ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF TWO DIFFERENTIAL EQUATIONS

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CHAPTER I

INTRODUCTION

The purpose of this paper is to study two differential equations. A method of approximation by iteration is used to define sequences of functions which converge to solutions of these equations. Some properties of the solutions are proved for general boundary conditions and certain special solutions are studied in detail.

If f is a function whose domain is the set of all real numbers, let $K(f) = f' - f$. In Chapter II the integral equation $f(x) = b + \int_a^x f(t)dt$ is studied and it is shown that this equation is equivalent to the differential equation $K(f) = \underline{0}$ with boundary condition $f(a) = b$, where $\underline{0}$ is the function whose domain is the domain of f such that if x is in the domain of f, then $(x, 0) \in \underline{0}$. Suppose y is a function whose domain is $[0, 1]$, p is a positive, continuous function over $[0, 1]$, and q is a continuous function over $[0, 1]$. Let $\mathfrak{L}(y) = (p \cdot y')' - q \cdot y$. In Chapter III the integral equation $y(x) = b + m\int_a^x \frac{1}{p(s)}ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot y(t)dtds$ is studied. It is shown that this equation is equivalent to the differential equation $\mathfrak{L}(y) = \underline{0}$ with boundary conditions $y(a) = b$ and $p(a) \cdot y'(a) = m$.

In this study a knowledge of the real number system will be assumed. The following definitions and theorems which are developed in standard advanced calculus courses, such as the one outlined by Pierpont (1), will be assumed and used in Chapters II and III.

Definition 1.1. Suppose a and b are real numbers such that $a < b$. Then,

(i) the closed interval $[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$, and

(ii) the open interval (a, b) is the set of all real numbers x such that $a < x < b$.

Definition 1.2. The statement that the set X is bounded means there is a positive number M such that if x belongs to X, then $|x| < M$. The notation "$x \in X$" will be used to mean that x is an element of X. The statement that U is an upper bound of X means if $x \in X$, then $x \leq U$. The statement that L is a lower bound of X means if $x \in X$, then $x \geq L$.

Definition 1.3. The statement that L is a least upper bound of the set X means

(i) L is an upper bound of X and

(ii) if u is an upper bound of X, then $L \leq u$.

Definition 1.4. The statement that G is a greatest lower bound of X means

(i) G is a lower bound of X and

(ii) if q is a lower bound of X, then $q \leq G$.

Definition 1.5. The statement that f is a relation means that f is a set of ordered pairs; the statement that f is a function means that f is a relation such that no two ordered pairs in f have the same first element. The domain of f, denoted by D_f, is the set of all x such that x is the first element of an ordered pair in f; the range of f, denoted by R_f, is the set of all y such that y is the second element of an ordered pair in f. If $(x, y) \in f$, then y will be denoted by $f(x)$.

Definition 1.6. Suppose each of f and g is a function such that there is an element common to their domains.

(i) The sum of f and g, indicated by $f + g$, is the function h such that $D_h = D_f \cap D_g$ and if $x \in D_h$, then $h(x) = f(x) + g(x)$.

(ii) The product of f and g, indicated by $f \cdot g$, is the function h such that $D_h = D_f \cap D_g$ and if $x \in D_h$, then $h(x) = f(x) \cdot g(x)$.

Definition 1.7. Suppose f is a function such that if $x \in D_f$, then $f(x) \neq 0$. Then the reciprocal of f, indicated by $\frac{1}{f}$, is the function h such that $D_h = D_f$ and if $x \in D_h$, then $h(x) = \frac{1}{f(x)}$.

Definition 1.8. The statement that f is a strictly increasing function means if $x_1 \in D_f$, $x_2 \in D_f$, and $x_1 < x_2$, then $f(x_1) < f(x_2)$; the statement that f is a strictly decreasing function means if $x_1 \in D_f$, $x_2 \in D_f$, and $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Definition 1.9. The statement that the function f is continuous at $(x_o, f(x_o))$ means if ε is a positive number, there is a positive number δ such that if $x \in D_f$ and $|x - x_o| < \delta$, then $|f(x) - f(x_o)| < \varepsilon$.

Definition 1.10. The statement that f is continuous means if $x \in D_f$, then f is continuous at $(x, f(x))$.

Definition 1.11. The statement that x_o is a limit point of the set M means if ε is a positive number, then there is an $x \in M$ such that $x \neq x_o$ and $|x - x_o| < \varepsilon$.

Definition 1.12. The statement that f is differentiable at $(x_o, f(x_o))$ means x_o is a limit point of D_f and there exists a real number a such that if ε is a positive number, there is a positive number δ such that if $x \in D_f$ and $0 < |x - x_o| < \delta$, then $\left|\frac{f(x) - f(x_o)}{x - x_o} - a\right| < \varepsilon$. Denote a by $f'(x_o)$.

Definition 1.13. The statement that f is differentiable over $[a, b]$ means if $x \in [a, b]$, then f is differentiable at $(x, f(x))$.

Definition 1.14. The statement that f is integrable over [a, b] means [a, b] is a subset of D_f and there exists a number I such that if ε is a positive number, there exists a positive number δ such that if $a = x_0 < x_1 < \ldots < x_n = b$, $x_{p-1} \leq \xi_p \leq x_p$, $p = 1, 2, \ldots, n$, and $x_i - x_{i-1} < \delta$, $i = 1, 2, \ldots, n$, then $\left|\sum_{p=1}^{n} f(\xi_p)(x_p - x_{p-1}) - I\right| < \varepsilon$. Denote I by $\int_a^b f(t)dt$.

Definition 1.15. If a is a real number, then $\int_a^a f(t)dt = 0$.

Definition 1.16. If f is integrable over [a, b], then $\int_b^a f(t)dt = -\int_a^b f(t)dt$.

Definition 1.17. A sequence is a function whose domain is the set of positive integers. A real sequence is a sequence whose range is a subset of the real numbers. Let $\{a_i\}_{i=1}^{\infty}$ denote the sequence $\{(1, a_1), (2, a_2), (3, a_3) \ldots\}$.

Definition 1.18. The statement that the sequence $\{a_p\}_{p=1}^{\infty}$ converges means there is a number a such that if ε is a positive number, there exists a positive integer N such that if $n > N$, then $|a_n - a| < \varepsilon$.

Definition 1.19. Suppose that for each positive integer i, f_i is a function and suppose $D_{f_i} = D_{f_j}$, $i, j = 1, 2, \ldots$.

The statement that $\{f_p\}_{p=1}^{\infty}$ converges uniformly means there exists a function f, $D_f = D_{f_i}$, such that if ε is a positive number, there exists a positive integer N such that if $n > N$ and $x \in D_f$, then $|f_n(x) - f(x)| < \varepsilon$.

Definition 1.20. The statement that the series $\sum_{p=1}^{\infty} a_p$ converges means that the sequence $\{a_1 + a_2 + \ldots + a_n\}_{n=1}^{\infty}$ converges.

Definition 1.21. The statement that $\sum a_p$ converges absolutely means the series $\sum |a_p|$ converges.

Definition 1.22. The statement that f and g are linearly independent functions over [a, b] means that [a, b] is a subset of D_f and [a, b] is a subset of D_g, and if each of c_1 and c_2 is a real number such that $c_1 \cdot f(x) + c_2 \cdot g(x) = 0$ for all $x \in [a, b]$, then $c_1 = c_2 = 0$.

Theorem 1.1. Suppose M is a set. If M is bounded below, M has a greatest lower bound; if M is bounded above, M has a least upper bound.

Theorem 1.2. If M is a bounded, infinite set, then M has a limit point.

Theorem 1.3. If f is continuous at $(x_o, f(x_o))$ and g is continuous at $(x_o, g(x_o))$, then

(i) $f + g$ is continuous at $(x_o, f(x_o) + g(x_o))$,

(ii) $f \cdot g$ is continuous at $(x_o, f(x_o) \cdot g(x_o))$, and

(iii) if $f(x_o) \neq 0$, $\frac{1}{f}$ is continuous at $(x_o, \frac{1}{f(x_o)})$.

Theorem 1.4. If f is continuous and D_f is closed and bounded, then f is bounded, i.e., R_f is a bounded set.

Theorem 1.5. If f is continuous over [a, b] and f is differentiable over (a, b), then there is a number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Theorem 1.6. If f is continuous over [a, b], $f(a) < f(b)$, and ξ is a real number such that $f(a) < \xi < f(b)$, then there is a number $c \in (a, b)$ such that $f(c) = \xi$.

Theorem 1.7. Suppose each of f and g is a differentiable function.

(i) If $D_f = D_g$, then $f' + g' = (f + g)'$ and $(g \cdot f)' = g \cdot f' + g' \cdot f$.

