PROPERTIES OF SEMIGROUPS

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CHAPTER I

INTRODUCTION

This paper is an introductory, algebraic study of semigroups. The primary function of this chapter is to establish and orient the ideas which are essential to the basic format of the paper.

The content of the paper is briefly described as follows: Left and right zero elements, zero elements, left and right zero semigroups, zero semigroups, identity elements, and cancellative semigroups are considered in Chapter Two. Chapter Three deals with a partial ordering of idempotents, nowhere commutative semigroups, bands, dominators, rectangular semigroups, regular elements and semigroups, inverses, and inverse semigroups. Chapter Four concerns a particular collection of semigroups. Each semigroup in this collection is formed by defining a binary operation on the collection of all transformations of a given set.

Attention in Chapter One will now be focused upon the ideas which are basic to the paper. The ideas include undefined terms, definitions, notations, and examples. The undefined terms are set, element, and ordered pair.

Intuitively, a set will be thought of as a collection of one or more objects called elements. Capital letters will be used to denote sets while lower-case letters will be used to denote elements.

Definition 1.1. Let each of A and B be a set. The statement that A is a subset of B, denoted by $A \subseteqq B$, means every element of A is an element of B.

Definition 1.2. Let each of A and B be a set. The statement that A and B are equal, denoted by $A = B$, means $A \subseteqq B$ and $B \subseteqq A$.

Definition 1.3. Let each of A and B be a set. The statement that A is a proper subset of B, denoted by $A \subset B$, means $A \subseteqq B$ and $A \neq B$.

Definition 1.4. Let A be a set. Then $a \in A$ means a is an element of A.

Definition 1.5. Let A be a set and let $a, b \in A$. The statement that a equals b, denoted by $a = b$, means a is b. Obviously, if $a = b$, then $b = a$.

Definition 1.6. Let A be a set. Then $\{x \mid x \in A\}$ is an example of the set builder notation that will be used. It denotes the set of all elements x such that x is an element of A.

Definition 1.7. Let each of A and B be a set. The union of A and B is denoted by $A \cup B$ and defined by the following. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. It is clear that $A \cup B$ is a set if each of A and B is a set.

Definition 1.8. Let each of A and B be a set. The intersection of A and B is denoted by $A \cap B$ and defined by the following. If there exists at least one element of A which is also an element of B, then $A \cap B$ is defined by: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. If no such element exists, then A and B are said to be mutually exclusive and $A \cap B$ has no meaning.

Definition 1.9. An ordered pair of elements, a and b, is denoted by (a,b).

Definition 1.10. The statement that R is a relation means R is a set of ordered pairs.

Definition 1.11. The inverse of a relation R is denoted by R^{-1} and is defined by the following. $R^{-1} = \{(x,y) \mid (y,x) \in R\}$.

Definition 1.12. Let R be a relation. The domain of R is denoted by D(R) and is defined by: $D(R) = \{x \mid (x,y) \in R \text{ for some } y\}$. The range of R is denoted by R(R) and $R(R) = D(R^{-1})$.

Definition 1.13. The statement that F is a function means F is a relation in which no two ordered pairs have the same first element. It will be convenient to use the notation $y = F(x)$ to mean that $(x,y) \in F$.

Definition 1.14. The statement that F is a reversible function means both F and F^{-1} are functions.

Definition 1.15. Let each of A and B be a set. Then $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

Definition 1.16. Let S be a set. The statement that O is a binary operation defined on S means O is a function whose domain is $S \times S$ and whose range is a subset of S.

Definition 1.17. Let O be a binary operation defined on a set S. The statement that O is associative means if $a \in S$, $b \in S$, $c \in S$, $((a,b),x) \in O$, $((x,c),y) \in O$, $((b,c),z) \in O$, and $((a,z),w) \in O$, then $y = w$.

Definition 1.18. Let O be an associative binary operation defined on a set S. The following notation will be used. If $((a,b),x) \in O$, then x will be denoted by ab. It follows that O is associative means if $a \in S$, $b \in S$, and $c \in S$, then $(ab)c = a(bc)$.

Definition 1.19. The statement that S is a semigroup means S is a set on which there is defined an associative binary operation.

Definition 1.20. The statement that a semigroup S is of finite order means there exists a positive integer n which corresponds to the number of elements in the set S. Also, S is said to be of order n. If no such positive integer exists, then S is said to be of infinite order.

Definition 1.21. The statement that a semigroup S is degenerate means S is of order 1. The statement that S is non-degenerate means there are at least two elements in the set S.

Definition 1.22. Let each of S and S' be a semigroup. The statement that S is isomorphic with S', denoted by

$S \cong S'$, means there exists a reversible function F whose domain is S and whose range is S' and such that if $(x,x') \in F$ and $(y,y') \in F$, then $(xy,x'y') \in F$.

Definition 1.23. Let S be a semigroup and let $e \in S$. The statement that e is an idempotent element means $e = ee$. Also, the statement that e is idempotent will be used.

Definition 1.24. Let S be a semigroup and let $S' \subseteq S$. The statement that S' is a subsemigroup of S means if $a,b \in S'$, then $ab \in S'$.

Definition 1.25. Let S be a semigroup, $A \subseteq S$, and $x \in S$. The sets xA and Ax are defined as follows. $xA = \{xa \mid a \in A\}$ and $Ax = \{ax \mid a \in A\}$.

Definition 1.26. Let S be a semigroup, $A \subseteq S$, and $B \subseteq S$. The set AB is defined by $AB = \{ab \mid a \in A \text{ and } b \in B\}$.

Definition 1.27. Let S be a semigroup and $A \subseteq S$. The statement that A is a left (right) ideal of S means if $x \in S$, then $xA \subseteq A$ $(Ax \subseteq A)$. The statement that A is an ideal of S means A is both a left and right ideal of S.

Definition 1.28. The statement that a semigroup S is left simple (right simple) means if A is a left (right) ideal of S, then $A = S$. The statement that A is simple means if A is an ideal of S, then $A = S$.

Definition 1.29. Let each of a and b be an element of a semigroup S. The statement that the elements a and b commute with each other means $ab = ba$. The statement that a is commutative means that a commutes with each element of S. The statement that S is commutative means if each of a and b is in S, then $ab = ba$.

<u>Theorem 1.1.</u> If S is a semigroup of finite order, then S contains an idempotent element.

<u>Proof.</u> The semigroup S is of finite order implies there exists a positive integer j which corresponds to the number of elements in S. Now let $a \in S$ and let $A = \{a, a^2, a^3, \ldots, a^j, a^{j+1}\}$. It is clear $A \subseteq S$ since $a \in S$ and S is a semigroup. Now there are j+1 representations for the elements in A and since $A \subseteq S$ and S contains only j elements, it follows that there exist positive integers n and k where $n < k \leq j+1$ and such that $a^n = a^k$. Now $n < k$ implies there exists a positive integer p such that $k = n+p$. Thus $a^n = a^k$ and $k = n+p$ imply $a^n = a^{n+p}$. The statement that $a^n = a^{n+mp}$ where m is a positive integer will be proved by mathematical induction. The statement is true for $m = 1$ since $a^n = a^{n+p}$ implies $a^n = a^{n+1p}$. Now assume $a^n = a^{n+tp}$ is true for the positive integer t and show $a^n = a^{n+(t+1)p}$ is true. Clearly $a^n = a^{n+tp}$ implies $a^n = (a^n)(a^{tp})$. Now $a^n = (a^n)(a^{tp})$ and $a^n = a^{n+p}$ imply $a^n = (a^{n+p})(a^{tp})$ which implies $a^n = a^{n+p+tp}$ which implies $a^n = a^{n+(t+1)p}$. Thus $a^n = a^{n+mp}$ is true for each positive integer m and, in particular, $a^n = a^{n+np}$. If $p = 1$, then $a^n = a^{n+np}$ and $p = 1$ imply $a^n = a^{n+n}$ which implies $a^n = a^{2n}$. If $p > 1$, then $a^n = a^{n+np}$ and $a^{np-n} = a^{np-n}$ imply $a^n(a^{np-n}) = a^{n+np}(a^{np-n})$ which implies

$a^{n+np-n} = a^{n+np+np-n}$ which implies $a^{np} = a^{2(np)}$. Hence, in either case, a^{np} is an idempotent element of S. Thus the theorem is proved.

