THE CONVOLUTION RING

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THE CONVOLUTION RING

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CHAPTER I

INTRODUCTION

This paper deals with the development of the convolution ring and the construction of a field from this ring. Since the development relies extensively on certain properties of integrals of continuous functions, these properties will be stated and proven in Chapter I. All other properties of continuous functions and all properties of the real number system are assumed. The development of the convolution ring and the construction of the field are contained in Chapter II.

Definition 1.1: The statement that \( D \) is a subdivision of the interval \([a, b]\) means: \( D \) is a finite sequence of points \( \{x_i\}_{i=0}^{n} \), such that \( a = x_0 < x_1 < \ldots < x_n = b \). The set \( \{x| x_{i-1} \leq x \leq x_i\} \) is called the \( i \)th subinterval of \( D \).

Definition 1.2: The norm of the subdivision \( D = \{x_i\}_{i=0}^{n} \) of the interval \([a, b]\), denoted by \(|D|\), is defined by

\[ |D| = \max \{|x_i - x_{i-1}| \mid 1 \leq i \leq n \} \]

Definition 1.3: Suppose \([a, b]\) is an interval. Then the statement that \( K \) is a marking of the subdivision \( D = \{x_i\}_{i=0}^{n} \) of \([a, b]\) means: \( K \) is a finite sequence of points \( \{k_i\}_{i=1}^{n} \), such that \( k_i \) belongs to the \( i \)th subinterval of \( D \). If \( f \) is continuous over \([a, b]\) then \( K = \{k_i\}_{i=1}^{n} \) is a maximal
marking of \( D \) with respect to \( f \) if \( f(k_i) \) is the maximal value of \( f \) over the \( i \)th subinterval of \( D \).

**Definition 1.4:** The statement that \( Y \) is the integral of \( f \) over the interval \([a,b]\) means: If \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( D = \{x_i\}_{i=0}^{n} \) is a subdivision of \([a,b]\) and \( |D| < \delta \), and \( \mathcal{M} = \{\xi_i\}_{i=1}^{n} \) is a marking of \( D \), then

\[
\left| \sum_{D} f(\xi_i)(x_i - x_{i-1}) - Y \right| < \epsilon.
\]

\( Y \) is denoted by \( \int_{a}^{b} f(x) \, dx \).

**Theorem 1.1:** If \( f \) is continuous over \([a,b]\), then

\( \int_{a}^{b} f(x) \, dx \) exists.

**Proof:** Consider the set

\[
X = \left\{ x \mid x = \sum_{D} f(\xi_i)(x_i - x_{i-1}) \right\}
\]

where \( D = \{x_i\}_{i=0}^{n} \) is a subdivision of \([a,b]\) and \( \{\xi_i\}_{i=1}^{n} \) a maximal marking of \( D \) with respect to \( f \). Since \( X \) is bounded below by \( \min_{a \leq x \leq b} f(x) (b - a) \), \( X \) has a greatest lower bound, \( Y \).

\( Y \) will be shown to be the integral of \( f \) over \([a,b]\).

Note that if \( D = \{x_i\}_{i=0}^{n} \) and \( D' = \{x'_i\}_{i=0}^{m} \) are subdivisions of \([a,b]\), \( D' \) is a subsequence of \( D \), \( \{\xi'_i\}_{i=1}^{m} \) is a maximal marking of \( D' \) and \( \{\xi_i\}_{i=1}^{n} \) is a maximal marking of \( D \), both with respect to \( f \), then

\[
\sum_{D} f(\xi_i)(x_i - x_{i-1}) \leq \sum_{D'} f(\xi'_i)(x'_i - x'_{i-1}).
\]
Suppose $\epsilon > 0$. Because of the note mentioned above and the fact that $\gamma$ is the greatest lower bound of $x$, there is a subdivision $D^* = \{x^*_i\}_{i=0}^m$ of $[a,b]$ such that if $D^*$ is a subsequence of the subdivision $D = \{x_i\}_{i=0}^n$ of $[a,b]$ and $\{r_i\}_{i=1}^n$ is a maximal marking of $D$, then

$$\left| \sum_D f(r_i) (x_i - x_{i-1}) - Y \right| < \epsilon/2.$$ 

Suppose $D^*$ is such a subdivision mentioned above. Since $f$ is uniformly continuous over $[a,b]$, there is $\delta' > 0$ such that if $x_1$ and $x_2$ are in $[a,b]$, then $|f(x_1) - f(x_2)| < \frac{\epsilon}{2(b-a)}$. Let $\delta^* = \min\{x^*_1 - x^*_1 - x^*_1, 1 \leq i \leq m\}$ and $\delta = \min\{\delta', \delta\}$. Let $D' = \{x'_i\}_{i=0}^j$ be a subdivision of $[a,b]$ such that $|D'| < \delta$.

Then $D' \cup D^* = \{z_i\}_{i=0}^k$ is a subdivision of $[a,b]$ such that if $\{\gamma_i\}_{i=1}^k$ is a maximal marking of $D' \cup D^*$, then

$$\left| \sum_{D' \cup D^*} f(\gamma_i) (z_i - z_{i-1}) - Y \right| < \epsilon/2.$$ 

Suppose $\{\gamma'_i\}_{i=1}^j$ is a marking of $D'$. Let $\{\gamma''_i\}_{i=1}^k$ be a sequence such that

$$\sum_{D'} f(\gamma'_i) (x'_i - x'_{i-1}) = \sum_{D' \cup D^*} f(\gamma''_i) (z_i - z_{i-1})$$

and $|\gamma''_i - \gamma_i| < \delta$ for $i = 1, 2, \ldots, k$.

Hence,

$$\left| \sum_{D' \cup D^*} f(\gamma_i) (z_i - z_{i-1}) - \sum_D f(\gamma_i) (x_i - x_{i-1}) \right|$$

$$= \left| \sum_{D' \cup D^*} f(\gamma_i) (z_i - z_{i-1}) - \sum_{D' \cup D^*} f(\gamma''_i) (z_i - z_{i-1}) \right|$$

$$= \left| \sum_{D' \cup D^*} (f(\gamma_i) - f(\gamma''_i)) (z_i - z_{i-1}) \right|.$$


\[ \sum_{D' \cup D^*} |f(\eta_i) - f(f'_{i'})| (z_i - z_{i-1}) \]

\[ \leq \frac{\varepsilon}{2(b-a)} \sum_{D' \cup D^*} (z_i - z_{i-1}) \]

\[ = \frac{\varepsilon}{2} . \]

Therefore \( |\sum_{D} f(f'_{i'})(x_i' - x_{i-1}') - y| < \varepsilon \). Hence \( \int_{a}^{b} f(x) \, dx \) exists.

