.4 there exists a θ_1, $0 < \theta_1 < 1$, so that

$$f(a+\Delta x, b) - f(a,b) = \Delta x f_1(a+\theta_1\Delta x, b).$$

Similarly, there exists a θ_2, $0 < \theta_2 < 1$, so that

$$f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) = \Delta y f_2(a+\Delta x, b+\theta_2\Delta y),$$

and the theorem is proved.

3.6. *Theorem*. *If* f *is defined on* $C((a,b); \delta_1)$ *and* f_1 *and* f_{12} *exist on* $C((a,b); \delta_1)$, *and if* f_{12} *is continuous at* (a,b), *and if for each* Δx *such that* $|\Delta x| < \delta_1$ $f_2(a+\Delta x, b)$ *exists, then* $f_{21}(a,b)$ *exists and* $f_{21}(a,b) = f_{12}(a,b)$.

Proof. Choose $\epsilon > 0$. Since f_{12} is continuous at (a,b), there exists a $\delta > 0$, $\delta \le \delta_1$, so that whenever Δx and Δy are chosen such that

$$\sqrt{\Delta^2 x + \Delta^2 y} < \delta,$$

then $f_{12}(a+\Delta x, b+\Delta y) - f_{12}(a,b) < \frac{\epsilon}{2}$.

Choose Δx such that $0 < |\Delta x| < \delta$.

Case I. Assume $\Delta x > 0$. Now, since $f_2(a+\Delta x,b)$ and $f_2(a,b)$ exist, there exists a $\delta_3 > 0$ so that whenever Δy is chosen such that $0 < |\Delta y| < \delta_3$, then

$$\left| \frac{f(a+\Delta x,b+\Delta y)-f(a+\Delta x,b)}{\Delta y} - f_2(a+\Delta x,b)\right| < \frac{\epsilon \Delta x}{4}$$

and

$$\left| \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y} - f_2(a,b)\right| < \frac{\epsilon \Delta x}{4}.$$

Choose Δy such that $0 < |\Delta y| < \delta_3$ and $\sqrt{\Delta^2 x + \Delta^2 y} < \delta$. For each x such that $a \leq x \leq a+\Delta x$, let

$$g(x) = f(x,b+\Delta y)-f(x,b).$$

For each such x, $f_1(x,b+\Delta y)$ and $f_1(x,b)$ exist; by 2.7 (i), $g'(x)$ exists and $g'(x) = f_1(x,b+\Delta y)-f_1(x,b)$. Consider $g(a+\Delta x)-g(a)$. By 3.4 there exists a θ_1, $0 < \theta_1 < 1$, such that $g(a+\Delta x)-g(a) = \Delta x g'(a+\theta_1 \Delta x)$, and

$\Delta x g'(a+\theta_1 \Delta x) = \Delta x\left[f_1(a+\theta_1 \Delta x,b+\Delta y)-f_1(a+\theta_1 \Delta x,b)\right]$;

there exists a θ_2, $0 < \theta_2 < 1$, so that

$$f_1(a+\theta_1 \Delta x,b+\Delta y)-f_1(a+\theta_1 \Delta x,b)$$
$$= \Delta y f_{12}(a+\theta_1 \Delta x,b+\theta_2 \Delta y).$$

Therefore,

$$g(a+\Delta x)-g(a) = \Delta x \Delta y f_{12}(a+\theta_1 \Delta x,b+\theta_2 \Delta y),$$

and by definition of g,

$$g(a+\Delta x)-g(a)$$
$$= \left[f(a+\Delta x,b+\Delta y)-f(a+\Delta x,b)\right] - \left[f(a,b+\Delta y)-f(a,b)\right].$$

Hence,

$$f_{12}(a+\theta_1\Delta x, b+\theta_2\Delta y)$$

$$= \frac{1}{\Delta x}\frac{f(a+\Delta x, b+\Delta y)-f(a+\Delta x, b)}{\Delta y} - \frac{1}{\Delta x}\frac{f(a, b+\Delta y)-f(a,b)}{\Delta y}.$$

Since $0 < \theta_1, \theta_2 < 1$, it follows that

$$f_{12}(a,b)-\frac{\epsilon}{2} < f_{12}(a+\theta_1\Delta x, b+\theta_2\Delta y) < f_{12}(a,b)+\frac{\epsilon}{2}.$$

Hence,

$$f_{12}(a,b)-\frac{\epsilon}{2} < f_{12}(a+\theta_1\Delta x, b+\theta_2\Delta y)$$

$$= \frac{1}{\Delta x}\frac{f(a+\Delta x, b+\Delta y)-f(a+\Delta x, b)}{\Delta y} - \frac{1}{\Delta x}\frac{f(a, b+\Delta y)-f(a,b)}{\Delta y}$$

$$< \frac{1}{\Delta x}\left[f_2(a+\Delta x, b)+\frac{\epsilon\Delta x}{4}\right] - \frac{1}{\Delta x}\left[f_2(a,b)-\frac{\epsilon\Delta x}{4}\right]$$

$$= \frac{f_2(a+\Delta x, b)-f_2(a,b)}{\Delta x}+\frac{\epsilon}{2}, \text{ and}$$

$$f_{12}(a,b)+\frac{\epsilon}{2} > f_{12}(a+\theta_1\Delta x, b+\theta_2\Delta y)$$

$$= \frac{1}{\Delta x}\frac{f(a+\Delta x, b+\Delta y)-f(a+\Delta x, b)}{\Delta y} - \frac{1}{\Delta x}\frac{f(a, b+\Delta y)-f(a,b)}{\Delta y}$$

$$> \frac{1}{\Delta x}\left[f_2(a+\Delta x, b)-\frac{\epsilon\Delta x}{4}\right] - \frac{1}{\Delta x}\left[f_2(a,b)+\frac{\epsilon\Delta x}{4}\right]$$

$$= \frac{f_2(a+\Delta x, b)-f_2(a,b)}{\Delta x}-\frac{\epsilon}{2}.$$

<u>Case</u> II. Assume $\Delta x < 0$. A similar argument will yield the same inequalities derived above; the details are omitted.

Since

$$-\epsilon < \frac{f_2(a+\Delta x,b)-f_2(a,b)}{\Delta x} - f_{12}(a,b) < \epsilon ,$$

in other words, since

$$\left| \frac{f_2(a+\Delta x,b)-f_2(a,b)}{\Delta x} - f_{12}(a,b) \right| < \epsilon ,$$

f_{21} exists and $f_{21}(a,b) = f_{12}(a,b)$.

3.7. Example. Consider the following function:

$$f(x,y) = 2xy\frac{x^2-y^2}{x^2+y^2}, \text{ when } (x,y) \neq (0,0);$$

$$f(0,0) = 0.$$

For each (x,y) such that $(x,y) \neq (0,0)$, $f_{12}(x,y)$ and $f_{21}(x,y)$ exist and $f_{12}(x,y) = f_{21}(x,y)$. But although $f_{12}(0,0)$ and $f_{21}(0,0)$ exist, $f_{12}(0,0) \neq f_{21}(0,0)$.