The preceding theorem was valuable in the search for examples of semigroups of finite order. This fact will be illustrated by outlining the procedure used in determining all semigroups of order two. In order for the set $S = \{a,b\}$ to be a semigroup, it was necessary to determine an associative binary operation on S. To do this it was convenient to define a binary operation O on S by means of an operation table such as

	a	b
a	a	b
b	a	a

where, in this case,

$O = \{((a,a),a),\ ((a,b),b),\ ((b,a),a),\ ((b,b),a)\}$. Then associativity was checked by a method developed by F.W. Light. In the above case, O is not associative since $(bb)b = ab = b$ and $b(bb) = ba = a$. Theorem 1.1 was useful in determining all semigroups of order two in that it decreased the number of possible operation tables by restricting the element a to be idempotent in each case.

The following examples of semigroups will serve to illustrate various concepts throughout the paper.

Example 1.1. $S1 = \{a\}$ with

	a
a	a

Example 1.2. $S2 = \{a,b\}$ with

	a	b
a	a	a
b	a	a

Example 1.3. $S3 = \{a,b\}$ with

	a	b
a	a	a
b	a	b

Example 1.4. $S4 = \{a,b\}$ with

	a	b
a	a	a
b	b	b

Example 1.5. $S5 = \{a,b\}$ with

	a	b
a	a	b
b	a	b

Example 1.6. $S6 = \{a,b\}$ with

	a	b
a	a	b
b	b	a

Example 1.7. $S7 = \{a,b,c\}$ with

	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

Example 1.8. $S8 = \{a,b,c\}$ with

	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

Example 1.9. $S9 = \{a,b,c\}$ with

	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Example 1.10. $S10 = \{a,b,c\}$ with

	a	b	c
a	a	b	a
b	a	b	a
c	a	b	a

Example 1.11. $S11 = \{a,b,c\}$ with

	a	b	c
a	a	b	b
b	b	a	a
c	b	a	a

Example 1.12. S12 = $\{a,b,c\}$ with

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Example 1.13. S13 = $\{a,b,c\}$ with

	a	b	c
a	a	a	a
b	a	b	b
c	a	b	b

Example 1.14. S14 = $\{a,b,c\}$ with

	a	b	c
a	a	b	a
b	b	a	b
c	a	b	a

Example 1.15. S15 = $\{a,b,c\}$ with

	a	b	c
a	a	a	a
b	b	b	b
c	a	a	a

Example 1.16. S16 = $\{a,b,c\}$ with

	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

<u>Example 1.17.</u> S17 = $\{a,b,c\}$ with

	a	b	c
a	a	a	a
b	a	b	c
c	a	c	a

<u>Example 1.18.</u> S18 = $\{a,b,c\}$ with

	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

<u>Example 1.19.</u> S19 = $\{a,b,c,d\}$ with

	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	b	a	d
d	d	b	b	d

<u>Example 1.20.</u> S20 = $\{a,b,c,d,e\}$ with

	a	b	c	d	e
a	a	d	a	d	e
b	e	b	c	e	e
c	c	b	c	b	e
d	e	d	a	e	e
e	e	e	e	e	e

Example 1.21. The set of positive integers with the binary operation of ordinary addition is a semigroup.

Example 1.22. The set of positive integers with the binary operation of ordinary multiplication is a semigroup.

Example 1.23. The set of integers with the binary operation of ordinary addition is a semigroup.

Example 1.24. The set of integers with the binary operation of ordinary multiplication is a semigroup.

Example 1.25. The set of positive integers with the binary operation $O = \{((a,b),a) \mid (a,b) \in S \times S\}$ is a semigroup.

Example 1.26. The set of positive integers with the binary operation $O = \{((a,b),b) \mid (a,b) \in S \times S\}$ is a semigroup.

CHAPTER II

ZEROES AND CANCELLATIVE SEMIGROUPS

Definition 2.1. An element z of a semigroup S is called a left zero element (right zero element) if for each x in S, $zx = z$ ($xz = z$). The element z is called a zero element of S if it is both a left and right zero element of S.

Definition 2.2. The statement that a semigroup S is a left zero semigroup (right zero semigroup) means every element of S is a left (right) zero element of S.

Definition 2.3. The statement that a semigroup S is a zero semigroup means there exists an element 0 in S such that if $a,b \in S$, then $ab = 0$.

Definition 2.4. An element e of a semigroup S is called a left identity element (right identity element) if for each a in S, $ea = a$ ($ae = a$). The element e is called an identity element of S if it is both a left and right identity element of S.

Definition 2.5. Let S be a semigroup. Then S^1 is defined to be S if S contains an identity element and is defined to be $S \cup \{1\}$ where 1 is an identity element for $S \cup \{1\}$ if S contains no identity element.

<u>Definition 2.6.</u> The statement that a semigroup S is left cancellative (right cancellative) means that if $a,x,y \in S$ such that $ax = ay$ ($xa = ya$), then $x = y$. The statement that S is cancellative means S is both left and right cancellative.

<u>Theorem 2.1.</u> If a semigroup S contains a left zero element a and a right zero element b, then $a = b$ and S contains a unique zero element.

<u>Proof.</u> The element b is in S and a is a left zero of S imply $ab = a$. The element a is in S and b is a right zero of S imply $ab = b$. Thus $a = b$. Now a is a zero element of S since it is both a left and right zero element. It follows that a is unique since if S contains a zero element it is a right zero element. Hence it is a by the previous argument.

<u>Theorem 2.2.</u> A zero semigroup S is a left (right) zero semigroup if, and only if, it is degenerate.

<u>Proof.</u> Suppose S is a zero semigroup which contains exactly one element and show S is a left zero semigroup. Let $a,b \in S$. Since S contains only one element, $a = b$ and $aa = a$. Thus $ab = a$ and S is a left zero semigroup.

Suppose S is both a zero and a left zero semigroup and show S contains exactly one element. Since S is a zero semigroup there exists an element 0 in S such that if $x,y \in S$, then $xy = 0$. Now let $a \in S$. The element a is in S and $0 \in S$ and S is a zero semigroup imply $a0 = 0$. Then

$a, 0 \in S$ and S is a left zero semigroup imply $a0 = a$. Now $a0 = 0$ and $a0 = a$ imply $a = 0$. Hence S contains only one element and is degenerate. This completes the proof of the theorem.

The semigroup S2 is an example of a zero semigroup which is not a left zero semigroup. The semigroup S4 is an example of a left zero semigroup which is not a zero semigroup.

Theorem 2.3. A left zero semigroup S is a right zero semigroup if, and only if, it is degenerate.

The proof of this theorem is very similar to that of the preceding theorem and will not be given.

Theorem 2.4. If e is an idempotent element of a left (right) cancellative semigroup S, then e is a left (right) identity element of S.

Proof. Let $a \in S$. Show $ea = a$. The element $ea = (ee)a$ since e is an idempotent element. Then $ea = e(ea)$ since the binary operation defined on S is associative. Now $ea = e(ea)$ and S is a left cancellative semigroup imply $a = ea$. Hence e is a left identity element of S. Similarly, e is a right identity element of S if S is a right cancellative semigroup.

Corollary 2.1. If e is an idempotent element of a cancellative semigroup S, then e is an identity element of S.

The proof is a direct consequence of Theorem 2.4.

Theorem 2.5. A cancellative semigroup can contain at most one idempotent element, namely an identity element.

Proof. Let S be a cancellative semigroup. If S contains no idempotent element, then it is clear that the theorem is true. Thus let S contain an idempotent element e and show S contains no other such element. Let f be an idempotent element of S and show f = e. Since e is an idempotent element of a cancellative semigroup S and $f \in S$, it follows from Corollary 2.1 that ef = f. Then ef = f and f is idempotent imply ef = ff. Now ef = ff and S is a cancellative semigroup imply e = f. It also follows from Corollary 2.1 that e is an identity element. Thus the theorem is proved.

It is interesting to note that the set of positive integers with the binary operation of ordinary addition is a cancellative semigroup which contains no idempotent element.

Theorem 2.6. If S is a cancellative semigroup, then S^1 is a cancellative semigroup.

Proof. By definition $S^1 = S$ if S has an identity element and $S^1 = S \cup \{1\}$ where 1 is an identity element of $S \cup \{1\}$ if S has no identity element. If $S^1 = S$, then it is clear S^1 is a cancellative semigroup. If $S^1 = S \cup \{1\}$, then it remains to show $S \cup \{1\}$ is a semigroup and $S \cup \{1\}$ is cancellative. To show $S \cup \{1\}$ is a semigroup, let each of a, b, and c be an arbitrary element of $S \cup \{1\}$ and show (ab)c = a(bc). Now either all of a, b, and c are elements of S or there is one or more of a, b, and c which is not an element of S. If all of a, b, and c are elements of S, then (ab)c = a(bc) since S is a semigroup. If one or more of a, b, and c is not an element of S then, without loss of generality, let one of these be a. Hence a = 1. Thus (ab)c = (1b)c = bc = 1(bc) = a(bc). Thus $S \cup \{1\}$ is a semigroup.

