TOPOLOGICAL SPACES, FILTERS AND NETS

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CHAPTER I

TOPOLOGICAL SPACES

Definition 1.1 A set $\mathcal{O}$ of subsets of a set E defines on E a topological structure (or more briefly, a topology) if it possesses the following properties (called axioms of the topological structure):

O_I: Every union of sets of $\mathcal{O}$ is a set of $\mathcal{O}$.

O_{II}: Every finite intersection of sets of $\mathcal{O}$ is a set of $\mathcal{O}$.

The sets of $\mathcal{O}$ are called open sets of the topological structure defined by $\mathcal{O}$.

Definition 1.2 A topological space is a set provided with a topological structure; its elements are then called points.

When one can exhibit that a set $\mathcal{O}$ of subsets of E satisfies O_{II} it is often convenient to establish separately that it satisfies the following two axioms, which together are equivalent to O_{II}.

O_{II_a}: The intersection of two sets of $\mathcal{O}$ belongs to $\mathcal{O}$.

O_{II_b}: E belongs to $\mathcal{O}$.

Examples of topologies: E being any set, the set of subsets of E consisting of E and $\emptyset$ (the empty set) satisfies axioms O_I and O_{II} and defines a topology on E. It is the same for the set P(E) of all subsets of E. The topology it defines is called the discrete topology.

Definition 1.3 In a topological space E, a neighborhood of a subset A of E is any set which contains an open set containing A.

The neighborhoods of a subset $\{x\}$ reduced to a single point are also called neighborhoods of the point x.

Proposition 1.1 In order that a set be a neighborhood of each of its points, it is necessary and sufficient that it be open.

Consider a set A. If A is open, and $a \in A$, then by Definition 1.3, A can qualify as the set containing an open set which contains a, i.e., $A \subset A$. Therefore A is a neighborhood of a.

Now, if A is a neighborhood of every point belonging to A, then for every $a \in A$ there exists an open set B_a such that $a \in B_a \subset A$. Now, A is contained in $\bigcup_{a \in A} B_a$ since every element of A is contained in $\bigcup_{a \in A} B_a$. However, $\bigcup_{a \in A} B_a$ is contained in A. Therefore $A = \bigcup_{a \in A} B_a$. By O_I, A is open.

Designate by V(x) the set of neighborhoods of x. V(x) has the following properties:

V_I: Every subset of E that contains a set of V(x) belongs to V(x).

V_{II} : Every finite intersection of sets of V(x) belongs to V(x).

V_{III} : The element x belongs to every set of V(x). (These three properties are in effect the immediate consequences of Definition 1.3 and of the axiom O_{II} .)

V_{IV} : If V belongs to V(x), there exists a set W belonging to V(x) and such that, for every $y \in W$, V belongs to V(y).

By Definition 1.3, it is seen that V_I is a justifiable statement. O_{II} is the reason that V_{II} is true. By Definition 1.3, x must belong to every neighborhood of x. Therefore, V_{III} is verified. By virtue of Proposition 1.1, if we take for W an open set containing x and contained in V, we see that V_{IV} is true. Hence, V_I, V_{II}, V_{III}, V_{IV} are verified.

<u>Definition 1.4</u> In a topological space E, closed sets are the complements of the open sets of E.

<u>Examples</u>: Let R be the real line provided with the usual topology (see exercise 1). Now $(0,1) \in \mathcal{O}$, so (0,1) is open. $\complement[(0,1)] = (\leftarrow, 0] \cup [1, \rightarrow)$ is closed by Definition 1.4.

Let R^2 be the Cartesian plane and let $\mathcal{O}$ be the usual topology for R^2. Now the disc $(x-0)^2 + (y-0)^2 < 2$ is an element of $\mathcal{O}$. $\complement[(x-0)^2 + (y-0)^2 < 2] = \{(x,y) \mid x^2 + y^2 \geq 2\}$ is closed by Definition 1.4.

For the real line R provided with the usual topology the set $(0,1]$ is neither open nor closed.

For the Cartesian plane R^2 the set $\{x,y \mid x^2 + y^2 < 2,$ or $x=1,\ y=1\}$ is neither open nor closed.

<u>Definition 1.5</u> In a topological space E, a point x is interior to a set A if A is a neighborhood of x. The set

of points interior to A is called the interior of A and is denoted by $\mathring{A}$.

We note that the interior of a non-empty set can be empty; this is the case for a set reduced to a single point when it is not open, for example in the real line.

Examples: In the plane consider the set $A = \{z \mid |z| \leq 2\}$. Now consider the monotonically decreasing sequence of closed sets $\{B_\alpha\}$ such that for each α, B_α is a closed disc with center at the origin and each $B_\alpha \subset A$. This sequence of closed sets will converge to the closed set which contains only one point, namely the origin. The interior of a closed set which contains only one point is empty.

For the discrete topology of the real line, each single point is considered as an open set; therefore its interior is a single point.

In the complex plane, consider the set $A = \{z \mid |z| < 1$ and the point $z = (0,1)\}$. $\mathring{A} = \{z \mid |z| < 2\}$ which does not contain $(0,1)$.

Definition 1.6 In a topological space E, a point x is adherent to a set A if every neighborhood of x contains at least one point of A. The set of points adherent to A is called the adherence of A and is denoted by $\bar{A}$.

(Note: In order that a set be closed, it is necessary and sufficient that it be identical with its adherence.)

Proposition 1.2 If A is an open set in E, for every subset B of E, $A \cap \bar{B} \subset \overline{A \cap B}$.

In effect, if $x \in A$ is adherent to B, i.e., $x \in \overline{B}$, for every neighborhood V of x, $V \cap A$ is still a neighborhood of x, since A is open. Then $V \cap A \cap B$ is not empty, which shows that x is adherent to $A \cap B$. Therefore $x \in \overline{A \cap B}$.

Definition 1.7 In a topological space E, a point x is called a frontier point of a set A, if it is at the same time adherent to A and to $\complement A$ (complement of A).

The frontier of a set A is that set of points which are interior to neither A nor $\complement A$ and is denoted by $Fr(A) = \overline{A} \cap \overline{\complement A}$.

Examples: Let R^2 be the entire plane. Obviously, every point in the plane is interior to R^2. Therefore $Fr(R^2) = \emptyset$.

Let $A = \{x | x < 0\}$. $Fr(A) = \overline{A} \cap \overline{\complement A} = [\leftarrow, 0] \cap [0, \rightarrow] = 0$.

Let $B = \{z \mid |z| = 1\}$. $Fr(B) = \overline{B} \cap \overline{\complement B} = \{z \mid |z| = 1\} = B$.

Definition 1.8 In a topological space E, a set A is said to be dense with respect to a set B, if every point of B is adherent to A (that is, if $B \subset \overline{A}$). A set A is said to be everywhere dense if it is dense with respect to the entire space E (that is, if $E = \overline{A}$).

Examples: Let $B = \{x | x = p/q$ where p and q are integers and $q \neq 0\}$. Then B is everywhere dense in the set R of real numbers. That is, $R = \overline{B}$.

In the plane let $A = \{z \mid |z| < 1$ and the point $(0,1)\}$. Let $B = \{z \mid |z| < 1\}$. Here we see that $B \subset A$ but $A \not\subset B$. However, B is dense with respect to A since $A \subset \overline{B}$ and also A is dense with respect to B since $B \subset \overline{A}$.

Exercise 1.1 Let R represent the real line. The usual topology for R is defined as follows. Let B be the set of all open intervals of R. Let $\mathcal{O}$ be the collection of all arbitrary unions of open intervals of R.

$$B = \{I_\alpha \mid I_\alpha \subset R \text{ and } I_\alpha = (x_\alpha, y_\alpha)\}$$
$$\mathcal{O} = \{U \mid U = \bigcup_{\alpha \in A} I_\alpha,\ I_\alpha \in B\}$$

Show $\mathcal{O}$ satisfies O_I and O_{II}.

Consider any $\bigcup_{\alpha \in A} U_\alpha$ where A is an arbitrary index set and $U_\alpha \in \mathcal{O}$ for every $\alpha \in A$. $\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A}(\bigcup_{\beta_\alpha \in B_A} I_{\beta_\alpha})$. Now there exists an index set $A \cup B_A$ such that $\bigcup_{\alpha \in A}(\bigcup_{\beta_\alpha \in B_A} I_{\beta_\alpha}) = \bigcup_{\gamma \in A \cup B_A} I_\gamma \in \mathcal{O}$. Therefore O_I is satisfied.

Now, if U_1 and U_2 are elements of $\mathcal{O}$, $U_1 \cap U_2 = \bigcup_{\alpha \in A} I_\alpha \cap \bigcup_{\beta \in B} I_\beta = \bigcup_{(\alpha,\beta) \in A \times B}(I_\alpha \cap I_\beta) \in \mathcal{O}$. Therefore O_{II_a} is satisfied. (Note: If $I_\alpha = I_\beta$, then $\alpha = \beta$.)