(ii) If $R_g \subset D_f$, then $[f(g)]' = f'(g) \cdot g'$.

Theorem 1.8. If f is differentiable over [a, b], then f is continuous over [a, b].

Theorem 1.9. If f is differentiable over [a, b] and $f' = \underline{0}$, then f is constant over [a, b].

Theorem 1.10. If f is continuous over [a, b], then f is integrable over [a, b].

Theorem 1.11. Suppose each of f and g is integrable over [a, b], then

(i) $f + g$ is integrable over [a, b] and

$$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt;$$

(i) if k is a real number, then $k \cdot f$ is integrable over [a, b] and $\int_a^b k \cdot f(t)dt = k \cdot \int_a^b f(t)dt$.

Theorem 1.12. If f is integrable over [a, b] and c is a number such that $a \leq c \leq b$, then f is integrable over [a, b], f is integrable over [c, b], and

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Theorem 1.13. If f is integrable over [a, b], then $|f|$ is integrable over [a, b] and $\int_a^b f(t)dt \leq \int_a^b |f(t)| dt$.

Theorem 1.14. If f and g are integrable over [a, b] and for each $x \in [a, b]$, $f(x) < g(x)$, then $\int_a^b f(t)dt < \int_a^b g(t)dt$.

Theorem 1.15. If f is continuous over [a, b], $f(x) \geq 0$ for all $x \in [a, b]$, and $f(x) > 0$ for at least one $x \in [a, b]$, then $\int_a^b f(t)dt > 0$.

Theorem 1.16. If g' is integrable over [a, b], then

$$\int_a^b g'(t)dt = g(b) - g(a).$$

Theorem 1.17. Suppose g is an integrable function whose domain is the set of all real numbers, a is a real number, and $f(x) = b + \int_a^x g(t)dt$ for all x. Then if g is continuous at $(x_o, g(x_o))$, f is differentiable at $(x_o, f(x_o))$ and $f'(x_o) = g(x_o)$.

Theorem 1.18. If f is integrable over [a, b], then f is bounded over [a, b].

Theorem 1.19. If f is integrable over [a, b] and continuous over (a, b), then there is a $c \in (a, b)$ such that $\int_a^b f(t)dt = f(c)\cdot(b - a)$.

Theorem 1.20. If $\{a_p\}$ converges to a and $\{a_p\}$ converges to b, then a = b.

Theorem 1.21. The following two statements are equivalent:

(i) $\{a_p\}$ converges.

(ii) If ε is a positive number, there exists a positive integer N such that if $n > N$ and $m < N$, then $|a_n - a_m| < \varepsilon$.

Theorem 1.22. Suppose that for each positive integer i, f_i is a continuous function whose domain is [a, b]. If $\{f_p\}_{p=1}^{\infty}$ converges uniformly to f, then f is a continuous function.

Theorem 1.23. The following two statements are equivalent:

(i) $\sum a_p$ converges.

(ii) If ε is a positive number, there exists a positive integer N such that if $m > n > N$, then $|a_n + \ldots + a_m| < \varepsilon$.

Theorem 1.24. If there exist a number ν and a positive integer N such that $0 < \nu < 1$ and $\left|\frac{a_{n+1}}{a_n}\right| < 1 - \nu$ for all $n > N$, then $\sum |a_p|$ converges.

Theorem 1.25. Suppose $\sum a_p$ and $\sum b_p$ are series such that if i is a positive integer, then $a_i \geq 0$ and $b_i \geq 0$, and there exists a positive integer N such that if $n > N$, then $b_n \geq a_n$. Then, if $\sum b_p$ converges, $\sum a_p$ converges.

Theorem 1.26. If $\sum a_p$ converges absolutely and k is a number, then $\sum k \cdot a_p$ converges absolutely.

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1. Pierpont, James, The Theory of Functions of Real Variables, I, II, New York, Dover Publications, Inc., 1905.

CHAPTER II

SOLUTION OF A CERTAIN FIRST ORDER DIFFERENTIAL EQUATION

Suppose f is a differentiable function whose domain is the set of all real numbers. Let $K(f) = f - f'$. The purpose of this chapter is to prove the existence of solutions to $K(f) = \underline{0}$ that satisfy certain boundary conditions and to study properties of these solutions.

Theorem 2.1. Suppose f is a differentiable function whose domain is the set of all real numbers. If $K(f) = \underline{0}$ and a is a real number, then $f(x) = f(a) + \int_a^x f(t)dt$ for all real numbers x.

Proof. Suppose $K(f) = \underline{0}$ and a is a real number. By Theorem 1.8, f is continuous; therefore by Theorem 1.10, f is integrable. If x is a real number and $x \neq a$, then f is integrable over [a, x], or [x, a], and $\int_a^x f'(t)dt = \int_a^x f(t)dt$. Therefore by Theorem 1.16, $f(x) = f(a) + \int_a^x f(t)dt$.

Theorem 2.2. Suppose f is continuous and

$f(x) = b + \int_a^x f(t)dt$ for all x, then $K(f) = \underline{O}$ and $f(a) = b$.

Proof. Since $f(a) = b + \int_a^a f(t)dt$, by Definition 1.15 $f(a) = b$. If x is a real number, by Theorem 1.17 $f'(x) = f(x)$. This completes the proof of Theorem 2.2.

Hence by Theorems 2.1 and 2.2 the differential equation $K(f) = \underline{O}$, together with the boundary condition $f(a) = b$, is equivalent to the integral equation $f(x) = b + \int_a^x f(t)dt$.

Suppose α and β are real numbers such that $\alpha < \beta$. Let y_o be a continuous function over (α, β). Suppose $a \in (\alpha, \beta)$. By Definition 1.10 and Theorems 1.10, 1.17, and 1.8 it is possible to define a sequence $\{y_n\}_{n=o}^{\infty}$ such that if $x \in [\alpha, \beta]$ and n is a positive integer, then $y_n(x) = b + \int_a^x y_{n-1}(t)dt$. It is noted that for each positive integer n, y_n is a continuous, differentiable, and integrable function whose domain is $[\alpha, \beta]$ and $y_n(a) = b$.

Consider the absolute values of the differences of successive terms in the sequence $\{y_n\}_{n=o}^{\infty}$. By Definitions 1.6 and 1.2 and Theorems 1.3 and 1.4 there is a positive number M such that if $t \in [\alpha, \beta]$, then $|y_2(t) - y_1(t)| < M$. Then by

Theorems 1.11, 1.13, and 1.14, if $x \in [\alpha, \beta]$, $|y_3(x) - y_2(x)| = \left|\int_a^x [y_2(t) - y_1(t)]dt\right| \leq \left|\int_a^x |y_2(t) - y_1(t)|dt\right| < \left|\int_a^x Mdt\right| = M|x - a|$. By induction it can be proved that if $n > 3$ and $x \in [\alpha, \beta]$, then $|y_n(x) - y_{n-1}(x)| < \frac{M|x - a|^{n-2}}{(n - 2)!}$.

Next, consider the series $\sum_{p=o}^{\infty} \frac{(\beta - \alpha)^p}{p!}$. Let ν be a number such that $0 < \nu < 1$. There is a positive integer N such that $N + 1 > \frac{\beta - \alpha}{1 - \nu}$. Suppose $n > N$. Then $n + 1 > \frac{\beta - \alpha}{1 - \nu}$. Therefore $\left|\frac{\frac{(\beta - \alpha)^{n+1}}{(n + 1)!}}{\frac{(\beta - \alpha)^n}{n!}}\right| = \left|\frac{\beta - \alpha}{n + 1}\right| < 1 - \nu$. Hence by Theorem 1.24, $\sum_{p=o}^{\infty} \left|\frac{(\beta - \alpha)^p}{p!}\right|$ converges. By Theorem 1.26 if M is a number, $\sum_{p=o}^{\infty} \left|\frac{M (\beta - \alpha)^p}{p!}\right|$ converges. These considerations lead to the following theorem.