To show $S \cup \{1\}$ is left cancellative let each of a, b, and c be an arbitrary element of $S \cup \{1\}$ such that ab = ac and show b = c. Since $a \in S \cup \{1\}$, then either a = 1 or $a \neq 1$. If a = 1, then ab = ac and a = 1 imply 1b = 1c which implies b = c since 1 is an identity element for $S \cup \{1\}$. If $a \neq 1$, then b = 1 or $b \neq 1$. If b = 1, then ab = ac implies a = ac. Now suppose by way of contradiction $c \neq 1$. The element a = ac implies ac = a(cc). Each of a, c, and cc is an element of S, ac = a(cc), and S is cancellative imply c = cc. Now c is an idempotent of a cancellative semigroup. Thus it follows by Corollary 2.1 that c is an identity element of S. This is a contradiction since S contains no identity element in this case. Thus c = 1 and it follows that b = c since b = 1 in this case. If $b \neq 1$, then either c = 1 or $c \neq 1$. If c = 1, then $b \neq 1$ and c = 1 lead to a contradiction similar to the preceding one. If $c \neq 1$, then it follows that b = c since ab = ac where $a,b,c \in S$ and S is a cancellative semigroup. Now it has been shown that $S \cup \{1\}$ is a left cancellative semigroup and by a similar argument it can be shown that $S \cup \{1\}$ is a right cancellative semigroup. Thus it follows that $S \cup \{1\}$ is a cancellative semigroup. Hence S^1 is a cancellative semigroup and the theorem is proved.

<u>Theorem 2.7.</u> If S is a non-degenerate left zero semigroup, then S is right cancellative and S^1 is not.

<u>Proof.</u> To show S is right cancellative let each of a, b, and c be an element of S such that ba = ca and show b = c. It follows that ba = b and ca = c since S is a left zero semigroup. Thus b = ba = ca = c and S is right cancellative.

By definition $S^1 = S$ or $S^1 = S \cup \{1\}$. If $S^1 = S$, then the desired conclusion would not be satisfied. Thus it is necessary to show $S^1 \neq S$. To show $S^1 \neq S$ it is necessary to show S contains no identity elements. Now suppose, by way of contradiction, that S does contain an identity element e. Now let $a \in S$. Then $e, a \in S$ and S is a left zero semigroup imply ea = e. Also ea = a since e is an identity element for S. Thus ea = e and ea = a imply e = a which implies that S contains exactly one element, namely e. But S contains exactly one element implies S is degenerate which contradicts the hypothesis. Thus S contains no identity element. Hence $S^1 = S \cup \{1\}$. Now to show S^1 is not right cancellative it will suffice to exhibit the existence of elements a, b, and c in S^1 such that ba = ca and $b \neq c$. Let $a \in S$. Thus, since S is a left zero semigroup, aa = a. The element 1 is an identity for S and $a \in S$ imply $1a = a$. Now $aa = a$ and $1a = a$ imply $aa = 1a$. Thus $aa = 1a$ and $a \neq 1$ since $a \in S$ and S contains no identity element. If $a = 1$, then a would be an identity element for S and this would be a contradiction. Thus S^1 is not right cancellative. This completes the proof of the theorem.

Theorem 2.8. If a is an element of a semigroup S and $A = \{x \in S \mid axa = a\}$ is a set, then Aa is a left zero subsemigroup of S and aA is a right zero subsemigroup of S.

Proof. To show Aa is a left zero subsemigroup of S let $x, y \in Aa$ and show $xy \in Aa$ and $xy = x$. The element x is in Aa implies there exists an element b in A such that $x = ba$. The element y is in Aa implies there exists an element c in A such that $y = ca$. Now $x = ba$ and $y = ca$ imply $xy = baca$. The element c is in A implies $aca = a$. Thus $xy = b(aca) = ba = x$. Hence $xy = x$ and $xy \in Aa$. Thus Aa is a left zero subsemigroup of S. By a similar argument it can be shown that aA is a right zero subsemigroup of S. This completes the proof of the theorem.

Theorem 2.9. If S is a left zero semigroup, then S is left simple and each element of S forms a right ideal of S.

Proof. To show S is left simple let A be a left ideal of S and show $A = S$. By the definition of a left ideal it is clear that $A \subseteq S$. Thus, to show $A = S$, it remains to show $S \subseteq A$. Let $x \in S$ and show $x \in A$. Let $a \in A$. Now a, $x \in S$ and S is a left zero semigroup imply $xa = x$. Then

$a \in A$, $x \in S$ and A is a left ideal of S imply $xa \in A$. Hence $x \in A$. Thus $S \subseteq A$ and $A = S$. Hence S is left simple.

To show each element of S forms a right ideal of S let $y \in S$ and show $\{y\}$ is a right ideal of S. Let $b \in S$. It is clear that $y, b \in S$ and S is a left zero semigroup imply $yb = y$. Hence $\{y\}$ is a right ideal of S. This completes the proof.

Theorem 2.10. Let S be a semigroup such that if $ab = cd$ $(a,b,c,d \in S)$ then either $a = c$ or $b = d$. Then S is either a left zero semigroup or a right zero semigroup.

Proof. To facilitate the proof it will be shown that each element of S is an idempotent element. Let $e \in S$. Clearly $ee \in S$ since $e \in S$. Since $ee \in S$, let $ee = x$. Now $ee = x$ implies $e(ee) = ex$ which implies $(ee)e = ex$. This implies $ee = e$ or $e = x$. Hence $ee = e$. It also follows that S is a band.

If S is a semigroup containing only one element, then it is clear that the hypothesis and conclusion are satisfied. Thus suppose S is a semigroup containing more than one element and let $a, b \in S$ such that $a \neq b$. The element ab is in S since $a, b \in S$. Let $ab = c$. Now $c \in S$ and S is a band imply $cc = c$. Now $ab = c$ and $c = cc$ imply $ab = cc$ and this implies $a = c$ or $b = c$.

Suppose $a = c$ and show S is a left zero semigroup. Now $a = c$ implies $ab = a$, a fact that will be used later. To show S is a left zero semigroup let $x, y \in S$ and show

$xy = x$. Again, if $x = y$, then $xy = x$ since $xx = x$; thus, consider $x \neq y$. Now $x,y \in S$ imply $xy \in S$. Let $xy = w$. Now $w \in S$ since $xy \in S$. The element w is in S and S is a band imply $w = ww$. Then $xy = w$ and $w = ww$ imply $xy = ww$. Now $xy = ww$ implies $x = w$ or $y = w$. If $x = w$, then $xy = x$ since $xy = w$. Thus $xy = x$ if $y \neq w$. To show $y \neq w$ suppose, by way of contradiction, $y = w$. Thus $xy = y$. This implies $(xy)a = ya$ which implies $x(ya) = ya$. Then $x(ya) = ya$ implies $x = y$ or $ya = a$. Thus it follows that $ya = a$ since $x \neq y$. Now $ya = a$ and $a = ab$ imply $ya = ab$. This implies $y = a$ or $a = b$. Now $y = a$ or $a = b$ implies $y = a$ since $a \neq b$. Hence $xy = y$ and $y = a$ imply $xy = a$. Then $xy = a$ and $a = ab$ imply $xy = ab$ which implies $x = a$ or $y = b$ which is a contradiction since if $x = a$ then $x = y$ and if $y = b$ then $a = b$. Thus $y \neq w$. Hence $x = w$ and $xy = x$. Thus, if $a = c$, then S is a left zero semigroup. It can be shown by a similar argument that S is a right zero semigroup if $b = c$. This concludes the proof.

Theorem 2.11. If S is a semigroup having a right zero element, then the set K of all right zero elements of S is a right zero subsemigroup of S and is an ideal of S contained in every ideal of S.

Proof. To show K is a right zero subsemigroup of S it is necessary to show K is a subsemigroup of S and K is a right zero semigroup.