Evidently the union of the collection of all open intervals contains R. Therefore O_{II_b} is satisfied.

Therefore, $\mathcal{O}$ defines a topology on R.

<u>Exercise 1.2</u> Let R^2 be the Cartesian plane. $R^2 = \{(x,y) \mid x \in R \text{ any } y \in R\}$. Let B be the collection of all open discs in R^2. Let $\mathcal{O}$ be the set of all arbitrary unions of sets of B.

$$B = \{D_\alpha \mid D_\alpha = \{(x,y) \mid (x-h_\alpha)^2 + (y-k_\alpha)^2 < r_{\alpha_\beta}^2\}\}$$
$$\mathcal{O} = \{U \mid U = \bigcup_{\alpha \in A} D_\alpha,\ D_\alpha \in B\}$$

Show that $\mathcal{O}$ satisfies O_I and O_{II}.

Consider any $\bigcup_{\alpha \in A} U_\alpha$. $\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A}(\bigcup_{\beta_\alpha \in B_A} D_{\beta_\alpha})$ which can be expressed as $\bigcup_{\gamma \in A \cup B_A^\gamma} D_\gamma$ which is an element of $\mathcal{O}$. Therefore O_I is satisfied.

Now consider any $U_1 \varepsilon \mathcal{O}$ and $U_2 \varepsilon \mathcal{O}$. $U_1 \bigcap U_2 = \bigcup_{\alpha \varepsilon A} D_\alpha \bigcap \bigcup_{\beta \varepsilon B} D_\beta$, which can be written as $\bigcup_{(\alpha,\beta) \varepsilon A \times B} (D_\alpha \bigcap D_\beta)$. Therefore O_{II_a} is satisfied.

Clearly the union of the collection of all D_α contains R^2, so O_{II_b} is satisfied. (Here again, if $D_\alpha = D_\beta$, $\alpha = \beta$).
Therefore $\mathcal{O}$ defines a topology on R^2.

Exercise 1.3 Let E be a set. Let $\mathcal{B}$ be a collection of subsets of E $[\mathcal{B} \subset P(E)]$ so that the following conditions are satisfied:

P_1: If B_1 and B_2 are elements of $\mathcal{B}$, then there is an element $B_3 \varepsilon \mathcal{B}$ so that $B_3 \subset B_1 \bigcap B_2$.

P_2: $\emptyset \in \mathcal{B}$

P_3: $E \varepsilon \mathcal{B}$

Show how to use $\mathcal{B}$ to generate a topology on E. Define from $\mathcal{B}$ a collection $\mathcal{O}$ of subsets of E which will satisfy O_I and O_{II}.

Let $\mathcal{O} = \{U \subset E \mid U = \bigcup_{\alpha \varepsilon A} B_\alpha,\ B_\alpha \varepsilon \mathcal{B}\}$

Consider any $\bigcup_{\alpha \varepsilon A} U_\alpha$. Then $\bigcup_{\alpha \varepsilon A} U_\alpha = \bigcup_{\alpha \varepsilon A} (\bigcup_{\beta \varepsilon B} B_\beta)$ which is $\bigcup_{\gamma \varepsilon A \cup B} B_\gamma$ and $\bigcup_{\gamma \varepsilon A \cup B} B_\gamma \varepsilon \mathcal{O}$. O_I is satisfied.

Let $U_1 \varepsilon \mathcal{O}$ and $U_2 \varepsilon \mathcal{O}$. $U_1 \bigcap U_2 = \bigcup_{\alpha \varepsilon A} B_\alpha \bigcap \bigcup_{\gamma \varepsilon C} B_\gamma$ but this is $\bigcup_{(\alpha,\gamma) \varepsilon A \times C} (B_\alpha \bigcap B_\gamma)$ which is an element of $\mathcal{O}$. Hence O_{II_a} is satisfied.

Since from P_3, $E \in \mathcal{B}$, $\mathcal{O}$ defines a topology on E.
(If $B_\alpha = B_\beta$, $\alpha = \beta$).

Exercise 1.4 If E is a set ordered by a relation of order that one writes $x \leqslant y$, one designates by $S_g(x)$ the set of all y such that $y \leqslant x$, by $S_d(x)$ the set of all y such that $x \leqslant y$.

a) Show that the set of all subsets of E containing $S_g(x)$ is the set of neighborhoods of x in a topology which is called the left topology.

If $S_g(x) \subset U$, then U is a neighborhood of x. Let $\mathcal{O}_x = \{U_x \mid S_g(x) \subset U_x\}$. $\mathcal{O} = \bigcup_{x \in E} \mathcal{O}_x$.

Consider $\bigcup_{\alpha \in A} U_\alpha$ where $A \subset E$. For each $\alpha \in A$, $U_\alpha = U_x \in \mathcal{O}$ for some $x \in E$, since $A \subset E$. Now $\bigcup_{\alpha \in A} U_\alpha = U_z \in \mathcal{O}_z$ for some $z \in E$. $S_g(z) \subset U_z$. Therefore $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$, so O_I is satisfied.

Now consider two elements of $\mathcal{O}$, say U_x and U_y. It is necessary to show that $U_x \cap U_y \in \mathcal{O}$ $(x \neq y)$. Either $x < y$ or $y < x$ since E is an ordered set and $x \neq y$. If $x < y$, $S_g(x) \subset S_g(y)$ and therefore, since $S_g(x) \subset U_x$ and $S_g(y) \subset U_y$, $U_x \cap U_y \neq \emptyset$. In either case $U_x \cap U_y = U_z$ where $z = \min(x,y)$. Now $U_z \supset S_g(x)$ if $x < y$ and $U_z \supset S_g(y)$ if $y < x$. Therefore $U_z \in \mathcal{O}$ and O_{II_a} is satisfied.

If $x \in E$, then $S_g(x) \subset E$ so $E \in \mathcal{O}_x$ and thus $E \in \mathcal{O}$. Therefore O_{II_b} is satisfied and $\mathcal{O}$ defines a topology on E.

b) For the left topology, show every intersection of open sets is open and every union of closed sets is closed.

Show $\bigcap_{\alpha \in A} U_\alpha$ is open if each U_α is open.

Consider the case where $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$. Then $\bigcap_{\alpha \in A} U_\alpha = V$ and $V \subset E$. Choose any $x \in V$. Then $S_g(x) \subset V$. Therefore $V \in \mathcal{O}$. Therefore V is open.

Now, it is necessary to show that $\bigcup_{\alpha \varepsilon A} V_\alpha$ is closed if each V_α is closed.

Let A be a subset of E so that A is not of the form $\emptyset$ or $\{x \mid x > a\}$ or $\{x \mid x \geq a\}$ or E. Suppose A is closed. Assume $A \neq \emptyset$ and $A \neq E$. Either A is bounded below or A is not bounded below.

<u>Case I</u>: A is not bounded below.
We know A is not bounded above (unless E is bounded above) so if A is not bounded below, A = E, a contradiction.

<u>Case II</u>: A is bounded below.
Let a be the lower bound of A. Either A contains a or A does not contain a. In either case $A = \{x \mid x > a\}$ or $A = \{x \mid x \geq a\}$, a contradiction.

Therefore we see that the only closed sets are of the form $\{x \mid x > a\}$ or $\{x \mid x \geq a\}$. It is obvious that if we take an arbitrary union of sets of this type we will get a set of the form $\{x \mid x > a\}$ or $\{x \mid x \geq a\}$; in either case it is closed.

c) For the left topology, show the closure of the set $\{x\}$ is $S_d(x)$.

To show that $S_d(x)$ is the closure of $\{x\}$ we need to show that if $y \varepsilon S_d(x)$, then every neighborhood of y contains x. Now the left topology is the set $\Theta = \bigcup_{z \varepsilon E} \Theta_z$ where $\Theta_z = \{U_z \mid S_g(z) \subset U_z\}$. Evidently if $y > x$ there is a $U_z \varepsilon \Theta_z$ for some $z \varepsilon E$ such that $y \varepsilon U_z$. However $x \varepsilon U_z$. Indeed, by the very way the left topology is defined, every neighborhood of y contains x, since every neighborhood of y contains $S_g(x)$ and $x \in S_g(x)$. Therefore the closure of $\{x\}$ is $S_g(x)$. ($x < y$ implies $x \in S_g(x) \subset S_g(y) \subset U_y$)

<u>Exercise 1.5</u> For every subset A of a topological space E, we let $\alpha(A) = \mathring{\overline{A}}$, and $\beta(A) = \overline{\mathring{A}}$.

a) Show that if A is open, one has $A \subset \alpha(A)$ and that if A is closed one has $A \supset \beta(A)$.

If A is open, $A = \mathring{A}$. Also $A \subset \overline{A}$. Now $A = \mathring{A} \subset \mathring{\overline{A}}$. Therefore $A \subset \alpha(A)$.