**Definition 1.5:**

\[ \int_{a}^{a} f(x) \, dx = 0 \text{ and } \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx. \]

**Theorem 1.2:** Suppose \( f \) is differentiable over \([a, b]\) and that \( \int_{a}^{b} f'(x) \, dx \) exists. Then \( \int_{a}^{b} f'(x) \, dx = f(x) \bigg|_{a}^{b} = f(b) - f(a) \).

**Proof:** Suppose \( \varepsilon > 0 \). Let \( \delta > 0 \) such that if \( D = \{x_i\}_{i=0}^{n} \) is a subdivision of \([a, b]\) such that \( |D| < \delta \) and \( \{s'_{i}\}_{i=1}^{n} \) is a marking of \( D \), then

\[ \left| \sum_{D} f'(s'_{i}) (x_i - x_{i-1}) - \int_{a}^{b} f'(x) \, dx \right| < \varepsilon . \]

Suppose \( D' = \{x'_i\}_{i=0}^{m} \) is a subdivision of \([a, b]\) such that \( |D'| < \delta \). Let \( \{s'_{i}\}_{i=1}^{m} \) be a marking of \( D' \) such that if \( 1 \leq i \leq m \) then

\[ f'(s'_{i}) = \frac{f(x'_i) - f(x'_{i-1})}{x'_i - x'_{i-1}} . \]

\[ \sum_{D} f'(s'_{i}) (x'_i - x'_{i-1}) = \sum_{D} f(x'_i) - f(x'_{i-1}) = f(b) - f(a) . \]

Therefore \( \int_{a}^{b} f(x) \, dx = f(b) - f(a) . \)
Lemma 1.1: Suppose \( \int_a^b f(x) \, dx \) exists and \( a < c < b \), then each of \( \int_a^c f(x) \, dx \) and \( \int_c^b f(x) \, dx \) exist and
\[
\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx.
\]

Lemma 1.2: \( \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx \).

Theorem 1.3: Suppose \( f \) is continuous over \([a, b]\). Then \( F(x) = \int_a^x f(t) \, dt \) is differentiable over \([a, b]\) and \( F'(x) = f(x) \) over \([a, b]\).

Proof: Suppose \( \epsilon > 0 \) and \( x_0 \in [a, b] \). Since \( f \) is continuous at \( x_0 \) there is \( \delta > 0 \) such that if \( x \in [a, b] \) and \( |x - x_0| \leq \delta \) then \( |f(x) - f(x_0)| \leq \epsilon \). If \( x \in [a, b] \), \( 0 < |x - x_0| \leq \delta \), and \( x_0 < x \), then
\[
\left| \int_a^x f(t) \, dt - \int_a^{x_0} f(t) \, dt \right| = \left| \int_{x_0}^x f(t) \, dt - f(x_0) \right| \leq \int_{x_0}^x |f(t) - f(x_0)| \, dt.
\]
\[
\leq \int_{x_0}^x \epsilon \, dt = \int_{x_0}^x \epsilon \, dx.
\]
since \(|t - x_0| < \delta\). But since
\[
\int_{x_0}^{x} \frac{c}{x - x_0} \, dt = \epsilon,
\]
then
\[
\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \epsilon.
\]

If \(x = x_0\), changing the integral to read \(\int_{x_0}^{x}\) will give the same result. Hence \(F\) is differentiable over \([a, b]\) and \(F(x) = f(x)\) over \([a, b]\).

**Theorem 1.4:** Suppose \(f\) is continuous over \([a, b]\), \(\phi\) is continuously differentiable and strictly monotone over \([\alpha, \beta]\), and \(\phi(\alpha) = a, \phi(\beta) = b\). Then each \(x\) in \([a, b]\) is determined by \(\phi\) at some \(x^*\) in \([\alpha, \beta]\), and
\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{\alpha} f(\phi(x^*)) \phi'(x^*) \, dx^*.
\]

**Proof:** Suppose \(\epsilon > 0\). Let \(Y\) denote the first integral and \(X\) the second. Let \(\delta > 0\) such that if \(D = \{x^*\}_{i=0}^{m} \) is a subdivision of \([\alpha, \beta]\) such that \(|D| < \delta\), and \(\{\delta_i\}_{i=1}^{m}, \{\delta_i^*\}_{i=1}^{m-1}\) are markings of \(D\), then
\[
\left| \sum_{D^*} f(\phi(\delta_i)) \phi'(\delta_i^*)(x^*_i - x^*_{i-1}) - x \right| < \epsilon/2,
\]
and such that \(D^* = \{\phi(x^*_i)\}_{i=0}^{m} = \{x_i\}_{i=0}^{m}\) introduces a subdivision of \([a, b]\) such that if \(\{\delta_i^*\}_{i=1}^{m}\) is a marking of \(D^*\), then
\[
\left| \sum_{D^*} f(\delta_i^*)(x_i - x_{i-1}) - Y \right| < \epsilon/2.\] The use of a double marking in this way follows easily from Definition 1.4 and Theorem 1.1.
Suppose \( \{ \phi_i \}_{i=1}^m = \{ \phi^{-1} \circ \phi_i \}_{i=1}^m \) and by the mean value theorem \( \{ \phi_i \}_{i=1}^m \) is a marking of \( D \) such that 
\[
\frac{\phi(x_i) - \phi(x_{i-1})}{x_i - x_{i-1}} = \phi'(x_i) \quad \text{for} \quad 1 \leq i \leq m.
\] Then 
\[
\sum_{D^*} f(\phi_i)(x_i - x_{i-1}) = \sum_{D} f(\phi(x_i)(\phi(x_i) - \phi(x_{i-1}))
\]
\[
= \sum_{D} f(\phi(x_i))\phi'(x_i)(x_i - x_{i-1}).
\]
Therefore \( |x - y| < \epsilon \). Hence \( x = y \).