Theorem 2.3. Suppose y_o is a function continuous over $[\alpha, \beta]$ and $y_n(x) = b + \int_a^x y_{n-1}(t)dt$ for $n > 0$ and $x \in [\alpha, \beta]$. Then $\{y_n\}_{n=o}^{\infty}$ converges uniformly over $[\alpha, \beta]$ to a continuous function y. Furthermore, for each $x \in [\alpha, \beta]$,

$$y(x) = b + \int_a^x y(t)dt.$$

Proof. Let ε be a positive number. By the work prior to the statement of this theorem, there is a positive number M such that if n is a positive integer and $x \in [\alpha, \beta]$, then

$$|y_n(x) - y_{n-1}(x)| < \frac{M|x - a|^{n-2}}{(n-2)!}.$$

Since $\sum_{p=o}^{\infty} \frac{(\beta - \alpha)^p}{p!}$ converges, by Theorem 1.23 there is a positive integer N such that if $m > n > N$, then

$$\sum_{p=n+1}^{m} \frac{(\beta - \alpha)^p}{p!} < \frac{\varepsilon}{M}.$$

Suppose $m > n > N + 2$ and $x \in [\alpha, \beta]$. Then

$$|y_m(x) - y_n(x)| \leq |y_{n+1}(x) - y_n(x)| +$$

$$|y_{n+2}(x) - y_{n+1}(x)| + \ldots + |y_m(x) - y_{m-1}(x)| <$$

$$\frac{M|x - a|^{n-1}}{(n-1)!} + \ldots + \frac{M|x - a|^m}{m!} < \sum_{p=n+1}^{m} \frac{M(\beta - \alpha)^p}{p!} < \varepsilon.$$

Therefore $\{y_n(x)\}_{n=o}^{\infty}$ converges to a number, call it $y(x)$. Hence by Definition 1.19, $\{y_n\}_{n=o}^{\infty}$ converges uniformly over $[\alpha, \beta]$ to the function y. By Theorem 1.22, y is a continuous function over $[\alpha, \beta]$. It will next be shown that if $x \in [\alpha, \beta]$, then $y(x) = b + \int_a^x y(t)dt$.

Suppose there is an $x_o \in [\alpha, \beta]$ such that $y(x_o) \neq b + \int_a^{x_o} y(t)dt$. Then $\{y_n(x_o)\}_{n=o}^{\infty}$ does not converge to

$b + \int_a^{x_o} y(t)dt$. Thus there is a positive number εo such that if N is a positive integer, then there is an $n > N$ such that $\left|y_n(x_o) - b - \int_a^{x_o} y(t)dt\right| \geq \varepsilon o$. Since $\{y_n\}_{n=o}^{\infty}$ converges uniformly to y there is a positive integer N_o such that if $n > N_o$ and $t \in [\alpha, \beta]$, then $\left|y_n(t) - y(t)\right| < \frac{\varepsilon o}{3|x_o - a| + 3}$. Let $n > N_o + 1$ and $\left|y_n(x_o) - b - \int_a^{x_o} y(t)dt\right| \geq \varepsilon o$. Then by definition of y_n and Theorems 1.11, 1.13, and 1.14,

$$\varepsilon o \leq \left|y_n(x_o) - b - \int_a^{x_o} y(t)dt\right| =$$

$$\left|b + \int_a^{x_o} y_{n-1}(t)dt - b - \int_a^{x_o} y(t)dt\right| = \left|\int_a^{x_o} [y_{n-1}(t) - y(t)]dt\right|$$

$$\leq \left|\int_a^{x_o} \left|y_{n-1}(t) - y(t)\right| dt\right| < \left|\int_a^{x_o} \frac{\varepsilon o}{3|x_o - a| + 3}dt\right|$$

$$= \frac{\varepsilon o|x_o - a|}{3|x_o - a| + 3} \leq \frac{\varepsilon o}{3}.$$

The assumption that there exists $x_o \in [\alpha, \beta]$ such that $y(x_o) \neq b + \int_a^{x_o} y(t)dt$ leads to the contradiction that $\varepsilon o < \frac{\varepsilon o}{3}$. Hence $y(x_o) = b + \int_a^{x_o} y(t)dt$. Therefore if $x \in [\alpha, \beta]$, then $y(x) = b + \int_a^{x} y(t)dt$. This completes the proof of Theorem 2.3.

By Theorem 2.2 the function y in Theorem 2.3 has the properties that $K(y) = \underline{0}$ and $y(a) = b$. This shows that the differential equation $K(f) = \underline{0}$ together with the boundary condition $f(a) = b$ has a solution over $[\alpha, \beta]$.

Theorem 2.4. Suppose y is a continuous function over $[\alpha, \beta]$ such that $y(x) = b + \int_a^x y(t)dt$ for all $x \in [\alpha, \beta]$. If $z(x) = b + \int_a^x z(t)dt$ for all $x \in [\alpha, \beta]$, then $z = y$.

Proof. Suppose $z(x) = b + \int_a^x z(t)dt$ for all $x \in [\alpha, \beta]$. Let ε be a positive number. By Theorem 1.18, z is bounded over $[\alpha, \beta]$. By Theorem 1.4, y is bounded over $[\alpha, \beta]$. Hence by Definition 1.6 there is a positive number K such that if $t \in [\alpha, \beta]$, then $|z(t) - y(t)| < K$. By the work prior to the statement of Theorem 2.3, if M is a real number, then $\sum_{p=o}^{\infty} \frac{M(\beta - \alpha)^p}{p!}$ converges. Hence by a variation of Theorem 1.23 there is a positive integer N such that if $n > N$, then $\frac{K(\beta - \alpha)^n}{n!} < \varepsilon$. Let $n > N$. Suppose $x \in [\alpha, \beta]$. Either $x = a$, or $x \neq a$. If $x = a$, by Definition 1.15, $z(x) = y(x)$. Suppose $x \neq a$. Then by Theorems 1.11, 1.13, and 1.14, $|z(x) - y(x)| = \left|\int_a^x [z(t) - y(t)]dt\right| \leq$

$$\left|\int_a^x |z(t) - y(t)|\ dt\right| < \left|\int_a^x Kdt\right| = K|x - a|.$$

Since $|z(x) - y(x)| < K|x - a|$, by Theorems 1.11, 1.13, and 1.14, $|z(x) - y(x)| = \left|\int_a^x [z(t) - y(t)]dt\right| \leq \left|\int_a^x |z(t) - y(t)| dt\right|$

$$< \left|\int_a^x K|t - a| dt\right| = \frac{K|x - a|^2}{2!}.$$

After n repetitions of this procedure,

$$|z(x) - y(x)| < \frac{K|x - a|^n}{n!} < \frac{K(\beta - \alpha)^n}{n!} < \varepsilon.$$

Hence $z(x) = y(x)$. Therefore $z = y$ over $[\alpha, \beta]$. Thus the solution of $K(f) = \underline{0}$ that contains the ordered pair (a, b) is unique. This completes the proof of Theorem 2.4.

Since in Theorems 2.1 - 2.4 the only restriction on the interval $[\alpha, \beta]$ was that it contained the number a, then in light of Theorem 2.4 it is clear that there exists one and only one function y such that D_y is the set of all real numbers, $y(a) = b$, and for each $x \in D_y$, $y(x) = b + \int_a^x y(t)dt$.

Theorem 2.5. Suppose y is the continuous function such that $y(x) = b + \int_a^x y(t)dt$ for all real x and $b = f(a)$. Then if x is a real number, $y(x) = \sum_{p=0}^{\infty} \frac{b(x - a)^p}{p!}$.

Proof. In Theorem 2.3, take $y_o = \underline{0}$. Then if x is a real number, $y_1(x) = b + \int_a^x 0dt = b$; $y_2(x) = b + \int_a^x y_1(t)dt =$

$b + \int_a^x bdt = b + b(x - a)$. Suppose there is a positive integer k such that if x is a real number, then $y_k(x) = \sum_{p=o}^{k-1} \frac{b(x - a)^p}{p!}$.

Thus $y_{k+1}(x) = b + \int_a^x \sum_{p=o}^{k-1} \frac{b(t - a)^p}{p!}dt = \sum_{p=o}^{k} \frac{b(x - a)^p}{p!}$.

Hence if n is a positive integer and x is a real number, then $y_n(x) = \sum_{p=o}^{n-1} \frac{b(x - a)^p}{p!}$. Since $\{y_n(x)\}_{n=o}^{\infty}$ converges to y(x), $\left\{\sum_{p=o}^{n-1} \frac{b(x - a)^p}{p!}\right\}_{n=1}^{\infty}$ converges to y(x). Therefore y(x) has the series representation $\sum_{p=o}^{\infty} \frac{b(x - a)^p}{p!}$.

Theorem 2.6. Suppose y is the continuous function such that y(a) = b and $y(x) = b + \int_a^x y(t)dt$ for all real numbers x. Then, if b is positive, y is a strictly increasing, positive function; if b is negative, y is a strictly decreasing, negative function; and if b = 0, y is the x-axis.

Proof. Suppose b = 0. By Theorem 2.5, $y(x) = \sum_{p=o}^{\infty} \frac{0(x - a)^p}{p!}$ for all x. Hence y(x) = 0 for all x.

