SOME PROPERTIES OF THE CANTOR SET

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SOME PROPERTIES OF THE CANTOR SET

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CHAPTER I

INTRODUCTION

The purpose of this paper is to explore some of the properties of the Cantor set and to extend the idea of this set to metric spaces, in general, and to other sets of real numbers and sets in N-dimensional Euclidean space, in particular. All of the well-known properties of the real numbers and the theorems of real analysis are used freely without proof in this paper.

Definition 1.1. The Cantor set is the set of real numbers contained in the closed interval from 0 to 1 whose ternary (base 3) expansion can be written without the use of the digit 1.

It is understood that whenever a number has two possible ternary expansions, as in the case of a fraction whose denominator is a power of 3, the number will belong to the Cantor set if either expansion contains only the digits 0 and 2.

Example 1.1. The number 1 can be written either as 1.000... or as 0.222...; the number 1/3 as 0.1000... or as 0.0222..., and so on. Thus 1, 1/3, 1/9, ..., 1/3^n, ... are elements of the Cantor set.
This example suggests another description of the Cantor set. It is the set obtained by removing from the closed interval from 0 to 1, the open middle third interval, \((1/3, 2/3)\), then removing the open middle thirds of the remaining two segments, and continuing this "middle third" process indefinitely.

Definition 1.2. The statement that the set \(M \subset \mathbb{R}_1\), the set of all real numbers, has outer length \(k\) means if \(C\) is a collection of open intervals covering \(M\), then the sum of the lengths of these intervals is greater than or equal to \(k\); and if \(\varepsilon > 0\), there exists a collection of disjoint open intervals covering \(M\) such that the sum of the lengths of these intervals is less than \(k + \varepsilon\).

Definition 1.3. Suppose \(M\) is a bounded subset of \(\mathbb{R}\), and \(I\) is the minimal interval containing all of \(M\). The inner length of \(M\) is the length of the interval \(I\) minus the outer length of the complement of \(M\) in \(I\).

Definition 1.4. The statement that the set \(M\) is measurable means the inner length of \(M\) is equal to the outer length of the set \(M\).

Definition 1.5. The statement that the set \(M\) has measure \(k\) means \(M\) is measurable, and the outer length of \(M\) is \(k\).

Theorem 1.1. The Cantor set has measure 0.

Proof: The complement of the Cantor set in \([0, 1]\) is the union of the open intervals \((1/3, 2/3), (1/9, 2/9), \ldots\)
(7/9, 8/9), (1/27, 2/27), ..., and has outer length $k$, where

$$k = \sum_{p=0}^{\infty} \frac{2^p}{3^{p+1}} = 1.$$

Let $\varepsilon > 0$. There exists a positive integer $n$ such that

$$\sum_{p=0}^{n} \frac{2^p}{3^{p+1}} > 1 - \varepsilon/2. $$

If $G$ denotes the open intervals of the complement of the Cantor set in $[0, 1]$ whose outer length is given by $\sum_{p=0}^{n} \frac{2^p}{3^{p+1}}$, and $G'$ denotes the complement of $G$ in $[0, 1]$, then $G'$ contains the Cantor set and is the union of $2^{n+1}$ closed intervals, each of length $1/3^{n+1}$. The length of $G' = (2/3)^{n+1} < \varepsilon/2$. If $g \in G'$, let $I_g$ be an open interval covering $g$, whose length is less than $1/3^{n+1} + \varepsilon/2^{n+2}$. Then $\bigcup_{g \in G'} I_g$ has length $2^{n-1}(1/3^{n+1} + \varepsilon/2^{n+2})$, which is equal to $(2/3)^{n+1} + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since for each $\varepsilon > 0$, there exists such a $G'$ containing the Cantor set, then the Cantor set has outer length 0.

The inner length of the Cantor set is 0. Hence the Cantor set is measurable and has measure 0. Henceforth the Cantor set will be denoted by $C$.

**Theorem 1.2.** $C$ contains no interval.

**Proof:** Suppose there exist points $a$ and $b$ such that $(a, b) \subset C$. There exists $\varepsilon > 0$ such that $\varepsilon < b - a$. Let $G$ be any collection of open intervals covering $C$. Since $C$ has measure 0, there exists $g \in G$ such that $\bigcup_{n \in G} I_n$ has length

$$k < \varepsilon.$$ 

But $(a, b) \subset C$; hence $b - a \leq k < \varepsilon < b - a$. This contradiction completes the proof.
Definition 1.6. The statement that $x$ is a limit point of the set $M$ means if $\varepsilon > 0$, there exists a point $y$, distinct from $x$, such that $y \in M$, and $|y - x| < \varepsilon$.

Definition 1.7. The statement that the set $M$ is closed means if $x$ is a limit point of $M$, then $x \in M$.

Definition 1.8. The statement that $M$ is a perfect set means $M$ is closed; and if $x \in M$, then $x$ is a limit point of $M$.

Theorem 1.3. $C$ is a perfect set.

Proof: Suppose there exists a limit point $x_0$ of $C$ such that $x_0 \notin C$. Then one of the following is true:

1. $x_0$ belongs to one of the open intervals removed in the construction of $C$, or
2. $x_0 < 0$, or
3. $x_0 > 1$.

Case (1). Suppose $x_0$ belongs to one of the open intervals removed in the construction of $C$. Let $I$ denote this interval. $I$ is an open set, so there exists $\varepsilon > 0$ such that if $0 < |x - x_0| < \varepsilon$, then $x \in I$, and thus $x \notin C$. Therefore $x_0$ is not a limit point of $C$.

Case (2). Suppose $x_0 < 0$. Then $-x_0 > 0$, and if $x$ is a point such that $|x - x_0| < -x_0$, then $x \notin C$ because $0$ is the greatest lower bound of $C$. Therefore $x_0$ is not a limit point of $C$.

The argument is similar if $x_0 > 1$. Therefore, if $x_0$ is a limit point of $C$, then $x_0 \in C$; hence $C$ is closed.
Suppose \( y \in C \), and \( \varepsilon > 0 \). Let \( 0.a_1a_2a_3\ldots \) denote the ternary expansion of \( y \). Since \( y \in C \), then \( a_i = 0 \) or \( 2 \) for \( i \geq 1 \). There exists a positive integer \( n \) such that \( \frac{1}{3^n} < \varepsilon \).

Then let \( x = 0.b_1b_2b_3\ldots \), where \( b_1 = a_1 \) for \( 1 \leq n \) and for \( i \geq n + 2; b_{n+1} = 0 \) if \( a_{n+1} = 2; \) and \( b_{n+1} = 2 \) if \( a_{n+1} = 0 \).

Then \( x \in C \), and \( |x - y| = \frac{2}{3^{n+1}} < \varepsilon \). Therefore \( y \) is a limit point of \( C \).

**Definition 1.9.** The statement that the set \( M \) is countable means there exists a reversible function \( f \) whose domain is \( M \) and whose range is a subset of the positive integers.

**Definition 1.10.** If each of \( A \) and \( B \) is a set, the statement that \( A \) and \( B \) have the same cardinal means there exists a reversible function \( f \) whose domain is \( A \) and whose range is \( B \).

**Definition 1.11.** Cardinals of real numbers:

(1). The cardinal of any finite set of real numbers is the number of elements in the set.

(2). The cardinal of the set of positive integers is denoted by \( \aleph_0 \) and is the smallest infinite cardinal.

(3). The cardinal of the set of real numbers, \( \mathbb{R}_1 \), is denoted by \( c \) and is equal to \( 2^{\aleph_0} \). The conjecture that \( c \) is the cardinal number next larger than \( \aleph_0 \) is known as the continuum hypothesis.

**Theorem 1.4.** \( C \) has cardinal \( c \).

**Proof:** Suppose \( x \in C \). Then \( x = 0.a_1a_2a_3\ldots \), where \( a_1 = 0 \) or \( 2 \) for \( i \geq 1 \). Let \( f(x) = 0.b_1b_2b_3\ldots \), where
Each element of the range of f is a binary representation of a number y ∈ [0, 1].

Suppose y ∈ [0, 1]. Then y has a binary representation, \(0.b_1b_2b_3\ldots\), where \(b_1 = 0\) or 1 for \(i \geq 1\). Let \(g(y) = 0.c_1c_2c_3\ldots\), where \(c_1 = 2b_1\) for \(i \geq 1\). Hence \(g(y) = x\) for some \(x \in C\).

Note: This is not a one-to-one correspondence, since \(f(1/3) = 0.0111\ldots = 0.1000\ldots\), and \(f(2/3) = 0.1000\ldots\). But this does show that \(C\) has cardinal \(k > c\). Since \(C \subseteq R_1\), then \(C\) has cardinal \(k \leq c\). These two equations complete the proof.

Definition 1.12. The statement that \(N\) is a neighborhood of the point \(x\) means there exists a number \(k > 0\) such that \((x - k, x + k) = N\).

Definition 1.13. The statement that the set \(K = S\) is nowhere dense in \(S\) means if \(x \in M\) and \(N\) is any neighborhood of \(x\), there exists a neighborhood \(P \subseteq N\) such that if \(y \in P\), then \(y \notin M\).