**Definition 1.6:** \( \int_0^\infty f(x)dx = A \) means that \( \{ \int_0^n f(x)dx \}_{n=1}^\infty \) converges to \( A \) if \( \{ x_n \}_{n=1}^\infty \) is any positive, increasing, and unbounded sequence. A similar statement holds for \( f(x) |_{0}^{\infty} = A \).

**Lemma 1.3:** If \( f \) is continuous and bounded over the set of non-negative real numbers, then \( \int_0^\infty e^{-x}f(x)dx \) exists.

**Theorem 1.5:** Suppose \( f \) is continuous over the set of non-negative real numbers, and that there is \( k > 0 \) and \( M > 0 \) such that \( |f(x)| \leq Me^{kx} \) for all \( x \geq 0 \). Then there is \( R > 0 \) such that if \( s \geq R \), \( \int_0^\infty e^{-xs}f(x)dx \) exists. Also there is \( R' > 0 \) such that if \( s \geq R' \), the function \( F(s) = \int_0^\infty e^{-xs}f(x)dx \) is differentiable and \( F'(s) = \int_0^\infty x e^{-xs}f(x)dx \).

**Proof:** The first statement is obvious, for suppose \( R = k + 1 \), then by Lemma 1.3 the integral does exist. The
next observation is pertinent to the question of differentiability of \( F \).

Suppose for some \( s_0 > 0 \) that 
\[
H(x, s) = \frac{e^{-xs} - e^{-xs_0}}{s - s_0} + xe^{-xs_0}.
\]
Then, according to the mean value theorem, for each \( x \) and \( s \) there is \( s' \) such that \( s' \) is between \( s \) and \( s_0 \) and such that
\[
\frac{e^{-xs} - e^{-xs_0}}{s - s_0} = -xe^{-xs'}.
\]
Hence there is
\[
H'(x, s') = -xe^{-xs'} + xe^{-xs_0} = H(x, s).
\]

Suppose \( \epsilon > 0 \) and \( a > 0 \). Then there is \( \delta > 0 \) such that if \( k_1 \) and \( k_2 \) are between 0 and \( a \) and such that \( |k_1 - k_2| < \delta \), then
\[
|e^{-k_1} - e^{-k_2}| < \epsilon/a.
\]
Therefore if \( |s' - s_0| < \delta/a \) and \( 0 < x \leq a \), then \( |xs' - xs_0| < \delta \) and hence
\[
|x(e^{-xS_0} - e^{-xs'})| < \frac{\epsilon}{a} |x| < \epsilon.
\]
Therefore if \( |s - s_0| < \delta/a \), then \( |H(x, s)| < \epsilon \) if \( 0 < x \leq a \).

Let \( R'' = \mathbb{R} \) such that if \( s > R'' \), then \( \int_0^{\infty} xe^{-xs} f(x)dx \) exists. Let \( s_0 > R'' \), \( \{x_n\}_{n=1}^\infty \) be any positive, increasing, and unbounded sequence, \( \{M_i\}_{i=1}^\infty \) be a sequence of positive numbers with the property that \( |f(x)| < M_n \) if \( x \in [0, x_n] \), and \( \{\delta_i\}_{i=1}^\infty \) be a sequence of positive numbers such that if \( 0 < |s - s_0| < \delta_i \), then
\[
\left| \frac{e^{-xs} - e^{-xs_0}}{s - s_0} xe^{-xs_0} \right| < \frac{1}{(M_i)_{x_i}^2} \quad \text{for } 0 < x \leq x_i.
\]
Note that the sequence \( \left\{ \int_{0}^{x_k} e^{-xs_0}(-x)f(x)dx \right\}_{k=1}^\infty \) converges to \( \int_0^{\infty} xe^{-xs_0} f(x)dx \). If \( \{s_n\}_{n=1}^\infty \) is a sequence such that \( 0 < |s_n - s_0| < \delta_n \) for \( n = 1, 2, 3, \ldots \), then the sequence
\[
\left\{ \int_{0}^{\infty} \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} f(x) \, dx \right\}_{n=1}^{\infty} \text{ converges to } \int_{0}^{\infty} -xe^{-xs_0}f(x) \, dx.
\]

This is evident, since,
\[
\left| \int_{0}^{\infty} \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} f(x) \, dx - \int_{0}^{\infty} -xe^{-xs_0}f(x) \, dx \right| = \left| \int_{0}^{\infty} f(x) \left( \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} \right) \, dx \right| 
= \int_{0}^{\infty} f(x) \left| \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} \right| \, dx < \frac{1}{n} \left( \frac{1}{(M_n)^2} \right) \leq \frac{1}{x_n}.
\]

Note that there is \( k > 0 \) such that if \( 0 < |s - s_0| < k \),
\[
\left\{ \int_{0}^{\infty} \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} f(x) \, dx \right\}_{n=1}^{\infty} \text{ converges uniformly to } 0 \text{ if } \{x_n\}_{n=1}^{\infty} \text{ is any positive, increasing, and unbounded sequence.}
\]

Hence the sequence \( \left\{ \int_{0}^{\infty} \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} f(x) \, dx \right\}_{n=1}^{\infty} \) converges to \( \int_{0}^{\infty} -xe^{-xs_0}f(x) \, dx \) on comparing it with the sequence
\[
\left\{ \int_{0}^{\infty} \frac{e^{-xs_n} - e^{-xs_0}}{s_n - s_0} f(x) \, dx \right\}_{n=1}^{\infty}.
\]

Therefore if \( R' = R'' \), then \( F \) is differentiable and
\[
F'(s) = \int_{0}^{\infty} -xe^{-xs}f(x) \, dx \text{ for } s \equiv R'.
\]

**Theorem 1.6:** Suppose \( f \) has the conditions stated in Theorem 1.5. Then there is \( R > 0 \) such that there is a Taylor's
expansion of $F(s) = \int_0^\infty e^{-xs}f(x)\,dx$ over an open interval of length two about each point $s' > R$.