Suppose b is positive and x is a real number. Either x = a, x > a, or a > x. If x = a, then y(x) = y(a) = b. Thus y(x) is positive. Suppose x > a. Suppose $y(x) \leq 0$. If y(x) < 0, then by Theorem 1.6 there is a $p \in (a, x)$ such that $y(p) = 0$. Let $M = \{c \mid y(c) = 0 \text{ and } c \in (a, x)\}$. Since M is bounded, by Theorem 1.1, M has a greatest lower bound, call it c_o, and by continuity $c_o \in M$. By Theorem 1.5 there is a $q \in (a, c_o)$ such that $y'(q) = \dfrac{y(c_o) - y(a)}{c_o - a} < 0$. Thus $y(q) < 0$. Hence by Theorem 1.6 there is a $q_1 \in (a, q)$ such that $y(q_1) = 0$. Thus $q_1 \in M$, but $q_1 < c_o$ and this is a contradiction. Therefore $y(x) \not< 0$. Suppose $x < a$. Suppose $y(x) \leq 0$. If $y(x) < 0$, then by Theorem 1.6 there is a $p \in (x, a)$ such that $y(p) = 0$. Let $M = \{c \mid y(c) = 0 \text{ and } c \in (x, a)\}$. Since M is bounded, by Theorem 1.1, M has a greatest lower bound, call it c_o, and by continuity $c_o \in M$. By Theorem 1.5 there is a $q \in (x, c_o)$ such that $y'(q) = \dfrac{y(c_o) - y(x)}{c_o - x} > 0$. Thus $y(q) > 0$. Hence by Theorem 1.6 there is a $q_1 \in (x, q)$ such that $y(q_1) = 0$. Thus $q_1 \in M$, but $q_1 < c_o$ and this is a contradiction. Hence $y(x) \not< 0$. Therefore if b is positive, then y is a positive function. Suppose y is not a strictly increasing function. Then there are real numbers x_1 and x_2 such that $x_1 < x_2$ and $y(x_1) \geq y(x_2)$. By Theorems 1.5 and 2.2 there exists a number $c \in (x_1, x_2)$ such that $y(c) = \dfrac{y(x_2) - y(x_1)}{x_2 - x_1} \leq 0$.

This contradicts the fact that y is a positive function. Hence $y(x_1) \not\geq y(x_2)$. Therefore if b is positive, then y is a positive, strictly increasing function.

By a similar proof if b is negative, then y is a negative, strictly decreasing function.

Theorem 2.7. Suppose y is the function such that

$y(a) = b$ and $y(x) = b + \int_a^x y(t)dt$ for all real numbers x.

Then, if b is positive, y is unbounded above and if b is negative, y is unbounded below.

Proof. Suppose b is positive. Suppose y is bounded above. By Theorem 1.1 there is a least upper bound of y, call it L. Since b is positive and by Theorem 2.6, y is strictly increasing, then there is an $x_o > a$ such that

$0 < L - y(x_o) < \frac{b}{3}$. Hence if $x > x_o$, then $0 < L - y(x) < \frac{b}{3}$. Let $x = x_o + \frac{1}{2}$. By Theorem 1.5 there is a $c \in (x_o, x)$ such that $y'(c) = \dfrac{y(x) - y(x_o)}{x - x_o} = 2[y(x) - y(x_o)]$. Thus $y(c) = 2[y(x) - y(x_o)]$. Therefore

$$0 < y(c) - y(x_o) < \frac{b}{3},$$

$$0 < y(x) - y(x_o) < \frac{b}{3}, \text{ and}$$

$$0 < y(x) - y(c) < \frac{b}{3}.$$

Addition of these inequalities shows that $2[y(x) - y(x_o)] < b$; therefore $y(c) < b$. However since $c > a$ by Theorem 2.6, then $y(c) > y(a) = b$. Hence the assumption that y is bounded above leads to a contradiction. Thus y is unbounded above.

If b is negative, then a similar proof shows that y is unbounded below.

Theorem 2.8. Suppose y is the function such that

$$y(a) = b \text{ and } y(x) = b + \int_a^x y(t)dt \text{ for all real numbers } x.$$

Then, if ε is a positive number, there is a real number x such that $|y(x)| < \varepsilon$.

Proof. Clearly the theorem is true if $b = 0$. Suppose $b \neq 0$. Suppose there is a positive number ε such that if x is a real number, then $|y(x)| > \varepsilon$.

Suppose b is positive. Since y is positive, then $y(x) > \varepsilon$ for all x. By Theorem 1.1 there is a greatest lower bound of y, call it εo. Since by Theorem 2.6, y is strictly increasing, then there is a number x_o such that $\varepsilon_o < y(x_o) < \frac{9}{8}\varepsilon o$. Let $x_1 = x_o - 1$. By Theorem 2.6 and the assumption that εo is a greatest lower bound of y, then $y(x_o) > y(x_1) > \varepsilon o$. Hence $0 < y(x_o) - y(x_1) < \frac{\varepsilon o}{8}$. By Theorem 1.5 there is a $\eta \in (x_1, x_2)$ such that $y'(\eta) = \dfrac{y(x_o) - y(x_1)}{x_o - x_1} = y(x_o) - y(x_1) < \frac{\varepsilon o}{8}$. Thus $y(\eta) < \frac{\varepsilon o}{8}$. Hence the assumption that y is bounded below by

ε_o leads to the contradiction that $y(\eta) < \varepsilon_o$. Thus if b is positive, then y is not bounded below by a positive number. Therefore if b is positive and ν is a positive number, then there is a real number x such that $|y(x)| < \nu$.

If b is negative the proof of the theorem is similar. Hence if ε is a positive number, then there is a real number x such that $|y(x)| < \varepsilon$. This completes the proof of Theorem 2.8.

Now the solution of $K(f) = \underline{0}$ that contains the point (0, 1) will be studied in detail.

Definition 2.1. Let E denote the function such that $E(0) = 1$ and if x is a real number, then $E(x) = 1 + \int_0^x E(t)dt$.

Theorem 2.9. If each of x and c is a real number, then $E(x)\ E(c) = E(x + c)$.

Proof. Suppose each of x and c is a real number. Then by Definition 2.1, $E(x)\cdot E(c) = [1 + \int_0^x E(t)dt]\cdot[1 + \int_0^c E(t)dt]$ and $E(x + c) = 1 + \int_0^{x+c} E(t)dt$. By Theorems 1.7 and 2.2,

$$\frac{d\ E(x + c)}{dx} = \frac{d\ E(x + c)}{d(x + c)}\cdot\frac{d(x + c)}{dx} = \frac{d\ E(x + c)}{d(x + c)}\cdot 1 = E(x + c).$$

By Theorems 1.7 and 2.2, $\dfrac{d\ E(c)\cdot E(x)}{dx} = E(c)\cdot\dfrac{d\ E(x)}{dx} = E(c)\cdot E(x)$.

Note that $\left.\dfrac{d\ E(x + c)}{dx}\right]_{x=0} = E(0 + c) = E(c)$, and

$\left.\dfrac{d\ E(x)\ E(c)}{dx}\right]_{x=0} = E(c)\cdot E(0) = E(c)$. Hence by Theorem 2.2,

$K[E(x)\cdot E(c)] = \underline{0}$, $K[E(x + c)] = \underline{0}$, $[E(0)\cdot E(c)]' = E(c)$, and $[E(0 + c)]' = E(c)$. Therefore by Theorem 2.4, $E(x)\cdot E(c) = E(x + c)$.

Theorem 2.10. If x is a real number, then $E(-x) = \frac{1}{E(x)}$.

Proof. Suppose x is a real number. Since -x is a real number, by Theorem 2.9, $E(x)\cdot E(-x) = E(x - x) = E(0) = 1$. Since $E(x) \neq 0$, $E(-x) = \frac{1}{E(x)}$.

Definition 2.2. Suppose each of a and b is a real number. Let $I_{a,b}$ be a relation such that if p is a real number and there is a real number x such that $p = b + \int_a^x y(t)dt$, then $(p, x) \in I_{a,b}$.

Theorem 2.11. Suppose $I_{a,b}$ is the relation defined in Definition 2.2. Then, if $b \neq 0$, $I_{a,b}$ is a function.

Proof. To show that $I_{a,b}$ is a function, prove that no two ordered pairs in $I_{a,b}$ have the same first element. Suppose $(p_1, x_1) \in I_{a,b}$, $(p_2, x_2) \in I_{a,b}$, and $p_1 = p_2$, but $x_1 \neq x_2$. Since $p_1 = p_2$, by Definition 2.2, $\int_a^{x_1} y(t)dt = \int_a^{x_2} y(t)dt$. By Definition 1.16 and Theorem 1.12,

$0 = \left| \int_a^{x_1} y(t)dt - \int_a^{x_2} y(t)dt \right| = \left| \int_{x_2}^{x_1} y(t)dt \right|$. Since $x_1 \neq x_2$, either y is zero over $[x_1, x_2]$ or over $[x_2, x_1]$, or y is positive for some numbers in the integrable interval and negative for others. Since by Theorem 2.6 neither case is possible, then $p_1 \neq p_2$; therefore $I_{a,b}$ is a function.