3.8. Theorem. If f is defined on $C((a,b); \delta_1)$, and if f is differentiable at (a,b), then $f_1(a,b)$ and $f_2(a,b)$ exist, and $f_1(a,b) = A_f$ and $f_2(a,b) = B_f$.

Proof. Choose $\epsilon > 0$. Since f is differentiable at (a,b), there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta ,$$

then

$$\left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_f\Delta x+B_f\Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| < \epsilon .$$

Choose Δx such that $0 < |\Delta x| < \delta$ and let $\Delta y = 0$.

Then

$$\left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_f\Delta x+B_f\Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right|$$

$$= \left| \frac{f(a+\Delta x,b)-f(a,b)}{\Delta x} - A_f \right| < \epsilon .$$

Therefore, $f_1(a,b)$ exists and $f_1(a,b) = A_f$. In a similar manner, it can be shown that $f_2(a,b)$ exists and $f_2(a,b) = B_f$.

3.9. __Theorem.__ __If__ f __is defined on__ $C((a,b);\delta_1)$, __and if__ f_2 __exists on__ $C((a,b);\delta_1)$ __and__ f_2 __is continuous at__ (a,b), __and if__ $f_1(a,b)$ __exists, then__ f __is differentiable at__ (a,b).

__Proof.__ Let $(A_f,B_f) = (f_1(a,b),f_2(a,b))$, and choose $\epsilon > 0$. Since $f_2(a,b)$ exists, there exists a $\delta_2 > 0$, $\delta_2 \leq \delta_1$, so that whenever Δy is chosen such that $0 < |\Delta y| < \delta_2$, then

$$\left| \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y} - f_2(a,b) \right| < \epsilon .$$

Since f_2 is continuous at (a,b), there exists a $\delta_3 > 0$, $\delta_3 \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$\sqrt{\Delta^2 x+\Delta^2 y} < \delta_3,$$

then

$$\left| f_2(a+\Delta x,b+\Delta y)-f_2(a,b) \right| < \frac{\epsilon}{2}.$$

Since $f_1(a,b)$ exists, there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_1$, so that whenever Δx is chosen such that $0 < |\Delta x| < \delta_4$, then

$$\left| \frac{f(a+\Delta x,b)-f(a,b)}{\Delta x} - f_1(a,b) \right| < \frac{\epsilon}{2}.$$

Choose $\delta = \min(\delta_2, \delta_3, \delta_4)$. Choose Δx and Δy such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta .$$

<u>Case</u> I. Assume $\Delta x = 0$. Then $\Delta y \neq 0$, and therefore,

$$\left| \frac{f(a+\Delta x, b+\Delta y) - f(a,b)}{\sqrt{\Delta^2 x + \Delta^2 y}} - \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right|$$

$$= \left| \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} - f_2(a,b) \right| < \epsilon .$$

<u>Case</u> II. Assume $\Delta x \neq 0$. Let

$$f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$= f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) + f(a+\Delta x, b) - f(a,b).$$

Now there exists a θ, $0 < \theta < 1$, such that

$$f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) = \Delta y f_2(a+\Delta x, b+\theta\Delta y).$$

Since $0 < \theta < 1$, $f_2(a,b) - \frac{\epsilon}{2} < f_2(a+\Delta x, b+\theta\Delta y) < f_2(a,b) + \frac{\epsilon}{2}$.

Now, further assume that $\Delta x > 0$ and $\Delta y \geq 0$. Then

$$f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta y f_2(a+\Delta x, b+\theta\Delta y) + f(a+\Delta x, b) - f(a,b)$$

$$< \Delta y \left[f_2(a,b) + \frac{\epsilon}{2} \right] + \Delta x \left[f_1(a,b) + \frac{\epsilon}{2} \right]$$

$$< f_1(a,b)\Delta x + f_2(a,b)\Delta y + \frac{\epsilon}{2}\sqrt{\Delta^2 x + \Delta^2 y} + \frac{\epsilon}{2}\sqrt{\Delta^2 x + \Delta^2 y}$$

$= f_1(a,b)\Delta x + f_2(a,b)\Delta y + \epsilon\sqrt{\Delta^2 x + \Delta^2 y}$, and

$$f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta y f_2(a+\Delta x, b+\theta\Delta y) + f(a+\Delta x, b) - f(a,b)$$

$$> \Delta y \left[f_2(a,b) - \frac{\epsilon}{2} \right] + \Delta x \left[f_1(a,b) - \frac{\epsilon}{2} \right]$$

$$> f_1(a,b)\Delta x + f_2(a,b)\Delta y - \frac{\epsilon}{2}\sqrt{\Delta^2 x + \Delta^2 y} - \frac{\epsilon}{2}\sqrt{\Delta^2 x + \Delta^2 y}$$

$$= f_1(a,b)\Delta x + f_2(a,b)\Delta y - \epsilon\sqrt{\Delta^2 x + \Delta^2 y} .$$

These same inequalities are obtained in a similar manner by considering the other allowable combinations of Δx and Δy.

In both cases

$$\left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{f_1(a,b)\Delta x+f_2(a,b)\Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| < \epsilon .$$

Therefore, f is differentiable at (a,b).

3.10 <u>Theorem</u>. <u>If</u> f <u>is defined on</u> $C((a,b); \delta_1)$, <u>and if</u> f <u>is differentiable at</u> (a,b), <u>then</u> f <u>is continuous at</u> (a,b).

<u>Proof</u>. Choose $\epsilon > 0$. Since f is differentiable at (a,b), there exists a $\delta_2 > 0$, $\delta_2 \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x+\Delta^2 y} < \delta_2,$$

then

$$\left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_f\Delta x+B_f\Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| < 1.$$

Choose $\delta = \min(\delta_2, \frac{\epsilon}{1+|A_f|+|B_f|})$. Choose Δx and Δy such that

$$\sqrt{\Delta^2 x+\Delta^2 y} < \delta .$$

If $\Delta x = \Delta y = 0$, then $|f(a+\Delta x, b+\Delta y)-f(a,b)| = 0 < \epsilon$. Assume that not both of Δx and Δy are zero. Then

$$|f(a+\Delta x, b+\Delta y)-f(a,b)| < \sqrt{\Delta^2 x+\Delta^2 y} + |A_f\Delta x+B_f\Delta y|$$

$$\leq \sqrt{\Delta^2 x+\Delta^2 y} + |A_f|\,|\Delta x|+|B_f|\,|\Delta y|$$

$$< \sqrt{\Delta^2 x+\Delta^2 y} + |A_f|\sqrt{\Delta^2 x+\Delta^2 y} + |B_f|\sqrt{\Delta^2 x+\Delta^2 y}$$

$$= \sqrt{\Delta^2 x + \Delta^2 y}\ (1+|A_f|+|B_f|)$$

$$< \delta\,(1+|A_f|+|B_f|)$$

$$\leq \frac{\epsilon}{1+|A_f|+|B_f|}(1+|A_f|+|B_f|)$$

$$= \epsilon\ .$$

Therefore, f is continuous at (a,b).