To show K is a subsemigroup of S let $a,b \in K$ and show $ab \in K$. Now $ab \in K$ if ab is a right zero element of S;

thus, let $x \in S$ and show $x(ab) = ab$. The element x is in S and a is a right zero element of S imply $xa = a$. Clearly $xa = a$ implies $x(ab) = ab$. Hence K is a subsemigroup of S. It is clear that K is a right zero semigroup since if $a, b \in K$, then $ab = b$ since b is a right zero of S and $a \in S$. Thus K is a right zero subsemigroup of S.

To show K is an ideal of S let $x \in S$ and show $xK \subseteqq K$ and $Kx \subseteqq K$. To show $xK \subseteqq K$ let $xk \in xK$ where $k \in K$ and show $xk \in K$. The element x is in S and k is a right zero element of S imply $xk = k$. Now $xk \in K$ since $xk = k$ and $k \in K$. Thus $xK \subseteqq K$. To show $Kx \subseteqq K$ let $hx \in Kx$ where $h \in K$ and show $hx \in K$. To show $hx \in K$ let $c \in S$ and show $c(hx) = hx$. The element c is in S and h is a right zero element of S imply $ch = h$ which implies $(ch)x = hx$. This implies $c(hx) = hx$. Thus $Kx \subseteqq K$ and K is an ideal of S.

To show K is contained in each ideal of S let A be an ideal of S and show $K \subseteqq A$. Let $k \in K$ and show $k \in A$. Let $y \in A$. Now A is an ideal of S implies $Ak \subseteqq A$; thus, $yk \in A$ since $y \in A$. Now $y \in S$ and k is a right zero element of S imply $yk = k$; hence, it follows $k \in A$ since $k = yk$ and $yk \in A$. Thus $K \subseteqq A$. This concludes the proof of the theorem.

CHAPTER III

RECTANGULAR SEMIGROUPS, BANDS, DOMINATORS, REGULAR ELEMENTS, AND INVERSES

Definition 3.1. The statement that S is a band means S is a semigroup in which every element is idempotent.

Definition 3.2. A semigroup S is said to be nowhere commutative provided the following condition is satisfied: If $a,b \in S$ such that $ab = ba$, then $a = b$.

Definition 3.3. Let X be a set and let P be a relation whose domain is X and whose range is a subset of X. Then P is called a partial ordering of X provided the following are true:

(1) If $a \in X$, then $(a,a) \in P$.

(2) If $a,b \in X$, $(a,b) \in P$, and $(b,a) \in P$, then $a = b$.

(3) If $a,b,c \in X$, $(a,b) \in P$, and $(b,c) \in P$, then $(a,c) \in P$.

Definition 3.4. Let S be a semigroup which contains at least one idempotent and let E be the set of all idempotents of S. Define the relation
$E^* = \{(e,f) \mid e,f \in E \text{ and } ef = fe = e\}$.

Theorem 3.1. If S is a semigroup containing an idempotent and E is the set of all idempotents of S, then the relation E^* is a partial ordering of E.

Proof. By definition, $E^* = \{(e,f) \mid e,f \in E$ and $ef = fe = e\}$. Let $e \in E$ and show $(e,e) \in E^*$. It is clear $(e,e) \in E^*$ since $e \in E$ and e is an idempotent implies $ee = ee = e$.

Let $(e,f) \in E^*$ and $(f,e) \in E^*$ and show $e = f$. Now $(e,f) \in E^*$ implies $ef = fe = e$. Similarly $(f,e) \in E^*$ implies $fe = ef = f$. Hence $fe = e$ and $fe = f$ imply $e = f$.

Let $(e,f) \in E^*$ and $(f,g) \in E^*$ and show $(e,g) \in E^*$. Now $(e,f) \in E^*$ implies $ef = fe = e$ and $(f,g) \in E^*$ implies $fg = gf = f$. Thus $eg = (ef)g = e(fg) = ef = e$ and $ge = g(fe) = (gf)e = fe = e$. Hence $eg = ge = e$ and $(e,g) \in E^*$. This concludes the proof of the theorem.

Corollary 3.1. If S is a band, then E^* is a partial ordering of S.

Proof. Since S is a band every element of S is idempotent; hence, $S = E$. By the preceding theorem E^* is a partial ordering of E and it follows that E^* is a partial ordering of S since $S = E$.

Definition 3.5. An idempotent element e of a semigroup S is said to be primitive if the following are satisfied whenever (x,e) is in E^*:

(1) e is not a zero element of S.

(2) $x = e$ if S contains no zero element.

(3) $x = e$ or $x = 0$ if 0 is a zero element of S.

<u>Theorem 3.2.</u> A non-degenerate semigroup S is nowhere commutative if, and only if, it is a band without zero in which every element is primitive.

<u>Proof.</u> Suppose S is nowhere commutative and show S is a band without zero in which every element is primitive. Let $a \in S$. Let $aa = x$ and it is clear $x \in S$ since $a \in S$ and S is a semigroup. Now $ax = a(aa) = (aa)a = xa$. Thus, since S is nowhere commutative, it follows that $x = a$. Hence $aa = a$ and S is a band. Suppose, by way of contradiction, 0 is a zero element of S. Since S is non-degenerate let $b \in S$ and $b \neq 0$. Then $b \in S$ and 0 is a zero element of S imply $0b = b0 = 0$. Now $0b = b0$ and S is nowhere commutative imply $0 = b$ which is a contradiction. Hence S contains no zero element. To show every element of S is primitive let $e \in S$. The element e is in S and S is a band imply e is an idempotent element. Let $(y,e) \in E^*$. It has already been shown that e is not a zero element of S since S contains no zero element. Thus it remains to show $y = e$. Now $(y,e) \in E^*$ implies by definition of E^* that $ye = ey = y$. Then $ye = ey$ and S is nowhere commutative imply $y = e$. Hence every element of S is primitive.

Suppose S is a band without zero in which every element is primitive and show S is nowhere commutative. Let $a,b \in S$ such that $ab = ba$ and show $a = b$. It is clear that $(ab)b = (ba)b = b(ab)$ since $ab = ba$. Also $(ab)b = a(bb) = ab$ since S is a band. Now $(ab)b = b(ab) = ab$ and ab and b are idempotents imply $(ab,b) \in E^*$ which implies ab = b since every element of S is primitive and S contains no zero element. Similarly ab = a. Hence a = b and S is nowhere commutative. Thus the theorem is proved.

Definition 3.6. The statement that a semigroup S is rectangular means if $x,y \in S$, then $xyx = x$.

Theorem 3.3. A semigroup S is nowhere commutative if, and only if, it is a rectangular band.

Proof. Suppose S is a rectangular band and show S is nowhere commutative. Let $a,b \in S$ such that $ab = ba$ and show $a = b$. Now $a,b \in S$ and S is a rectangular band imply $aba = a$ and $bab = b$. Also $aaa,b \in S$ and S is a rectangular band imply $b(aaa)b = b$. Now $a = aba = ababa = (ab)a(ba) = (ba)a(ab) = b(aaa)b = b$. Thus S is nowhere commutative.

Suppose S is nowhere commutative and show S is a rectangular band. Let $a,b \in S$ and show $aba = a$. It has been shown in Theorem 3.2 that S is nowhere commutative implies S is a band. Since S is a band it is clear that $a(aba) = (aba)a$. Now S is nowhere commutative and $a(aba) = (aba)a$ imply $a = aba$. Thus S is a rectangular band. Hence the theorem is proved.

Definition 3.7. The statement that an element d of a semigroup S is a dominator means if $a \in S$, then $dad = d$. If S is a semigroup containing a dominator, then the set D of all dominators of S is called the dominator of S.

Theorem 3.4. A semigroup S contains a dominator if, and only if, it contains an ideal A which is a rectangular band. Also, A is the dominator of S.

Proof. Suppose S contains a dominator and show S contains an ideal which is a rectangular band. Since S contains a dominator, let D be the dominator of S and show D is an ideal which is a rectangular band. To show D is an ideal of S let $x \in S$ and show $xD \subseteq D$ and $Dx \subseteq D$. Let $xd \in xD$ where $d \in D$. Let $a \in S$. Now $ax \in S$ and $d \in D$ imply $daxd = d$ which implies $xdaxd = xd$. Then $xdaxd = xd$ implies $xd(a)xd = xd$ which implies xd is a dominator of S. Hence $xd \in D$ and $xD \subseteq D$. Now let $dx \in Dx$ where $d \in D$ and show $dx \in D$. Let $b \in S$. Then $xb \in S$ and $d \in D$ imply $dxbd = d$. This implies $dxbdx = dx$. Now $dxbdx = dx$ implies $dx(b)dx = dx$ and it follows that dx is a dominator of S. Hence $dx \in D$ and $Dx \subseteq D$. Thus D is an ideal of S. It remains to show that the ideal D is a rectangular band. First, it is necessary to show D is a subsemigroup of S. Let $d,t \in D$ and show $dt \in D$. Let $a \in S$. Now $ta \in S$ and $d \in D$ imply $dtad = d$. This implies $dtadt = dt$ which implies $dt \in D$. Hence D is a semigroup. To show D is a rectangular band it will first be shown that D is a band. Let $d \in D$ and show $dd = d$. The element d is in S and $d \in D$ imply $ddd = d$. Then $dd \in S$ and $d \in D$ imply $dddd = d$. Thus $d = dddd = d(ddd) = dd$. Clearly D is rectangular since if $d,t \in D$, then $dtd = d$ since $t \in S$ and $d \in D$. Thus D is a rectangular band.