Theorem 2.12. The function $I_{a,b}$ of Theorem 2.11 is unbounded.

Proof. Suppose $I_{a,b}$ is a bounded function. By Definition 1.2 there is a positive number Q such that if $p \in D_{I_{a,b}}$, then $|I_{a,b}(p)| < Q$. There is a real number x such that $|x| > Q$. Since $R_{I_{a,b}}$ is the set of all real numbers, then there is a number q such that $q \in D_{I_{a,b}}$ and $I_{a,b}(q) = x$. Then $|I_{a,b}(q)| < Q$. However $|I_{a,b}(q)| = |x| > Q$. Since the assumption that $I_{a,b}$ is bounded by Q leads to the contradiction that $|I_{a,b}(q)| > Q$, then $I_{a,b}$ is not bounded.

Theorem 2.13. Suppose $I_{a,b}$ is the function of Theorem 2.11 and $(p, c) \in I_{a,b}$. Then, if $q \in D_{I_{a,b}}$, $I_{a,b}(q) = c + \int_p^q \frac{1}{t}dt$.

Proof. Suppose $q \in D_{I_{a,b}'}$. Suppose $p < q$. Note that if $p > q$, then the proof is similar. Since $0 \notin D_{I_{a,b}'}$, by Definition 1.7 and Theorems 1.3 and 1.10, $\int_p^q \frac{1}{t}dt$ exists. Let ε be a positive number. There is a positive number δ such that if $p = s_0 < s_1 < \ldots < s_n = q$ and $s_i - s_{i-1} < \delta$ and $s_{i-1} \leq \xi_i \leq s_i$, $i = 1, 2, \ldots, n$, then $\left|\sum_{i=1}^{n} \frac{1}{\xi_i} (s_i - s_{i-1}) - \int_p^q \frac{1}{t}dt\right| < \varepsilon$. It is necessary to prove that $\int_p^q \frac{1}{t}dt = I_{a,b}(q) - c$. Suppose $p = s_0 < \ldots < s_n = q$ and $s_i - s_{i-1} < \delta$, $i = 1, 2, \ldots, n$. Suppose $I_{a,b}(s_0) = x_0$, $I_{a,b}(s_1) = x_1, \ldots, I_{a,b}(s_n) = x_n$. By Definitions 2.2 and 1.16 and Theorem 1.12 if $1 \leq i \leq n$, then $s_i - s_{i-1} = \int_{x_{i-1}}^{x_i} y(t)dt$. By Theorem 1.19 there is a number c in the interval of integration such that $y(c) = \frac{s_i - s_{i-1}}{x_i - x_{i-1}}$. By Theorem 2.6 $y(c) \in (s_{i-1}, s_i)$. Therefore if $1 \leq i \leq n$, let $\xi_i = \frac{s_i - s_{i-1}}{x_i - x_{i-1}}$. Then $\left|\sum_{i=1}^{n} \frac{1}{\xi_i} \cdot (s_i - s_{i-1}) - I_{a,b}(q) + c\right| =$

$$\left|\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{s_i - s_{i-1}} \cdot (s_i - s_{i-1}) - I_{a,b}(q) + c\right| =$$

$\left|x_n - x_0 - I_{a,b}(q) + c\right| = 0 < \varepsilon$. Therefore

$I_{a,b}(q) = c + \int_p^q \frac{1}{t}dt$.

Theorem 2.14. The function $I_{a,b}$ of Theorem 2.11 is continuous.

Proof. Suppose $q \in D_{I_{a,b}}$. Consider an interval $[s, t]$ such that $q \in (s, t)$. Since $0 \notin [s, t]$ by Definitions 1.7 and 1.2 and Theorems 1.3 and 1.4 there is a positive number T such that if $x \in [s, t]$, then $\left|\frac{1}{x}\right| < T$. Let ε be a positive number. Let $\delta = \text{minimum}\left\{\frac{\varepsilon}{T}, (q - s), (t - q)\right\}$. Suppose $q_1 \in D_{I_{a,b}}$ such that $|q_1 - q| < \delta$. Then by Definition 1.16 and Theorems 1.12, 1.13, 1.14, and 2.13,

$\left|I_{a,b}(q_1) - I_{a,b}(q)\right| = \left|\int_p^{q_1} \frac{1}{t}dt - \int_p^q \frac{1}{t}dt\right| \leq \left|\int_{q_1}^q \left|\frac{1}{t}\right|dt\right| <$

$T|q - q_1| < T\cdot\delta \leq \varepsilon$. Hence $I_{a,b}$ is continuous at $(q, I_{a,b}(q))$. Therefore $I_{a,b}$ is a continuous function.

Definition 2.3. Denote $I_{0,1}$ by L. Note that D_L is the set of all positive numbers.

Theorem 2.15. If s and t are positive numbers, then $L(s\cdot t) = L(s) + L(t)$.

Proof. Suppose s and t are positive numbers. Then $s \in D_L$ and $t \in D_L$. Suppose $L(s) = x_1$ and $L(t) = x_2$, then

$s = E(x_1)$ and $t = E(x_2)$. By Theorem 2.9 and Definition 2.2, $L(s\ t) = L[E(x_1)\cdot E(x_2)] = L[E(x_1 + x_2)] = x_1 + x_2 = L[E(x_1)] + L[E(x_2)] = L(s) + L(t)$.

Theorem 2.16. Suppose $p \in D_L$. If k is a real number, then $L[(p)^k] = k\cdot L(p)$.

Proof. Suppose $p \in D_L$ and $L(p) = x$. Suppose k is a positive integer. Then by Definition 2.2 and Theorem 2.9, $L[(p)^k] = L\{[E(x)]^k\} = L\{[E(x)]\cdot[E(x)]\cdot\ .\ .\ .\cdot[E(x)]\} = L[E(x + x + \ .\ .\ .\ + x)] = L[E(k\cdot x)] = k\cdot x = k\cdot L(p)$. Hence if k is a positive integer, then $L[(p)^k] = k\cdot L(p)$.

Suppose k is a negative integer. Then $-k$ is a positive integer. By Definition 2.2 and Theorems 2.9 and 2.10, $L[(p)^k] = L\{[E(x)]^k\} = L\left\{\frac{1}{[E(x)]^{-k}}\right\} = L\left\{\frac{1}{E(-k\ x)}\right\} = L[E(k\cdot x)] = k\cdot x = k\cdot L(p)$. Hence if k is a negative integer, then $L[(p)^k] = k\cdot L(p)$.

Suppose k is a positive rational number. There are positive integers s and q such that $k = \frac{s}{q}$. Thus

$L[(p)^k] = L\left\{(p)^{\frac{s}{q}}\right\}$. By previous work $q\cdot L\left[(p)^{\frac{s}{q}}\right] = L\left\{\left[(p)^{\frac{s}{q}}\right]^q\right\} = L[(p)^s] = s\cdot L(p)$. Thus $\frac{s}{q}\cdot L(p) = L\left[(p)^{\frac{s}{q}}\right]$. Hence if k is a positive rational number, then $L[(p)^k] = k\cdot L(p)$.

Suppose k is a negative rational number. There are positive integers s and q such that $k = \frac{-s}{q}$. Thus $L[(p)^k] = L\left[(p)^{\frac{-s}{q}}\right]$. By previous work,

$q \cdot L\left[(p)^{\frac{-s}{q}}\right] = L\left\{\left[(p)^{\frac{-s}{q}}\right]^q\right\} = L[(p)^{-s}] = -s \cdot L(p)$. Thus

$\frac{-s}{q} \cdot L(p) = L\left[(p)^{\frac{-s}{q}}\right]$. Hence if k is a negative rational number, then $L[(p)^k] = k \cdot L(p)$.

Suppose k is a real number. Let ε be a positive number. By continuity of L there is a positive number δ_1 such that if $q \in D_L$ and $\left|q - (p)^k\right| < \delta_1$, then $\left|L[(p)^k] - L(q)\right| < \frac{\varepsilon}{2}$. By a property of the real number system there is a positive number δ_2 such that if r is a rational number and $|r - k| < \delta_2$, then $\left|(p)^{r-k} - 1\right| < \delta_1 \cdot (p)^{-k}$. Let $\delta = \text{minimum}\left\{\frac{\varepsilon}{2|x| + 1}, \delta_1, \delta_2\right\}$. Suppose r is a rational number such that $|r - k| < \delta$, then $\left|(p)^{r-k} - 1\right| < \delta_1 \cdot (p)^{-k}$. Hence $\left|(p)^r - (p)^k\right| < \delta_1$ and $\left|L[(p)^k] - L[(p)^r]\right| < \frac{\varepsilon}{2}$. By previous work, $\left|L[(p)^k] - k \cdot L(p)\right|$ $= \left|L[(p)^k] - r \cdot x + r \cdot x - k \cdot x\right| \leq \left|L[(p)^k] - L[(p)^r]\right| + |x||k - r|$ $< \frac{\varepsilon}{2} + \delta|x| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore if k is a real number, then $L[(p)^k] = k \cdot L(p)$.