3.11. Example. Consider the following function:

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}},\ (x,y) \neq (0,0);$$

$$f(0,0) = 0.$$

This function is continuous at (0,0), and $f_1(0,0)$ and $f_2(0,0)$ exist, but f is not differentiable at (0,0).

3.12. Example. Consider the following function:

$$f(x,y) = x^2+y^2,\ x \text{ and } y \text{ both rational;}$$

$$f(x,y) = 0,\ \text{otherwise.}$$

This function is differentiable at (0,0) only.

3.13. Theorem. If f is defined on $C((a,b);\delta_1)$, and if f is continuous at (a,b), then there exists a $\delta > 0$, $\delta \leq \delta_1$, so that $E = \{f(x,y) \mid (x,y) \in C((a,b);\delta)\}$ is bounded.

Proof. Since f is continuous at (a,b), there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$\sqrt{\Delta^2 x + \Delta^2 y} < \delta,$$

then $|f(a+\Delta x, b+\Delta y) - f(a,b)| < 1$. Choose $M = 1+|f(a,b)|$ and let $(x,y) \in C((a,b);\delta)$. Then for some Δx and Δy such that

$$\sqrt{\Delta^2 x + \Delta^2 y} < \delta,$$

$(x,y) = (a+\Delta x, b+\Delta y)$. Therefore, $|f(x,y)-f(a,b)| < 1$, and hence, $|f(x,y)|-|f(a,b)| < 1$, i.e. $|f(x,y)| < 1+|f(a,b)| = M$. Therefore, E is bounded.

3.14. Theorem. If f and g are defined on $C((a,b);\delta_1)$, and if f and g are differentiable at (a,b), then

i) $f \pm g$ are defined on $C((a,b);\delta_1)$ and $f \pm g$ are differentiable at (a,b);

ii) fg is defined on $C((a,b);\delta_1)$ and fg is differentiable at (a,b); and

iii) if $g(a,b) \neq 0$, then there exists a $\delta' > 0$ so that $\frac{f}{g}$ is defined on $C((a,b);\delta')$, and $\frac{f}{g}$ is differentiable at (a,b).

Proof. Consider f+g. Since f and g are differentiable at (a,b), by 3.8, $f_1(a,b)$, $f_2(a,b)$, $g_1(a,b)$, and $g_2(a,b)$ exist and $f_1(a,b) = A_f$, $f_2(a,b) = B_f$, $g_1(a,b) = A_g$, and $g_2(a,b) = B_g$.

If f+g is differentiable at (a,b), then by 3.8,

$$(f+g)_1(a,b)$$

and $(f+g)_2(a,b)$ will exist and $(f+g)_1(a,b) = A_{f+g}$ and

$$(f+g)_2(a,b) = B_{f+g}.$$

Therefore, in view of 2.7 (i), it seems evident that A_f+A_g and B_f+B_g would be natural choices for A_{f+g} and B_{f+g}.

Let $(A_{f+g}, B_{f+g}) = (A_f+A_g, B_f+B_g)$. Choose $\epsilon > 0$. Now there exists a $\delta > 0$, $\delta \le \delta_1$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta,$$

then

$$\left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_f \Delta x + B_f \Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| < \frac{\epsilon}{2}$$

and

$$\left| \frac{g(a+\Delta x, b+\Delta y)-g(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_g \Delta x + B_g \Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| < \frac{\epsilon}{2}.$$

Choose Δx and Δy such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta.$$

Then

$$\left| \frac{(f+g)(a+\Delta x, b+\Delta y)-(f+g)(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{(A_f+A_g)\Delta x + (B_f+B_g)\Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right|$$

$$= \left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)+g(a+\Delta x, b+\Delta y)-g(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_f \Delta x + B_f \Delta y + A_g \Delta x + B_g \Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right|$$

$$\leq \left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_f \Delta x + B_f \Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| + \left| \frac{g(a+\Delta x, b+\Delta y)-g(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{A_g \Delta x + B_g \Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right|$$

$$< \epsilon .$$

Therefore, f+g is differentiable at (a,b).

The above proof is very similar to the proof of 2.7 (i). Such will also be the case for $f-g$, fg, and $\frac{f}{g}$. For example, choose $(A_{fg}, B_{fg}) = (g(a,b)A_f + f(a,b)A_g, g(a,b)B_f + f(a,b)B_g)$; then the required inequality will follow by using the same type of argument used to obtain the inequality in the proof of 2.7 (ii).

3.15. <u>Theorem</u>. <u>If</u> f <u>is defined on</u> $C((a,b); \delta_1)$, <u>and if</u> f_1 <u>and</u> f_2 <u>exist on</u> $C((a,b); \delta_1)$, <u>and if</u> f_1 <u>and</u> f_2 <u>are differentiable at</u> (a,b), <u>then</u> $f_{12}(a,b)$ <u>and</u> $f_{21}(a,b)$ <u>exist and</u> $f_{12}(a,b) = f_{21}(a,b)$.

<u>Proof</u>. Choose $\epsilon > 0$. Since f_1 and f_2 are differentiable at (a,b), by 3.8, $f_{11}(a,b)$, $f_{12}(a,b)$, $f_{21}(a,b)$, and $f_{22}(a,b)$ exist and $f_{11}(a,b) = A_{f_1}$, $f_{12}(a,b) = B_{f_1}$, $f_{21}(a,b) = A_{f_2}$ and $f_{22}(a,b) = B_{f_2}$; therefore, there exists a $\delta_2 > 0$, $\delta_2 \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta_2,$$

then

$$\left| \frac{f_1(a+\Delta x, b+\Delta y) - f_1(a,b)}{\sqrt{\Delta^2 x + \Delta^2 y}} - \frac{f_{11}(a,b)\Delta x + f_{12}(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right| < \frac{\epsilon}{4\sqrt{2}}$$

and

$$\left| \frac{f_2(a+\Delta x, b+\Delta y) - f_2(a,b)}{\sqrt{\Delta^2 x + \Delta^2 y}} - \frac{f_{21}(a,b)\Delta x + f_{22}(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right| < \frac{\epsilon}{4\sqrt{2}}.$$

Since $f_{11}(a,b)$ and $f_{22}(a,b)$ exist, there exists a

$$\delta_3 > 0, \ \delta_3 \leq \delta_1,$$

so that whenever Δx is chosen such that $0 < |\Delta x| < \delta_3$, then

$$\left| \frac{f_1(a+\Delta x,b)-f_1(a,b)}{\Delta x} - f_{11}(a,b) \right| < \frac{\epsilon}{4},$$

and whenever Δy is chosen such that $0 < |\Delta y| < \delta_3$, then

$$\left| \frac{f_2(a,b+\Delta y)-f_2(a,b)}{\Delta y} - f_{22}(a,b) \right| < \frac{\epsilon}{4}.$$

Choose $\delta = \min(\delta_2, \delta_3)$. Choose Δk such that

$$0 < \sqrt{\Delta^2 k + \Delta^2 k} < \delta;$$

in other words, $0 < |\Delta k|\sqrt{2} < \delta$.