Suppose S contains an ideal A which is a rectangular band and show S contains a dominator D and $D = A$. To show

S contains a dominator, let $d \in A$ and let $a \in S$. The element da is in A since A is an ideal of S. Now $da, d \in A$ and A is a rectangular band imply $d(da)d = d$ which implies $dad = d$ since $dd = d$. Thus d is a dominator of S. Let D be the dominator of S and it follows from the above that $A \subseteq D$. In order to show $A = D$ it remains to show $D \subseteq A$. Let $d \in D$ and let $a \in A$. The element d is a dominator of S and $a \in S$ imply $dad = d$. This implies $d \in A$ since $a \in A$ implies $da \in A$ which implies $dad \in A$. The preceding follows from the fact that A is an ideal of S. Thus $d \in A$ and $D \subseteq A$. Hence S contains a dominator $D = A$ and the theorem is proved.

Theorem 3.5. A semigroup S contains a unique dominator if, and only if, it contains a zero element.

Proof. Suppose S contains a unique dominator d and show S contains a zero element. The element d is a unique dominator of S implies that the dominator of S is the set $D = \{d\}$. By Theorem 3.4 it follows that D is an ideal of S which implies that if $x \in S$, then $xD \subseteq D$ and $Dx \subseteq D$. Thus it follows that $xd = d$ and $dx = d$ for every x in S. This implies d is a zero element of S.

Suppose S contains a zero element z and show z is a unique dominator of S. Let $a \in S$. It is clear that $zaz = z$ since z is a zero element of S. Thus z is also a dominator of S. To show z is unique let d be a dominator of S and show $d = z$. Now $z \in S$ and d is a dominator of S imply $dzd = d$. It follows that $dzd = z$ since z is a zero element of S. Hence $d = z$ and z is the unique dominator of S.

Theorem 3.6. An element x of a semigroup S is a commutative dominator of S if, and only if, it is a zero element of S. Hence, x is a commutative dominator of S if, and only if, it is a unique dominator of S.

Proof. Suppose x is a zero element of S and show x is a commutative dominator of S. By Theorem 3.5 this is equivalent to supposing x is a unique dominator of S and showing x is a commutative dominator of S. Thus it only remains to show x is commutative. Let $a \in S$. Since x is a zero element of S, it follows that $xa = x$ and $ax = x$. Thus $xa = ax$.

Suppose $x \in S$ is a commutative dominator of S and show x is a zero element of S. Also show x is a unique dominator of S. By Theorem 3.5 it will suffice to show x is unique. Let d be a dominator of S and show $d = x$. Now $x \in S$ and d is a dominator of S imply $dxd = d$. Then $d \in S$ and x is a dominator of S imply $xdx = x$. Now $xd = dx$ since d is a commutative element. Thus it follows that $d = dxd = ddx = dx = dxx = xdx = x$. Therefore the theorem is true.

Theorem 3.7. A simple semigroup S is rectangular if, and only if, it contains a dominator and in both cases S is the dominator of S.

Proof. Suppose S contains a dominator d and show S is rectangular. Since d is a dominator of S, let D be the dominator of S. By Theorem 3.4, D is an ideal of S. Thus $D = S$ since D is an ideal of S and S is simple. It is clear that S is rectangular since if $x, y \in S$, then $x, y \in D$ and this implies $xyx = x$. Conversely it is clear that each element of S is a dominator of S if S is rectangular. Thus if D is the dominator of S it follows that $D = S$. Thus the theorem is proved.

Theorem 3.8. If d is a dominator of a semigroup S, then the set of all elements of S which commute with d is a subsemigroup with a zero element.

Proof. Let Z be the set of all elements of S which commute with d. Clearly Z has meaning since $dd = dd$ implies $d \in Z$. To show Z is a subsemigroup of S, let $x, y \in Z$ and show $xy \in Z$. To show $xy \in Z$ it is necessary to show xy commutes with d. Now it follows that $(xy)d = x(yd) = x(dy) = (xd)y = (dx)y = d(xy)$. This implies $xy \in Z$. Thus Z is a subsemigroup of S. To show d is a zero element of Z let $a \in Z$ and show $ad = d$ and $da = d$. The element a is in Z implies $ad = da$. The element a is in S and d is a dominator of S imply $dad = d$. It is clear that $d = dd$ since d is a dominator. Now $ad = add = dad = d$ and $da = dda = dad = d$. Hence d is a zero element of S and the theorem is proved.

Definition 3.8. The statement that an element a of a semigroup S is regular means there exists an element x in S such that a = axa. The statement that S is regular means each element of S is regular.

Definition 3.9. Let a be an element of a semigroup S. Then $Sa \cup \{a\}$ is called the principal left ideal of S generated by a and $aS \cup \{a\}$ is called the principal right ideal of S generated by a. The left ideal $Sa \cup \{a\}$ will be denoted by L(a) and the right ideal $aS \cup \{a\}$ will be denoted by R(a).

Definition 3.10. The statement that the elements a and b of a semigroup S are inverses of each other means aba = a and bab = b.

Definition 3.11. The statement that a semigroup S is an inverse semigroup means each element of S has a unique inverse in S.

Theorem 3.9. A semigroup S is regular if, and only if, each element of S has an inverse in S.

Proof. It is clear from the definition of inverses that S is regular if each element of S has an inverse in S.

Suppose S is regular and show each element of S has an inverse in S. Let $a \in S$. There exists an element x in S such that a = axa since S is regular and $a \in S$. Clearly a = axa implies axa = axaxa. Thus a = a(xax)a. To show a and xax are inverses of each other it remains to show xax = xax(a)xax. Clearly a = axa implies xax = xaxax = xaxaxax. Thus xax = xax(a)xax. Hence a and xax are inverses of each other. This completes the proof of the theorem.

<u>Theorem 3.10.</u> A semigroup S is regular if, and only if, $A \cap B = AB$ for every right ideal A and every left ideal B of S.

<u>Proof.</u> Suppose S is regular. Let A be a right ideal of S and let B be a left ideal of S. Show $A \cap B = AB$. Let $ab \in AB$ where $a \in A$ and $b \in B$ and show $ab \in A \cap B$. The element ab is in A since $a \in A$ and A is a right ideal of S. The element ab is in B since $b \in B$ and B is a left ideal of S. Thus $ab \in A$ and $ab \in B$ imply $ab \in A \cap B$. Thus $AB \subseteq A \cap B$. From the above it is clear that $A \cap B$ is meaningful. Let $x \in A \cap B$ and show $x \in AB$. Now $x \in A \cap B$ implies $x \in A$ and $x \in B$. The element x is in S and S is regular imply there exists an element y in S such that $xyx = x$. Now $yx \in B$ since $x \in B$ and B is a left ideal of S. Thus it follows that $x \in AB$ since $x = x(yx)$ where $x \in A$ and $yx \in B$. Thus $A \cap B \subseteq AB$ and it follows that $A \cap B = AB$.

To show the converse let $a \in S$ and show a is regular. Consider R(a) and L(a). By hypothesis $R(a) \cap L(a) = R(a)L(a)$. It is clear that $a \in R(a) \cap L(a)$ since $a \in R(a)$ and $a \in L(a)$. Thus $a \in R(a)L(a)$. This implies there exist an element x in R(a) and an element y in L(a) such that $a = xy$. Now $x \in R(a)$ implies $x = a$ or there exists an element x' in S such that $x = ax'$. The element y is in L(a) implies $y = a$ or there exists an element y' in S such that $y = y'a$. If

$x = a$ and $y = a$, then $a = aa$ and this implies $a = aaa$. If $x = a$ and $y = y'a$, then it follows that $a = ay'a$. If $x = ax'$ and $y = a$, then it follows that $a = ax'a$. If $x = ax'$ and $y = y'a$, then it follows that $a = a(x'y')a$. Thus a is regular. This completes the proof of the theorem.