If A is closed $\mathring{A} \subset A$. Then $\overline{\mathring{A}} \subset \overline{A}$. But since A is closed $A = \overline{A}$. Therefore $\beta(A) \subset A$.

b) Show that for every subset A of E, one has $\alpha(\alpha(A)) = \alpha(A)$ and $\beta(\beta(A)) = \beta(A)$.

$$\mathring{\overline{A}} \subset \overline{A}$$
$$\overline{\mathring{\overline{A}}} \subset \overline{\overline{A}} = \overline{A}$$
$$\mathring{\overline{\mathring{\overline{A}}}} \subset \mathring{\overline{A}}$$

Thus $\alpha(\alpha(A)) \subset \alpha(A)$

But $A \subset \overline{A}$

$$\overline{A} \subset \overline{\overline{A}} = \overline{A}$$
$$\mathring{\overline{A}} \subset \mathring{\overline{A}}$$
$$\mathring{\overline{A}} \subset \overline{\mathring{\overline{A}}}$$
$$\mathring{\mathring{\overline{A}}} \subset \mathring{\overline{\mathring{\overline{A}}}}$$
$$\mathring{\overline{A}} \subset \mathring{\overline{\mathring{\overline{A}}}}$$

Thus $\alpha(A) \subset \alpha(\alpha(A))$

Therefore $\alpha(\alpha(A)) = \alpha(A)$

$\mathring{A} \subset \overline{\mathring{A}}$

$\mathring{\mathring{A}} = \mathring{A} \subset \mathring{\overline{\mathring{A}}}$

$\overline{\mathring{A}} \subset \overline{\mathring{\overline{\mathring{A}}}}$

Thus $\beta(A) \subset \beta(\beta(A))$

$\mathring{\overline{\mathring{A}}} \subset \overline{\mathring{A}}$

$\overline{\mathring{\overline{\mathring{A}}}} \subset \overline{\overline{\mathring{A}}} = \overline{\mathring{A}}$

Thus $\beta(\beta(A)) \subset \beta(A)$,

Therefore $\beta(\beta(A)) = \beta(A)$

c) Give an example of a set A such that the seven sets A, $\mathring{A}$, $\overline{A}$, $\mathring{\overline{A}}$, $\overline{\mathring{A}}$, $\overline{\mathring{\overline{A}}}$, $\mathring{\overline{\mathring{A}}}$ are distinct.

Let $A = \{x \mid x \in (0,1),\ x \in (1,2),\ x$ is a rational number between 2 and 3, or $x = 4\}$

$\mathring{A} = (0,1) \cup (1,2)$

$\overline{A} = [0,3] \cup \{4\}$

$\overline{\mathring{A}} = [0,2]$

$\mathring{\overline{A}} = (0,3)$

$\overline{\mathring{\overline{A}}} = [0,3]$

$\mathring{\overline{\mathring{A}}} = (0,2)$

<u>Exercise 1.6</u> Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$; give an example where A is open and where the three sets $A \cap \overline{B}$, $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ are distinct.

$A \subset \overline{A}$ and $B \subset \overline{B}$

$A \cap B \subset \overline{A} \cap \overline{B}$

$\overline{A \cap B} \subset \overline{\overline{A} \cap \overline{B}} = \overline{A} \cap \overline{B}$

so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Let $A = \{x \mid x \in (2,3)\}$.

Let $B = \left\{x \mid x \in (1,2) \text{ and } x \in \left(\frac{5}{2},4\right)\right\}$.

$$A \cap \overline{B} = \left\{x \mid x \in \left[\tfrac{5}{2},3\right)\right\}$$

$$\text{and } \overline{A \cap B} = \left\{x \mid x \in \left[\tfrac{5}{2},3\right]\right\}$$

$$\text{and } \overline{A} \cap \overline{B} = \left\{x \mid x \in \left[\tfrac{5}{2},3\right] \text{ and } x = 2\right\}$$

all are distinct.

Give an example where A is not open and where $A \cap \overline{B}$ is not contained in $\overline{A \cap B}$.

Let $A = \left\{x \mid x \in [2,3)\right\}$

$B = \left\{x \mid x \in (1,2) \text{ and } x \in \left(\frac{5}{2},4\right)\right\}$

$A \cap \overline{B} = \left\{x \mid x = 2 \text{ or } x \in \left[\frac{5}{2},3\right)\right\}$

$\overline{A \cap B} = \left\{x \mid x \in \left[\frac{5}{2},3\right]\right\}$

$A \cap \overline{B} \not\subset \overline{A \cap B}$ since 2 is not an element of $\overline{A \cap B}$.

On the same set E, it is possible (if it has more than one element) to define different topological structures by means of different sets of subsets of E satisfying axioms O_I and O_{II}. The topological spaces thus defined are considered different.

<u>Definition 1.9</u> Being given two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ defined on the same set E by means of two sets of subsets $\mathcal{O}_1$, $\mathcal{O}_2$ (of which the elements are the respective open sets of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$), $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ (or $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$) if $\mathcal{O}_2 \subset \mathcal{O}_1$; moreover, if $\mathcal{O}_1 \neq \mathcal{O}_2$ one says that $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_2$.

<u>Example</u>: The set of topologies on any set E is ordered

by the relation $\mathcal{T}$ is coarser than $\mathcal{T}'$; the topology of which the open sets are E and $\emptyset$ is coarser than any of the others, and is called the smallest element of the set of topologies; the discrete topology is finer than all the others, and is called the largest element of the set of topologies.

<u>Proposition 1.3</u> Being given two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on a set E, the following statements are equivalent:

a) $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

b) Whenever $x \in E$, every neighborhood of x for $\mathcal{T}_2$ is a neighborhood of x for $\mathcal{T}_1$.

It will be shown first that (a) implies (b). If V is a neighborhood of x for the topology $\mathcal{T}_2$, there exists an open set A for $\mathcal{T}_2$ such that $x \in A \subset V$; since A is also open for $\mathcal{T}_1$, V is a neighborhood of x for $\mathcal{T}_1$.

Conversely, (b) implies (a): because if A is an open set for $\mathcal{T}_2$, it is a neighborhood of each of its points for $\mathcal{T}_2$, and also for $\mathcal{T}_1$, which shows A is open for $\mathcal{T}_1$.

Let $\mathcal{S}$ be any set of subsets of any set E and consider the topologies on E for which all the sets of $\mathcal{S}$ are open. There exist such topologies; for example, the discrete topology.

The set $\mathcal{O}$ of the open sets for the topology $\mathcal{T}$ generated by $\mathcal{S}$ can be defined in the following fashion: $\mathcal{O}$ must contain, by virtue of O_{II}, the set $\mathcal{S}'$ of the finite intersections of sets of $\mathcal{S}$ (which contains E, the intersection of the empty set of $\mathscr{G}$); in view of O_I, $\mathscr{O}$ must also contain $\mathscr{G}''$, any arbitrary union of sets of $\mathscr{G}'$.

Definition 1.10 If it can be shown that every set of $\mathscr{G}''$ is a union of sets of $\mathscr{G}$, one says that $\mathscr{G}$ is a base for the topology it generates.

Exercise 1.7 On an ordered set E, the right topology has for a base the set of subsets $S_d(x)$; the left topology has for a base the set of subsets $S_g(x)$.

For the left topology, $\mathscr{O} = \bigcup_{x\in E} \mathscr{O}_x$ where $\mathscr{O}_x = \{U_x \mid S_g(x) \subset U_x\}$.

If it can be shown that every set of $\mathscr{O}$ is a union of sets of $\{S_g(x)\}$, then $\{S_g(x)\}$ is a base for $\mathscr{O}$.

It is necessary to show that if $U \in \mathscr{O}$, $U = \bigcup_{x\in A} S_g(x)$

Let V be the union of all $S_g(x)$ where $S_g(x) \subset U$. Suppose $y \in U$. Then there is a $S_g(x)$ such that $y \in S_g(x) \subset U$. Therefore $y \in V$. Hence $U \subset V$. But $V \subset U$ by the very way V is defined.

Therefore, since we have $U \subset V$ and $V \subset U$, $V = U$.

The proof would be the same for the right topology.

Exercise 1.8 On an ordered set E, the upper bound of the right topology and the left topology is the discrete topology.

$$\mathscr{O}_l = \{U_x \mid S_g(x) \subset U_x\}; \quad S_g(x) = \{y \mid y \leq x\}$$
$$\mathscr{O}_r = \{U_y \mid S_d(y) \subset U_y\}; \quad S_d(y) = \{x \mid x \geq y\}$$

Define by $\mathscr{O}$ the set of open sets for the topology generated by $\mathscr{O}_l \cup \mathscr{O}_r$. Let x be any point in E. Now $\{x\}$ is also in $\mathscr{O}$, since $S_g(x) \cap S_d(x) = \{x\}$, and

$S_g(x) \varepsilon\ \theta_\ell$ and $S_d(x) \varepsilon\ \theta_r$. Therefore we see that $\{x\}$, where $x \varepsilon E$, is contained in θ . Therefore $\theta_\ell \cup \theta_r$ generates the discrete topology on the ordered set E.