Assume $\Delta k > 0$; the procedure will be similar if $\Delta k < 0$. For each x such that $a \le x \le a+\Delta k$, let

$$g(x) = f(x,b+\Delta k)-f(x,b).$$

By 2.7 (i), $g'(x)$ exists and $g'(x) = f_1(x,b+\Delta k)-f_1(x,b)$. Consider $g(a+\Delta k)-g(a)$. By 3.4 there exists a θ, $0 < \theta < 1$, so that $g(a+\Delta k)-g(a) = \Delta k g'(a+\theta\Delta k)$. Therefore,

$$\begin{aligned} & g(a+\Delta k)-g(a) \\ = & \left[f(a+\Delta k,b+\Delta k)-f(a+\Delta k,b)\right]-\left[f(a,b+\Delta k)-f(a,b)\right] \\ = & \Delta k\left[f_1(a+\theta\Delta k,b+\Delta k)-f_1(a+\theta\Delta k,b)\right]. \end{aligned}$$

From the inequalities of the first paragraph, it follows that

$$\Delta k\left[\theta f_{11}(a,b)+f_{12}(a,b)-\frac{\epsilon}{4}\right] < f_1(a+\theta\Delta k,b+\Delta k)-f_1(a,b)$$
$$< \Delta k\left[\theta f_{11}(a,b)+f_{12}(a,b)+\frac{\epsilon}{4}\right]$$

and

$$\theta \Delta k f_{11}(a,b) - \Delta k \frac{\epsilon}{4} < f_1(a+\theta\Delta k, b) - f_1(a,b)$$

$$< \theta \Delta k f_{11}(a,b) + \Delta k \frac{\epsilon}{4}.$$

Therefore,

$$f_1(a+\theta\Delta k, b+\Delta k) - f_1(a+\theta\Delta k, b)$$

$$= \left[f_1(a+\theta\Delta k, b+\Delta k) - f_1(a,b)\right] - \left[f_1(a+\theta\Delta k, b) - f_1(a,b)\right]$$

$$< \Delta k f_{12}(a,b) + \Delta k \frac{\epsilon}{2},$$

and similarly,

$$\Delta k f_{12}(a,b) - \Delta k \frac{\epsilon}{2} < f_1(a+\theta\Delta k, b+\Delta k) - f_1(a+\theta\Delta k, b).$$

Hence,

$$\Delta^2 k\left[f_{12}(a,b) - \frac{\epsilon}{2}\right] < \Delta k\left[f_1(a+\theta\Delta k, b+\Delta k) - f_1(a+\theta\Delta k, b)\right]$$

$$< \Delta^2 k\left[f_{12}(a,b) + \frac{\epsilon}{2}\right];$$

in other words,

$$\Delta^2 k\left[f_{12}(a,b) - \frac{\epsilon}{2}\right] < \left[f(a+\Delta k, b+\Delta k) - f(a+\Delta k, b)\right]$$

$$-\left[f(a, b+\Delta k) - f(a,b)\right]$$

$$< \Delta^2 k\left[f_{12}(a,b) + \frac{\epsilon}{2}\right].$$

For each y such that $b \leq y \leq b+\Delta k$, let

$$h(y) = f(a+\Delta k, y) - f(a,y).$$

By using the same procedure on h that was used for g, it follows that

$$\Delta^2 k\left[f_{21}(a,b) - \frac{\epsilon}{2}\right]$$

$$< \left[f(a+\Delta k, b+\Delta k) - f(a, b+\Delta k)\right] - \left[f(a+\Delta k, b) - f(a,b)\right]$$

$$< \Delta^2 k\left[f_{21}(a,b) + \frac{\epsilon}{2}\right].$$

Therefore,

$$\Delta^2 k\left[f_{12}(a,b)-\frac{\epsilon}{2}\right] < \Delta^2 k\left[f_{21}(a,b)+\frac{\epsilon}{2}\right]$$

and

$$\Delta^2 k\left[f_{21}(a,b)-\frac{\epsilon}{2}\right] < \Delta^2 k\left[f_{12}(a,b)+\frac{\epsilon}{2}\right];$$

in other words,

$$-\epsilon < f_{21}(a,b)-f_{12}(a,b) < \epsilon \quad .$$

Since $\epsilon > 0$ was chosen arbitrarily, it is clear that

$$f_{21}(a,b) = f_{12}(a,b)$$

and the proof is complete.

3.16. __Theorem__. __If__ g __and__ h __are defined on__ $I(\zeta;\delta_1)$, __and__ $g'(\zeta)$ __and__ $h'(\zeta)$ __exist__, __and if__ f __is defined on__

$$C((a,b);\delta_2),$$

$a = g(\zeta)$ __and__ $b = h(\zeta)$, __and__ f __is differentiable at__ (a,b), __then there exists a__ $\delta_3 > 0$ __so that__ $F = f(g,h)$ __is defined on__ $I(\zeta;\delta_3)$, $F'(\zeta)$ __exists__, __and__

$$F'(\zeta) = f_1(a,b)g'(\zeta)+f_2(a,b)h'(\zeta).$$

__Proof__. By 2.5, g and h are continuous at ζ, and with this fact it will follow that there exists a $\delta_3 > 0$ so that $F = f(g,h)$ is defined on $I(\zeta;\delta_3)$.

__Case__ I. Assume $g'(\zeta) = 0 = h'(\zeta)$. Choose $\epsilon > 0$. Since f is differentiable at (a,b), by 3.8, $f_1(a,b)$ and $f_2(a,b)$ exist and $f_1(a,b) = A_f$ and $f_2(a,b) = B_f$; therefore, there exists a $\delta_4 > 0$, $\delta_4 \le \delta_2$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x+\Delta^2 y} < \delta_4,$$

then

$$\left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\ \Delta^2 y}} - \frac{f_1(a,b)\Delta x+f_2(a,b)\Delta y}{\sqrt{\Delta^2 x+\ \Delta^2 y}} \right| < 1.$$

Since $g'(\zeta) = 0 = h'(\zeta)$, there exists a $\delta_5 > 0$, $\delta_5 \leq \delta_1$, so that whenever t is chosen such that $0 < |t-\zeta| < \delta_5$, then

$$\left| \frac{g(t)-g(\zeta)}{t-\zeta} \right| < \frac{\epsilon}{2(|f_1(a,b)|+|f_2(a,b)|+1)}$$

and

$$\left| \frac{h(t)-h(\zeta)}{t-\zeta} \right| < \frac{\epsilon}{2(|f_1(a,b)|+|f_2(a,b)|+1)}.$$