CHAPTER III

SOLUTION OF A STURM-LIOUVILLE TYPE SECOND ORDER DIFFERENTIAL EQUATION

In this chapter a second order differential equation of the Sturm-Liouville type with certain boundary conditions will be studied. Suppose p is a positive, continuous function over [0, 1] and q is a continuous function over [0, 1]. If y is a function such that y is differentiable over [0, 1] and $(p \cdot y')$ is differentiable over [0, 1], let $\mathcal{L}(y) = (p \cdot y')' - q \cdot y$ and $D_y = [0, 1]$. The purpose of this chapter is to prove the existence of solutions to $\mathcal{L}(y) = \underline{0}$ that satisfy certain boundary conditions and to study properties of these solutions.

Theorem 3.1. Suppose y is a function such that y is differentiable and $(p \cdot y')$ is differentiable over [0, 1]. If $\mathcal{L}(y) = \underline{0}$ and $a \in [0, 1]$, then

$$y(x) = y(a) + p(a) \cdot y'(a) \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot y(t) dt ds$$

for all $x \in [0, 1]$.

Proof. Suppose $\mathcal{L}(y) = \underline{0}$ and $a \in [0, 1]$. By Theorem 1.8, y is continuous over [0, 1]; therefore by Theorem 1.3, $q \cdot y$ is continuous over [0, 1]. By Theorem 1.10, $q \cdot y$ is integrable over [0, 1]. By Theorem 1.16 if $s \in [0, 1]$, then

$$\int_a^s [p(t)\cdot y'(t)]'dt = p(s)\cdot y'(s) - p(a)\cdot y'(a) = \int_a^s q(t)\cdot y(t)dt.$$

Since p is positive over [0, 1],

$y'(s) = p(a)\cdot y'(a)\cdot \frac{1}{p(s)} + \frac{1}{p(s)} \int_a^s q(t)\cdot y(t)dt.$ Clearly $y'(s)$ is integrable over [0, 1]. Then by Theorem 1.11 if $x \in [0, 1]$,

$$\int_a^x y'(s)ds = p(a)\cdot y'(a) \int_a^x \frac{1}{p(s)}ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t)\cdot y(t)dtds.$$

Thus by Theorem 1.16,

$$y(x) = y(a) + p(a)\cdot y'(a) \int_a^x \frac{1}{p(s)}ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t)\cdot y(t)dtds$$

for all $x \in [0, 1]$.

Theorem 3.2. Suppose y is continuous and

$y(x) = b + m \int_a^x \frac{1}{p(s)}ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t)\cdot y(t)dtds$ for all $x \in [0, 1]$, then $\mathcal{L}(y) = \underline{0}$, $b = y(a)$, and $m = p(a)\cdot y'(a)$.

Proof. By Definition 1.15, $y(a) = b$. By Theorems 1.7 and 1.17, $y'(x) = m\cdot\frac{1}{p(x)} + \frac{1}{p(x)} \int_a^x q(t)\cdot y(t)dt.$ Therefore by Definition 1.15, $y'(a) = m\cdot\frac{1}{p(a)}$, or $p(a)\cdot y'(a) = m$. Since $p(x)\cdot y'(x) = m + \int_a^x q(t)\cdot y(t)dt$, by Theorem 1.17, $[p(x)\cdot y'(x)]' = q(x)\cdot y(x)$. Therefore $\mathcal{L}(y) = \underline{0}$. This completes the proof of Theorem 3.2.

By Theorems 3.1 and 3.2 the differential equation $\mathcal{L}(y) = \underline{0}$ together with the boundary conditions $y(a) = b$ and $p(a) \cdot y'(a) = m$ is equivalent to the integral equation

$$y(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot y(t) dt ds.$$

A sequence of functions that will be used to prove the existence of a solution of $\mathcal{L}(y) = \underline{0}$ which satisfies certain boundary conditions is defined as follows. Suppose $a \in [0, 1]$. Let f_o be a continuous function over $[0, 1]$. By Definitions 1.6 and 1.7 and Theorems 1.3, 1.10, 1.17, and 1.11 a sequence $\{f_i\}_{i=o}^{\infty}$ of functions over $[0, 1]$ exists such that if $x \in [0, 1]$ and n is a positive integer, then

$$f_n(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot f_{n-1}(t) dt ds$$ where b and m are real numbers. It is noted that for each positive integer n, f_n is a continuous, differentiable, and integrable function whose domain is $[0, 1]$, $f_n(a) = b$, and $p(a) \cdot f_n'(a) = m$.

Consider the absolute value of the differences of successive terms of $\{f_n\}_{n=o}^{\infty}$. By Definitions 1.6 and 1.2 and Theorems 1.3 and 1.4 there is a positive number M such that if $t \in [0, 1]$, then $|f_1(t) - f_o(t)| < M$. Since q and p are continuous and p is positive over $[0, 1]$, by Definitions 1.7 and 1.2 and Theorems 1.3 and 1.4 there are positive numbers

$\frac{1}{P}$ and Q such that if $t \in [0, 1]$, then $\left|\frac{1}{p(t)}\right| < \frac{1}{P}$ and $|q(t)| < Q$. Therefore by Theorems 1.11, 1.13, and 1.14 if $x \in [0, 1]$, then $|f_2(x) - f_1(x)| =$

$\left|\int_a^x \frac{1}{p(s)} \int_a^s q(t)\cdot[f_1(t) - f_o(t)]dtds\right| \leq$

$\left|\int_a^x \left|\frac{1}{p(s)}\right| \int_a^s |q(t)|\cdot|f_1(t) - f_o(t)|dtds\right| < \left|\int_a^x \frac{1}{P}\int_a^s Q\cdot Mdtds\right| =$

$\frac{Q\cdot M}{P}\left|\int_a^x \int_a^s dtds\right| = \frac{Q\cdot M}{2\cdot P}(x - a)^2$. By induction it can be shown that if $n \geq 2$ and $x \in [0, 1]$, then

$$|f_n(x) - f_{n-1}(x)| < \frac{Q^{n-1}\ M|x - a|^{2(n-1)}}{P^{n-1}\ [2(n - 1)]!}.$$

Consider the series $\sum_{n=1}^{\infty} \frac{Q^{n-1}\ M}{P^{n-1}\ [2(n - 1)]!}$. Let ν be a number such that $0 < \nu < 1$. There is a positive integer N such that if $n > N$, then $\frac{Q}{P} < 2n(2n - 1)(1 - \nu)$. Hence

$\left|\dfrac{\dfrac{Q^n\ M}{P^n\ (2n)!}}{\dfrac{Q^{n-1}\ M}{P^{n-1}\ [2(n - 1)]!}}\right| = \left|\frac{Q}{P\cdot 2n(2n - 1)}\right| < 1 - \nu$. Therefore by Theorem 1.24, $\sum_{n=1}^{\infty} \frac{Q^{n-1}\ M}{P^{n-1}\ [2(n - 1)]!}$ converges. If $x \in [0, 1]$ and k is a positive integer, then $0 \leq (x - a)^{2k} \leq 1$. Therefore for any positive integer i,

$0 < \frac{Q^{i-1} M(x - a)^{2(i-1)}}{P^{i-1} [2(i - 1)]!} \leq \frac{Q^{i-1} M}{P^{i-1} [2(i - 1)]!}$. Hence if $x \in [0, 1]$, then by Theorem 1.25, $\sum_{n=1}^{\infty} \frac{Q^{n-1} M(x - a)^{2(n-1)}}{P^{n-1} [2(n - 1)]!}$ converges. These considerations lead to the following theorem.

Theorem 3.3. Suppose f_o is a continuous function over $[0, 1]$ and $f_n(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot f_{n-1}(t) dt ds$ for $n > 0$ and $x \in [0, 1]$. Then $\{f_n\}_{n=o}^{\infty}$ converges uniformly over $[0, 1]$ to a continuous function f. Furthermore for each $x \in [0, 1]$, $f(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot f(t) dt ds$.