Exercise 1.9 Being given a topological space E, consider the following properties:

D_1: The topology of E possesses a denumerable base.

D_2: There exists a denumerable subset of E everywhere dense.

D_3: Every subset of E where all the points are isolated is denumerable.

D_4: Every set of non-empty open sets of E, two by two without a point in common, is denumerable.

Show that D_1 implies D_2 and D_3, and that each of D_2 and D_3 implies D_4.

Proof: D_1 implies D_2.

Let $\mathcal{B} = \{B_i \mid i = 1,2,3,\ldots\}$ be a denumerable base for E. Define $A = \{x_i \mid x_i \varepsilon B_i,\ i = 1,2,3,\ldots\}$. It is necessary to show $\overline{A} = E$.

Choose $x \varepsilon E$ so that $x \neq x_i$, for any i. Let U be a neighborhood of x. Now there exists a B_i such that $x \varepsilon B_i \subset U$. Now $x_i \varepsilon B_i \subset U$ and $x \neq x_i$. Therefore every neighborhood U of x contains a point of A distinct from x; hence $x \varepsilon \overline{A}$.

Therefore $E \subset \overline{A}$, and since $A \subset E$, $\overline{A} \subset \overline{E} = E$; thus A is a denumerable subset of E everywhere dense.

D_1 implies D_3

Let $\mathcal{B} = \{B_i \mid i = 1,2,3,\ldots\}$. Let A be any subset of

E so that all the points are isolated; i.e., for every point x of A there exists a neighborhood U of x so that if $y \neq x$ and $y \in U$, then $y \notin A$. Let U_j be a neighborhood of x so that $U_j \subset B_i$ for some i, and U_j contains no element of A other than x. Now if $x \in A$, $x \in B_i$ for some i. If $x, y \in A$ and $x \neq y$, it is possible to find U_{j_1} and U_{j_2} so that U_{j_1} and U_{j_2} have no points in common.

Consider $P = \{U_j\}$ where each U_j is a neighborhood of a point of A, and no two elements of P have a point in common. Clearly P is denumerable. Hence A is denumerable.

<u>D_3 implies D_4</u>

Let $V = \{U_\alpha \subset E \mid U_\alpha$ is open and nonempty and if $x \in U_\alpha$, $x \in U_\beta$, then $\alpha = \beta\}$. Let $P = \{x_\alpha \mid$ one and only one x_α is chosen from each $U_\alpha\}$. Clearly, P is a set which contains only isolated points, since for any $x_\beta \in P$ there is a neighborhood of x_β that contains no other points of P, namely the U_β from which x_β was chosen. Therefore P is denumerable. If P is denumerable, certainly V is denumerable.

<u>D_2 implies D_4</u>

D_2 states that there exists a denumerable subset A of E such that $\bar{A} = E$. It is necessary to show that this implies that every set of nonempty open sets of E, two by two without a point in common, is denumerable.

Let $V = \{U_j \subset E \mid U_j$ is open and nonempty and if $y \in U_n$, $y \in U_m$, the $m = n\}$. Define $P = \{y_j \mid y_j \in U_j, j = 1,2,3,\ldots\}$.

We recall from the proof of D_1 implies D_3 that $A = \{x_i \mid x_i \in B_i,$ $i = 1,2,3,\ldots\}$. Now let W be a mapping of P onto A where $x_i \to y_j$, when $i = j$. Since this is a one to one mapping, P is denumerable. Since P is denumerable, certainly V is denumerable.

<u>Definition 1.11</u> A topology induced on A by the topology of E is the topology where the open sets are the traces on A of the open sets of E. The set A, provided with this topology, is called a subspace of E.

<u>Proposition 1.4</u> If A and B are two subsets of a topological space E such that $B \subset A$, the adherence of B with respect to the subspace A is the trace on A of the adherence of B with respect to E.

In effect, if $x \in A$, every neighborhood of x with respect to A is of the form $V \cap A$, where V is a neighborhood of x in E. Or, $V \cap B = (V \cap A) \cap B$; then, in order that x be adherent to B with respect to A, it is necessary and sufficient that it be adherent to B with respect to E.

<u>Exercise 1.10</u> If A and B are two subsets of a topological space E such that $B \subset A$, show that:

(a) the interior of B with respect to E is contained in the interior of B with respect to the subspace A. Give an example where these two sets are distinct.

It is necessary to show $\overset{o}{B}_E \subset \overset{o}{B}_A$. Let $x \in \overset{o}{B}_E$ and show $x \in \overset{o}{B}_A$. Now, $x \in \overset{o}{B}_E$ implies there exists a neighborhood U of x such that $U \subset \overset{o}{B}_E$. Then $U \cap A$ is a neighborhood of x in A. Since $x \in U \cap A \subset B$, then $x \in \mathring{B}_A$.

Example: Let E be the real line with the usual topology. Let $A = [x,y]$. Let $B = (\frac{x+y}{2}, y]$. Now $\mathring{B}_E = (\frac{x+y}{2}, y)$, but $\mathring{B}_A = (\frac{x+y}{2}, y]$. Therefore $\mathring{B}_E \subset \mathring{B}_A$.

(b) the frontier of B with respect to A is contained in the trace on A of the frontier of B with respect to E. Give an example where these two sets are distinct.

$Fr(B_E) = \overline{B_E} \cap \overline{\complement B_E}$

$Fr(B_A) = \overline{B_A} \cap \overline{\complement B_A}$

It is necessary to show $\overline{B_A} \cap \overline{\complement B_A} \subset \overline{B_E} \cap \overline{\complement B_E} \cap A$.

It is known that $\overline{B_A} = \overline{B_E} \cap A$ by Proposition 1.4. Now $(\overline{B_E} \cap A) \cap \overline{\complement B_A} \subset (\overline{B_E} \cap A) \cap \overline{\complement B_E}$ since $\overline{\complement B_A} \subset \overline{\complement B_E}$.

Therefore $(\overline{B_E} \cap A) \cap \overline{\complement B_A} \subset \overline{B_E} \cap \overline{\complement B_E} \cap A$ or $Fr(B_A) \subset Fr(B_E) \cap A$.

Example: Let $A = [0,1]$ and $B = (\frac{1}{2},1]$

$Fr(B_A) = \{\frac{1}{2}\}$

$Fr(B_E) = \{\frac{1}{2},1\}$

Therefore $Fr(B_A) \subset Fr(B_E) \cap A$.

Exercise 1.11 Let A and B be any two subsets of E.

(a) Show that the trace on A of the interior of B with respect to E is contained in the interior of $B \cap A$ with respect to A. Give an example where these two sets are distinct. (Assume $B \cap A \neq \emptyset$.)

Let $x \in \mathring{B}_E \cap A$. Now A is open with respect to the subspace A. Since $x \in \mathring{B}_E$, x is not a frontier point of B with respect to E. Therefore $x \in (\widehat{B \cap A})^{\circ}_E$. However, $(\widehat{B \cap A})^{\circ}_E \subset (\widehat{B \cap A})^{\circ}_A$. Therefore $x \in (\widehat{B \cap A})^{\circ}_A$. Hence $\mathring{B}_E \cap A \subset (\widehat{B \cap A})^{\circ}_A$.

<u>Example</u>: Let $A = [0,1]$. Let $B = [0,\frac{1}{2})$. $\overset{o}{B}_E = (0,\frac{1}{2})$, so $\overset{o}{B}_E \cap A = (0,\frac{1}{2}) \cap [0,1] = (0,\frac{1}{2})$. But $\overset{o}{(B \cap A)}_A = [0,\frac{1}{2})$. Therefore $\overset{o}{B}_E \cap A \subset \overset{o}{(B \cap A)}_A$.

(b) Show that the trace on A of the closure of B with respect to E contains the closure of $B \cap A$ with respect to A. Give an example.

Since A is closed with respect to itself, $(\overline{B \cap A})_A = \overline{B}_A \cap A$. Now certainly $\overline{B}_A \subset \overline{B}_E$. Therefore $(\overline{B \cap A})_A = \overline{B}_A \cap A \subset \overline{B}_E \cap A$.

<u>Example</u>: Let $A = [0,1]$, $B = (1,2)$. Now $\overline{B}_E \cap A = 1$, but $(\overline{B \cap A})_A = \emptyset$.

<u>Exercise 1.12</u> If each point of a subset A of a topological space E is isolated, the topology induced on A by that of E is discrete, and conversely.

If $x \in A$ there exists a neighborhood U of x such that U does not contain any point of A other that x. Now $U \subset E$ and U contains an open set $U' \subset E$ so that $U' \cap A = x$. Hence we see that the topology induced on A is discrete.

Now if the topology induced on A is discrete, we know that if $x \in A$, then $x \in \mathcal{O}$ where $\mathcal{O}$ is the set of traces on A by open sets of E. That is, there exists an open set in E, say U, such that $U \cap A = X$. By definition, x is isolated.