Since g and h are continuous at ζ, there exists a

$$\delta_6 > 0,\ \delta_6 \leq \delta_1,$$

so that whenever t is chosen such that $|t-\zeta| < \delta_6$, then

$$\left| g(t)-g(\zeta) \right| < \frac{\delta_4}{\sqrt{2}}$$

and

$$\left| h(t)-h(\zeta) \right| < \frac{\delta_4}{\sqrt{2}}.$$

Choose $\delta = \min(\delta_3, \delta_5, \delta_6)$. Choose t such that

$$0 < |t-\zeta| < \delta .$$

Let $\Delta t = t-\zeta$, $\Delta x = g(\zeta+\Delta t)-g(\zeta)$ and $\Delta y = h(\zeta+\Delta t)-h(\zeta)$. Then $g(\zeta+\Delta t) = g(\zeta)+[g(\zeta+\Delta t)-g(\zeta)] = a+\Delta x$ and

$$h(\zeta+\Delta t) = h(\zeta)+[h(\zeta+\Delta t)-h(\zeta)] = b+\Delta y.$$

Now

$$|\Delta x| < \frac{\delta_4}{\sqrt{2}}$$

and

$$|\Delta y| < \frac{\delta_4}{\sqrt{2}}.$$

Thus, $\Delta^2 x + \Delta^2 y < \delta_4^2$, and therefore,

$$\sqrt{\Delta^2 x + \Delta^2 y} < \delta_4.$$

Now, if $\Delta x = \Delta y = 0$, then

$$\left| \frac{F(t)-F(\zeta)}{t-\zeta} - \left[f_1(a,b)g'(\zeta)+f_2(a,b)h'(\zeta)\right] \right|$$

$$= \left| \frac{F(\zeta+\Delta t)-F(\zeta)}{t-\zeta} \right|$$

$$= \left| \frac{f(g(\zeta+\Delta t),h(\zeta+\Delta t))-f(g(\zeta),h(\zeta))}{t-\zeta} \right|$$

$$= \left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{t-\zeta} \right|$$

$$= 0 < \epsilon .$$

Assume that not both of Δx and Δy are zero; then

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta_4,$$

and therefore,

$$\left| \frac{F(t)-F(\zeta)}{t-\zeta} - \left[f_1(a,b)g'(\zeta)+f_2(a,b)h'(\zeta)\right] \right|$$

$$= \left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{t-\zeta} \right|$$

$$= \left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} \quad \frac{\sqrt{\Delta^2 x+\Delta^2 y}}{t-\zeta} \right|$$

$$\leq \left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| \left| \frac{\Delta x}{t-\zeta} \right| + \left| \frac{f(a+\Delta x,b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| \left| \frac{\Delta y}{t-\zeta} \right|$$

$$< \left[\left| \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right| + 1 \right] \left| \frac{g(t)-g(\zeta)}{t-\zeta} \right|$$

$$+ \left[\left| \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right| + 1 \right] \left| \frac{h(t)-h(\zeta)}{t-\zeta} \right|$$

$$< \left[|f_1(a,b)| + |f_2(a,b)| + 1 \right] \frac{\epsilon}{2\left[|f_1(a,b)| + |f_2(a,b)| + 1 \right]}$$

$$+ \left[|f_1(a,b)| + |f_2(a,b)| + 1 \right] \frac{\epsilon}{2\left[|f_1(a,b)| + |f_2(a,b)| + 1 \right]}$$

$$= \epsilon .$$

<u>Case</u> II. Assume that not both of $g'(\zeta)$ and $h'(\zeta)$ are zero; and for simplicity, suppose it is $g'(\zeta)$ that is not zero. Choose $\epsilon > 0$ and let

$$\epsilon' = \min\left(\frac{\epsilon}{6(|g'(\zeta)|+1)}, \frac{\epsilon}{6(|h'(\zeta)|+1)}, 1 \right).$$

Since f is differentiable at (a,b), by 3.8, $f_1(a,b)$ and $f_2(a,b)$ exist and $f_1(a,b) = A_f$ and $f_2(a,b) = B_f$; therefore, there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_2$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta_4,$$

then

$$\left| \frac{f(a+\Delta x, b+\Delta y) - f(a,b)}{\sqrt{\Delta^2 x + \Delta^2 y}} - \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right| < \epsilon'.$$

Since $g'(\zeta)$ and $h'(\zeta)$ exist, $g'(\zeta) \neq 0$, there exists a $\delta_5 > 0$, $\delta_5 \leq \delta_1$, so that whenever t is chosen such that

$$0 < |t-\zeta| < \delta_5,$$

then

$$\left| \frac{g(t)-g(\zeta)}{t-\zeta} - g'(\zeta) \right| < \min\left(\frac{\epsilon}{6(|f_1(a,b)|+1)}, \frac{|g'(\zeta)|}{2}\right)$$

and

$$\left| \frac{h(t)-h(\zeta)}{t-\zeta} - h'(\zeta) \right| < \frac{\epsilon}{6(|f_2(a,b)|+1)}.$$

By 2.5, g and h are continuous at ζ; therefore, there exists a $\delta_6 > 0$, $\delta_6 \leq \delta_1$, so that whenever t is chosen such that $|t-\zeta| < \delta_6$, then

$$|g(t)-g(\zeta)| < \frac{\delta_4}{\sqrt{2}}$$

and

$$|h(t)-h(\zeta)| < \frac{\delta_4}{\sqrt{2}}.$$

Choose $\delta = \min(\delta_3, \delta_5, \delta_6)$. Choose t such that

$$0 < |t-\zeta| < \delta .$$

Let $\Delta t = t-\zeta$, $\Delta x = g(\zeta+\Delta t)-g(\zeta)$ and $\Delta y = h(\zeta+\Delta t)-h(\zeta)$. Then $g(\zeta+\Delta t) = g(\zeta)+[g(\zeta+\Delta t)-g(\zeta)] = a+\Delta x$ and

$$h(\zeta+\Delta t) = h(\zeta)+[h(\zeta+\Delta t)-h(\zeta)] = b+\Delta y.$$

Now $\Delta x \neq 0$, for $\Delta x = 0$ contradicts the fact that

$$\left| \frac{\Delta x}{t-\zeta} - g'(\zeta) \right| < \frac{|g'(\zeta)|}{2}.$$

Since

$$0 < |\Delta x| < \frac{\delta_4}{\sqrt{2}}$$

and

$$|\Delta y| < \frac{\delta_4}{\sqrt{2}},$$

it follows that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta_4.$$

Assume $t - \xi > 0$; the procedure will be similar if $t - \xi$ is assumed to be negative. Now