Proof. Let ε be a positive number. By the work prior to the statement of this théorem there are positive numbers Q, M, and P such that if $n \geq 2$ and $x \in [0, 1]$, then

$$|f_n(x) - f_{n-1}(x)| < \frac{Q^{n-1} M(x - a)^{2(n-1)}}{P^{n-1} [2(n - 1)]!}.$$

Since $\sum_{n=1}^{\infty} \frac{M Q^{n-1}}{P^{n-1} [2(n - 1)]!}$ converges, by Theorem 1.23 there is a positive integer N such that if $m > n > N$, then

$$\sum_{i=n+1}^{m} \frac{M Q^{i-1}}{P^{i-1} [2(i - 1)]!} < \varepsilon.$$

Suppose $m > n > N$ and $x \in [0, 1]$. Then

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_{m-1}(x)| +$$

$$|f_{m-1}(x) - f_{m-2}(x)| + \ . \ . \ . + |f_{n+1}(x) - f_n(x)|$$

$$< \sum_{i=n+1}^{m} \frac{Q^{i-1}\ M(x - a)^{2(i-1)}}{P^{i-1}\ [2(i - 1)]!} < \sum_{i=n+1}^{m} \frac{Q^{i-1}\ M}{P^{i-1}\ [2(i - 1)]!} < \varepsilon.$$

Therefore $\{f_n(x)\}_{n=o}^{\infty}$ converges to a number, call it $f(x)$. Hence by Definition 1.19, $\{f_n\}_{n=o}^{\infty}$ converges uniformly over $[0, 1]$ to the function f. By Theorem 1.22, f is a continuous function. It will next be shown that if $x \in [0, 1]$, then

$$f(x) = b + m \int_a^x \frac{1}{p(s)}ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds.$$

Suppose there is an $x_o \in [0, 1]$ such that

$$f(x_o) \neq b + m \int_a^{x_o} \frac{1}{p(s)}ds + \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds.$$ Then

$\{f_n(x_o)\}_{n=o}^{\infty}$ does not converge to

$b + m \int_a^{x_o} \frac{1}{p(s)}ds + \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds$. Clearly $x_o \neq a$. Hence there is a positive number εo such that if N is a positive integer, then there is an $n > N$ such that

$$\left| f_n(x_o) - b - m \int_a^{x_o} \frac{1}{p(s)}ds - \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds \right| \geq \varepsilon o.$$

By work prior to the statement of this theorem, there are positive numbers Q and P such that if $t \in [0, 1]$, then $|q(t)| < Q$ and $\left|\frac{1}{p(t)}\right| < \frac{1}{P}$. Since $\{f_n\}_{n=o}^{\infty}$ converges uniformly to f there is a positive integer N_o such that if $n > N_o$ and $t \in [0, 1]$, then $|f_n(t) - f(t)| < \frac{\varepsilon_o \cdot P}{3(x_o - a)^2 \cdot Q}$.

Let $n > N_o + 1$ and

$$\left|f_n(x_o) - b - m \int_a^{x_o} \frac{1}{p(s)}ds - \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds\right| \geq \varepsilon_o.$$

Then by definition of f_n and Theorems 1.11, 1.13, and 1.14,

$$\varepsilon_o \leq \left|f_n(x_o) - b - m \int_a^{x_o} \frac{1}{p(s)}ds - \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds\right| =$$

$$\left|b + m \int_a^{x_o} \frac{1}{p(s)}ds + \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f_{n-1}(t)dtds - b - m \int_a^{x_o} \frac{1}{p(s)}ds\right.$$

$$\left.- \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds\right| = \left|\int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)[f_{n-1}(t) - f(t)]dtds\right|$$

$$\leq \left|\int_a^{x_o}\left|\frac{1}{p(s)}\right| \int_a^s |q(t)||f_{n-1}(t) - f(t)|\,dtds\right| <$$

$$\left|\int_a^{x_o} \frac{1}{P} \int_a^s \frac{Q\cdot\varepsilon_o\cdot P}{3(x_o - a)^2\, Q}dtds\right| = \frac{1}{3(x_o - a)^2}\left|\int_a^{x_o} \int_a^s \varepsilon_o dtds\right| = \frac{\varepsilon_o}{3}.$$

The assumption that there exists an $x_o \in [0, 1]$ such that $f(x_o) \neq b + m \int_a^{x_o} \frac{1}{p(s)}ds + \int_a^{x_o} \frac{1}{p(s)} \int_a^s q(t)\cdot f(t)dtds$ leads to the contradiction that $\varepsilon_o < \frac{\varepsilon_o}{3}$. Hence

$f(x_o) = b + m \int_a^{x_o} \frac{1}{p(s)}ds + \int_a^{x_o} \frac{1}{p(s)}ds \int_a^s q(t)\cdot f(t)dtds$. Therefore if $x \in [0, 1]$, then

$$f(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot f(t) dt ds.$$ This completes the proof of Theorem 3.3.

By Theorem 3.2 the function f in Theorem 3.3 has the properties that $\mathfrak{L}(f) = \underline{0}$, $f(a) = b$, and $p(a) \cdot f'(a) = m$. This shows that the differential equation $\mathfrak{L}(y) = \underline{0}$ together with the boundary conditions $y(a) = b$ and $p(a) \cdot y'(a) = m$ has a solution over $[0, 1]$.

Theorem 3.4. Suppose f is a continuous function over $[0, 1]$ such that $f(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot f(t) dt ds$ for all $x \in [0, 1]$. If for all $x \in [0, 1]$

$$g(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot z(t) dt ds,$$ then $g = f$.

Proof. Let ε be a positive number. Suppose for all

$$x \in [0, 1], \; z(x) = b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot z(t) dt ds.$$

By Theorems 3.2 and 1.8, g is continuous over $[0, 1]$. By Theorems 1.3 and 1.4 there is a positive number M' such that if $t \in [0, 1]$, then $|f(t) - g(t)| < M'$. Since the series $\sum_{n=1}^{\infty} \frac{Q^{n-1} M}{P^{n-1} [2(n-1)]!}$ used in Theorem 3.3 converges, by Theorem 1.26, $\frac{M'}{M} \sum_{n=1}^{\infty} \frac{Q^{n-1} M}{P^{n-1} [2(n-1)]!}$ converges. Furthermore since $|x - a| \leq 1$ for all $x \in [0, 1]$, then by Theorem 1.25, $\sum_{n=1}^{\infty} \frac{Q^{n-1} M'(x - a)^{2(n-1)}}{P^{n-1} [2(n - 1)]!}$ converges for all $x \in [0, 1]$. By a variation of Theorem 1.23 there is a positive integer N such that if $n > N$, then $\frac{M' Q^{n-1}(x - a)^{2(n-1)}}{P^{n-1} [2(n - 1)]!} < \varepsilon$. Let $n > N$. Suppose $x \in [0, 1]$. Then by Theorems 1.11, 1.13, and 1.14, $|g(x) - f(x)| =$

$$\left| b + m \int_a^x \frac{1}{p(s)} ds + \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot g(t) dt ds - b - m \int_a^x \frac{1}{p(s)} ds - \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot f(t) dt ds \right| = \left| \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot [g(t) - f(t)] dt ds \right|$$

$$\leq \left| \int_a^x \left| \frac{1}{p(s)} \right| \int_a^s |q(t)| |g(t) - f(t)| dt ds \right| < \frac{Q M'}{P} \left| \int_a^x \int_a^s dt ds \right|$$

$= \frac{Q M'}{2 P} (x - a)^2$. Hence $|g(x) - f(x)| < \frac{Q M'}{2 P} (x - a)^2$. Thus by Theorems 1.11, 1.13, and 1.14, $|g(x) - f(x)| =$

$$\left| \int_a^x \frac{1}{p(s)} \int_a^s q(t) \cdot [g(t) - f(t)] dt ds \right| < \frac{Q}{P} \left| \int_a^x \int_a^s \frac{Q M'(t - a)^2}{2 P} dt ds \right|$$

$= \frac{Q^2 M'(x - a)^4}{4! P^2}$. After n repetitions of this procedure,

$|g(x) - f(x)| < \frac{M' Q^{n-1}(x - a)^{2(n-1)}}{P^{n-1} [2(n - 1)]!} < \varepsilon$. Hence $g(x) = f(x)$.

Therefore $g = f$ over $[0, 1]$. Thus the solution of $\mathcal{X}(y) = \underline{0}$ that contains the point (a, b) and has a slope of $m \cdot \frac{1}{p(a)}$ at a is unique.

Theorem 3.5. Let u and v be any two solutions of $\mathcal{L}(y) = \underline{0}$. There exists a real number γ such that if $x \in [0, 1]$, then $p(x)\cdot[u(x)\cdot v'(x) - u'(x)\cdot v(x)] = \gamma$.