Theorem 3.11. If a is a regular element of a semigroup S, then $L(a) = Sa$ and $R(a) = aS$.

Proof. By definition, $L(a) = Sa \cup \{a\}$; thus, it will suffice to show $a \in Sa$. Since a is a regular element of S there exists an element x in S such that $a = axa$. Thus $a = axa$ implies $a \in Sa$ since $(ax)a \in Sa$. Similarly $R(a) = aS$. This concludes the proof.

Theorem 3.12. If e, f, ef, and fe are idempotent elements of a semigroup S, then ef and fe are inverses of each other.

Proof. The element ef is idempotent implies $ef = (ef)(ef)$. Then $ef = (ef)(ef)$ implies $ef = e(ff)(ee)f$ since f and e are idempotent elements. Now $ef = e(ff)(ee)f$ implies $ef = (ef)(fe)(ef)$. Similarly $fe = (fe)(ef)(fe)$. Hence ef and fe are inverses of each other and the theorem is proved.

Theorem 3.13. An element a of a semigroup S is regular if, and only if, there exists an idempotent element e in S such that $L(a) = L(e)$.

Proof. Suppose a is regular and show there exists an idempotent e in S such that $L(a) = L(e)$. The element a is regular implies there exists an element x in S such that $a = axa$. Then $a = axa$ implies $xa = (xa)(xa)$. Thus xa is idempotent. Now it remains to show $L(a) = L(xa)$. To show $L(a) \subseteq L(xa)$, let $y \in L(a)$ and show $y \in L(xa)$. Now $y \in L(a)$ implies $y = a$ or $y \in Sa$. If $y = a$, then $y = axa$ since $a = axa$. Thus $y = axa$ implies $y \in Sxa$ since $a(xa) \in Sxa$. If $y \in Sa$, then there exists an element y' in S such that $y = y'a$. Then $y = y'a$ and $a = axa$ imply $y = (y'a)xa$ which implies $y \in Sxa$. Now $y \in Sxa$ implies $y \in L(xa)$. Thus $L(a) \subseteq L(xa)$. To show $L(xa) \subseteq L(a)$, let $z \in L(xa)$ and show $z \in L(a)$. Now $z \in L(xa)$ implies $z = xa$ or $z \in Sxa$. If $z = xa$, then $z \in Sa$. If $z \in Sxa$, then there exists an element z' in S such that $z = z'za$. This implies $z \in Sa$ since $(z'z)a \in Sa$. Then $z \in Sa$ implies $z \in L(a)$ which implies $L(xa) \subseteq L(a)$. Hence $L(a) = L(xa)$ and xa is idempotent.

Suppose there exists an idempotent element e in S such that $L(a) = L(e)$ and show a is regular. The element a is in $L(a)$ and $L(a) = L(e)$ imply $a = e$ or $a \in Se$. If $a = e$, then $a = aaa$ since e is idempotent implies $e = eee$. If $a \in Se$ then there exists an element a' in S such that $a = a'e$. Then $a = a'e$ and e is idempotent imply $a = ae$. Now $e \in L(e)$ and $L(e) = L(a)$ imply $e = a$ or $e \in Sa$. If $e = a$, then $a = aaa$ since e is idempotent. If $e \in Sa$, then there exists an element e' in S such that $e = e'a$. Thus $a = ae$ and $e = e'a$ imply $a = ae'a$. Therefore a is regular and the theorem is proved.

Theorem 3.14. The following conditions on a semigroup S are equivalent:

(1) S is regular and any two idempotents commute with each other.

(2) If $a \in S$, then there exist unique idempotent elements e and f in S such that $L(a) = L(e)$ and $R(a) = R(f)$.

(3) S is an inverse semigroup.

Proof. To prove the equivalence of the three statements it will suffice to show (1) implies (3), (3) implies (2), and (2) implies (1).

Suppose (1) is true and show (3) is true. Suppose S is regular and any two idempotents of S commute with each other and show S is an inverse semigroup. To show S is an inverse semigroup it is necessary to show each element of S has a unique inverse in S. Let $a \in S$. The element a is in S and S is regular imply by Theorem 3.9 there exists an element b in S such that a and b are inverses of each other. Hence $aba = a$ and $bab = b$. Thus it remains to show b is unique with respect to a. Suppose a and c are inverses of each other and show $b = c$. Now a and c are inverses of each other implies $aca = a$ and $cac = c$. Clearly $aba = a$ and $aca = a$ imply ab, ba, ac, and ca are idempotent elements. It follows from the hypothesis that any two of these idempotents commute with each other. This fact and the above equations are used to show $b = c$. Now $b = bab = b(aca)b = (ba)c(ab) = (ba)cac(ab) = (ba)(ca)(cab) = (ca)(ba)(cab) = (caca)(ba)(cab) = (cac)(aba)(cab) = (c)(a)(cab) = c(ac)(ab) = c(ab)(ac) = c(aba)c = cac = c$. Thus b is unique and (3) is true.

Suppose (3) is true and show (2) is true. Let $a \in S$ and show there exists a unique idempotent e in S such that $L(a) = L(e)$. Now S is an inverse semigroup implies there exists a unique element b in S such that $a = aba$ and $b = bab$. Now $a = aba$ implies a is regular and it follows from Theorem 3.13 that there exists an idempotent element $e = ba$ such that $L(a) = L(e)$. It remains to show e is unique. Suppose f is an idempotent such that $L(a) = L(f)$ and show $f = e$. Each of a, ba, and f is a regular element implies by Theorem 3.11 that $L(a) = Sa$, $L(ba) = Sba$, and $L(f) = Sf$. Clearly $L(a) = L(ba)$ and $L(a) = L(f)$ imply $Sba = Sf$. Then $ba \in Sba$ since $ba \in S$ and ba is idempotent. Now $ba \in Sba$ and $Sba = Sf$ imply there exists an element x in S such that $ba = xf$. It is clear that $ba = xf$ and f is idempotent imply $ba = baf$. Clearly $f \in Sf$ since $f \in S$ and f is idempotent. Now $f \in Sf$ and $Sf = Sba$ imply there exists an element y in S such that $f = yba$. The element ba is its own inverse since ba is idempotent. To show $f = ba$ it will suffice to show that f and ba are inverses of each other since S is an inverse semigroup. To show f and ba are inverses of each other it is necessary to show $ba = ba(f)ba$ and $f = f(ba)f$. The equalities $ba = baf$, $f = yba$, and $baba = ba$ are used to show this. Clearly $ba = (ba)(ba) = (baf)(ba) = ba(f)ba$. Clearly $f = yba = y(ba) = y(ba)(ba) = (yba)(ba) = f(ba) = f(baf) = f(ba)f$. Hence f and ba are inverses of each other and $f = ba$. Thus $e = ba = f$. By a similar argument it can be shown that there exists a unique idempotent element g such that $R(a) = R(g)$. Thus (2) is true.

Suppose (2) is true and show (1) is true. To do this it is convenient to show first that (3) is true. Let $a \in S$. By hypothesis there exist unique idempotent elements e and f in S such that $L(a) = L(e)$ and $R(a) = R(f)$. By Theorem 3.13 it follows that each element of S is regular. This implies by Theorem 3.11 that $L(a) = Sa$, $L(e) = Se$, $R(a) = aS$, and $R(f) = fS$. Thus $Sa = Se$ and $aS = fS$. Now S is regular also implies by Theorem 3.9 that each element of S has an inverse in S. Thus let b be an inverse of a and by the definition of inverses it follows that $a = aba$ and $bab = b$. To show b is unique with respect to a let c be an element of S such that $a = aca$ and $cac = c$ and show $b = c$. Clearly $Sa = Sba$ and ba is idempotent. Hence $e = ba$. Clearly $Sa = Sca$ and ca is idempotent. Hence $e = ca$ and it follows that $ba = ca$. Similarly it can be shown that $ab = ac$. Now $b = bab = cab = cac = c$. Thus each element of S has a unique inverse in S. Thus (3) is true. To show (1) is true let g and h

be idempotent elements of S and show $gh = hg$. The element gh is in S and S is an inverse semigroup imply there exists a unique element x in S such that $gh = ghxgh$ and $xghx = x$. Clearly $gh(hx)gh = gh$ and $hx(gh)hx = hxghx = hx$. Thus $x = hx$ since x and hx are both inverses of gh. Similarly it can be shown that $x = xg$. Now $xx = xghx = x(gh)x = x$ which implies x is idempotent. The element x is idempotent implies x is its own inverse. Thus it follows that $x = gh$ since both x and gh are inverses of x. Thus gh is idempotent. Similarly it can be shown that hg is idempotent. Now since g, h, gh, and hg are idempotent elements, then, by Theorem 3.12, gh and hg are inverses of each other. Thus gh is its own inverse since gh is idempotent. Thus $gh = hg$ and (1) is true. Therefore the theorem is true.