$$\begin{aligned} F(t) - F(\xi) &= F(\xi + \Delta t) - F(\xi) \\ &= f(g(\xi + \Delta t), h(\xi + \Delta t)) - f(g(\xi), h(\xi)) \\ &= f(a + \Delta x, b + \Delta y) - f(a, b). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{F(t) - F(\xi)}{t - \xi} &= \frac{f(a + \Delta x, b + \Delta y) - f(a, b)}{t - \xi} \\ &< \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y + \epsilon' \sqrt{\Delta^2 x + \Delta^2 y}}{t - \xi} \\ &< f_1(a,b)\frac{\Delta x}{t - \xi} + f_2(a,b)\frac{\Delta y}{t - \xi} + \epsilon' \frac{|\Delta x|}{t - \xi} + \epsilon' \frac{|\Delta y|}{t - \xi} \\ &= f_1(a,b)\frac{g(t) - g(\xi)}{t - \xi} + f_2(a,b)\frac{h(t) - h(\xi)}{t - \xi} \\ &\quad + \epsilon' \left| \frac{g(t) - g(\xi)}{t - \xi} \right| + \epsilon' \left| \frac{h(t) - h(\xi)}{t - \xi} \right| \\ &< f_1(a,b)g'(\xi) + f_2(a,b)h'(\xi) \\ &\quad + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \\ &= f_1(a,b)g'(\xi) + f_2(a,b)h'(\xi) + \epsilon, \end{aligned}$$

and by a similar argument it follows that

$$\frac{F(t)-F(\zeta)}{t-\zeta} > f_1(a,b)g'(\zeta)+f_2(a,b)h'(\zeta)-\epsilon .$$

Therefore,

$$\left| \frac{F(t)-F(\zeta)}{t-\zeta} - \left[f_1(a,b)g'(\zeta)+f_2(a,b)h'(\zeta)\right] \right| < \epsilon .$$

Cases I and II verify that $F'(t)$ exists and

$$F'(t) = f_1(a,b)g'(\zeta)+f_2(a,b)h'(\zeta).$$

3.17. Theorem. If r_0 and s_0 are real numbers and for each $r \in I(r_0; \delta_1)$ $g(r,s_0)$ and $h(r,s_0)$ are defined, and $g_1(r_0,s_0)$ and $h_1(r_0,s_0)$ exist, and if f is defined on $C((a,b); \delta_2)$, $a = g(r_0,s_0)$ and $b = h(r_0,s_0)$, and f is differentiable at (a,b), then there exists a $\delta_3 > 0$ so that for each $r \in I(r_0; \delta_3)$, $F(r,s_0) = f(g(r,s_0),h(r,s_0))$ is defined, $F_1(r_0,s_0)$ exists, and

$$F_1(r_0,s_0) = f_1(a,b)g_1(r_0,s_0)+f_2(a,b)h_1(r_0,s_0).$$

Proof. The theorem follows by 3.15.

3.18. It should be clear that an appropriate change in the hypothesis of 3.16 concerning g and h would be sufficient for $F_2(r_0,s_0)$ to exist and for $F_2(r_0,s_0)$ to be equal to

$$f_1(a,b)g_2(r_0,s_0)+f_2(a,b)h_2(r_0,s_0).$$

3.19. Theorem. Let $\Delta x, \Delta y > 0$. If P is a set of ordered pairs of real numbers and f is defined on P, and if for each t such that $0 \le t \le 1$, $(a+t\Delta x, b+t\Delta y)$ is an interior point of P and f is differentiable at $(a+t\Delta x, b+t\Delta y)$, then there exists a θ, $0 < \theta < 1$, so that

$$f(a+\Delta x, b+\Delta y)-f(a,b)$$
$$= \Delta x f_1(a+\theta\Delta x, b+\theta\Delta y)+\Delta y f_2(a+\theta\Delta x, b+\theta\Delta y).$$

Proof. Since (a,b) and $(a+\Delta x, b+\Delta y)$ are interior points of P, there exist $\delta_1, \delta_2 > 0$ so that $C((a,b);\delta_1) \subset P$ and $C((a+\Delta x, b+\Delta y);\delta_2) \subset P$. For each t such that

$$-\delta_1 < t < 1+\delta_2,$$

let $g(t) = a+t\Delta x$, $h(t) = b+t\Delta y$, and $F(t) = f(g(t),h(t))$; by 3.16, $F'(t)$ exists and

$$F'(t) = f_1(g(t),h(t))g'(t)+f_2(g(t),h(t))h'(t).$$

By 3.4 there exists a θ, $0 < \theta < 1$, so that

$$F(1)-F(0) = F'(0+\theta) = F'(\theta).$$

Now since

$$F'(\theta) = f_1(a+\theta\Delta x, b+\theta\Delta y)\Delta x+f_2(a+\theta\Delta x, b+\theta\Delta y)\Delta y$$

and

$$F'(\theta) = F(1)-F(0) = f(a+\Delta x, b+\Delta y)-f(a,b),$$

the theorem is proved.

3.20. It should be clear that 3.18 is a generalization of 2.8.

3.21. *Theorem*. *If* F *is defined on* $N((a,b);\delta_1;\delta_1)$, *and if* F_2 *exists on* $N((a,b);\delta_1;\delta_1)$, $F(a,b) = 0$ *and* $F_2(a,b) \neq 0$, *and if* F_2 *is continuous at* (a,b), *and if for each* y *such that* $|y-b| < \delta_1$, F *is continuous at* (a,y) *with respect to the first variable, then each of the following statements is true:*

1) *there exist* δ' *and* δ'', $0 < \delta', \delta'' \leq \delta_1$, *so that for each* x *such that* $|x-a| < \delta'$, *there exists exactly one* y,

<u>denoted by</u> $f(x)$, <u>such that</u> $|y-b| < \delta''$ <u>and</u> $F(x,y) = 0$;

ii) f <u>is continuous at</u> a; <u>and</u>

iii) <u>if</u> $F_1(a,b)$ <u>exists, then</u> F <u>is differentiable at</u> (a,b), $f'(a)$ <u>exists, and</u>

$$f'(a) = -\frac{F_1(a,b)}{F_2(a,b)},$$

<u>i.e.</u>

$$f'(a) = -\frac{F_1(a,f(a))}{F_2(a,f(a))}.$$

<u>Proof of</u> (i). Assume $F_2(a,b) > 0$. Since F_2 is continuous at (a,b), there exists a $\delta_2 > 0$, $\delta_2 \leq \delta_1$, so that whenever x and y are chosen such that $|x-a| < \delta_2$ and $|y-b| < \delta_2$, then

$$\left| F_2(x,y) - F_2(a,b) \right| < \frac{F_2(a,b)}{2};$$

in other words,

$$\frac{F_2(a,b)}{2} < F_2(x,y) < \frac{3F_2(a,b)}{2}.$$

Since $F_2(a,b)$ exists and $F_2(a,b)$ 0, there exists a

$$\delta'' > 0, \ \delta'' \leq \delta_2,$$

so that whenever y is chosen such that $0 < y-b \leq \delta''$, then

$$\left| \frac{F(a,y)-F(a,b)}{y-b} - F_2(a,b) \right| < F_2(a,b);$$

it follows that if $0 < y-b \leq \delta''$, then $F(a,y) > 0$, and if

$$-\delta'' \leq y-b < 0,$$

then $F(a,y) < 0$.