Proof: Let $k > 0$ and $M > 0$ such that $|f(x)e^{-kx}| < M$ for all $x \geq 0$. Therefore if $s = k + 1$, then $\int_0^\infty x^n e^{-xs}f(x)\,dx$ exists and $\left|\int_0^\infty x^n e^{-xs}f(x)\,dx\right| \leq \int_0^\infty Mx^ne^{-x}\,dx = Mn!$ holds for $n = 0, 1, 2, 3, \ldots$, since if $R > 0$, then $\left|\int_0^R x^n e^{-xs}f(x)\,dx\right| \int_0^R x^n e^{-x(s-1)}\,dx \leq \int_0^R x^n e^{-kx}\,dx$. Then by successive applications of Theorem 1.5, $F$ has derivatives of all orders at each $s$ such that $s = k + 1$. Denoting the $n^{th}$ derivative of $F$ by $F^{(n)}$, then $F^{(n)}(s) = (-1)^n\int_0^\infty x^n e^{-xs}f(x)\,dx$ for $n = 0, 1, 2, 3, \ldots$.

By Taylor's theorem, $F(s) = F^{(0)}(s') + F^{(1)}(s')(s - s') + \frac{F^{(2)}(s')}{2!} (s - s')^2 + \ldots + \frac{F^{(n)}(s'')}{n!} (s - s')^n$, where $s''$ is between $s'$ and $s$.

Now if $s$ and $s' > R$ and $|s - s'| < 1$, and $s''$ is between $s$ and $s'$, then $\left|\frac{F^{(n)}(s'')}{n!} (s - s')^n\right| \leq \frac{n!}{n!} (s - s')^n$ for $n = 0, 1, 2, 3, \ldots$. Therefore $\left\{\frac{F^{(n)}(s'')}{n!} (s - s')^n\right\}_{n=1}^\infty$ converges to 0. Therefore, $F(s) = F^{(0)}(s') + F^{(1)}(s')(s - s') + \frac{F^{(2)}(s')}{2!} (s - s')^2 + \ldots$ for $|s - s'| < 1$ and $s, s' > R$ since the remainder converges to 0.
Definition 1.7: Suppose $R$ is the rectangle, $a \leq x \leq b$, $c \leq y \leq d$, $D_1 = \{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, and $D_2 = \{y_j\}_{j=0}^m$ is a subdivision of $[c, d]$. Then if $A_{ij}$ is the rectangle $x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j, 1 \leq i \leq n, 1 \leq j \leq m$, the sequence $D = \{A_{ij}\}_{i=0}^n, j=0^m$ is a subdivision of $R$. Denote the area of $A_{ij}$ by $\Delta A_{ij}$.

Definition 1.8: Suppose $D = \{A_{ij}\}_{i=0}^n, j=0^m$ is a subdivision of $R: a \leq x \leq b, c \leq y \leq d$. Then $\{(x_i', y_j')\}_{i=1}^n, j=1^m$ is a marking of $D$ if $(x_i', y_j') \in A_{ij}$. If $F$ is continuous over $R$, then $\{(k_i, k_j')\}$ is a maximal marking of $D$ with respect to $F$ if $F(k_i, k_j')$ is the maximal value of $F$ over $A_{ij}$.

Definition 1.9: The norm of a subdivision $D = \{A_{ij}\}_{i=0}^n, j=0^m$ of $R: a \leq x \leq b, c \leq y \leq d$, denoted by $|D|$, is the maximal diameter among all the $A_{ij}$ of $D$.

Definition 1.10: $Y$ is said to be the double integral of $F$ over the rectangle $R: a \leq x \leq b, c \leq y \leq d$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $D = \{A_{ij}\}_{i=0}^n, j=0^m$ is a subdivision of $R$, $|D| < \delta$, and $\{(x_i', y_j')\}_{i=1}^n, j=1^m$ is a marking of $D$, then

$$\sum_D F(x_i', y_j') \Delta A_{ij} - Y < \varepsilon.$$ 

$Y$ is denoted by $\iiint_R F(x, y) dA$.

Theorem 1.7: Suppose $F$ is continuous over $R: a \leq x \leq b, c \leq y \leq d$. Then $\iiint_R F(x, y) dA$ exists.
Proof: The proof would be a modification of the one for Theorem 1.1.

**Lemma 1.5:** Suppose \( F \) is continuous over \( R: a \leq x \leq b, c \leq y \leq d \). Then \( h(x) = \int_c^d F(x, y) \, dy \) exists and is continuous over \([a, b]\).

**Theorem 1.2:** Suppose \( F \) is continuous over \( R: a \leq x \leq b, c \leq y \leq d \). Then each of \( \int_a^b \left( \int_c^d F(x, y) \, dy \right) \, dx \) and \( \int_c^d \left( \int_a^b F(x, y) \, dx \right) \, dy \) equals \( \iint_R F(x, y) \, dA \).

Proof: Suppose \( \epsilon > 0 \). Let \( \delta_1 > 0 \) so that if \( D = \{ A_{ij} \}_{i=0}^n, \, \{ (c_i, c'_i) \}_{i=1}^m \) is a subdivision of \( R \), \( |D| < \delta_1 \), and \( \{ (c'_i, c''_i) \}_{i=1}^m \) is a marking of \( D \), then

\[
\sum_{D} F(c'_i, c''_i) \Delta A_{ij} - \sum_{R} F(x, y) \, dA < \frac{\epsilon}{2}.
\]

Let \( \delta_2 > 0 \) and \( \delta_2 < \frac{\delta_1}{\sqrt{2}} \) such that if \( D = \{ x_i \}_{i=0}^n \) is a subdivision of \([a, b]\), \( |D| < \delta_2 \), and \( \{ (c'_i) \}_{i=1}^n \) is a marking of \( D \), then

\[
\sum_{D} \left( \int_c^d F(c'_i, y) \, dy \right) (x_i - x_{i-1}) - \int_a^b \left( \int_c^d F(x, y) \, dy \right) \, dx < \frac{\epsilon}{2}.
\]

Suppose \( D_1 = \{ x'_i \}_{i=0}^n \) is a subdivision of \([a, b]\) and \( |D_1| < \delta_2 \). Let \( \delta_3 > 0 \) and \( \delta_3 < \delta/\sqrt{2} \) such that if \( D = \{ y_i \}_{i=0}^n \) is a subdivision of \([c, d]\), \( |D| < \delta_3 \), and \( \{ (c'_i) \}_{i=1}^n \) is a marking of \( D_1 \) and \( \{ (c''_i) \}_{i=1}^n \) is a marking of \( D \), then
\[ \sum_{D_1} \left( \int_c^d F(x,y) \, dy \right) (x'_{i+1} - x'_{i-1}) - \sum_{D_1} \left( \sum_{D} F(x_i', y_j') (y_{i+1} - y_{i-1}) (x_{i+1} - x_{i-1}) \right) \leq \varepsilon/2. \]