Theorem 1.5. \(C\) is nowhere dense in \(R_1\).

Proof: Suppose \(x \in C\) and \(N\) is a neighborhood of \(x\). Since \(x \in C\), then \(x\) is a limit point of \(C\). There exists a point \(y \notin x\) such that \(y \in N\) and \(y \in C\). Let \(0.a_1a_2a_3\ldots\) denote the ternary expansion of \(x\), and let \(0.b_1b_2b_3\ldots\) denote the ternary expansion of \(y\). Since \(y \notin x\), there exists a positive integer \(n\) such that \(a_n \neq b_n\). Let \(n\) denote the least such integer. For convenience, suppose \(x < y\). Then \(a_n = 0\), and \(b_n = 2\).
Suppose \( p \in \mathbb{R}^n \) such that if \( 0.p_1p_2p_3\ldots \) denoted the ternary expansion of \( p \), then \( p_i = a_i \) for \( 1 \leq n - 1 \); \( p_n = 1 \); and \( p_i = 0, 1, \) or \( 2 \) for \( i \geq n + 1 \). Let \( P \) denote the set of all such points \( p \), with the exception of the point \( p_0 \) whose ternary expansion is \( 0.a_1a_2a_3\ldots a_{n-1}l22\ldots \). \( P \subset \mathbb{N} \), and if \( p \in P \), then \( x < p < y \), and \( p \notin C \). Thus \( C \) is nowhere dense in \( \mathbb{R}^n \).

**Definition 1.14.** Suppose each of \( M \) and \( N \) is a set. The statement that \( M \) and \( N \) are mutually separated means:

1. \( M \) and \( N \) are disjoint; and
2. no point of \( M \) is a limit point of \( N \), and no point of \( N \) is a limit point of \( M \).

**Definition 1.15.** The statement that the set \( M \) is connected means if \( M_1 \) and \( M_2 \) are disjoint, non-empty subsets of \( M \) such that \( M_1 \cup M_2 = M \), then either

1. there exists a point of \( M_1 \) which is a limit point of \( M_2 \), or
2. there exists a point of \( M_2 \) which is a limit point of \( M_1 \).

**Definition 1.16.** The statement that the set \( M \) is totally disconnected means no nondegenerate subset of \( M \) is connected.

**Theorem 1.6.** \( C \) is totally disconnected.

**Proof:** Suppose there exists a nondegenerate subset \( K \) of \( C \) such that \( K \) is connected. \( K \) contains at least two points. Let \( a \) and \( b \) be two points of \( M \) such that \( a < b \).
Suppose \( K = \{a, b\} \). Let \( M_1 = \{a\} \), \( M_2 = \{b\} \). Then
\( M_1 \cup M_2 = M \), and \( M_1 \) and \( M_2 \) are mutually separated sets. Thus
\( M \) is not connected.

Suppose there exists a point \( c \in C - M \) such that
\( a < c < b \). Let \( M_1 = M \cap [0, c) \), and let \( M_2 = M \cap (c, 1] \). \( M_1 \)
and \( M_2 \) are disjoint, non-empty sets, and \( M_1 \cup M_2 = M \).

Obviously, \( 0 \) is not a limit point of \( M_2 \). Let \( x \in M_1 \)
such that \( x < 0 \). Then \( x \) is an interior point of \( (0, c) \).
There exists a neighborhood \( N \) of \( x \) such that \( N \subseteq M_1 \), and thus
\( x \) is not a limit point of \( M_2 \).

Similarly, no point of \( M_2 \) is a limit point of \( M_1 \). Thus
\( M_1 \) and \( M_2 \) are mutually separated sets, and \( M \) is not a connected set.

Therefore, if \( M \) is connected, then \( [a, b] \subseteq M \subseteq C \). But \( C \)
contains no interval, and the theorem is proved.

**Definition 1.17.** The addition set of the set \( M \) is
\( \{(a + b) \mid a \in M \text{ and } b \in M\} \), and will be denoted by \( A(M) \).

**Definition 1.18.** The difference set of the set \( M \) is
\( \{(a - b) \mid a \in M \text{ and } b \in M\} \), and will be denoted by \( D(C) \).

**Theorem 1.7.** \( A(C) = [0, 2] \).

**Proof:** Let \( x \in [0, 2] \), and let \( x_0.x_1x_2x_3\ldots \) denote the
ternary expansion of \( x \). The proof of this theorem lies in the
construction of two points \( a, b \in C \) such that if the ternary expansions of \( a \) and \( b \) are denoted by \( 0.a_1a_2a_3\ldots \) and
\( 0.b_1b_2b_3\ldots \), respectively, then \( a + b = x \). The numbers \( a \) and
\( b \) will be constructed one digit at a time, from left to right,
with appropriate provisions being made for carrying when necessary.

If $x = 2$, let $a = b = 1$. Since $1 \in C$, then $2 \in A(C)$. Suppose $x < 2$. Consider first $x_0$. It is either $0$ or $1$. If $x_0 = 0$, let $a_0 = b_0 = 0$, and arrange, according to the directions below, $a_1$ and $b_1$ so that nothing will be carried over from the addition. If $x_0 = 1$, let $a_0 = b_0 = 0$, and arrange $a_1$ and $b_1$ so that $1$ will be carried over from the addition. Now, for $i \geq 1$, $x_i = 0$, $1$, or $2$.

Case (1). Suppose $x_i = 0$. If nothing was carried over to add to $a_{i-1} + b_{i-1}$, let $a_i = b_i = 0$. Arrange $a_{i+1}$ and $b_{i+1}$ so that nothing will be carried over to add to $a_i + b_i$.

If $1$ was carried over to add to $a_{i-1} + b_{i-1}$, let $a_i = 2$, $b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that $1$ is carried over to add to $a_i + b_i$.

Case (2). Suppose $x_i = 1$. If nothing was carried over to add to $a_{i-1} + b_{i-1}$, let $a_i = b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that $1$ is carried over to add to $a_i + b_i$.

If $1$ was carried over to add to $a_{i-1} + b_{i-1}$, let $a_i = b_i = 2$, and arrange $a_{i+1}$ and $b_{i+1}$ so that nothing is carried over to add to $a_i + b_i$.

Case (3). Suppose $x_i = 2$. If nothing was carried over to add to $a_{i-1} + b_{i-1}$, let $a_i = 2$, $b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that nothing will be carried over to add to $a_i + b_i$. 
If 1 was carried over to add to $a_{i-1} + b_{i-1}$, let $a_i = b_i = 2$, and arrange $a_{i+1}$ and $b_{i+1}$ so that 1 will be carried over to add to $a_i + b_i$.

Since $a, b \in C$, and $x = a + b$, then $x \in A(C)$.

**Theorem 1.8.** $D(C) = [-1, 1]$.

**Proof:** Suppose $x \in [0, 1]$. If $x = 1$, then $x \in D(C)$, since $1, 0 \in C$ and $1 - 0 = 1$. Let $x_0x_1x_2x_3\ldots$ denote the ternary expansion of $x$ for $x < 1$. As in Theorem 1.7., the proof depends upon the construction of two numbers $a, b \in C$ such that $a - b = x$. In this case, provisions are made for borrowing when necessary.

Since $x \in [0, 1)$, then $x_0 = 0$. Let $a_0 = b_0 = 0$, and arrange $a_1$ and $b_1$, according to the directions below, so that nothing will be borrowed from $a_0$.

For $i \geq 1$, $x_i = 0, 1, \text{ or } 2$.

**Case (1).** Suppose $x_1 = 0$. If nothing was borrowed from $a_{i-1}$, let $a_i = b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that nothing will be borrowed from $a_i$.

If 1 was borrowed from $a_{i-1}$, let $a_i = 0, b_i = 2$, and arrange $a_{i+1}$ and $b_{i+1}$ so that 1 will be borrowed from $a_i$.

**Case (2).** Suppose $x_1 = 1$. If nothing was borrowed from $a_{i-1}$, let $a_i = 2, b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that 1 will be borrowed from $a_i$.

If 1 was borrowed from $a_{i-1}$, let $a_i = b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that nothing will be borrowed from $a_i$. 
Case (3). Suppose $x_i = 2$. If nothing was borrowed from $a_{i-1}$, let $a_i = 2$, $b_i = 0$, and arrange $a_{i+1}$ and $b_{i+1}$ so that nothing will be borrowed from $a_i$.

If 1 was borrowed from $a_{i-1}$, let $a_i = b_i = 2$, and arrange $a_{i+1}$ and $b_{i+1}$ so that 1 will be borrowed from $a_i$.

Since $a_i$, $b_i \in C$, and $x = a_i - b_i$, then if $x \in [0, 1]$, then $x \in D(C)$. Suppose $y \in [-1, 0]$. There exists $x \in D(C)$ such that $y = -x$. Since $x = a_i - b_i$ for some $a_i, b_i \in C$, then $y = b_i - a_i$. Hence $y \in D(C)$, and $D(C) = [-1, 1]$.

Definition 1.19. The statement that the function $f$ is increasing at the point $x$ means there exists a neighborhood $N$ of $x$ such that if $y \in N$, then $f(y) < f(x)$ if $y < x$, and $f(y) > f(x)$ if $y > x$.