CHAPTER II

FILTERS

Definition 2.1 A filter on a set E is a set of subsets of E which possesses the following properties:

F_I: Every set containing a set of $\mathcal{F}$ belongs to $\mathcal{F}$.

F_{II}: Every finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$.

F_{III}: The empty subset of E does not belong to $\mathcal{F}$.

Axiom F_{II} is equivalent to the following two axioms:

F_{IIa}: The intersection of any two sets of $\mathcal{F}$ belongs to $\mathcal{F}$.

F_{IIb}: E belongs to $\mathcal{F}$.

Definition 2.2 Being given two filters $\mathcal{F}$ and $\mathcal{F}'$ on the same set E, $\mathcal{F}'$ is finer than $\mathcal{F}$, or $\mathcal{F}$ is coarser than $\mathcal{F}'$, if $\mathcal{F} \subset \mathcal{F}'$. If $\mathcal{F} \neq \mathcal{F}'$, $\mathcal{F}'$ is strictly finer than $\mathcal{F}$, or $\mathcal{F}$ is strictly coarser than $\mathcal{F}'$.

Theorem 2.1 In order that there exist a filter containing a set $\mathcal{G}$ of subsets of E, it is necessary and sufficient that none of the finite intersections of sets of $\mathcal{G}$ be empty.

If such a filter exists it must contain, according to F_{II}, the set $\mathcal{G}'$ of finite intersections of sets of $\mathcal{G}$. If such a filter exists, it is necessary that the empty set does not belong to $\mathcal{G}'$. Now show this condition is sufficient.

Every filter containing $\mathcal{G}'$ (if one exists) contains also, according to F_I, the set $\mathcal{G}''$ of subsets of E which contain a set of $\mathcal{G}'$. Or, $\mathcal{G}''$ satisfies F_I. It satisfies F_{II} according to the definition of $\mathcal{G}'$. Finally, $\mathcal{G}''$ satisfies F_{III} , since the empty set of E does not belong to $\mathcal{G}'$. $\mathcal{G}''$ is then a filter containing $\mathcal{G}$, and every filter containing $\mathcal{G}$ is finer than $\mathcal{G}''$. One says that $\mathcal{G}''$ is generated by $\mathcal{G}$.

Corollary: Let $\mathcal{F}$ be a filter on E and A be a subset of E. In order that there exist a filter $\mathcal{F}'$ finer than $\mathcal{F}$ and such that $A \varepsilon \mathcal{F}'$, it is necessary and sufficient that A intersect every set of $\mathcal{F}$.

Let $\mathcal{F} \subset \mathcal{F}'$ and $A \varepsilon \mathcal{F}'$. We know the intersection of A and any other set of $\mathcal{F}'$ is not empty. Now since $\mathcal{F} \subset \mathcal{F}'$, evidently A intersects every set of $\mathcal{F}$.

Now, assume A intersects every set of $\mathcal{F}$. Then, by Theorem 2.1, there exists a filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$.

Proposition 2.1 Let $\mathcal{B}$ be a set of subsets of E. In order that the set of subsets of E containing a set of $\mathcal{B}$ be a filter, it is necessary and sufficient that $\mathcal{B}$ possess the following two properties:

B_I: The intersection of two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$.

B_{II}: $\mathcal{B}$ is not empty, and the empty set of E does not belong to $\mathcal{B}$.

These properties are evidently necessary from the definition of a filter. It is necessary to show they are sufficient. If such a set $\mathcal{F}$ exists, we see immediately that this set satisfies F_I. It is clear that every element of $\mathcal{B}$ is an element of $\mathcal{F}$. Therefore the intersection of any two elements of $\mathcal{F}$ is an element of $\mathcal{F}$. Hence F_{II_a} is satisfied. Evidently $E \in \mathcal{F}$, so F_{II_b} is satisfied. Since the empty set does not belong to $\mathcal{B}$, neither does it belong to $\mathcal{F}$. Therefore $\mathcal{F}$ is a filter.

<u>Definition 2.3</u> A set of subsets of a set E which satisfies axioms B_I and B_{II} is a base of the filter that it generates. Two bases of a filter are called equivalent if they generate the same filter.

<u>Proposition 2.2</u> In order that a subset $\mathcal{B}$ of a filter $\mathcal{F}$ be a base of that filter, it is necessary and sufficient that every set of $\mathcal{F}$ contain a set of $\mathcal{B}$.

The condition is evidently necessary; it is sufficient, because if it is fulfilled, the set of subsets of E containing a set of $\mathcal{B}$ is identical to $\mathcal{F}$, in view of F_I.

<u>Proposition 2.3</u> In order that a filter $\mathcal{F}'$ of base $\mathcal{B}'$ be finer than a filter $\mathcal{F}$ of base $\mathcal{B}$, it is necessary and sufficient that every set of $\mathcal{B}$ contain a set of $\mathcal{B}'$.

This results immediately from Definition 2.2 and Definition 2.3.

<u>Definition 2.4</u> Every base of the filter of the neighborhoods of a point (or a subset) of a topological space is called the fundamental system of neighborhoods of that point (subset).

<u>Proposition 2.4</u> In order that a finite set Φ of filters on E have an upper bound, it is necessary and sufficient that, when one takes arbitrarily a set in each filter of Φ , the intersection of these sets is never empty.

In order that any set Φ of filters on E have an upper bound, it is necessary and sufficient that every finite subset of Φ have an upper bound.

<u>Proof of second statement</u>:

Let $\mathcal{G} = \{F \mid F \in \mathcal{F}_\alpha$ for some $\mathcal{F}_\alpha \in \Phi\}$. Consider any finite subset of $\mathcal{G}$, say $\{F_1, F_2, F_3, \ldots, F_n\}$. Consider $\bigcap_{i=1}^{n} F_i$. Assume $\bigcap_{i=1}^{n} F_i = \emptyset$. We know that every finite subset of Φ has an upper bound; therefore if $\bigcap_{i=1}^{n} F_i = \emptyset$ the subset of Φ from which $\{F_i\}_{i=1}^{n}$ was taken would not have an upper bound. This is obvious because for some filter $\mathcal{F}$ to be an upper bound of $\{\mathcal{F}_i\}$ from which $\{F_i\}_{i=1}^{n}$ was taken, it must be that for each $\mathcal{F}_i$, every set of $\mathcal{F}_i$ must belong to $\mathcal{F}$. Apparently this cannot be so, for then $\mathcal{F}$ would not be a filter if $\mathcal{F}$ contained $\{F_i\}$. Hence $\bigcap_{i=1}^{n} F_i \neq \emptyset$, and by Theorem 2.1, there exists a filter containing $\mathcal{G}$, and hence containing every filter in Φ . Therefore Φ has an upper bound.

<u>Definition 2.5</u> An ultrafilter on a set E is a filter such that there does not exist another filter strictly finer than it. (In other words, it is a maximum element of the ordered set of filters on E.)

The following theorem will be accepted without proof.

Theorem 2.2 If $\mathfrak{F}$ is a filter on a set E, there exists an ultrafilter finer than $\mathfrak{F}$.

Proposition 2.5 Let $\mathfrak{F}$ be an ultrafilter on a set E. If A and B are two subsets of E such that $A \cup B \in \mathfrak{F}$, then $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.

Suppose that $A \notin \mathfrak{F}$, $B \notin \mathfrak{F}$, and $A \cup B \in \mathfrak{F}$. $\mathfrak{G}$ is the set of subsets $X \subset E$ such that $A \cup X \in \mathfrak{F}$. One verifies immediately that $\mathfrak{G}$ is a filter on E. $\mathfrak{G}$ is strictly finer than $\mathfrak{F}$ because $B \in \mathfrak{G}$. But this contradicts the hypothesis that $\mathfrak{F}$ is an ultrafilter.

Proposition 2.6 Let $\mathfrak{G}$ be a system of generators of a filter on a set E; if for every $X \subset E$, $X \in \mathfrak{G}$ or $\complement X \in \mathfrak{G}$, then $\mathfrak{G}$ is an ultrafilter on E.

In effect, every filter containing $\mathfrak{G}$ (it exists by hypothesis) is identical to $\mathfrak{G}$; because, if $X \in \mathfrak{F}$, $\complement X \notin \mathfrak{F}$, so $\complement X \notin \mathfrak{G}$ which implies $X \in \mathfrak{G}$.

Definition 2.6 If the trace, on a nonempty subset A of a set E, of a filter $\mathfrak{F}$ on E, is a filter on A, one says that this filter is induced by $\mathfrak{F}$ on A.

Definition 2.7 Let $\{x_n\}_{n \in N}$ be an infinite sequence of elements of a set E. The elementary filter associated with the sequence $\{x_n\}$ is the filter generated by the image of the filter of Frechet by the mapping $n \rightarrow x_n$ of N into E.