Suppose \( D_2 = \{ y'_{i} \}_{i=0}^k \) is a subdivision of \([c,d]\) and \( |D_2| < \delta_3 \). Consider the subdivision \( D \) of \( R \) formed by \( D_1 = \{ x'_{i} \}_{i=0}^m \) and \( D_2 = \{ y'_{i} \}_{i=0}^k \), \( |D| < \delta \), since \( |D_1| = \frac{\delta_1}{\sqrt{2}} \) and \( |D_2| < \frac{d_1}{\sqrt{2}} \). If \( \{(x'_{i}, y'_{j})\}_{i=1}^m, \{y'_{j}\}_{j=1}^k \) is a marking of \( D \), such that \( \{(x'_{i})_{i=1}^m \) is a marking of \( D_1 \) and \( \{y'_{j}\}_{j=1}^k \) is a marking of \( D_2 \), then

\[ \sum_{D} F(x_i', y_j') \Delta A_{ij} = \sum_{D_1} \left( \sum_{D_2} F(x_i', y_j') (y_{i+1} - y_{i-1}) (x_{i+1} - x_{i-1}) \right). \]

Hence

\[ \left| \sum_{a} \left( \int_{c}^{d} F(x,y) \, dy \right) \, dx - \sum_{R} F(x,y) \, dA \right| \leq \varepsilon. \]

Therefore

\[ \int_{a}^{b} \left( \int_{c}^{d} F(x,y) \, dy \right) \, dx = \sum_{R} F(x,y) \, dA. \]

Similarly

\[ \int_{c}^{d} \left( \int_{a}^{b} F(x,y) \, dx \right) \, dy = \sum_{R} F(x,y) \, dA. \]

Note that \( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} F(x,y) \, dy \right) \, dx = A \) means that

\[ \left\{ \int_{-\infty}^{x_n} \left( \int_{-\infty}^{y_n} F(x,y) \, dy \right) \, dx \right\}_{n=1}^{\infty} \]

converges to \( A \) if \( \{x_n\}_{n=1}^{\infty} \) and

\[ \{y_n\}_{n=1}^{\infty} \] are two positive, increasing, and unbounded sequences.

Since the sequence in the preceding sentence is the same sequence as \( \left\{ \int_{-\infty}^{y_n} \left( \int_{-\infty}^{x_n} F(x,y) \, dx \right) \, dy \right\}_{n=1}^{\infty} \), then
Theorem 1.2: Suppose $f$ is continuously differentiable over $[a, b]$ and $g$ is continuous over $[a, b]$. Then
\[ \int_a^b f(x)g(x)\,dx = f(x) \int_a^x g(t)\,dt \bigg|_a^b - \int_a^b f'(x) \int_a^x g(t)\,dt\,dx. \]

Proof: If $F(x) = f(x) \int_a^x g(t)\,dt$ over $[a, b]$, then
\[ F'(x) = f(x)g(x) + f'(x) \int_a^x g(t)\,dt \text{ over } [a, b]. \]
By applying Theorem 1.2 and rearranging terms the theorem follows.

Lemma 1.6: If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ for $|x| < R$, then $f$ is continuous over the set $\{x \mid |x| < R\}$.

Theorem 1.10: Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ for $|x| < R$, and that $\{x_i\}_{i=1}^\infty$ is a sequence of distinct points such that $|x_i| < R$ and $f(x_i) = 0$ for $i = 1, 2, \ldots$, and such that $\{x_i\}_{i=1}^\infty$ converges to 0. Then $a_n = 0$ for $n = 0, 1, 2, \ldots$.

Proof: By continuity of $f$ at 0, $f(0) = 0$. Hence $a_0 = 0$. Letting $g_1(x) = a_1 + a_2 x + a_3 x^2 + \ldots$ for $|x| < R$, it has the same properties stated for $f$, hence $a_1 = 0$. If this process is continued it is found that $a_2 = 0, a_3 = 0, \ldots$. The theorem follows by applying the principle of mathematical induction.

Theorem 1.11: Suppose $f$ is continuous over $[0, 1]$ and $f(0) = f(1) = 0$. Then $f$ can be uniformly approximated by a polynomial over $[0, 1]$. 

\[ \int_0^\infty \left( \int_0^\infty F(x, y)\,dy \right)\,dx = A \text{ also.} \]
Proof: Consider the sequence of polynomials \( \{p_n\}_{n=1}^{\infty} \)

where

\[
p_n(t) = \frac{\int_0^1 \left[1 - (t - x)^2\right]^n f(x) \, dx}{\int_{-1}^1 (1 - x^2)^n \, dx}
\]

for \( n = 1, 2, 3 \),

\[ \cdots \text{. If } z = t - x, \text{ then} \]

\[
\int_0^1 \left[1 - (t - x)^2\right]^n f(x) \, dx = -\int_t^{t-1} (1 - z^2)^n f(t - z) \, dz = \int_{t-1}^t (1 - z^2)^n f(t - z) \, dz.
\]

Now if \( f \) is defined to be 0 for \( x \) outside the interval \([0,1]\),

then

\[
\int_{t-1}^t (1 - z^2)^n f(t - z) \, dz = \int_{-1}^1 (1 - z^2)^n f(t - z) \, dz.
\]

Hence \( p_n(t) = \frac{\int_{-1}^1 (1 - x^2)^n f(t - x) \, dx}{\int_{-1}^1 (1 - x^2)^n \, dx} \) also.

Note that

\[
\int_{-1}^1 (1 - x^2)^n \, dx = 2 \int_0^1 (1 - x^2)^n \, dx = 2 \int_0^1 x(1 - x^2)^n \, dx = \frac{1}{n+1}
\]

and \( \{(n+1)(1 - x^2)^n\}_{n=1}^{\infty} \) converges to 0 if \( x = 0 \) and \( x = 1 \)

since \( \{nk^n\}_{n=1}^{\infty} \) converges to 0 if \( k < 1 \). The sequence \( \{p_n\}_{n=1}^{\infty} \)

is an approximating sequence of polynomials for \( f \) over \([0,1]\).

Suppose \( \epsilon > 0 \). Let \( \delta > 0 \) such that if \( |x_1 - x_2| < \delta \), then

\[ |f(x_1) - f(x_2)| < \epsilon/2. \]

Let \( M > 0 \) such that \( |f(x)| < M \) for all
x and let $N > 0$ such that if $n > N$, then $(n+1)(1 - \delta^2)^n < \frac{\epsilon}{3M}$.