Theorem 1.9. If $f$ is a non-decreasing function continuous on $I = [a, b]$, and $f(a) < f(b)$, then there exists a point $x$ such that $a < x < b$, and $f$ is increasing at $x$.

Proof: Assume there is no such point. Then if $a < x < b$, $x$ belongs to some closed interval $J \subseteq I$ such that $f$ is constant over $J$.

Let $I_1 = \{x \in I \mid f(x) = f(a)\}$, and let $I_2 = \{x \in I \mid f(x) = f(b)\}$. Since $f$ is non-decreasing, then $I_1$ and $I_2$ are closed and bounded. Let $a_1$ denote the least upper bound of $I_1$, and let $b_1$ denote the greatest lower bound of $I_2$. Since $I_1$ and $I_2$ are closed sets, then $a_1 \in I_1$ and $b_1 \in I_2$. Furthermore, $f(a_1) = f(a) < f(b) = f(b_1)$. Hence $a_1 \neq b_1$. 

Let \( M_0 = (a_1, b_1) \).

Let \( I' = \{ J \subseteq M_0 \mid J \text{ is a maximal closed interval for which } f \text{ is constant over } J \} \).

There exists a positive integer \( n_1 \) such that for some \( J \in I' \), \( J \) has length \( \geq 1/n_1 \).

Let \( N_1 = M_0 - \bigcup_{J \in I'} J \mid J \text{ has length } \geq 1/n_1 \). \( N_1 \subseteq M_0 \), and \( N_1 \) is the union of at least two open intervals.

**Notation:** Let the symbol \( \overline{M} \) denote the closure of the set \( M \).

Suppose \( a_1 \not\in \overline{N}_1 \). \( \overline{N}_0 = [a_1, b_1] \), and \( a_1 \in \overline{N}_0 \). Thus there exists a positive integer \( i \) such that \( a_1 \in J_1 \subseteq \overline{N}_0 \), and \( f \) is constant over \( J_1 \). Furthermore, there exists \( x_0 \in M_0 \) such that \( f(x_0) = f(a_1) = f(a) \); hence \( x_0 \in I_1 \). But \( x_0 > a_1 \), and this contradicts the fact that \( a_1 \) is the least upper bound of \( I_1 \). Therefore \( a_1 \not\in \overline{N}_1 \).

Let \( N_1 \) denote the maximal open interval of \( N_1 \) having \( a_1 \) as its left end point, and let \( b_2 \) denote the right end point of \( N_1 \). \( N_1 = (a_1, b_2) \), and \( M_1 \subseteq N_1 \subseteq M_0 \). Thus \( b_2 \not= b_1 \).

There exists a positive integer \( n_2 > n_1 \) such that for some \( J \subseteq N_1 \), \( J \) has length \( \geq 1/n_2 \).

Let \( N_2 = M_1 - \bigcup_{J \subseteq M_1} J \mid J \text{ has length } \geq 1/n_2 \). Then \( N_2 \subseteq M_1 \), and \( N_2 \) is the union of at least two open intervals. By the same argument used to show that \( a_1 \not\in \overline{N}_1 \), it can be shown that \( b_2 \not\in \overline{N}_2 \).
Let $M_2$ denote the maximal open interval of $N_2$ having $b_2$ as its right end point, and let $a_2$ denote its left end point. $M_2 = (a_2, b_2)$, and $M_2 \subseteq M_1 \subseteq M_0$. Therefore $a_2 \neq a_1$.

There exists a positive integer $n_2 > n_1$ such that for some $J \subset M_2$, $J$ has length $\geq 1/n_2$.

Let $N_3 = M_2 - \bigcup_{J \subset M_2, J \text{ has length } \geq 1/n_2} J$. $N_3 \subset M_2$, and $N_3$ is the union of at least two open intervals. Let $M_3 = (a_3, b_3)$, where $b_3$ is the right end point chosen in the same manner as $b_2$.

There exists a positive integer $n_3 > n_2$ such that for some $J \subset M_3$, $J$ has length $\geq 1/n_3$.

Let $N_4 = M_3 - \bigcup_{J \subset M_3, J \text{ has length } \geq 1/n_3} J$. Let $M_4 = (a_4, b_4)$, where $a_4$ is chosen in the same manner as $a_2$.

Continue in this fashion, alternating left and right end points.

Consider $\left\{M_{2p}\right\}_{p=0}^{\infty}$. This is a nest of closed, bounded sets such that $M_{2p+2} \subseteq M_{2p}$. $M_{2p} = [a_{p+1}, b_{p+1}]$, and if $i > j$, then $a_i > a_j$; $b_i < b_j$; hence $f(a_i) > f(a_j)$, and $f(b_i) < f(b_j)$. Furthermore, there exists a point $x$ such that $x \in M_{2p}$ for $p \geq 0$.

Consider the sequences $\left\{a_{p+1}\right\}_{p=0}^{\infty}$ and $\left\{b_{p+1}\right\}_{p=0}^{\infty}$. Now, $\left\{a_{p+1}\right\}_{p=0}^{\infty}$ is a strictly increasing sequence converging to $x$, and $\left\{b_{p+1}\right\}_{p=0}^{\infty}$ is a strictly decreasing sequence converging to $x$. Similarly, $\left\{f(a_{p+1})\right\}_{p=0}^{\infty}$ and $\left\{f(b_{p+1})\right\}_{p=0}^{\infty}$ are strictly increasing and decreasing, respectively. Furthermore,
\( \{ f(a_{p+1}) \}_{p=0}^{\infty} \) and \( \{ f(b_{p+1}) \}_{p=0}^{\infty} \) both converge to \( f(x) \). Hence \( x \in \mathbb{M}_{2p} \), for \( p \geq 0 \).

Let \( y_0 < x \). There exists \( a_n \in \{ a_{p+1} \}_{p=0}^{\infty} \) such that \( y_0 < a_n < x \). Then \( f(y_0) \leq f(a_n) < f(a_{n+1}) < f(x) \).

Let \( y_1 > x \). There exists \( b_n \in \{ b_{p+1} \}_{p=0}^{\infty} \) such that \( x < b_n < y_1 \), and \( f(y_1) \geq f(b_n) > f(b_{n+1}) > f(x) \).

Thus if \( J_1 \subseteq \mathbb{M}_0 \) such that \( f \) is constant over \( J_1 \), then \( x \notin J_1 \). If \( y < x \), then \( f(y) < f(x) \); and if \( y > x \), then \( f(y) > f(x) \). Hence \( f \) is increasing at \( x \), and the theorem is proved.

One might be led to believe that for such a well-behaved function as the one described in Theorem 1.9., there would be at least one interval over which the function is strictly increasing. The example below shows that this is not the case.

**Example 1.2.** A non-decreasing function \( f \), continuous on \([0, 1]\), such that \( f \) is increasing only at the points of the Cantor set, \( C \), which are not end points of the open intervals removed in the construction of \( C \).

Let \( I_1 = [1/3, 2/3] \). If \( x \in I_1 \), let \( f(x) = 1/2 \).

Let \( I_2 = [1/9, 2/9] \). If \( x \in I_2 \), let \( f(x) = 1/4 \).

Let \( I_3 = [7/9, 8/9] \). If \( x \in I_3 \), let \( f(x) = 3/4 \).

Let \( I_4 = [1/27, 2/27] \). If \( x \in I_4 \), let \( f(x) = 1/8 \).

Continue in this fashion for the closure of all open intervals removed in the construction of \( C \).
Let $I = \bigcup_{p \geq 1} I_p$, and let $M = [0, 1] - I$. $M \subset C$, and $M$ is not empty since $C$ is uncountable. Since $C$ is nowhere dense in $[0, 1]$, then if $x \in M$, $x$ is a limit point of $I$. There exists a strictly increasing sequence $\{c_p\}_{p=0}^\infty$ such that $c_i \in I$ for $i \geq 0$, and $\{c_p\}_{p=0}^\infty$ converges to $x$. Now, let $f(x) = \lim_{p \to \infty} \{f(c_p)\}_{p=0}^\infty$. This limit exists because $\{f(c_p)\}_{p=0}^\infty$ is a non-decreasing sequence which is bounded above. Also, the limit is independent of the choice of the sequence $\{c_p\}_{p=0}^\infty$.

Let $\varepsilon > 0$. Let $n$ denote the least positive integer such that $1/2^n < \varepsilon$. If $x_1, x_2 \in [0, 1]$ and $|x_1 - x_2| < 1/2^n$, then $|f(x_1) - f(x_2)| < 1/2^n < \varepsilon$. Hence $f$ is continuous over $[0, 1]$.

If $x \in M$, and $y_1 < x$, then $f(y_1) < f(x)$. If $x \in M$, and $y_2 > x$, then $f(y_2) > f(x)$. Hence $f$ is increasing at $x$.

The set $M$ of this example is the smallest set over which a non-decreasing function, continuous on a closed interval, is increasing.
CHAPTER II

SETS HOMEOMORPHIC TO THE CANTOR SET

Definition 2.1. A topology is a collection $\mathcal{J}$ of sets such that:

1. If $T \subseteq \mathcal{J}$, then $\bigcup_{t \in T} t \in \mathcal{J}$.
2. If $T$ is a finite subset of $\mathcal{J}$, then $\bigcap_{t \in T} t \in \mathcal{J}$.
3. $\emptyset$ (the null set) $\in \mathcal{J}$.