That is to say, the elementary filter associated with the sequence $\{x_n\}$ is the set of $X \subset E$ such that $x_n \in X$

except for a finite number of values of n. If S_n designates the set of the x_p such that $p \geq n$, the sets S_n form a base of the elementary filter associated with the sequence $\{x_n\}$.

<u>Proposition 2.7</u> If a filter $\mathcal{F}$ possesses a denumerable base, there exists an elementary filter finer than $\mathcal{F}$, and $\mathcal{F}$ is the filter intersection of all the elementary filters finer than $\mathcal{F}$.

In effect, arrange the denumerable base of $\mathcal{F}$ in a sequence $\{A_n\}$. If one places $B_n = \bigcap_{p=1}^{n} A_p$, the B_n's form a base of $\mathcal{F}$, and one has $B_{n+1} \subset B_n$. Let a_n be any point of B_n. $\mathcal{F}$ is coarser than the filter associated with the sequence $\{a_n\}$.

<u>Exercise 2.1</u> If the intersection of all the sets of a filter $\mathcal{F}$ on a set E is empty, show that $\mathcal{F}$ is finer than the filter of the complements of the finite subsets of E.

Evidently the sets of $\mathcal{F}$ must be infinite; otherwise $\mathcal{F}$ could not be a filter. Therefore E must be an infinite set. Since the complements of the finite subsets of E form a filter, say $\mathcal{F}'$, let us show that if $\complement A \in \mathcal{F}'$, $\complement A \in \mathcal{F}$, where A is finite.

Assume $\complement A \notin \mathcal{F}$; that is, $\complement A$ does not contain any set of $\mathcal{F}$. If $\complement A$ does not belong to $\mathcal{F}$, then no subset of $\complement A$ belongs to $\mathcal{F}$. Therefore, every set of $\mathcal{F}$ must have at least one point of A in common, for if they (the sets of $\mathcal{F}$) did not, there would exist sets U and V of $\mathcal{F}$

such that $U \cap V \subset \complement A$, which implies that $\complement A \in \mathcal{F}$, contrary to our assumption. However, if there is a point of A common to every set of $\mathcal{F}$, the intersection of all the sets of $\mathcal{F}$ would not be empty, a contradiction to our hypothesis. Therefore $\complement A \in \mathcal{F}$. Hence $\mathcal{F}' \subset \mathcal{F}$.

<u>Exercise 2.2</u> The intersection filter of two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on a set E is identical with the set of subsets of the form $A \cup B$, where A is an arbitrary set of $\mathcal{F}_1$ and B is an arbitrary set of $\mathcal{F}_2$.

Let $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$. Then $\mathcal{F} \subset \mathcal{F}_1$ and $\mathcal{F} \subset \mathcal{F}_2$. Now let $\mathcal{F}' = \{A \cup B \mid A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\}$. Let $C \in \mathcal{F}$. Then $C \in \mathcal{F}_1$ and $C \in \mathcal{F}_2$. Certainly $C = C \cup C \in \mathcal{F}'$. Therefore $\mathcal{F} \subset \mathcal{F}'$.

Let $D \in \mathcal{F}'$. Then $D = A \cup B$, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Since $A \subset A \cup B$ and $B \subset A \cup B$, we know from the definition of a filter that $D = A \cup B \in \mathcal{F}_1$ and $D = A \cup B \in \mathcal{F}_2$. Therefore $D \in \mathcal{F}$ and we have that $\mathcal{F}' \subset \mathcal{F}$. Hence $\mathcal{F} = \mathcal{F}'$.

<u>Exercise 2.3</u> Show that on an infinite set E, the filter of the complements of the finite subsets is the filter intersection of the elementary filters associated with the infinite sequence of elements of E of which the terms are all distinct.

Let $\mathcal{F}$ be the filter of the complements of the finite subsets of E. Then $\mathcal{F}$ has for a base the set $\mathcal{B} = \{\complement A \mid A \subset E$ and A is finite$\}$. Therefore, for each A, there are at most n elements of A where n is a positive integer. Let $\{A_x\}_{x=1}^{n}$ denote the finite subsets of E; i.e., A_x is finite for each x. Then $\{x_i, i = 1,2,3,\ldots,n \mid x_i \in A_x$ where A_x is finite and $A_x \subset E\}$ must be denumerable; because if it were not there would be at least one A_x for which A_x would not be finite, a contradiction.

Now if $\{A_x\}_{x=1}^{n}$ is denumerable, certainly $\{\complement A_x\}_{x=1}^{n}$ is denumerable. Hence $\mathcal{B}$ is denumerable.

Then, by Proposition 2.7, there exists an elementary filter finer than $\mathcal{F}$, and by Proposition 2.7, $\mathcal{F}$ is the intersection of all the elementary filters finer than it.

<u>Exercise 2.4</u> If two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on a set E have an upper bound in the set of filters on E, show that this upper bound is identical with the set of subsets of the form $A \cap B$, where A is an arbitrary set of $\mathcal{F}_1$ and B is an arbitrary set of $\mathcal{F}_2$.

Let $\mathcal{F} = \{A \cap B \mid A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2\}$.

Choose any $C \in \mathcal{F}_1$. Let $B = E \in \mathcal{F}_2$. Then $C \cap E = C \in \mathcal{F}$. Therefore $\mathcal{F}_1 \subset \mathcal{F}$.

Choose any $D \in \mathcal{F}_2$. Let $A = E \in \mathcal{F}_1$. Then $D \cap E = D \in \mathcal{F}$. Therefore $\mathcal{F}_2 \subset \mathcal{F}$.

Now assume there exists an $\mathcal{F}'$ such that $\mathcal{F}' \subsetneq \mathcal{F}$ and such that $\mathcal{F}_1 \subset \mathcal{F}'$ and $\mathcal{F}_2 \subset \mathcal{F}'$. Then there exists a $V \in \mathcal{F}$ such that $V \notin \mathcal{F}'$. However, $V = A \cap B$ where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Since $\mathcal{F}_1 \subset \mathcal{F}'$ and $\mathcal{F}_2 \subset \mathcal{F}'$, then $A \in \mathcal{F}'$ and $B \in \mathcal{F}'$. Then by the definition of a filter $A \cap B = V \in \mathcal{F}'$, a contradiction to our assumption. Therefore there does not exist a $V \in \mathcal{F}$ such that $V \notin \mathcal{F}'$, so $\mathcal{F}$ is indeed the upper bound of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Exercise 2.5 In a topological space E, the intersection filter of the filters of neighborhoods of all the points of a subset A of E is the filter of the neighborhoods of A.

Let $\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha$ where $\mathcal{F}_\alpha$ is the filter of neighborhoods of α. Let U be a neighborhood of A. Choose arbitrarily $\alpha \in A$. Then $U \in \mathcal{F}_\alpha$ since U is certainly a neighborhood of α. Therefore $U \in \mathcal{F}_\alpha$ for every $\alpha \in A$. Then $U \in \mathcal{F}$.

If V is not a neighborhood of A, then there is at least one $\alpha \in A$ such that V is not a neighborhood of α, which implies $V \notin \mathcal{F}_\alpha$. Then $V \notin \mathcal{F}$. Therefore $\mathcal{F}$ is the filter of neighborhoods of A.

Exercise 2.6 Show that every filter $\mathcal{F}$ is the intersection of the ultrafilters which are finer than $\mathcal{F}$. (If $A \notin \mathcal{F}$ but intersects every set of $\mathcal{F}$, notice that $\complement A \notin \mathcal{F}$ and that $\complement A$ intersects every set of $\mathcal{F}$.)

Let $\mathcal{F}$ be a filter. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be such that for each α, $\mathcal{F}_\alpha$ is an ultrafilter and $\mathcal{F} \subset \mathcal{F}_\alpha$. This implies $\mathcal{F} \subset \bigcap_{\alpha \in A} \mathcal{F}_\alpha = \mathcal{F}'$. It is necessary to show that if $\mathcal{F} \subsetneq \mathcal{F}'$ is assumed, a contradiction occurs.

Assume $\mathcal{F} \subsetneq \mathcal{F}'$. Then there is at least one $A \in \mathcal{F}'$ such that $A \notin \mathcal{F}$. $A \in \mathcal{F}_\alpha$ for every α. (It is necessary to show that there is an ultrafilter $\mathcal{F}_\beta$ which contains $\mathcal{F}$ and such that $A \notin \mathcal{F}_\beta$.) Note that if $A \notin \mathcal{F}$, A must intersect every set of $\mathcal{F}$, since $\mathcal{F} \subsetneq \mathcal{F}'$. Therefore $\complement A \in \mathcal{F}$, but $\complement A$ intersects every set of $\mathcal{F}$.

Therefore, there is an ultrafilter $\mathcal{F}_\beta$ which contains $\mathcal{F}$ and such that $A \notin \mathcal{F}_\beta$. This ultrafilter $\mathcal{F}_\beta$ is generated by the collection $\mathcal{B} = \{X | X \in \mathcal{F}$ or $X = \complement A\}$. Note that $A \notin \mathcal{F}_\beta$ and $\mathcal{F} \subset \mathcal{F}_\beta$; hence a contradiction that $\mathcal{F}'$ contains a set $A \notin \mathcal{F}$. Therefore $\mathcal{F}$ is not a proper subset of $\mathcal{F}'$. Then $\mathcal{F} = \mathcal{F}'$.