Then if $n > N$,

\[
\left| \frac{\int_{-1}^{1} (1 - (t - x)^2)^n f(x) dx}{\int_{-1}^{1} (1 - x^2)^n dx} - f(t) \right| = \left| \frac{\int_{-1}^{1} (1 - x^2)^n f(t - x) dx - \int_{-1}^{1} f(t)(1 - x^2)^n dx}{\int_{-1}^{1} (1 - x^2)^n dx} \right|
\]

\[
= \left| \frac{\int_{-1}^{1} (1 - x^2)^n f(t - x) dx - f(t) dx}{\int_{-1}^{1} (1 - x^2)^n dx} \right|
\]

\[
\leq \frac{\int_{-1}^{1} (1 - x^2)^n f(t - x) - f(t) dx + \int_{-1}^{1} (1 - x^2)^n f(t - x) - f(t) dx}{\int_{-1}^{1} (1 - x^2)^n dx}
\]

\[
+ \frac{\int_{-1}^{1} (1 - x^2)^n[f(t - x) - f(t)] dx}{\int_{-1}^{1} (1 - x^2)^n dx}
\]

\[
\leq 2M(n+1)(1 - \delta^2)^n + \frac{\epsilon}{2} \frac{\int_{-1}^{1} (1 - x^2)^n dx}{\int_{-1}^{1} (1 - x^2)^n dx} + 2M(n+1)(1 - \delta^2)^n
\]

\[
\leq \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4}
\]

\[
= \epsilon.
\]
Theorem 1.12: If \( f \) is continuous over \([0,1]\),
\[
f(1) = f(0) = 0 \quad \text{and} \quad \int_0^1 x^n f(x) \, dx = 0 \quad \text{for} \quad n = 0, 1, 2, 3, \ldots,
\]
then \( f(x) = 0 \) if \( x \in [0,1] \).

Proof: Suppose \( \epsilon > 0 \). By Theorem 1.11 there is a polynomial \( P \) such that
\[
\left| \int_0^1 f^2(x) \, dx - \int_0^1 P(x) f(x) \, dx \right| < \epsilon.
\]
Hence
\[
\int_0^1 f^2(x) \, dx < \epsilon.
\]
This implies that \( \int_0^1 f^2(x) \, dx = 0 \) which implies that \( f \) is 0 over \([0,1]\).
CHAPTER II

THE CONVOLUTION RING

This chapter consists of the definition and development of the convolution ring and the construction of a field from the convolution ring.

Definition 2.1: Suppose C is the set of functions which are continuous over the set of positive real numbers, continuous from the right at 0, and have the value of 0 at each negative real number. If \( f \) and \( g \) belong to C and \( t \) is a real number, define

\[
[f + g](t) = f(t) + g(t) \quad \text{and} \quad [f \circ g](t) = \int_0^t f(x)g(t - x)dx.
\]

Theorem 2.1: If \( f \) and \( g \) belong to C, then \( f + g \) and \( f \circ g \) belong to C.

Proof: Clearly, \( f + g \) belongs to C and \( f \circ g \) has the desired property at each negative real number, hence only the continuity of \( f \circ g \) over the positive real numbers and the right hand continuity of \( f \circ g \) at 0 need to be considered.

Suppose \( t_0 > 0 \) and \( \varepsilon > 0 \). Let \( N \) be a number such that \( |f(x)| < N \) and \( |g(x)| < N \) if \( x \leq 2t_0 \). Let \( \delta_1 > 0 \) such that if \( x_1 \) and \( x_2 \leq 2t_0 \) and \( |x_1 - x_2| < \delta_1 \), then \( |g(x_1) - g(x_2)| < \frac{\varepsilon}{4Nt_0} \).
Now let $\delta = \min \left\{ \delta_1, t_0, \frac{\epsilon}{2M^2} \right\}$. Then if $|t_0 - t_1| < \delta$ and $|t_1| < 2t_0$,

$$\left| \int_0^{t_0} f(x)g(t_0 - x)\,dx - \int_0^{t_1} f(x)g(t_1 - x)\,dx \right|
\leq \left| \int_0^{t_1} f(x)(g(t_0 - x) - g(t_1 - x))\,dx \right| + \int_0^{t_1} |f(x)||g(t_0 - x) - g(t_1 - x)|\,dx
\leq \left| \int_0^{t_1} M \frac{\epsilon}{4Mt_0}\,dx \right| + \int_0^{t_1} M^2\,dx
= \frac{|t_1|N\epsilon}{4Mt_0} + M^2|t_0 - t_1|< \epsilon.
$$

Hence $f \circ g$ is continuous at each positive real number.

Let $N$ be a number such that if $t < 1$, then $|f(t)| < N$ and $|g(t)| < N$. If $0 < t_1 < \frac{\epsilon}{N^2}$ and $t_1 < 1$, then

$$\left| \int_0^{t_1} f(x)g(t_1 - x)\,dx \right| < t_1 \cdot \frac{\epsilon}{N^2} \cdot N^2 < \epsilon.$$ 

Hence $f \circ g$ is continuous from the right at 0.

**Theorem 2.2:** If $f$ and $g$ belong to $C$, then $f + g = g + f$ and $f \cdot g = g \cdot f$.

**Proof:** $f + g = g + f$ since the range of $f$ and $g$ is the set of real numbers and they are commutative with respect to addition. In the case of $f \cdot g$ this is not as evident and needs to be considered.
\[ [f \cdot g](t) = \int_0^t f(x)g(t-x)\,dx. \] If \( t = t - x \), then
\[ \int_0^t f(x)g(t-x)\,dx = -\int_0^t f(t-z)g(z)\,dz = \int_0^t f(t-z)g(z)\,dz. \]
Hence \( f \cdot g \) is commutative.