Definition 2.2. $\bigcup_{T \in \mathcal{J}} T$ is called the space of the topology and is denoted by $X$. The ordered pair $(X, \mathcal{J})$ is called a topological space.

Definition 2.3. Suppose $X$ is a non-empty set. A metric for $X$ is a function $d$ on $X \times X$ into $\mathbb{R}_+$ such that:

1. If $x, y \in X$, then $d(x, y) \geq 0$.
2. If $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$.
3. If $x, y \in X$, then $d(x, y) = d(y, x)$.
4. If $x, y, z \in X$, then $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 2.4. Suppose $X$ is a set, and $d$ is a metric for $X$. If $p \in X$ and $r \in \mathbb{R}_+$, then $\{x \in X \mid d(p, x) < r\}$ is called an open sphere with center $p$ and radius $r$.

Definition 2.5. If $X$ is a set, $d$ is a metric for $X$, $G$ is the set of all open spheres, and $Q$ is the collection of all unions of subsets of $G$, then $(X, Q)$ is called a metric space, and will be denoted by $(X, d)$.
From these definitions, it follows that every metric space is a topological space. The set \( \mathbb{R} \), with the usual concept of distance and the natural topology of open intervals, is a metric space, and the definitions in Chapter I will be extended to this broader sense when appropriate.

**Definition 2.6.** If each of \((X, \mathcal{T})\) and \((Y, \mathcal{A})\) is a topological space and \(f\) is a function on \(X\) into \(Y\), then if \(p \in X\), the statement that \(f\) is continuous at \(p\) means if \(d\) is a neighborhood of \(f(p)\), there exists a neighborhood \(n\) of \(p\) such that if \(x \in n\), then \(f(x) \in d\).

**Definition 2.7.** If \(f\) is a function on \(X\) into \(Y\), the statement that \(f\) is continuous means if \(p \in X\), then \(f\) is continuous at \(p\).

**Definition 2.8.** The statement that the set \(M\) is homeomorphic to the set \(P\) means there exists a reversible function \(f\) on \(M\) onto \(P\) such that both \(f\) and \(f^{-1}\) are continuous.

**Definition 2.9.** A relation \(R\) is antisymmetric if and only if, if \((a,b) \in R\) and \((b,a) \in R\), then \(a = b\).

**Definition 2.10.** A linear ordering is an antisymmetric transitive relation \(R\) such that if \(D_R\) and \(R_R\) denote the domain and range of \(R\), respectively, and if \(a, b \in D_R \cup R_R\), then \((a,b) \in R\) or \((b,a) \in R\).

**Definition 2.11.** A well-ordering is a linear ordering \(R\), such that if \(X = D_R \cup R_R\), then there is an element \(a \in X\) such that if \(x \in X\) and \(x \neq a\), then \((a,x) \in R\).
Theorem 2.1. Suppose \((X, d)\) is a metric space, and \(M\) is a set in \(X\). \(M\) is homeomorphic to the Cantor set, \(C\), if and only if \(M\) is a perfect, totally disconnected subset of an open sphere \(G \subseteq X\).

Proof: Suppose \(M\) is a perfect, totally disconnected subset of an open sphere \(G \subseteq X\). Let \(W\) be a well-ordering of \(M\), and let \(x\) denote the first element of \(M\) according to \(W\).

Since \(M\) is totally disconnected, no nondegenerate subset of \(M\) is connected. Thus \(M\) is the union of a countable collection of totally disconnected sets, and there exist two non-empty, mutually separated perfect sets, \(S_0\) and \(S_2\), such that \(S_0 \cup S_2 = M\).

Either \(x \in S_0\), or \(x \in S_2\). No loss of generality occurs by assuming \(x \in S_0\). \(S_0\) is perfect, totally disconnected, and there exists a positive integer \(k\) such that if \(y \in S_0\) and \(y \neq x\), then \(d(x, y) \leq k\). There exists a perfect, totally disconnected proper subset, \(S_{00}\) of \(S_0\), such that \(x \in S_{00}\), and if \(y \in S_{00}\), then \(d(x, y) \leq k/2\). Let \(S_{02} = S_0 - S_{00}\). Let \(x_1 \in S_{02}\). Since \(S_{00}\) and \(S_{02}\) are mutually separated sets, then \(x_1 \notin S_{00}\).

\(S_{00}\) is bounded, perfect, and totally disconnected. There exists a perfect, totally disconnected proper subset, \(S_{000}\) of \(S_{00}\), such that \(x \in S_{000}\), and if \(y \in S_{000}\), then \(d(x, y) \leq k/4\). Let \(S_{002} = S_{00} - S_{000}\). \(S_{000}\) and \(S_{002}\) are mutually separated sets. Let \(x_2 \in S_{002}\).
Let the symbol $\alpha_{i1}$ denote the subscript whose first $i$ terms are 0. The symbol $\alpha_{i2}$ will mean the first $i$ terms are 0 and the next term is 2.

Continue in the above fashion, each time selecting a perfect proper subset, $S_{\alpha_{i1}}$ of $S_{\alpha_{i-1}}$, such that $x \in S_{\alpha_{i1}}$ and if $y \in S_{\alpha_{i1}}$, then $d(x, y) \leq k/2^i$. Define $S_{\alpha_{i-1}}^{2}$ as $S_{\alpha_{i-1}} - S_{\alpha_{i1}}$ and let $x_{i-1} \in S_{\alpha_{i-1}}^{2}$, so that $x_{i-1} \in S_{\alpha_{i-1}}^{2}$, but $x_{i-1} \notin S_{\alpha_{i1}}$, for $i \geq 1$.

Let $\varepsilon > 0$. There exists a positive integer $n$ such that $k/2^n < \varepsilon$. Suppose $m > n$. Then $x_m \in S_{\alpha_n}$ and $x \in S_{\alpha_n}$, and $d(x_m, x) \leq k/2^n < \varepsilon$. Hence $\{x_p\}_1^\infty$ converges to $x$.

Since for $i \geq 2$, $S_{\alpha_{i1}} \subset S_{\alpha_{i-1}}$, and $S_{\alpha_{i-1}}$ is closed and bounded, there exists a point $x'$ such that $x' \in S_{\alpha_{i1}}$ for $i \geq 1$. By the nature of the construction, $x$ is one such point.

Suppose $x' \neq x$. $d(x', x) = \lambda > 0$. There exists a positive integer $L$ such that $k/2^L < \lambda$, and if $L' > L$ and $y \in S_{\alpha_{L'}}$, then $d(x, y) < k/2^L$. Now, $x' \in S_{\alpha_{L}}$ and $x' \neq x$. Therefore, $\lambda = d(x', x) < k/2^L < \lambda$. This gives the desired contradiction. Hence $x$ is the only such point.

Let $S_{000...} = x$, and let $f(x) = 0.000... \in C$.

Suppose $m \in M$ such that if $m'$ precedes $m$ in $W$, then $m'$ has been mapped into an element of $C$ in the same manner as the point $x$. 

Let $\beta_1$ denote the first $i$ terms of an infinite sequence made up of the digits 0 and 2, and let $S_{\beta_1}$ denote the set containing the point $p$ at the $i$th step of the construction.

Either $m \in S_Q$, or $m \in S_2$. There is a least positive integer $j$ such that $m \in S_{\beta_j}^m$ and if $m'$ is any element of $M$ which precedes $m$ in $W$, then $m' \notin S_{\beta_j}^m$. $S_{\beta_j}^m$ has the property that if $y \in S_{\beta_j}^m$, there exists $m' \in M$ which precedes $m$ in $W$, such that $m' \in S_{\beta_j}^m$ and $d(m', m) < k/2^{j-1}$.

Let $S_{\beta_j}^m$ be a perfect, totally disconnected subset of $S_{\beta_j}$ such that if $p \in S_{\beta_j}^m$, then $d(p, m) < k/2^{j+1}$. Let $m_{j-1} \in S_{\beta_{j-1}}^m$ such that $m_{j-1} \notin S_{\beta_j}^m$. Let $S_{\gamma_j} = S_{\beta_{j-1}}^m - S_{\beta_j}^m$, where $\gamma_{j-1} = \beta_{j-1}$, and

$$\gamma_j = \begin{cases} \gamma_{j-1}^0 & \text{if } \beta_j = \beta_{j-1}^2 \\ \gamma_{j-1}^2 & \text{if } \beta_j = \beta_{j-1}^0. \end{cases}$$

Proceed as with $x$.

Let $S_{\beta_m} = \{m\}$, where $\beta_m$ denotes the infinite sequence created by the $\beta_1$'s. Let $f(m)$ be the number whose ternary expansion corresponds to $\beta_m$, and let $\beta_m$ denote this number. Then $\beta_m \in C$.