<u>Exercise 2.7</u> Show that every ultrafilter finer than the intersection of a finite number of filters is finer than at least one of them.

Consider $\bigcap_{i=1}^{n} \mathcal{F}_i$ where $\mathcal{F}_i$ is a filter on E for each i. Let $\bigcap_{i=1}^{n} \mathcal{F}_i = \mathcal{F}$. Obviously $\mathcal{F}$ is a filter on E, and $\mathcal{F} \subset \mathcal{F}_i$ for each i. Let $\mathcal{F} \subset \mathcal{F}'$ where $\mathcal{F}'$ is an ultrafilter on E.

Assume for each i, $\mathcal{F}_i \not\subset \mathcal{F}'$. Thus for each i, there exists at least one $A_i \in \mathcal{F}_i$ such that $A_i \notin \mathcal{F}'$. However $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_i$ for each i. Therefore $\bigcup_{i=1}^{n} A_i \in \mathcal{F}'$. Now $\bigcup_{i=1}^{k} A_i$ is an element of every $\mathcal{F}_i$, where $i = 1,2,3,\ldots k$, and $\bigcup_{i=k+1}^{n} A_i$ is an element of every $\mathcal{F}_i$, $i = k+1,\ldots,n$, so $\left[\bigcup_{i=1}^{k} A_i\right] \cup \left[\bigcup_{i=k+1}^{n} A_i\right] \in \mathcal{F} \subset \mathcal{F}'$. Therefore, by Proposition 2.5, either $\bigcup_{i=1}^{k} A_i$ or $\bigcup_{i=k+1}^{n} A_i$ is an element of $\mathcal{F}'$. Since the A_i's are finite, it is easily seen that if this process is continued we will finally arrive at the union of two of the A_i's of which one of them must be an element of $\mathcal{F}''$, a contradiction to our assumption.

Therefore there does exist an $\mathcal{F}_i$, for some i , such that $\mathcal{F}_i \subset \mathcal{F}'$.

(b) Give an example of an ultrafilter finer than the intersection of an infinite family of ultrafilters, but which is not identical to any of the ultrafilters of that family.

Let E = $[0,1]$. Consider $\bigcap_{\alpha \in [\frac{1}{2},1]} \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ is generated by α for each $\alpha \in [\frac{1}{2},1]$. Obviously $\bigcap_{\alpha \in [\frac{1}{2},1]} \mathcal{F}_\alpha = [0,1] = E$.

The ultrafilter generated by $\{0\}$ is finer than $\bigcap_{\alpha \in [\frac{1}{2},1]} \mathcal{F}_\alpha$ but is not contained in $\mathcal{F}_\alpha$, $\alpha \in [\frac{1}{2},1]$.

<u>Exercise 2.8</u> Show that the intersection of the sets of an ultrafilter contains at most one point, and that, if it is a single point, the ultrafilter is formed from the sets containing that point.

Let $\mathcal{F}$ be an ultrafilter and let $\{A_\alpha\}_{\alpha \in I}$ be the sets of the ultrafilter $\mathcal{F}$.

Let $\bigcap_{\alpha \in I} A_\alpha$ = A and assume A is not empty and not a single point.

Now if A $\notin \mathcal{F}$, A does not contain any set in $\mathcal{F}$, but every set in $\mathcal{F}$ contains A. However, if A is not in $\mathcal{F}$ then there exists a filter which contains A and also $\mathcal{F}$, namely $\mathcal{F}' = \mathcal{F} \cup \{A\}$, which contradicts the fact that $\mathcal{F}$ is an ultrafilter.

Therefore A $\in \mathcal{F}$. Then there must be distinct subsets U and V of E such that U $\cup$ V = A, and Proposition 2.5

says that either U or V must be an element of $\mathcal{F}$. Therefore $\bigcap_{\alpha \in I} A_\alpha \neq A$ where A is not a single point and not empty.

Hence $\bigcap_{\alpha \in I} A_\alpha = \{x\}$ where x is a single point, or $\bigcap_{\alpha \in I} A_\alpha = \emptyset$. The filter formed by taking every set which contains x is an ultrafilter, since there is not a filter $\mathcal{F}'$ such that $\mathcal{F} \subsetneq \mathcal{F}'$.

<u>Exercise 2.9</u> Show that, if a subset A of a set E does not belong to an ultrafilter $\mathcal{U}$ on E, the trace of $\mathcal{U}$ on A is the set of all the subsets of A.

If $A \notin \mathcal{U}$, then there must be at least one $W \in \mathcal{U}$ such that $W \cap A = \emptyset$.

Let $V \in \mathcal{U}$, where $V \cap A \neq \emptyset$. $V = (A \cap V) \cup (\complement A \cap V)$. Now $A \cap V \notin \mathcal{U}$, so $\complement A \cap V \in \mathcal{U}$ by Proposition 2.5. However, $(\complement A \cap V) \cap A = \emptyset$. Let $\complement A \cap V = Y$.

Consider this Y. Let X be any subset of A. Let $W = \{x \mid x \in Y \text{ or } x \in X\}$. Obviously $W \in \mathcal{U}$ since $Y \subset W$. Now $Y \cap A = X$. Hence it is possible to get any subset of A desired. Therefore the trace of $\mathcal{U}$ on A is the set of all subsets of A.

<u>Exercise 2.10</u> On an infinite set, show that an elementary filter associated with a sequence whose terms are all distinct is not an ultrafilter.

Let $\{x_n\}$ be an infinite sequence of distinct elements in an infinite set E. Assume the elementary filter associated with $\{x_n\}$ is an ultrafilter $\mathcal{F}$, where $\mathcal{F} = \{X \mid X$ contains all but a finite number of the x_n's$\}$. Now $E = \{x_1, x_3, x_5, \ldots\} \cup \complement\{x_1, x_3, x_5, \ldots\}$. Now according to Proposition 2.5, either $\{x_1, x_3, x_5, \ldots x_{2n-1}, \ldots\}$ or $\complement\{x_1, x_3, x_5, \ldots, x_{2n-1}, \ldots\}$ must be an element of $\mathcal{F}$. Obviously neither of them is, since neither contains all but a finite number of the x_n's. Therefore $\mathcal{F}$ is not an ultrafilter.

<u>Exercise 2.11</u> Let Φ be a totally ordered denumerable set of elementary filters. Show that there exists an elementary filter finer than all the filters of Φ. (Show that the union of all the filters of Φ has a denumerable base.)

Let $\{x_n\}$ be the sequence associated with $\mathcal{F}_m$, where $\mathcal{F}_m \in \Phi$. Now the base of $\mathcal{F}_m$ is $\{S_n\}$ where $S_n = \{x_p \mid p \geq n\}$. Certainly $\{S_n\}$ is denumerable, since $\{x_n\}$ is denumerable. Therefore every elementary filter in Φ has a denumerable base.

Consider $\bigcup_{n \in N} \mathcal{F}_m = \mathcal{F}$. Now $\mathcal{F}$ is a filter and $\mathcal{F}$ has as a base $\bigcup_{n \in N} S_n$. ($\mathcal{F}$ is a filter, since Φ is a totally ordered set of filters and the upper bound of Φ is the union of all the filters in Φ.) Certainly a denumerable union of denumerable sets is denumerable. Therefore, by Proposition 2.7, there exists an elementary filter finer than $\mathcal{F}$.

<u>Exercise 2.12</u> We have the definition for a filter base $\mathcal{B}$ on E as follows:

(1) $\mathcal{B} \subset P(E)$

(2) $\emptyset \notin \mathcal{B}$

(3a) if $U, V \in \mathcal{B}$, there exists a $W \in \mathcal{B}$ such that

$W \subset U \cap V$.

If E is finite show the following is an equivalent definition for a filter base:

(1) $\mathcal{B} \subset P(E)$, $\mathcal{B} = \{B_1, B_2, B_3, \ldots, B_k\}$

(2) $\emptyset \notin \mathcal{B}$

(3b) $\bigcap_{p=1}^{k} B_p = B_r$ for some $r = 1,2,3,\ldots, k$.

First assume $\bigcap_{p=1}^{k} B_p = B_r$ for some $r = 1,2,3,\ldots, k$. Then $B_r \in \mathcal{B}$. This means that if $B_n, B_m \in \mathcal{B}$, $B_n \cap B_m \supset B_r$. This implies condition (3a).