**Theorem 2.3:** If \( f, g, \) and \( h \) belong to \( C \), then
\[ (f + g) + h = f + (g + h) \quad \text{and} \quad (f \circ g) \circ h = f \circ (g \circ h). \]

**Proof:** Clearly, \( (f + g) + h = f + (g + h) \) because the real numbers are associative with respect to addition. Again, the fact that \( (f \circ g) \circ h = f \circ (g \circ h) \) is not as obvious.

\[ [(f \cdot g) \cdot h](t) = \int_0^t \left[ \int_0^x f(s)g(x-s)\,ds \right] h(t-x)\,dx \]
\[ = \int_0^t \left[ \int_0^t f(s)g(x-s)\,ds \right] h(t-x)\,dx, \]
since \( g(x) = 0 \) whenever \( x < 0 \). Then by Theorem 1.8,
\[ [(f \cdot g) \cdot h](t) = \int_0^t f(s) \left[ \int_0^t g(x-s)h(t-x)\,dx \right] \,ds. \]

Letting \( z = x - s \), then
\[ [(f \cdot g) \cdot h](t) = \int_0^t f(s) \left[ \int_0^{t-s} g(z)h(t-s-z)\,dz \right] \,ds \]
\[ = \int_0^t f(s) \left[ \int_0^{t-s} g(x)h(t-s-x)\,dx \right] \,ds, \]
since \( g(x) = 0 \) if \( x < 0 \). Hence
\[ [(f \cdot g) \cdot h](t) = [f \circ (g \circ h)](t). \]

**Definition 2.2:** Let \( \sigma \) denote the function which is 0 at each real number.
Definition 2.3: $f$ is said to have exponential order if there is $k > 0$ and $\alpha > 0$ such that if $x \geq 0$, then $|f(x)| \leq Me^{kx}$.

Theorem 2.4: If $f$, $g$, and $h$ belong to $C$, then
$$f \circ (g + h) = f \circ g + f \circ h.$$ 

Proof:
$$\int_0^t f(x)(g(t-x) + h(t-x))dx = \int_0^t f(x)g(t-x)dx + \int_0^t f(x)h(t-x)dx.$$ 

Theorem 2.5: If $f$ and $g$ belong to $C$, and $f \circ g = \Theta$, then either $f = \Theta$ or $g = \Theta$.

The set of functions $C$ can be divided into two classes of functions, those of exponential order and those not of exponential order. A proof of Theorem 2.5 for the set of functions not of exponential order will not be given. However, in Chapter I, the properties of functions of exponential order necessary to prove Theorem 2.5 were developed, and a proof of Theorem 2.5 will be offered for these functions.

The following theorems are needed before this can be done.

Theorem 2.6: If $f$ and $g$ belong to $C$ and each has exponential order, then $f \circ g$ has exponential order.

Proof: Let $k_1 > k_2 > 0$, $\alpha > 0$, and $\beta > 0$ such that $|f(x)| \leq Me^{k_1x}$ and $|g(x)| \leq He^{k_2x}$ for all $x \geq 0$. Therefore
$$\left| \int_0^t f(x)g(t-x)dx \right| \leq \int_0^t Me^{k_1x} \cdot He^{k_2(t-x)}dx$$
$$= \frac{HHe^{k_2t}}{Me^{k_2}} \int_0^t e^{(k_1-k_2)x}dx$$
which is easily seen to have exponential order; hence \( f \circ g \) has exponential order.

**Theorem 2.7:** If \( f \) and \( g \) belong to \( C \) and each of \( f \) and \( g \) have exponential order, then there is \( R > 0 \) such that if \( s > R \), then

\[
\int_0^\infty \left[ \int_0^x f(t)g(x-t) \, dt \right] \, dx = \left( \int_0^\infty e^{-ts} f(t) \, dt \right) \left( \int_0^\infty e^{-xs} g(x) \, dx \right).
\]

**Proof:** Let \( R > C \) such that if \( s > R \), then \( \int_0^\infty e^{-ts} f(t) \, dt \), \( \int_0^\infty e^{-xs} g(x) \, dx \), and \( \int_0^\infty e^{-xs} \left[ f \circ g \right](x) \, dx \) each exists. Then

\[
\int_0^\infty \left[ \int_0^x f(t)g(x-t) \, dt \right] \, dx = \int_0^\infty \left[ e^{-xs} \int_0^\infty f(t)g(x-t) \, dt \right] \, dx.
\]

Letting \( z = x - t \), then

\[
\int_0^\infty \left[ e^{-xs} \int_0^x f(t)g(x-t) \, dt \right] \, dx = \int_0^\infty \left[ f(t) \int_0^\infty e^{-s(z-t)} \, dz \right] \, dt
\]

\[
= \int_0^\infty \left[ f(t) \int_0^\infty e^{-s(z-t)} \, dz \right] \, dt,
\]

since \( g(z) = 0 \) if \( z < 0 \).

Therefore

\[
\int_0^\infty \left[ e^{-xs} \int_0^x f(t)g(x-t) \, dt \right] \, dx = \left( \int_0^\infty e^{-st} f(t) \, dt \right) \left( \int_0^\infty e^{-sz} g(z) \, dz \right).
\]
Theorem 2.8: Suppose $f$ belongs to $C$ and there is $k > 0$ such that $F(s) = \int_0^\infty e^{-sx}f(x)dx$ for $s \geq k$. If $F$ has the property that for each positive integer $M$, there exists an interval $[a, b]$ such that $b > a > M$ and $F(x) = 0$ if $x \in [a, b]$, then there is $R > 0$ such that if $x > R$, then $F(x) = 0$.

Proof: By Theorem 1.6, there is $R' > 0$ such that if $s_0 > R'$, there is a Taylor's expansion of $F$ about $s_0$ over $(\alpha, \beta)$ where $\alpha > R'$, $\beta - \alpha = 2$, $s_0 \in (\alpha, \beta)$, and if $s' \in (\alpha, \beta)$, then $|s' - s_0| < 1$.

Suppose $[a, b]$ is an interval such that $b > a > (R' + 1)$ and such that if $x \in [a, b]$, then $F(x) = 0$. If $|x - b| < 1$, then

$$F(x) = F(0)(b) + F(1)(b)(x - b) + \frac{F(2)(b)}{2!} (x - b)^2 + \cdots.$$  

By Theorem 1.10, $F(x) = 0$ if $|x - b| < 1$. If $|x - (b + 1)| < 1$, then $F(x) = F(0)(b + 1) + F(1)(b + 1)(x - (b + 1)) + \frac{F(2)(b + 1)}{2!} (x - (b + 1))^2 + \cdots$. Again by Theorem 1.10, $F(x) = 0$ if $|x - (b + 1)| < 1$. Continuing this process, $F(x) = 0$ if $|x - (b + n)| < 1$ for $n = 1, 2, 3, \ldots$. Hence $F(x) = 0$ for all $x \geq R' + 1$. If $R = R' + 1$ the theorem is proved.