This process defines a function $f$ on $M$ onto $C$. If $m \in M$, then $f(m) = \beta_m$, and $f(m') = f(m)$ if and only if $m' = m$. Hence $f$ is reversible. Now, if $f$ and $f^{-1}$ are both continuous, then $M$ is homeomorphic to $C$. 
Let \( \alpha_1 \sigma_{i,p} \) denote the infinite sequence whose first \( i \) terms are 0, and thereafter are identical to the terms following the \( i \)th term of \( \beta_p \).

Let \( \varepsilon > 0 \), and let \( m_1 \in M \). There exists a positive integer \( r \) such that \( 1/3^r < \varepsilon \). Suppose \( m_2 \in M \) such that \( d(m_1, m_2) < k/2^p \). Then \( m_1, m_2 \in S_{\beta_r} \) and

\[
|f(m_1) - f(m_2)| = |\beta_{m_1} - \beta_{m_2}|
= |\alpha_r \sigma_{r_{m_1}} - \alpha_r \sigma_{r_{m_2}}|
= |\alpha_r \sigma_{r_{m_1}} - \alpha_r \sigma_{r_{m_2}}|
< 1/3^r < \varepsilon.
\]

Hence if \( m_1 \) is any element of \( M \), then \( f \) is continuous at \( m_1 \); and therefore \( f \) is continuous.

Let \( \varepsilon' > 0 \). There exists a positive integer \( p \) such that \( k/2^p < \varepsilon' \). Then if \( c_1, c_2 \in C \) such that \( |c_1 - c_2| < 1/3^p \), then \( |f^{-1}(c_1) - f^{-1}(c_2)| = d(x_1, x_2) \) for some \( x_1, x_2 \in M \).

Since \( |c_1 - c_2| < 1/3^p \), the first \( p \) terms of \( c_1 \) and \( c_2 \) are identical. Let \( \xi_p \) denote these terms. Then \( x_1, x_2 \in S_{\xi_p} \), and \( d(x_1, x_2) \leq k/2^p < \varepsilon' \). Hence \( f^{-1} \) is continuous, and \( M \) is homeomorphic to \( C \).

Conversely, suppose \( M \) is homeomorphic to \( C \). There exists a reversibly continuous function \( f \) on \( C \) onto \( M \). Since \( f \) is continuous on \( C \), and \( C \) is closed and bounded, then \( M \) is closed and bounded.

Suppose there exists a point \( x \in M \) such that \( x \) is not a limit point of \( M \). Then there exists \( \varepsilon > 0 \) such that if \( x' \in M \)
and \( x' \neq x \), then \( d(x, x') \geq \varepsilon \). Furthermore, there exists a point \( c \in C \) such that \( f(c) = x \). Since \( C \) is a perfect set, then \( c \) is a limit point of \( C \).

Let \( \delta > 0 \). There exists \( c' \in C \) such that \( d(c, c') < \delta \), and there exists \( x' \in M \) such that \( f(c') = x' \). Then \( d(f(c), f(c')) = d(x, x') \geq \varepsilon \). Hence \( f \) is not continuous at \( c \), which is the desired contradiction.

Thus if \( x \in M \), then \( x \) is a limit point of \( M \), and since \( M \) is also a closed set, then \( M \) is a perfect set.

Suppose there exists a nondegenerate subset \( S \) of \( M \) such that \( S \) is a connected set. Then if \( S_1 \) and \( S_2 \) are disjoint subsets of \( S \) such that \( S_1 \cup S_2 = S \), there exists a point of \( S_1 \) which is a limit point of \( S_2 \) or a point of \( S_2 \) which is a limit point of \( S_1 \).

Suppose \( x_1 \in S_1 \) is a limit point of \( S_2 \). Let

\[
C_1 = \{ c \in C \mid f(c) \in S_1 \} \\
C_2 = \{ c \in C \mid f(c) \in S_2 \}.
\]

\( C_1 \) and \( C_2 \) are disjoint, and there exists \( c_1 \in C_1 \) such that \( f(c_1) = x_1 \).

Since \( M \) is homeomorphic to \( C \), then \( f^{-1} \) is continuous at \( x_1 \). Let \( \varepsilon > 0 \). There exists \( \delta > 0 \) such that if \( y \in S_2 \) and \( d(x_1, y) < \delta \), then \( d(f^{-1}(x_1), f^{-1}(y)) < \varepsilon \). Let \( \{ y_p \}_{p=0}^{\infty} \) be a sequence in \( S_2 \) which converges to \( x_1 \). By the continuity property \( \{ f^{-1}(y_p) \}_{p=0}^{\infty} \) converges to \( f^{-1}(x_1) = c_1 \). But for \( i \geq 0 \), \( f^{-1}(y_i) \in C_2 \); hence \( c_1 \) is a limit point of \( C_2 \). Thus \( C_1 \cup C_2 \) is a connected set, which is a contradiction.
Hence if $M$ is homeomorphic to $C$, then $M$ is bounded (a subset of some open sphere $G\subseteq X$), perfect, and totally disconnected. This completes the proof.

**Theorem 2.2.** If $M$ is a bounded, closed set in $\mathbb{R}^1$, then there exists a continuous function $f$ on $C$ onto $M$.

**Proof:** Suppose $M$ is a bounded, closed set in $\mathbb{R}^1$. There exists a minimal closed interval $I$ such that $M \subseteq I$.

Case (1). Suppose $M = I$. $I$ is homeomorphic to $[0, 1]$, and there exists a reversible function $g$ on $[0, 1]$ onto $I$.

Let $c \in C$, and let $0.c_1c_2c_3\ldots$ denote the ternary expansion of $c$. Let $h(c) = 0.b_1b_2b_3\ldots$, where $b_i = c_i/2$ for $i \geq 1$. Since $c_1 = 0$ or 2, then $b_1 = 0$ or 1 for $i \geq 1$. Thus the range of $h$ is $[0, 1]$.

Let $\varepsilon > 0$. There exists a positive integer $n$ such that $1/2^n < \varepsilon$. Let $\delta = 1/3^n$. If $c_1 \in C$ such that $|c_1 - c| < \delta$, then $|h(c_1) - h(c)| = |0.b_1b_2b_3\ldots - 0.b_1b_2b_3\ldots|$. There exists a positive integer $m > n$ such that $b_{i1} = b_i$ for $0 < i \leq m$; and hence $|h(c_1) - h(c)| < 1/2^n < \varepsilon$.

Thus $h$ is continuous on $C$, and the range of $h$ is $[0, 1]$. Let $f(c) = g(h(c))$. Since both $g$ and $h$ are continuous, then $f$ is continuous on $C$, and the range of $f$ is $I$.

Case (2). Suppose $M \subseteq I$. $M$ is closed; hence $I - M$ is open. Let $\{U_p\}_{p=0}^\infty$ be a sequence of maximal disjoint open intervals whose union is $I - M$. Let $a$ denote the left end point of $I$; let $b$ denote the right end point of $I$; and let $a_0$ and $b_0$ denote the left and right end points of $U_0$. 
Either \( a_0 = a \), or \( a_0 > a \). If \( a_0 = a \), let
\[ S_1 = \mathcal{C} \cap [0, 1/3]. \]
If \( x \in S_1 \), let \( f(x) = a \). If \( a_0 > a \), let
\[ f(0) = a \text{ and } f(1/3) = a_0. \]

Similarly, either \( b_0 = b \), or \( b_0 < b \). If \( b_0 = b \), let
\[ S_2 = \mathcal{C} \cap [2/3, 1]. \]
If \( x \in S_2 \), let \( f(x) = b \). If \( b_0 < b \), let
\[ f(2/3) = b_0 \text{ and } f(1) = b. \]

If \( a_0 \neq a \), let \( P_1 = (I - M) \cap [a, a_0] \). Either \( P_1 = \emptyset \), or there exists a least positive integer \( n_1 \) such that
\[ U_{n_1} \subseteq P_1. \]
If \( P_1 = \emptyset \), then use the method described in Case (1) to map \( \mathcal{C} \cap [0, 1/3] \) onto \([a, a_0]\). If \( P_1 \neq \emptyset \), then let \( a_1 \) and \( b_1 \) denote the left and right end points, respectively, of \( U_{n_1} \). Then \( a \leq a_1 < b_1 \leq a_0 \). If \( a = a_1 \), let
\[ S_3 = \mathcal{C} \cap [0, 1/9]. \]
If \( x \in S_3 \), let \( f(x) = a \). If \( a < a_1 \), let
\[ f(1/9) = a_1. \]
If \( b_1 = a_0 \), let \( S_4 = \mathcal{C} \cap [2/9, 1/3]. \) If \( x \in S_4 \), let
\[ f(x) = a_0. \]
If \( b_1 < a_0 \), let \( f(2/9) = b_1 \).