Theorem 3.15. If e and f are idempotent elements of an inverse semigroup, then $Se \cap Sf = Sef$ and $Sef = Sfe$.

Proof. Let $a \in Sef$ and show $a \in Se \cap Sf$. The element a is in Sef implies there exists an element b in S such that $a = bef$. It follows from Theorem 3.14 that $ef = fe$ since e and f are idempotent elements of an inverse semigroup. Thus $a = bfe$. Now it is clear that $a \in Se$ and $a \in Sf$ since $a = bfe$ and $a = bef$. Thus $a \in Se \cap Sf$. Hence $Sef \subseteq Se \cap Sf$.

From the preceding part of the proof it is clear that $Se \cap Sf$ is meaningful. Thus let $a \in Se \cap Sf$ and show $a \in Sef$. Then $a \in Se$ and $a \in Sf$. The element a is in Se implies there exists an element x in S such that $a = xe$. Clearly

$a = xe$ and e is idempotent imply $a = ae$. The element a is in Sf implies there exists an element y in S such that $a = yf$. Now $a = yf$ and f is idempotent imply $a = af$. Then $a = af$ and $a = ae$ imply $a = aef$. Clearly $a \in Sef$ since $a = aef$. Thus $Se \cap Sf \subseteq Sef$. Hence $Se \cap Sf = Sef$. It is clear that $Sef = Sfe$ since $ef = fe$. This completes the proof of the theorem.

Theorem 3.16. A non-degenerate right zero semigroup S has the following properties:

(1) If $a \in S$, then there exists a unique idempotent element e in S such that $L(a) = L(e)$.

(2) S in not an inverse semigroup.

Proof. The element a is in S and S is a right zero semigroup imply $aa = a$. Let $e = a$ and it follows that $L(a) = L(e)$ where e is an idempotent element. To show $e = a$ is unique, let f be an idempotent element of S such that $L(a) = L(f)$ and show $f = a$. Clearly a and f are idempotent elements and $L(a) = L(f)$ imply $Sa = Sf$. Now $a \in Sa$ since a is idempotent. Thus $a \in Sa$ and $Sa = Sf$ imply there exists an element x in S such that $a = xf$. It follows that $xf = f$ since S is a right zero semigroup. Hence $f = a = e$ and e is unique.

To show (2) suppose, by way of contradiction, that S is an inverse semigroup. Let $a \in S$. Now $a \in S$ and S is an inverse semigroup imply there exists a unique element x in S such that $axa = a$ and $xax = x$. Since S is non-degenerate, let $y \in S$ such that $y \neq x$. Now $ay, a \in S$ and a is a right zero semigroup imply $aya = a$. Similarly $yay = y$. Thus y and a are inverses of each other and x and a are inverses of each other. This implies $x = y$ since an element of an inverse semigroup has a unique inverse. But $x = y$ is a contradiction. Thus it follows that S is not an inverse semigroup. Hence the theorem is proved.

CHAPTER IV

TRANSFORMATION SEMIGROUPS

Definition 4.1. Let X be a set. The statement that F is a transformation of X means F is a function whose domain is X and whose range is a subset of X.

Definition 4.2. Let X be a set. Then $\mathcal{T}_X$ is defined to be the collection of all transformations of X.

Definition 4.3. Let X be a set and let each of F and G be an element of $\mathcal{T}_X$. The product of F and G is denoted by FG and is defined by $FG = \{(x, G(F(x))) \mid x \in X\}$.

Definition 4.4. Let X be a set. The statement that F is a constant transformation means $F \in \mathcal{T}_X$ and the range of F is a set consisting of a single element.

Theorem 4.1. Let X be a set. If $F \in \mathcal{T}_X$ and $G \in \mathcal{T}_X$, then $FG \in \mathcal{T}_X$.

Proof. By definition $FG = \{(x, G(F(x))) \mid x \in X\}$. To show FG is a function, let $(x, G(F(x))) \in FG$ and $(y, G(F(y))) \in FG$ where $x = y$ and show $G(F(x)) = G(F(y))$. Clearly $x, y \in X$, $x = y$, and F is a function whose domain is X imply $F(x) = F(y)$. Now $F(x), F(y) \in X$ since the range of F is a subset of X. Thus $F(x) = F(y)$, $F(x), F(y) \in X$ and G is a function whose domain is X imply $G(F(x)) = G(F(y))$.

Hence FG is a function. It is clear from the above definition that the domain of FG is X. Also it is clear that the range of FG is a subset of X since each element G(F(x)) in the range of FG is an element of X. FG is a function whose domain is X and whose range is a subset of X implies FG is a transformation of X. Hence $FG \in \mathcal{T}_X$. This concludes the proof.

Theorem 4.2. Let X be a set. Then $\mathcal{T}_X$ is a semigroup.

Proof. By Theorem 4.1 it follows that the product relation is a binary operation defined on $\mathcal{T}_X$; thus, it remains to show that this binary operation is associative. Let $F,G,H \in \mathcal{T}_X$ and show (FG)H = F(GH). By definition $(FG)H = \{(x,H(FG(x))) \mid x \in X\}$ and $F(GH) = \{(x,GH(F(x))) \mid x \in X\}$. To show these two sets are equal it will suffice to show H(FG(x)) = GH(F(x)). By definition H(FG(x)) = H(G(F(x))). Also, by definition, GH(F(x)) = H(G(F(x))); hence, the two are equal and (FG)H = F(GH). Therefore $\mathcal{T}_X$ is a semigroup.

Theorem 4.3. Let X be a set. An element F of $\mathcal{T}_X$ is idempotent if, and only if, y = F(y) for each y in R(F).

Proof. By definition $F = \{(x,F(x)) \mid x \in X\}$ and $FF = \{(x,F(F(x))) \mid x \in X\}$. Suppose y = F(y) for each y in R(F) and show F = FF. To do this it will suffice to show F(x) = F(F(x)) for each x in X. Now for each x in X, $F(x) \in R(F)$. $F(x) \in R(F)$ and y = F(y) for each y in R(F) imply F(x) = F(F(x)). Hence F = FF and F is idempotent. Now to show the only if part suppose F = FF and show

$y = F(y)$ for each y in $R(F)$. The above definitions of F and FF clearly imply $F(x) = F(F(x))$ since $F = FF$ and both F and FF are functions. Thus $y = F(y)$ for each y in $R(F)$. Therefore the theorem is proved.

Theorem 4.4. Let X be a set. Then F is a right zero element of $\mathcal{T}_X$ if, and only if, F is a constant transformation. Also, there are no left zero elements in $\mathcal{T}_X$ if X is non-degenerate.

Proof. Let F be a constant transformation of X and show F is a right zero element of $\mathcal{T}_X$. Let $G \in \mathcal{T}_X$ and show $GF = F$. Now F is a constant transformation of X implies there exists an element a in S such that $F = \{(x,a) \mid x \in X\}$. By the definition of the product of two transformations, $GF = \{(x,F(G(x))) \mid x \in X\}$. Now $G(x) \in X$ for every x in X. Hence $F(G(x)) = a$ and it is clear that $GF = F$. Thus F is a right zero element of $\mathcal{T}_X$.

Let F be a right zero element of $\mathcal{T}_X$ and show F is a constant transformation of X. To show F is a constant transformation of X it is necessary to show that the range of F is a set consisting of a single element. Let $b \in S$ and consider $G = \{(x,b) \mid x \in X\}$. It is clear that $G \in \mathcal{T}_X$ since G is a function whose domain is X and whose range is $\{b\} \subseteq X$. Now $G \in \mathcal{T}_X$ and F is a right zero element of $\mathcal{T}_X$ imply $GF = F$. By definition, $GF = \{(x,F(G(x))) \mid x \in X\}$ and since $G(x) = b$ for every $x \in X$, it follows that $GF = \{(x,F(b)) \mid x \in X\}$. Now

$GF = F$ implies $F = \{(x, F(b)) \mid x \in X\}$ which implies that the range of F consists entirely of the single element $F(b)$ of X. Thus F is a constant transformation.