Theorem 2.9: Suppose $f$ belongs to $C$ and there is $k > 0$ such that $F(s) = \int_0^\infty e^{-sx}f(x)dx = 0$ for $s \geq k$. Then $f(x) = 0$ for $x \geq 0$. 

Proof: Let \( \psi(x) = \int_0^x e^{-kt} f(t) \, dt \) and \( s \geq 1 \). Then integrating by parts,

\[
\int_0^\infty e^{-x(s+k)} f(x) \, dx = \int_0^\infty e^{-xs} \cdot e^{-xk} f(x) \, dx = e^{-xs} \int_0^x e^{-kt} f(t) \, dt \bigg|_0^\infty + \int_0^\infty \left[ se^{-xs} \int_0^x e^{-kt} f(t) \, dt \right] \, dx
\]

\[
= s \int_0^\infty e^{-xs} \varphi(x) \, dx,
\]

since \( \left\{ e^{-x_n s} \int_0^x e^{-kt} f(t) \, dt \right\}_{n=1}^\infty \) converges to 0 if \( \{x_n\}_{n=1}^\infty \)

is any positive, increasing, and unbounded sequence. If

\[ t = e^{-x} \quad \text{and} \quad \varphi(t) = \varphi(-\ln t), \]

then

\[
s \int_0^\infty e^{-xs} \varphi(x) \, dx = -s \int_0^1 t^s \varphi(t) (t^{-1} \, dt) = s \int_0^1 t^{s-1} \varphi(t) \, dt.
\]

Since \( \varphi(1) = \varphi(0) = 0 \), \( \varphi \) is continuous over \([0,1]\), and

\[
\int_0^1 t^n \varphi(t) \, dt = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots, \text{by Theorem 1.12} \quad \varphi
\]

is 0 over \([0,1]\). Hence \( \varphi(x) = 0 \) for all \( x \geq 0 \). Since

\[
\psi(x) = \int_0^x e^{-kt} f(t) \, dt, \]

this implies that \( f(x) = 0 \) for all \( x \geq 0 \).

**Theorem 2.5':** Suppose \( f \) and \( g \) belong to \( C \) and each of \( f \) and \( g \) has exponential order. If \( f \circ g = \Theta \), then either \( f = \Theta \) or \( g = \Theta \).
Proof: By Theorem 2.6 there is \( k > 0 \) so that
\[
\int_0^\infty e^{-xs} [f \circ g](x) \, dx = \left( \int_0^\infty e^{-ts} f(t) \, dt \right) \left( \int_0^\infty e^{-xs} g(x) \, dx \right)
\]
for \( s > k \). Hence
\[
\left( \int_0^\infty e^{-ts} f(t) \, dt \right) \left( \int_0^\infty e^{-xs} g(x) \, dx \right) = 0
\]
for \( s > k \). If \( F(s) = \int_0^\infty e^{-ts} f(t) \, dt \) and \( G(s) = \int_0^\infty e^{-xs} g(x) \, dx \), then \( F(s) \) and \( G(s) \) are both continuous, and since \( F(s)G(s) = 0 \) for \( s > k \), either \( F \) or \( G \) has the properties of Theorem 2.8.

Hence there is \( R > 0 \) such that if \( x > R \) then \( F(x) = 0 \) or \( G(x) = 0 \). Therefore, either \( f \) or \( g \) has the properties of the function in Theorem 2.9. Hence either \( f(x) = 0 \) or \( g(x) = 0 \) for \( x \geq 0 \).

Hence, either \( f = \emptyset \) or \( g = \emptyset \).

Theorem 2.10: The set \( C \) together with the two operations 
"+" and "*" is a commutative integral domain.

Proof: The proof follows easily from the preceding
theorems.

Definition 2.4: Suppose \( Q = \{(x,y) \mid x, y \in C \text{ and } y \neq \emptyset \} \).
\( a \) is said to be an equivalence class of \( Q \) if and only if
for each pair of elements, \((x,y)\) and \((x',y')\), of \( a \),
\( x \circ y' = x' \circ y \).

Definition 2.5: Suppose \( C^* \) is the set of all equivalence
classes of \( Q \). If \( a \) and \( b \) belong to \( C^* \), define
Theorem 2.11: The set \( C^* \) together with the two operations "+" and "\( \circ \)" is a field.

Proof: If \( a, b, \) and \( c \) belong to \( C^* \), then the properties:

1) \( a + b \) and \( a \circ b \) belong to \( C^* \),
2) \( a + b = b + a \) and \( a \circ b = b \circ a \),
3) \( (a + b) + c = a + (b + c) \) and \( (a \circ b) \circ c = a \circ (b \circ c) \),
4) \( a \circ (b + c) = a \circ b + a \circ c \), and
5) \( \Theta^* = \{(\theta, x) \mid x \in C \text{ and } x \neq \theta \} \), are directly inherited from \( C \).

If \( U = \{(x, x) \mid x \in C \text{ and } x \neq \theta \} \) and \( a \) belongs to \( C^* \), then
\[ a \cdot U = a. \] Hence \( U \) is the unity of the field \( C^* \). If \( a \) belongs to \( C^* \), then the inverse of \( a \) with respect to the operation "+", denoted by \(-a\), is the set \( \{(-a, \alpha') \mid (\alpha, \alpha') \in a \} \). If \( a \) belongs to \( C^* \) and \( a \neq \Theta^* \), then the inverse of \( a \) with respect to the operation "\( \circ \)" denoted by \( a^{-1} \), is the set
\[ \{(\alpha, \alpha') \mid (\alpha', \alpha) \in a \} \]. Hence \( C^* \) together with the two operations "+" and "\( \circ \)" have the properties of a field.

Note also that the field \( C^* \) contains a subring isomorphic to the ring \( C \). For suppose that \( f \) belongs to \( C \) and \( f \neq \theta \). If \( a \) belongs to \( C \), then the association between \( a \) and that element of \( C^* \) which contains \((a \circ f,f)\) is a mapping of the ring \( C \) onto the subring of \( C^* \), which consists of equivalence classes of ordered pairs of \( C \) where the second term is a multiple of \( f \).