Similarly, if \( b_0 \neq b \), let \( P_2 = (I - M) \cap [b_0, b] \). If \( P_2 = \emptyset \), use the method described in Case (1) to map
\( \mathcal{C} \cap [2/3, 1] \) onto \([b_0, b]\). If \( P_2 \neq \emptyset \), there exists a least positive integer \( n_2 \) such that \( U_{n_2} \subseteq P_2 \). Let \( a_2 \) and \( b_2 \) denote the left and right end points, respectively, of \( U_{n_2} \).
Then \( b_0 \leq a_2 < b_2 \leq b \). If \( b_0 = a_2 \), let \( S_5 = \mathcal{C} \cap [2/3, 7/9] \). If \( x \in S_5 \), let \( f(x) = b_0 \). If \( b_0 < a_2 \), let \( f(7/9) = a_2 \). If \( b_2 = b \), let \( S_6 = \mathcal{C} \cap [8/9, 1] \). If \( x \in S_6 \), let \( f(x) = b \). If \( b_2 < b \), let \( f(8/9) = b_2 \).
If \( a_1 \neq a \), let \( P_2 = (a - M) \cap [a, a_1] \), and proceed in this manner indefinitely. This process defines a function \( f \) on \( C \) having the property that if \( c_1, c_2 \in C \), and \( c_1 < c_2 \), then \( f(c_1) \leq f(c_2) \).

Now, suppose \( m \in M \). Then one and only one of the following statements is true of \( m \):

(1). If \( N \) is a neighborhood of \( m \), then there exists a point \( m' \in M \) such that \( m' \neq m \), and \( m' \in N(m) \); hence \( m \) is a limit point of \( M \).

(2). There exists a neighborhood \( N \) of \( m \) such that \( N(m) \cap M = \{m\} \), in which case \( m \) will be referred to as an isolated point of \( M \).

Suppose \( m \) is an isolated point of \( M \). Then for some \( i \) and \( j \), \( U_i \) and \( U_j \) have \( m \) as a common end point; or \( m = a \) or \( m = b \), and for some \( k \), \( U_k \) has \( m \) as a left or right end point. There exists a positive integer \( n \) such that \( S_n \subset C \); and if \( x \in S_n \), then \( f(x) = m \). Furthermore, there exists a positive integer \( k \) such that \( 2^k - 2 < n \leq 2^{k+1} - 2 \). \( S_n \) is the intersection of \( C \) with some closed interval \( J \subset [0, 1] \) such that \( J \) has length \( 1/3^k \).

Suppose \( x \in S_n \). Let \( \epsilon > 0 \), and let \( \delta = 1/3^k \). If \( x' \in C \) such that \( |x' - x| < \delta \), then \( x' \in S_n \); and hence \( f(x') - f(x) = 0 < \epsilon \). Thus if \( x \in S_n \), then \( f \) is continuous at \( x \).

Suppose \( m \) is a limit point of \( M \). There exists a point \( c \in C \) such that \( f(c) = m \). Let \( \{m_p\}_{p=0}^{\infty} \) be a sequence such
that \( m_1 \in M \) for \( i \geq 0 \), and \( \{m_p\}_{p=0}^{\infty} \) converges to \( m \). Let 
\( \{c_p\}_{p=0}^{\infty} \) be the sequence such that \( c_i \in C \), and \( f(c_i) = m_i \) for \( i \geq 0 \). Since \( f \) is monotone, then \( \{c_p\}_{p=0}^{\infty} \) converges to \( c \).

Consider \( \{c_p\}_{p=0}^{\infty} \). Either there exists both a strictly increasing subsequence \( \{c'_p\}_{p=0}^{\infty} \) and a strictly decreasing subsequence \( \{c''_p\}_{p=0}^{\infty} \) of \( \{c_p\}_{p=0}^{\infty} \); or there exists a strictly increasing subsequence and no strictly decreasing subsequence, or a strictly decreasing subsequence and no strictly increasing subsequence of \( \{c_p\}_{p=0}^{\infty} \). The proof is given for the case in which both subsequences exist; the other two cases are similar.

Let \( m'_i \) and \( m''_i \) denote \( f(c'_i) \) and \( f(c''_i) \), respectively, for \( i \geq 0 \). Then \( \{m'_p\}_{p=0}^{\infty} \) and \( \{m''_p\}_{p=0}^{\infty} \) converge to \( m \). Let \( \epsilon > 0 \). There exists a positive integer \( Q_1 \) such that if \( q > Q_1 \), then \( |m'_q - m| < \epsilon \); and there exists a positive integer \( Q_2 \) such that if \( q > Q_2 \), then \( |m''_q - m| < \epsilon \).

Let \( Q = \max(Q_1, Q_2) \). Suppose \( q > Q \). Let \( \delta_1 = |c'_q - c| \); \( \delta_2 = |c''_q - c| \); and let \( \delta = \min(\delta_1, \delta_2) \). If \( c_0 \in C \) such that \( |c_0 - c| < \delta \), then \( |f(c_0) - f(c)| < \epsilon \). Hence \( f \) is continuous at \( c \).

**Definition 2.12.** The statement that \( M \) is a Cantor set means \( M \) is homeomorphic to \( C \).

**Theorem 2.3.** If \( 0 < \epsilon < 1 \), then there exists a Cantor set contained in \([0, 1]\) of measure \( k \geq \epsilon \).

**Proof:** Let \( 0 < \epsilon < 1 \). There exists a positive integer \( n \) such that \( 1 - 1/n \geq \epsilon \). On the interval \([0, 1]\), remove the
open interval \( I_1 = (1/2 - 1/4n, 1/2 + 1/4n) \). Then from the
remaining two segments, remove the open middle intervals,
\( I_2 \) and \( I_3 \), each of length \( 1/8n \). Next, remove from the re-
maining four segments the open middle intervals, \( I_4, I_5, I_6, \)
and \( I_7 \), each of length \( 1/32n \). Continue in this fashion in-
definitely.

The remaining set, \( K \), is measurable and has outer
length \( \kappa = 1 - 1/n \sum_{p=0}^{\infty} 2^p/(2 \cdot 4^p) = 1 - 1/n \geq \varepsilon \).

From Theorem 2.1, \( M \) is a Cantor set if and only if \( M \) is
bounded, perfect, and totally disconnected. The proof of
this is similar to the proof for \( C \).

**Theorem 2.4.** If \( M \) is an uncountable set in \( R_1 \), there
exists a point \( x \in M \) such that \( x \) is a limit point of \( M \).

**Proof:** Assume that no point of \( M \) is a limit point of
\( M \). Then if \( x \in M \), there exists \( \varepsilon_x > 0 \) such that if \( y \in M \)
and \( y \neq x \), then \( |x - y| \geq \varepsilon_x \).

Let \( x \in M \). There exists a neighborhood \( N \) of \( x \) such
that \( N(x) \) has outer length \( < \varepsilon_x/4 \). Then \( N(x) \cap M \neq \emptyset \).

Let \( P = \bigcup_{x \in M} N(x). \) Since \( N(x_i) \cap N(x_j) = \emptyset \) if \( i \neq j \), then \( P \) is
a collection of disjoint open intervals covering \( M \), and
hence is countable. Since \( N(x_i) \cap M = \{ x_i \} \) for \( i \geq 1 \), then
\( M \) is a countable set. This completes the proof.

**Corollary 2.1.** If \( M \) is an uncountable set in \( R_1 \), and
\( P = \{ x \in M \mid x \is a limit point of M \} \), then \( P \) is an uncount-
able set.
Proof: Suppose P is a countable set. If M - P is countable, then M = P U (M - P) is the union of two countable sets, and thus M is countable. Therefore M - P is uncountable. There exists a point x ∈ M - P such that x is a limit point of M - P. But (M - P) ⊆ M, and hence x is a limit point of M. This contradiction completes the proof.

Theorem 2.5. If M is a closed, uncountable set in R^1, then M contains a Cantor set.

Proof: Suppose M is a closed, uncountable set in R^1. Either M is a Cantor set, or it is not. If M is a Cantor set, then M ⊆ M, and the theorem is proved.

Suppose M is not a Cantor set. Then from Theorem 2.1, at least one of the following is true:

1. M is unbounded.
2. There exists a point x ∈ M such that x is not a limit point of M.
3. M contains an interval.

Suppose M is unbounded. Let a and b be two points of M such that a < b, and there exists an uncountable number of points of M between a and b. Let M_1 = \{x ∈ M | a ≤ x ≤ b\}. M_1 is closed, bounded, and uncountable. If M_1 is a Cantor set, then the theorem is proved.

If M_1 is not a Cantor set, or if M is bounded but not a perfect set, let M_2 = \{x ∈ M_1 | x is a limit point of M_1\}. Since M_1 is closed, then M_2 is a perfect set, and M_2 ⊆ M_1 ⊆ M. If M_2 is a Cantor set, then the theorem is proved.
If \( M_2 \) is not a Cantor set, then \( M_2 \) must contain some closed interval \([c, d]\). But \([c, d]\) is homeomorphic to \([0, 1]\), which contains \( C \). Hence there exists a perfect subset \( P \) of \([c, d]\) which is homeomorphic to \( C \). \( P \subset M \), and the theorem is complete.

An immediate consequence of Theorem 2.5 is that every closed, uncountable set in \( R_1 \) contains a perfect set, and every perfect set in \( R_1 \) contains a Cantor set.

**Theorem 2.6.** If \( M \) is an uncountable set in \( R_1 \) having measure \( k > 0 \), then \( M \) contains a perfect set, and hence a Cantor set.

**Proof:** Consider \( J = [0, 1] \). If the theorem is true for \( M \subset J \), then it is true for \( M \subset R_1 \).