Let X be a non-degenerate set. Now suppose, by way of contradiction, that F is a left zero element of $\mathcal{T}_X$. Since X is non-degenerate, let a and b be distinct elements of X. It is clear that $G = \{(x,a) \mid x \in X\}$ and $H = \{(x,b) \mid x \in X\}$ are both elements of $\mathcal{T}_X$. It is also clear that $G \neq H$ since $a \neq b$. Thus one of G and H is not F so, without loss of generality, let $G \neq F$. Now $G \in \mathcal{T}_X$ and F is a left zero element of $\mathcal{T}_X$ imply $FG = F$. By definition $FG = \{(x, G(F(x))) \mid x \in X\}$. Now $F(x) \in X$ for each x in X and $G(x) = a$ for each x in S. Thus $G(F(x)) = a$ for each $x \in X$ and it follows that $FG = \{(x,a) \mid x \in X\}$. Thus $FG = G$. Now $FG = F$ and $FG = G$ imply $F = G$. But $F = G$ contradicts the assumption that F and G are distinct. Hence F is not a left zero element of $\mathcal{T}_X$ and $\mathcal{T}_X$ contains no left zero element. This concludes the proof.

<u>Theorem 4.5.</u> Let S be a semigroup with a right zero element and let K be the set of all right zero elements of S. Then $S \cong \mathcal{T}_K$ if the following are satisfied:

(1) If $a,b \in S$ such that $ka = kb$ for all k in K, then $a = b$.

(2) If $F \in \mathcal{T}_K$, then there exists an element a in S such that $F = \{(k,ka) \mid k \in K\}$.

Also, if $S \cong \mathcal{T}_K$, then (1) is true.

<u>Proof.</u> Suppose (1) and (2) are true and show $S \cong \mathcal{T}_K$. Consider $\phi = \{(x,F) \mid a \in S \text{ and } F = \{(k,ka) \mid k \in K\}\}$. To show ϕ is a function, let $(a,F) \in \phi$ and $(b,H) \in \phi$ such that $a = b$ and show $F = H$. Clearly $(a,F) \in \phi$ implies $F = \{(k,ka) \mid k \in K\}$ and $(b,H) \in \phi$ implies $H = \{(k,kb) \mid k \in K\}$. It is clear that $F = H$ since $a = b$. Hence ϕ is a function. To show ϕ is a reversible function, let $(a,F) \in \phi$ and $(b,H) \in \phi$ such that $F = H$ and show $a = b$. Now $(a,F) \in \phi$ implies $F = \{(k,ka) \mid k \in K\}$ and $(b,H) \in \phi$ implies $H = \{(k,kb) \mid k \in K\}$. Now $F = H$ implies $ka = kb$ for all k in K. But $ka = kb$ for all k in K implies by (1) that $a = b$. Hence ϕ is a reversible function. It is clear from the definition of ϕ that the domain of ϕ is S. It remains to show that the range of ϕ is $\mathcal{T}_K$ and if $a,b \in S$, then $\phi(ab) = \phi(a)\phi(b)$. To show that $R(\phi) = \mathcal{T}_K$ it is necessary to show $R(\phi) \subseteq \mathcal{T}_K$ and $\mathcal{T}_K \subseteq R(\phi)$. Let $F \in R(\phi)$ and show $F \in \mathcal{T}_K$. To show $F \in \mathcal{T}_K$ it is necessary to show F is a function whose domain is K and whose range is a subset of K. Let $(k,ka) \in F$ and $(h,ha) \in F$ such that $k = h$ and it is clear that this implies $ka = ha$. Thus F is a function. By (2) it is clear that the domain of F is K. Thus it remains to show that $R(F) \subseteq K$. Let $ka \in R(F)$ and show $ka \in K$. Let $x \in S$. Now k is a right zero element of S since $k \in K$. Then $x \in S$ and k is a right zero element of S imply $xk = k$. This implies $x(ka) = ka$ which implies $ka \in K$. Thus $F \in \mathcal{T}_K$ and $R(\phi) \subseteq \mathcal{T}_K$. To show $\mathcal{T}_K \subseteq R(\phi)$ let $F \in \mathcal{T}_K$ and show $F \in R(\phi)$.

Now $F \in \mathcal{T}_K$ implies by (2) that there exists an element a in S such that $F = \{(k,ka) \mid k \in K\}$. Since $a \in S$, it follows from the definition of ϕ that $(a,F') \in \phi$ where $F' = \{(k,ka) \mid k \in K\}$. Hence $F = F'$ and $F \in R(\phi)$. Now to show $S \cong \mathcal{T}_K$ it remains to show the following. If $a,b \in S$, then $\phi(ab) = \phi(a)\phi(b)$. Now by construction of ϕ, $\phi(ab) = F$ where $F = \{(k,k(ab)) \mid k \in K\}$, $\phi(a) = F'$ where $F' = \{(k,ka) \mid k \in K\}$, and $\phi(b) = F''$ where $F'' = \{(k,kb) \mid k \in K\}$. By the definition of product, $\phi(a)\phi(b) = F'F'' = \{(k,F''(F'(k))) \mid k \in K\}$. Now to show $\phi(ab) = \phi(a)\phi(b)$ it remains to show $F''(F'(k)) = k(ab)$. Now $F''(F'(k)) = F''(ka)$ by definition of F'. It has already been shown that if $k \in K$ and $a \in S$, then $ka \in K$. Thus $ka \in K$ implies $F''(ka)$ is meaningful and by definition $F''(ka) = (ka)b = k(ab)$. Hence $\phi(ab) = \phi(a)\phi(b)$ and $S \cong \mathcal{T}_K$.

Suppose $S \cong \mathcal{T}_K$ and show (1) is true. Since $S \cong \mathcal{T}_K$, there exists a reversible function ϕ whose domain is S and whose range is $\mathcal{T}_K$ and such that if $(a,F) \in \phi$ and $(b,G) \in \phi$, then $(ab,FG) \in \phi$. To show (1) is true, suppose, by way of contradiction, that there exist elements a and b in S such that $ka = kb$ for all k in K and $a \neq b$. The domain of ϕ is S and $a,b \in S$ imply there exist transformations F and G in $\mathcal{T}_K$ such that $(a,F) \in \phi$ and $(b,G) \in \phi$. Now $a \neq b$ implies $F \neq G$ since ϕ is a reversible function. Thus, since $F \neq G$ it follows that there exists an element y in K such that $F(y) \neq G(y)$. Now consider $H = \{(k,y) \mid k \in K\}$. It is clear that $H \in \mathcal{T}_K$ since H is a function whose domain is K and whose range is a subset of K. Now $H \in \mathcal{T}_K$ and ϕ is a reversible function whose domain is S and whose range is $\mathcal{T}_K$ imply there exists an element h in S such that $(h,H) \in \phi$. It will be necessary to show that $h \in K$. By Theorem 4.4 it follows that H is a right zero element of $\mathcal{T}_K$ since H is a constant transformation. To show $h \in K$, let $a \in S$ and show $ah = h$. Since $a \in S$, it is clear that there exists an element T in $\mathcal{T}_K$ such that $(a,T) \in \phi$. Now $(a,T) \in \phi$ and $(h,H) \in \phi$ imply $(ah,TH) \in \phi$. But $TH = H$ since H is a right zero element of $\mathcal{T}_K$. Thus $(ah,H) \in \phi$. Hence it follows that $ah = h$ since (h,H), $(ah,H) \in \phi$ and ϕ is a reversible function. Thus $h \in K$. Now $(h,H) \in \phi$ and $(a,F) \in \phi$ imply $(ha,HF) \in \phi$. Similarly $(hb,HG) \in \phi$. Now $ha = hb$ since $h \in K$. Hence $HF = HG$ since ϕ is a function. By the definition of the product of transformations, it follows that $HF = \{(k,F(H(k))) \mid k \in K\}$ and $HG = \{(k,G(H(k))) \mid k \in K\}$. Thus $F(H(k)) = G(H(k))$ for all k in K. But $H(k) = y$ for all k in K implies $F(y) = G(y)$ and this is a contradiction. Therefore $a = b$ and (1) is true.

BIBLIOGRAPHY

Books

Clifford, A. H. and G.B. Preston, The Algebraic Theory of Semigroups, Providence, Rhode Island, American Mathematical Society, 1961.

Articles

Latimer, Jack, "The Dominators of a Semigroup," American Mathematical Monthly, LXXI (1964), 1104-1107.