Since $F(a,b+\delta'') > 0$ and $F(a,b-\delta'') < 0$, and since F is continuous at $(a,b+\delta'')$ and $(a,b-\delta'')$ with respect to the first variable, there exists a $\delta' > 0$, $\delta' \le \delta_2$, so that whenever x is chosen such that $|x-a| < \delta'$, then

$$|F(x,b+\delta'')-F(a,b+\delta'')| < F(a,b+\delta''),$$

i.e. $F(x,b+\delta'') > 0$, and $|F(x,b-\delta'')-F(a,b-\delta'')| < -F(a,b-\delta'')$, *i.e.* $F(x,b-\delta'') < 0$.

Choose x such that $|x-a| < \delta'$. Now $F(x,b+\delta'') > 0$, $F(x,b-\delta'') < 0$, and since F_2 exists on N, then, by 2.5, F is continuous at (x,y) with respect to the second variable for each y such that $b-\delta'' < y < b+\delta''$. By 2.1 there exists a y such that $b-\delta'' < y < b+\delta''$ and $F(x,y) = 0$. Assume there exists a y_1 such that $y_1 \neq y$, $b-\delta'' < y_1 < b+\delta''$, and $F(x,y_1) = 0$. Then, by 2.8, assuming $y < y_1$, there exists a y_2 such that $y < y_2 < y_1$ and $F(x,y_1)-F(x,y) = (y_1-y)F_2(x,y_2)$; but this implies that $F_2(x,y_2) = 0$, a contradiction of an earlier restriction imposed on F_2. Therefore, y is uniquely determined. The proof is similar if $F_2(a,b)$ is assumed to be negative.

Proof of (ii). Choose $\epsilon > 0$. Let

$$N' = \{(x,y) \mid |x-a| < \delta'$$

and $|y-b| < \delta''\}$. Then, by applying (i), there exists a δ'_a, $0 < \delta'_a < \delta'$, and a δ''_b, $0 < \delta''_b \le \min(\delta'', \epsilon)$, so that for each x such that $|x-a| < \delta'_a$, there exists exactly one $y = g(x)$ such that $|y-b| < \delta''_b$ and $F(x,y) = 0$. Clearly, $g(x) = f(x)$ for each such x. Thus, $|f(x)-f(a)| < \epsilon$, and therefore, f is continuous at a.

Proof of (iii). The function F is differentiable at (a,b) by 3.9.

Choose $\epsilon > 0$. Since $F_2(a,b) \neq 0$, and F_2 is continuous at (a,b), it follows that $\frac{1}{F_2}$ is continuous at (a,b). Therefore, there exists a $\delta_3 > 0$, $\delta_3 \leq \min(\delta', \delta'')$, where δ' and δ'' are chosen as in (i), so that whenever Δx and Δy are chosen such that $|\Delta x| < \delta_3$ and $|\Delta y| < \delta_3$, then

$$\left| \frac{1}{F_2(a+\Delta x, b+\Delta y)} - \frac{1}{F_2(a,b)} \right| < \frac{\epsilon}{2(|F_1(a,b)|+1)}.$$

It follows from (i) that there exists a positive number M so that whenever Δx and Δy are chosen such that $|\Delta x| < \delta_3$ and $|\Delta y| < \delta_3$, then

$$\left| \frac{1}{F_2(a+\Delta x, b+\Delta y)} \right| < M.$$

Since f is continuous at a, there exists a $\delta_4 > 0$, $\delta_4 \leq \delta_3$, so that whenever x is chosen such that $|x-a| < \delta_4$, then $|f(x)-f(a)| < \delta_3$. Since $F_1(a,b)$ exists, there exists a $\delta > 0$, $\delta \leq \delta_4$, so that whenever Δx is chosen such that $0 < |\Delta x| < \delta$, then

$$\left| \frac{F(a+\Delta x, b)-F(a,b)}{\Delta x} - F_1(a,b) \right| < \frac{\epsilon}{2M}.$$

Choose x such that $0 < |x-a| < \delta$, and let $y = f(x)$. Let $\Delta x = x-a$ and $\Delta y = y-b = f(x)-f(a)$. Consider

$$0 = F(x,y)-F(a,b) = F(a+\Delta x, b+\Delta y)-F(a,b)$$
$$= [F(a+\Delta x, b+\Delta y)-F(a+\Delta x, b)]+[F(a+\Delta x, b)-F(a,b)].$$

Now there exists a θ, $0 < \theta < 1$, so that

$$F(a+\Delta x, b+\Delta y)-F(a+\Delta x, b) = \Delta y F_2(a+\Delta x, b+\theta\Delta y).$$

Thus, $0 = \Delta y F_2(a+\Delta x, b+\Theta \Delta y)+\left[F(a+\Delta x,b)-F(a,b)\right]$, i.e.

$$\frac{\Delta y}{\Delta x} = -\frac{F(a+\Delta x,b)-F(a,b)}{\Delta x}\,\frac{1}{F_2(a+\Delta x,b+\theta\Delta y)}.$$

Now

$$\left|\frac{f(x)-f(a)}{x-a}+\frac{F_1(a,b)}{F_2(a,b)}\right| = \left|\frac{\Delta y}{\Delta x}+\frac{F_1(a,b)}{F_2(a,b)}\right|$$

$$= \left|\frac{F_1(a,b)}{F_2(a,b)} - \frac{F(a+\Delta x,b)-F(a,b)}{\Delta x}\,\frac{1}{F_2(a+\Delta x,b+\theta\Delta y)}\right|$$

$$= \left|\frac{F_1(a,b)}{F_2(a,b)} - F_1(a,b)\frac{1}{F_2(a+\Delta x,b+\theta\Delta y)} + F_1(a,b)\frac{1}{F_2(a+\Delta x,b+\theta\Delta y)} - \frac{F(a+\Delta x,b)-F(a,b)}{\Delta x}\,\frac{1}{F_2(a+\Delta x,b+\theta\Delta y)}\right|$$

$$\leq \left|F_1(a,b)\right|\left|\frac{1}{F_2(a,b)} - \frac{1}{F_2(a+\Delta x,b+\theta\Delta y)}\right| + \left|\frac{1}{F_2(a+\Delta x,b+\theta\Delta y)}\right|\left|F_1(a,b) - \frac{F(a+\Delta x,b)-F(a,b)}{\Delta x}\right|$$

$$< \left|F_1(a,b)\right|\frac{\epsilon}{2(\left|F_1(a,b)\right|+1)} + M\frac{\epsilon}{2M}$$

$$< \epsilon .$$

Therefore, $f'(a)$ exists and

$$f'(a) = -\frac{F_1(a,b)}{F_2(a,b)};$$

in other words,

$$f'(a) = -\frac{F_1(a,f(a))}{F_2(a,f(a))}.$$

3.22. Theorem. If f is defined on $C((a,b);\delta_1)$, and if f is differentiable at (a,b), then each of the following statements is true:

i) for each α such that $0 \leq \alpha \leq 2\pi$, $D(f;(a,b);\alpha)$ exists and $D(f;(a,b);\alpha) = f_1(a,b)\cos\alpha + f_2(a,b)\sin\alpha$;

ii) if $E = \{F(\alpha) = f_1(a,b)\cos\alpha + f_2(a,b)\sin\alpha \mid 0 \leq \alpha \leq 2\pi\}$ and not both of $f_1(a,b)$ and $f_2(a,b)$ are 0, then there exists a unique α_1 such that $0 \leq \alpha_1 \leq 2\pi$ and $F(\alpha_1)$ is the least upper bound of E, and there exists a unique α_2 such that $0 \leq \alpha_2 \leq 2\pi$ and $F(\alpha_2)$ is the greatest lower bound of E, and

$$D(f;(a,b);\alpha_1) = \sqrt{f_1^2(a,b)+f_2^2(a,b)}$$

and

$$D(f;(a,b);\alpha_2) = -\sqrt{f_1^2(a,b)+f_2^2(a,b)};$$

and

iii) for each α such that $0 \leq \alpha \leq 2\pi$,

$$D(f;(a,b);\alpha) = -D(f;(a,b);\alpha+\pi).$$

Proof of (i). Choose α such that $0 \leq \alpha \leq 2\pi$. Choose $\epsilon > 0$. Since f is differentiable at (a,b), by 3.8, $f_1(a,b)$ and $f_2(a,b)$ exist, and $f_1(a,b) = A_f$ and $f_2(a,b) = B_f$; therefore, there exists a $\delta > 0$, $\delta \leq \delta_1$, so that whenever Δx and Δy are chosen such that

$$0 < \sqrt{\Delta^2 x + \Delta^2 y} < \delta,$$

then

$$\left| \frac{f(a+\Delta x, b+\Delta y)-f(a,b)}{\sqrt{\Delta^2 x+\Delta^2 y}} - \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y}{\sqrt{\Delta^2 x+\Delta^2 y}} \right| < \epsilon .$$

Choose Δs such that $0 < \Delta s < \delta$, and let $\Delta x = \Delta s \cos \alpha$ and $\Delta y = \Delta s \sin \alpha$. Then

$$0 < \Delta s = \sqrt{\Delta^2 x + \Delta^2 y} < \delta .$$

Therefore,

$$\left| \frac{f(a+\Delta x, b+\Delta y) - f(a,b)}{\sqrt{\Delta^2 x + \Delta^2 y}} - \frac{f_1(a,b)\Delta x + f_2(a,b)\Delta y}{\sqrt{\Delta^2 x + \Delta^2 y}} \right|$$

$$= \left| \frac{f(a+\Delta s \cos\alpha, b+\Delta s \sin\alpha) - f(a,b)}{\Delta s} - \frac{f_1(a,b)\Delta s \cos\alpha + f_2(a,b)\Delta s \sin\alpha}{\Delta s} \right|$$

$$= \left| \frac{f(a+\Delta s \cos\alpha, b+\Delta s \sin\alpha) - f(a,b)}{\Delta s} - \left[f_1(a,b)\cos\alpha + f_2(a,b)\sin\alpha\right] \right| < \epsilon .$$

Thus, $D(f;(a,b);\alpha)$ exists and

$$D(f;(a,b);\alpha) = f_1(a,b)\cos\alpha + f_2(a,b)\sin\alpha .$$

<u>Proof of</u> (ii). It was assumed in the hypothesis that not both of $f_1(a,b)$ and $f_2(a,b)$ were zero, for if they were, then $F(\alpha)$ would be zero for each α such that $0 \leq \alpha \leq 2\pi$, and this would have been a trivial result.

If F is defined for all angles α, then it is clear from trigonometry that F is of period 2π. Certainly, F is a continuous function of α. By 2.3,

$$E = \{F(\alpha) = f_1(a,b)\cos\alpha + f_2(a,b)\sin\alpha \mid \alpha \in [0, 2\pi]\}$$

has a unique least upper bound K and a unique greatest lower bound k, and there exists an $\alpha_1 \in [0, 2\pi]$ such that

$$F(\alpha_1) = K$$

and there exists an $\alpha_2 \in [0, 2\pi]$ such that $F(\alpha_2) = k$.

Clearly, $F'(\alpha)$ exists and

$$F'(\alpha) = -f_1(a,b)\sin\alpha + f_2(a,b)\cos\alpha$$

for each α such that $0 \leq \alpha \leq 2\pi$. By 2.4, $F'(\alpha_1) = F'(\alpha_2) = 0$.

Consider

$$F'(\alpha) = -f_1(a,b)\sin\alpha + f_2(a,b)\cos\alpha = 0.$$

It follows that there are but two solutions β_1 and β_2 of this equation, $0 \leq \beta_1, \beta_2 \leq 2\pi$, determined by the equations

$$\sin\beta_1 = \frac{f_2(a,b)}{\sqrt{f_1^2(a,b)+f_2^2(a,b)}}$$

and

$$\cos\beta_1 = \frac{f_1(a,b)}{\sqrt{f_1^2(a,b)+f_2^2(a,b)}}$$

and

$$\sin\beta_2 = -\frac{f_2(a,b)}{\sqrt{f_1^2(a,b)+f_2^2(a,b)}}$$

and

$$\cos\beta_2 = -\frac{f_1(a,b)}{\sqrt{f_1^2(a,b)+f_2^2(a,b)}},$$

respectively; it follows that

$$D(f;(a,b);\beta_1) = \sqrt{f_1^2(a,b)+f_2^2(a,b)}$$

and

$$D(f;(a,b);\beta_2) = -\sqrt{f_1^2(a,b)+f_2^2(a,b)}.$$

Therefore, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, and hence, α_1 and α_2 are unique.

Proof of (iii). Since

$$F(\alpha) = f_1(a,b) \cos\alpha + f_2(a,b) \sin\alpha ,$$

it follows from trigonometry that

$$f_1'(a,b) \cos\alpha + f_2(a,b) \sin\alpha$$
$$= -f_1(a,b) \cos(\alpha + \pi) - f_2(a,b) \sin(\alpha + \pi),$$

i.e. $D(f;(a,b);\alpha) = -D(f;(a,b);\alpha + \pi)$.

3.23. Example. Consider the function f of 3.11. For each α such that $0 \leq \alpha \leq 2\pi$, $D(f;(0,0);\alpha)$ exists and

$$D(f;(0,0);\alpha) = \frac{\sin 2\alpha}{2}.$$

But in 3.11 it was pointed out that f was not differentiable at (0,0).

3.24. The directional derivative could have been defined for a function of one real variable. Clearly, a function of one real variable could have at most two directional derivatives, and if these two directional derivatives differ only in sign, then the function is necessarily differentiable.

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