Suppose \( M \) is an uncountable subset of \( J \) having measure \( k \) such that \( 0 < k \leq 1 \). Then \( J - M \) has measure \( 1 - k \geq 0 \).

There exists a countable collection \( Q \) of disjoint open intervals covering \( J - M \) such that if \( I_n \in Q \), and \( z_n \) denotes the length of \( I_n \), then \[ \sum_{p=0}^{\infty} z_p < 1 - k + k/2 = 1 - k/2. \]

Let \( M' = \left\{ x \in M \mid \text{if } I_n \in Q, \text{then } x \notin I_n \right\} \). \( Q \) is an open set, and \( M' \) is the union of a countable collection of disjoint closed sets. \( M' \) has measure \( \geq k/2 \); hence there exists a closed, uncountable set \( P \subset M' \) such that \( P \) has measure \( m > 0 \). By Theorem 2.5, \( P \) contains a Cantor set. Since \( P \subset M' \subset M \), then \( M \) contains a Cantor set, and the proof is complete.
Example 2.1. Theorem 2.6 guarantees that there exists a Cantor set which is a subset of the set of irrational numbers. The following is a method of constructing such a set.

Suppose $J$ is an interval. Let the symbol $Z(J)$ denote the length of $J$.

Let $I = [-\pi, \pi]$, and let $\{\mathbf{r}_i\}_{i=1}^\infty$ be a sequence of all of the rational elements of $I$. Let $I_0 \subseteq I$ be an open interval having the following properties:

(1). $I_0 = (a_0, b_0)$, where each of $a_0$ and $b_0$ is an irrational number.

(2). $r_1 \in I_0$.

(3). $0 < Z(I_0) \leq Z(I)/3$.

Let $C_1 = I - I_0$. $C_1 \subseteq I$, and $C_1$ is the union of two closed intervals. Let $J_1$ and $J_2$ denote these intervals. There exists a positive integer $k$ such that $r_k \in J_1$. Let $k_1$ denote the least such integer. Let $I_1 \subseteq J_1$ be an open interval having the properties that:

(1). $I_1 = (a_1, b_1)$, where $a_1$ and $b_1$ are irrational.

(2). $r_{k_1} \in I_1$.

(3). $0 < Z(I_1) \leq Z(J_1)/3$.

Let $k_2$ denote the least positive integer such that $r_{k_2} \in J_2$. Let $I_2 \subseteq J_2$ be an open interval having properties defined in the same manner as for $I_1$.

Let $C_2 = C_1 - (I_1 \cup I_2)$. $C_2 \subseteq I$, and $C_2$ is the union of four closed intervals. Let $J_3$, $J_4$, $J_5$, and $J_6$ denote
these intervals. Let $k_n$, $n = 3, 4, 5, 6$, denote the least positive integer such that $r_{k_n} \in J_n$, and choose $I_n$ in the same manner as $I_1$. Let $C_2 = C_2 - \bigcup_{n=3}^6 I_n$.

Continue in this fashion, each time picking irrational end points and the first element in $\{r_p\}_{p=1}^\infty$ contained in each of the intervals. Then $\bigcap_{n=1}^\infty C_n$ is a Cantor set whose elements are irrational numbers.

In Theorem 2.6, the restriction is made that the uncountable set be of positive measure. The following example shows why this restriction is necessary.

**Example 2.2.** A set of measure 0 which contains no perfect set.

Let $C$ denote the Cantor set, and let $S$ denote the collection of all perfect subsets of $C$. Let $k$ denote the cardinal of $S$. Suppose $H$ is the collection of all closed sets in $R_1$, and $Q$ is the collection of all open sets in $R_1$. Then $S \subseteq H$, and if $h$ denotes the cardinal of $H$ and $Q$, then $k \leq h$.

Let $G$ be the collection of all open intervals having rational end points. $G$ has cardinal $\mathfrak{m}_0$. Now if $M \subseteq Q$, then $M$ is the union of a countable collection of subsets of $G$. Thus the collection of all subsets of $G$ is $Q$. By the continuum hypothesis, $Q$ has cardinal $c$. Thus $h = c$, and $k \leq c$. 
Suppose \( x \in C \) such that \( x \neq 1 \). Then \( C \cap [x, 1] \) is a perfect subset of \( C \). Since \( \{ x \in C \mid x \neq 1 \} \) has cardinal \( c \), then \( k = c \).

Let \( W \) be a well-ordering of \( S \) such that each element of \( S \) is preceded by at most a countable number of elements. Let \( S_1 \) denote the first element of \( S \) according to \( W \), and let each of \( x_1 \) and \( y_1 \) denote a point of \( S_1 \).

The symbol \( a < b \) will mean that \( a \) precedes \( b \) in \( W \). Suppose \( s \in S \) such that if \( s' < s \), then \( x_s \) and \( y_s \) are distinct points of \( s' \); and if \( s'' < s \), then no two of the points \( x_{s''}, y_{s''}, x_s, y_s \), are the same.

Let \( x_s \) and \( y_s \) be distinct points of \( s \) such that if \( s' < s \), then no two of the points \( x_s, y_s, x_{s''}, y_{s''} \), are the same.

Let \( X = \{ x_s \mid s \in S \} \), and let \( Y = \{ y_s \mid s \in S \} \). Since each of \( X \) and \( Y \) contain one and only one element from each set of \( S \), and \( S \) has cardinal \( c \), then each of \( X \) and \( Y \) has cardinal \( c \).

\( X \) cannot be a perfect set, nor can \( X \) contain a perfect set, since \( S \) contains all perfect subsets of \( C \), and if \( s \in S \) there exists \( y \in s \) such that \( y \notin X \). Since \( C \) has measure 0, and \( X \subset C \), then \( X \) has outer length 0. But \( C - X \subset C \) and hence has outer length 0. Thus \( X \) is measurable and has measure 0.

Substituting \([0, 1]\) for \( C \) in Example 2.2 leads to the construction of an immeasurable set which contains no perfect set.
Theorem 2.1 guarantees that any bounded, perfect, totally disconnected set in $\mathbb{R}^2$, the set of all ordered pairs of real numbers, is a Cantor set. The following example gives an easy method for constructing such a set in $\mathbb{R}^2$.

**Example 2.3.** A Cantor set in $\mathbb{R}^2$.

Let $S$ denote the square bounded by the points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Let $C$ denote the Cantor set.

Let $M_1$ denote the set of points $s \in S$ such that $s$ lies between the vertical lines $x = 1/3$ and $x = 2/3$. Let $M_2$ denote the set of points $s \in S$ such that $s$ lies between the horizontal lines $y = 1/3$ and $y = 2/3$.

Let $S_1 = S - (M_1 \cup M_2)$. $S_1$ consists of four mutually separated squares. From each square in $S_1$, remove the points $s \in S$ such that $s$ lies between any pairs of the lines $x = 1/9$ and $x = 2/9$, $x = 7/9$ and $x = 8/9$, $y = 1/9$ and $y = 2/9$, or $y = 7/9$ and $y = 8/9$.

Continue removing the "middle thirds" of the remaining squares indefinitely.

Let $S^*$ denote the remaining set. If $s \in S^*$, then $x = (a,b)$, where $a, b \in C$. Thus $S^* = C \times C$.

Suppose $x \in S^*$. Then for some $a \in C$ and $b \in C$, $x = (a,b)$. In ternary expansion,

$$x = (0.a_1a_2a_3\ldots,0.b_1b_2b_3\ldots)$$

where $a_i$ and $b_i = 0$ or 2 for $i \geq 1$. Let $f(x) = 0.a_1b_1a_2b_2\ldots$. Then $f(x) \in C$, and if $Ra(f)$ denotes the range of $f$, then $Ra(f) \subseteq C$. 
Suppose \( y \in C \). Let \( 0.y_1y_2y_3 \ldots \) denote the ternary expansion of \( y \). There exists a point \( x \in S^* \) such that \( x = (0.y_1y_3y_5 \ldots, 0.y_2y_4y_6 \ldots) \). Since 
\[ f(x) = 0.y_1y_2y_3 \ldots = y, \]
then \( y \in \text{Ra}(f) \). Hence \( C \subseteq \text{Ra}(f) \). Since \( \text{Ra}(f) \subseteq C \), then \( \text{Ra}(f) = C \), and \( f \) is reversible.

The proof that \( f \) and \( f^{-1} \) are continuous is similar to the proof in Theorem 2.1.

This process is easily extended to \( n \)-dimensional Euclidean space by taking the product of \( n \) copies of \( C \) as the bounded, perfect, totally disconnected set. If 
\[ x = (a_1, a_2, \ldots, a_n) \]
is an element of this product space and \( 0.a_{11}a_{12}a_{13} \ldots \), \( 0.a_{21}a_{22}a_{23} \ldots \), \( 0.a_{n1}a_{n2}a_{n3} \ldots \) denote the ternary expansions of the coordinates of \( x \), let 
\[ f(x) = 0.a_{11}a_{21} \ldots a_{n1}a_{12}a_{22} \ldots a_{n2} \ldots \]
CHAPTER III

SOME TOPOLOGICAL PROPERTIES RELATED TO THE CANTOR SET

Definition 3.1. The statement that the set M is of Category I means M is the union of a countable collection of nowhere dense sets.