Now assume that if B_n and $B_m \in \mathcal{B}$, then $B_n \cap B_m \supset B_r$ for some $r = 1,2,3,\ldots, k$. Obviously then $\bigcap_{p=1}^{k} B_p \supset B_r$ for some $r = 1,2,3,\ldots, k$. Assume $\bigcap_{p=1}^{k} B_p = W \supsetneq B_r$ for some $r = 1,2,3,\ldots, k$. Then there is a subset $A \subset W$ such that $A \not\subset B_r$, and $A \subset \mathcal{B}$ for every r, which is impossible if $A \not\subset B_r$ for some r. Therefore $B_r = W$. Then (3a) implies (3b).

Now consider the arbitrary case $\bigcap_{\alpha \in \mathcal{B}} B_\alpha = B_\beta$, where $B_\beta \in \mathcal{B}$.

This cannot be the case. Consider the set $E = (0,1)$. Define $\mathcal{B}$ as follows: $\mathcal{B} = \{(0, \frac{1}{n}) \mid 0 < \frac{1}{n} < 1\}$. Obviously (3a) does not hold since $\bigcap_{\alpha \in (0,1)} B_\alpha = \emptyset \notin \mathcal{B}$. Hence (3b) does not hold either.

<u>Exercise 2.13</u> Let $n \to f(n)$ be a mapping of N onto itself such that $\overset{-1}{f}(n)$ is finite for each $m \in N$. Show that for every sequence of elements $\{x_n\}$ of E, if one places $y_n = x_{f(n)}$, the elementary filters associated with $\{x_n\}$ and

$\{y_n\}$ are identical.

Let $\mathcal{F} = \{X \mid x_n \in X$ for all but a finite number of x_n's$\}$ and $\mathcal{F}' = \{Y \mid x_{f(n)} \in Y$ for all but a finite number of $x_{f(n)}$'s.$\}$ Let $X \in \mathcal{F}$. It is necessary to show $X = Y \in \mathcal{F}'$ for some Y. Consider $\complement X = \{x_{n_1}, x_{n_2}, \ldots, x_{n_j}\}$. Now for each $n \in N$, there is an $m \in N$ such that $n = f(m)$; $f^{-1}(m) = \{n_1', n_2', \ldots, n_p'\}$ which is a finite set. This means then that $f(n_p)_{p=1}^{n} = m$ for a finite number of $n \in N$. Then $\{x_{n_1}, x_{n_2}, \ldots, x_{n_j}\} = \{(x_{f(m_1)}, x_{f(m_2)}, x_{f(m_3)}, \ldots, x_{f(m_i)}), (x_{f(m_{21})}, \ldots, x_{f(m_{2j})}), \ldots, (x_{f(m_{j1})} \ldots x_{f(m_{jp})})\}$ where $x_{n_1} = (x_{f(m_1)}, x_{f(m_2)}, x_{f(m_3)}, \ldots, x_{f(m_i)})$, and so on for each $x_{n_j} \in \complement X$. Obviously $\{(x_{f(m_1)}, x_{f(m_2)}, \ldots, x_{f(m_i)}) \ldots\}$ is a finite set, since each element of the set is finite and there is only a finite number of them. Then $\{(x_{f(m_1)}, x_{f(m_2)}, \ldots, x_{f(m_i)}), (x_{f(m_{21})}, x_{f(m_{22})}, \ldots x_{f(m_{2j})}), \ldots\} = \complement Y$ for some $Y \in \mathcal{F}'$. Then, $\complement X = \complement Y$, so $X = Y$.

Now, let $Y \in \mathcal{F}'$ and show $Y = X \in \mathcal{F}$ for some X. Consider $\complement Y = \{x_{f(m_1)}, x_{f(m_2)}, \ldots, x_{f(m_p)}\}$. Now, for each $m \in N$, there is an $n \in N$ such that $f(m) = n$. Then $\complement Y$ can be written as $\{x_{n_1}, x_{n_2}, \ldots, x_{n_p}\}$ which is $\complement X$ for some $X \in \mathcal{F}$. Hence $Y = X \in \mathcal{F}$.

Therefore $\mathcal{F} = \mathcal{F}'$.

CHAPTER III

NETS

Definition 3.1 A set A is directed by a relation $\succ$ if $\succ$ is a binary relation on A with the properties:

(1) if a,b, and c are elements of A such that $a \succ b$ and $b \succ c$, then $a \succ c$.

(2) if a and b are elements of A, there exists an element c of A such that $c \succ a$ and $c \succ b$.

Definition 3.2 If f is a function which assigns to each element a of a directed set A a functional value f(a) in a set M, we shall call the function a "net" of elements of M.

Definition 3.3 Let f(a), a in A, be a net of real numbers, and let k be a real number. Then $\lim_{a \text{ in } A, \succ} f(a) = k$ means that for every positive ϵ there is an element a_ϵ of A such that $|f(a) - k| < \epsilon$ whenever $a \succ a_\epsilon$.

Definition 3.4 A filter base $\mathcal{B}$ is ultimately in a subset E of X if E contains some set from $\mathcal{B}$. If X is a topological space, $\mathcal{B}$ converges to an $x_0 \in X$ if it is ultimately in every neighborhood of x_0.

Definition 3.5 If $\mathcal{D}$ and $\mathcal{B}$ are two filter bases, we say that $\mathcal{D}$ is a refinement of $\mathcal{B}$ if every set in $\mathcal{B}$ contains some set in $\mathcal{D}$.

Definition 3.6 If E is a subset of X, the net $\mathcal{X}$ is ultimately in E if there is some index α_0 (depending on E) such that if $\alpha \geq \alpha_0$ then $x_\alpha \in E$. If X is a topological space, the net $\mathcal{X}$ converges to an element $x_0 \in X$ if $\mathcal{X}$ is ultimately in every neighborhood of x_0.

<u>Definition 3.7</u> Suppose we have a net $\mathcal{X} = \{x_\alpha\}_{\alpha \in A}$. A net $\mathcal{Y} = \{y_\beta\}_{\beta \in B}$ is said to be a subnet of $\mathcal{X}$ in case there is a mapping $\pi : B \to A$ with the properties:

(i) $y_\beta = x_{\pi(\beta)}$ for all $\beta \in B$;

(ii) given any $\alpha_0 \in A$, there is a $\beta_0 \in B$ such that if $\beta \geq \beta_0$, then $\pi(\beta) \geq \alpha_0$.

<u>Proposition 3.1</u> (a) If $\mathcal{X} = \{x_\alpha\}_{\alpha \in A}$ is a net in an abstract set X, and if $E(\alpha) = \{x_\lambda \mid \lambda \geq \alpha\}$, then the collection $\mathcal{B}(\mathcal{X}) = \{E(\alpha)\}$ is a filter base in X, called the filter base associated with the net $\mathcal{X}$.

(b) If the net $\mathcal{X}$ is ultimately in some set E, then $\mathcal{B}(\mathcal{X})$ is ultimately in E.

(c) If $\mathcal{Y} = \{y_\beta\}_{\beta \in B}$ is a subnet of $\mathcal{X}$ and if $\mathcal{B}(\mathcal{Y})$ is the filter base associated with $\mathcal{Y}$, then $\mathcal{B}(\mathcal{Y})$ is a refinement of $\mathcal{B}(\mathcal{X})$.

Let $E(\alpha_1)$ and $E(\alpha_2)$ be arbitrary sets in $\mathcal{B}(\mathcal{X})$. Since $A = \{\alpha\}$ is a directed set, there is an α_3 such that $\alpha_1 \leq \alpha_3$ and $\alpha_2 \leq \alpha_3$. Now $E(\alpha_3) = \{x_\lambda \mid \lambda \geq \alpha_3\}$; $E(\alpha_1) = \{x_\lambda \mid \lambda \geq \alpha_1\}$; $E(\alpha_2) = \{x_\lambda \mid \lambda \geq \alpha_2\}$.

Obviously $\{x_\lambda \mid \lambda \geq \alpha_3\} \subset \{x_\lambda \mid \lambda \geq \alpha_1\} \cap \{x_\lambda \mid \lambda \geq \alpha_2\}$, or, $E(\alpha_3) \subset E(\alpha_1) \cap E(\alpha_2)$. Therefore $E(\alpha)$ is a filter base, and (a) is proved.

To prove (b), we note that there exists an α_0 such that if $\alpha \geq \alpha_0$ then $x_\alpha \in E$. Consequently $E(\alpha_0) = \{x_\lambda \mid \lambda \geq \alpha_0\} \subset E$, and $\mathcal{B}(\mathcal{X})$ is ultimately in E.

To prove (c), let $E(\alpha_0) \in \mathcal{B}(\mathcal{X})$. By condition (ii) in the definition of a subnet, there exists a β_0 such that if $\beta \geq \beta_0$, then $\pi\beta \geq \alpha_0$. Since $F(\beta_0) = \{y_\beta \mid \beta \geq \beta_0\} = \{x_{\pi\beta} \mid \beta \geq \beta_0\}$ we conclude that $F(\beta_0) \subset E(\alpha_0)$. This shows that $\mathcal{B}(\mathcal{Y})$ is a refinement of $\mathcal{B}(\mathcal{X})$ and (c) is proved.

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