It follows from the definition that any Cantor set is of Category I, and any countable set is of Category I. Since any countable set is of Category I and has measure 0, and C is of Category I and has measure 0, one might suppose that every set of measure 0 is of Category I. The following example shows that this is not the case.

Example 3.1. A set of measure 0 which is not of Category I.

Suppose n is a positive integer, and J = [0, 1]. There exists a Cantor set $C_0 \subseteq J$ such that $C_0$ has measure $k$, where $1 > k > 1 - 1/n$. Let $\{C_p\}_{p=1}^{\infty}$ be a sequence of Cantor sets such that $C_i$ has measure $k_i > 1 - 1/i$ for $i \geq 1$. Let

$$M = \bigcup_{i \geq 1} C_i,$$

and let $L = J - M$.

$M$ is the union of a countable collection of measurable sets and hence is measurable. The sequence $\{k_p\}_{p=1}^{\infty}$ of outer lengths of $\{C_p\}_{p=1}^{\infty}$ converges to 1. Therefore $M$ has measure 1. $J$ has measure 1, and $L = J - M$ has measure 0.
Suppose $L$ is of Category I. Then $L$ is the union of a countable collection of nowhere dense sets. $M$ is the union of a countable collection of nowhere dense sets. Therefore, since $M \cup L = J$, $J = [0, 1]$ is of Category I. This is the desired contradiction.

**Definition 3.2.** The statement that the set $M$ is an $F_\infty$ set means $M$ is the union of a countable collection of closed sets.

**Definition 3.3.** The statement that the set $M$ is a $G_0$ set means $M$ is the intersection of a countable collection of open sets.

**Theorem 3.1.** A Cantor set is an $F_\infty$ set.

Proof: This follows immediately from the definition and the fact that a Cantor set is a closed set.

**Theorem 3.2.** Suppose $M$ is a point set in $R_1$ and $S$ is the minimal interval containing all of $M$. Let $M'$ denote the complement of $M$ in $S$. $M$ is an $F_\infty$ set if and only if $M'$ is a $G_0$ set.

Proof: Suppose $M$ is an $F_\infty$ set. Then $M$ is the union of a countable collection $G$ of closed sets. Let $M_1$, $M_2 \in G$. Then $M_1 \subseteq M$, and $M_2 \subseteq M$. Since $M_1$ and $M_2$ are closed sets, then $M_1'$ and $M_2'$, the complements of $M_1$ and $M_2$ in $S$, are open sets.

From DeMorgan's laws, $(M_1 \cup M_2)' = M_1' \cap M_2'$, and

$$(\bigcup_{M_1 \in G} M_1)' = \bigcap_{M_1 \in G} M_1' = M'.$$

Hence $M'$ is a $G_0$ set.
Conversely, suppose $M'$ is a $G_\delta$ set. $M'$ is the intersection of a countable collection $Q$ of open sets. Suppose $M'_1, M'_2 \in Q$. $M'_1$ and $M'_2$ are open sets; hence $M'_1$ and $M'_2$, the complements of $M'_1$ and $M'_2$ in $S$, are closed sets.

From DeMorgan's laws, $M'_1 \cap M'_2 = (M'_1 \cup M'_2)'$, and

$$\bigcap_{M'_1 \in Q} M'_1 = (\bigcup_{M'_1 \in Q} M'_1)' .$$

Then

$$(\bigcap_{M'_1 \in Q} M'_1)' = \bigcup_{M'_1 \in Q} M'_1 = K .$$

Hence $K$ is an $F_\sigma$ set, and the theorem is proved.

**Corollary 3.1.** Suppose $C'$ is the complement of $C$ in $[0, 1]$. $C'$ is a $G_\delta$ set.

Proof: This follows immediately from Theorem 3.1 and Theorem 3.2.

**Theorem 3.3.** $C'$ is an $F_\sigma$ set.

Proof: Let $S_i = [(2^{i+1} + 1)/(2^i \cdot 5), (2^{i+2} - 1)/(2^i \cdot 6)]$ for $i \geq 1$. Let $S = \bigcup_{i \geq 1} S_i = (1/3, 2/3)$. Then $S$ is an $F_\sigma$ set.

$C'$ is the union of a countable collection $Q$ of disjoint open intervals, each homeomorphic to $S$. Thus if $q \in Q$, then $q$ is an $F_\sigma$ set. Therefore, $C' = \bigcup_{q \in Q} q$ is the union of a countable collection of closed sets, and hence is an $F_\sigma$ set.

**Theorem 3.4.** $C$ is a $G_\delta$ set.

Proof: This follows immediately from Theorems 3.3 and 3.2.
The preceding theorems show that $C$ and $C'$ are both $F_r$ and $G_\delta$ sets. It is not true, however, that any set is both an $F_r$ and a $G_\delta$ set, as the following examples will show.

**Example 3.2.** A set which is not an $F_r$ set.

Let $I$ denote the set of irrational numbers. Suppose $I$ is an $F_r$ set. Then there exists a countable collection $S$ of closed sets $S_i$ such that $\bigcup_{S_i \in S} S_i = I$.

Since $I$ is an uncountable set, there exists a positive integer $n$ such that $S_n$ is an uncountable set. Now, let $P = \{S_1 \in S \mid S_1 \text{ is uncountable}\}$. If $S_j \in P$, then either some subset of $S_j$ is dense in $R_1$, or $S_j$ is nowhere dense in $R_1$.

Suppose there exists $S_j \in P$ such that for some subset $M$ of $S_j$, $M$ is dense in $R_1$. Then if $x \in M$ and $N$ is a neighborhood of $x$, there exists $x' \in N(x)$ such that $x' \neq x$, and $x' \in M$.

Let $x_1, x_2 \in M$ such that $x_1 < x_2$. There exists a rational number $r$ such that $x_1 < r < x_2$. Let $R$ denote the set of rational numbers, and let $Q = \{r \in R \mid x_1 < r < x_2\}$. There exists $r_0 \in Q$ such that if $N(r_0)$ is any neighborhood of $r_0$, there exists $x_0 \in N(r_0)$ such that $x_0 \in M$; for if no such $r_0$ exists, then $M$ is nowhere dense in $R_1$. Hence $r_0$ is a limit point of $M \subset S_j$, and $r_0 \notin S_j$. Therefore $S_j$ is not closed. But $S_j$ is closed, by definition; thus if $I$ is an $F_r$ set, then $S_i$ is nowhere dense in $R_1$, for $i \geq 1$. 
Since $I = \bigcup_{i \geq 1} S_i$, and $S_i$ is nowhere dense in $\mathbb{R}$ for $i \geq 1$, then $I$ is of Category I. $\mathbb{R}$ is of Category I, and $I \cup \mathbb{R} = \mathbb{R}_I$. Thus $\mathbb{R}_I$ is of Category I. This contradiction completes the proof.

**Example 3.3.** A set which is not a $G_δ$ set.

Let $\mathbb{R}$ denote the set of rational numbers, and suppose $\mathbb{R}$ is a $G_δ$ set. Then by Theorem 3.2, the set $I$ of irrational numbers is an $F_δ$ set. This contradicts the proof given in Example 3.2. Hence $\mathbb{R}$ is not a $G_δ$ set.

**Example 3.4.** A set which is neither an $F_δ$ set nor a $G_δ$ set.

Let $X$ be the set constructed in Example 2.2. Suppose $X$ is an $F_δ$ set. Let \( \{N_p\}_{p=0}^\infty \) be a sequence of closed sets whose union is $X$. Since $X$ is uncountable, there exists a positive integer $n$ such that $N_n$ is uncountable. Then by Theorem 2.5, $M_n$ contains a perfect set. But $M_n \subseteq X$, and $X$ contains no perfect set. Hence $X$ is not an $F_δ$ set.

Suppose $X$ is a $G_δ$ set. Let $I = [0, 1]$. By Theorem 3.2, $I - X$ is an $F_δ$ set. Let \( \{S_p\}_{p=0}^\infty \) be a sequence of closed sets whose union is $I - X$. Let $C$ denote the Cantor set. $X \subseteq C$, and $C - X$ is uncountable. Since $(C - X) \subseteq (I - X)$, there exists a positive integer $m$ such that $S_m$ contains an uncountable number of points of $C - X$. If no such $m$ exists, then $C - X$ is countable.

Let $P = \{x \in C - X \mid x \in S_m\}$. $P$ is uncountable, and since $S_m$ is closed, then $\overline{P}$, the closure of $P$, is a subset of
Since $G$ is a perfect set, and $C - X \subseteq C$, then if $x$ is a
limit point of $C - X$, either $x \in C - X$, or $x \in X$. Since
$S_m \cap X = \emptyset$, then $P = C - X$; and since $P \subseteq S_m$, then $P = P$.

$P$ is closed and uncountable; hence by Theorem 2.5, there
exists a perfect set $K$ such that $K \subseteq P \subseteq C$. Because of the
method of constructing $X$, there exists a point $x \in K$ such
that $x \in X$. Hence $I - X$ is not an $F_\sigma$ set, and $X$ cannot be a
$G_\delta$ set.
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