Proof. Denote $p\cdot(u\cdot v' - u'\cdot v)$ by g. Since $\mathcal{L}(v) = \underline{0}$ and $\mathcal{L}(u) = \underline{0}$, then $(p\cdot v')' = q\cdot v$ and $(p\cdot u')' = q\cdot u$. Therefore by Theorem 1.7,
$g' = (p\cdot v')'\cdot u + (p\cdot v')\cdot u' - (p\cdot u')'\cdot v - (p\cdot u')\cdot v' =$
$q\cdot v\cdot u + p\cdot v'\cdot u' - q\cdot u\cdot v - p\cdot u'\cdot v' = 0$. Hence by Theorem 1.9 there is a real number γ such that $g = \gamma$ over $[0, 1]$. To determine γ, evaluate g at a. Therefore,
$g(a) = p(a)\cdot[u(a)\cdot v'(a) - u'(a)\cdot v(a)] = \gamma$. Thus if $x \in [0, 1]$, then $p(x)\cdot[u(x)\cdot v'(x) - u'(x)\cdot v(x)] = \gamma$.

Theorem 3.6. Suppose ξ and η are the solutions of $\mathcal{L}(y) = \underline{0}$ such that $\xi(0) = 1$, $\eta(0) = 0$, $p(0)\cdot\xi'(0) = 0$, and $p(0)\cdot\eta'(0) = 1$. If $x \in [0, 1]$, then
$p(x)\cdot[\xi(x)\cdot\eta'(x) - \xi'(x)\cdot\eta(x)] = 1$.

Proof. By Theorem 3.5 there is a real number ν such that if $x \in [0, 1]$, then $p(x)\cdot[\xi(x)\cdot\eta'(x) - \xi'(x)\cdot\eta(x)] = \nu$. Since $0 \in [0, 1]$, $\nu = p(0)\cdot[\xi(0)\cdot\eta'(0) - \xi'(0)\cdot\eta(0)] = 1$. Therefore if $x \in [0, 1]$, then
$p(x)\cdot[\xi(x)\cdot\eta'(x) - \xi'(x)\cdot\eta(x)] = 1$.

Theorem 3.7. Suppose f is the solution of $\mathcal{L}(f) = \underline{0}$ such that $0 \leq a \leq 1$, $f(a) = b$, and $p(a)\cdot f'(a) = m$. Then, if $x \in [0, 1]$, $f(x) = [b\cdot p(a)\cdot\eta'(a) - m\cdot\eta(a)]\cdot\xi(x) -$
$[b\cdot p(a)\cdot\xi'(a) - m\cdot\xi(a)]\cdot\eta(x)$ where ξ and η are the solutions of $\mathcal{L}(f) = \underline{0}$ described in Theorem 3.6.

Proof. Let g denote $[b\cdot p(a)\cdot\eta'(a) - m\cdot\eta(a)]\cdot\xi - [b\cdot p(a)\cdot\xi'(a) - m\cdot\xi(a)]\cdot\eta$. If $g(a) = b$, $p(a)\cdot g'(a) = m$, and $\mathfrak{L}(g) = \underline{0}$, then by Theorems 3.2 and 3.4, $g = f$ over $[0, 1]$. First by Theorem 3.6,
$g(a) = [f(a)\cdot p(a)\cdot\eta'(a) - p(a)\cdot f'(a)\cdot\eta(a)]\cdot\xi(a) - [f(a)\cdot p(a)\cdot\xi'(a) - p(a)\cdot f'(a)\cdot\xi(a)]\cdot\eta(a) = f(a)\cdot p(a)\cdot[\xi(a)\cdot\eta'(a) - \xi'(a)\cdot\eta(a)] = f(a)\cdot 1 = b$. Therefore $g(a) = b$. Next by Theorems 1.7 and 3.6, $p(a)\cdot g'(a) = p(a)\cdot[f(a)\cdot p(a)\cdot\eta'(a) - p(a)\cdot f'(a)\cdot\eta(a)]\cdot\xi'(a) - p(a)\cdot[f(a)\cdot p(a)\cdot\xi'(a) - p(a)\cdot f'(a)\cdot\xi(a)]\cdot\eta'(a) = [p(a)]^2\cdot f(a)\cdot\eta'(a)\cdot\xi'(a) - [p(a)]^2\cdot f'(a)\cdot\eta(a)\cdot\xi'(a) - [p(a)]^2\cdot f(a)\cdot\xi'(a)\cdot\eta'(a) + [p(a)]^2\cdot f'(a)\cdot\eta'(a)\cdot\xi(a) = p(a)\cdot f'(a)\cdot\{p(a)\ [\xi(a)\ \eta'(a) - \xi'(a)\ \eta(a)]\} = p(a)\cdot f'(a)\cdot 1$. Therefore $p(a)\cdot g'(a) = m$. Finally since each of f, ξ, and η is a solution of $\mathfrak{L}(y) = \underline{0}$ by Theorems 1.7 and 3.5, $(p\ g')' = \{p\cdot[b\cdot p(a)\cdot\eta'(a) - m\cdot\eta(a)]\cdot\xi' - p\cdot[b\cdot p(a)\cdot\xi'(a) - m\cdot\xi(a)]\cdot\eta'\}' = q\cdot[b\cdot p(a)\cdot\eta'(a) - m\cdot\eta(a)]\cdot\xi - q\cdot[b\cdot p(a)\cdot\xi'(a) - m\cdot\xi(a)]\cdot\eta = q\cdot g$. Therefore $\mathfrak{L}(g) = (p\cdot g')' - q\cdot g = \underline{0}$. Hence $g = f$ over $[0, 1]$. Thus if $x \in [0, 1]$, then $f(x) = [b\cdot p(a)\cdot\eta'(a) - m\cdot\eta(a)]\cdot\xi(x) - [b\cdot p(a)\cdot\xi'(a) - m\cdot\xi(a)]\cdot\eta(x)$.

Theorem 3.8. Suppose $\mathcal{L}(u) = \underline{0}$, $\mathcal{L}(v) = \underline{0}$, and u and v are linearly independent functions over [0, 1].

Part A. The graph of neither u nor v is tangent to the x-axis.

Part B. Each of the functions, u and v, has at most a finite number of roots in [0, 1].

Part C. The functions u and v have no common root in [0, 1].

Part D. The derivatives u' and v' of the functions u and v have no common root in [0, 1].

Proof of Part A. Suppose one of the functions, say u, is tangent to the x-axis. Then there is an $x_o \in [0, 1]$ such that $u(x_o) = u'(x_o) = 0$. By Theorem 3.7 if $x \in [0, 1]$, then $u(x) = [u(x_o) \cdot p(x_o) \cdot \eta'(x_o) - p(x_o) \cdot u'(x_o) \cdot \eta(x_o)] \cdot \xi(x) - [u(x_o) \cdot p(x_o) \cdot \xi'(x_o) - p(x_o) \cdot u'(x_o) \cdot \xi(x_o)] \cdot \eta(x) = 0$. Thus if $c_1 = 0$ and c_2 is a real number, not zero, then $c_1 \cdot v + c_2 \cdot u = \underline{0}$ over [0, 1]. By Definition 1.22, c_2 would have to be zero. Since the assumption that u is tangent to the x-axis leads to a contradiction of the fact that u and are linearly independent functions over [0, 1], then neither function is tangent to the x-axis.

Proof of Part B. Suppose one of the functions, say u, has infinitely many roots in [0, 1]. Let $M = \{x \mid u(x) = 0 \text{ and } x \in [0, 1]\}$. Since M is an infinite, bounded set by Theorem 1.2 it has a limit point, call it x_0. Thus $u(x_0) = 0$ since u is continuous at $(x_0, u(x_0))$. Let ε be a positive number. There is a positive number δ such that if $x \in [0, 1]$ and $0 < |x - x_0| < \delta$, then

$$\left|\frac{u(x) - u(x_0)}{x - x_0} - u'(x_0)\right| < \varepsilon.$$ By Definition 1.11 there is an $x \in M$ such that $0 < |x - x_0| < \delta$. Therefore,

$$\left|\frac{u(x) - u(x_0)}{x - x_0} - u'(x_0)\right| = |u'(x_0)| < \varepsilon.$$ Thus $u'(x_0) = 0$.

Hence u is tangent to the x-axis at x_0; but by Part A of Theorem 3.8, this is not possible. Therefore u has at most a finite number of roots in $[0, 1]$.

Proof of Part C. Suppose u and v have a common root, x. Since $u(x) = v(x) = 0$, by Theorem 3.5, $p \cdot (u \cdot v' - u' \cdot v) = \underline{0}$ over $[0, 1]$. Since p is positive $u \cdot v' - u' \cdot v = \underline{0}$ over $[0, 1]$. By Part A of Theorem 3.8 neither function is tangent to the x-axis. Thus for each $x \in [0, 1]$, $u(x) = 0$ if and only if $v(x) = 0$. Let $x_1 < x_2 < \ldots < x_n$ be the common roots of u and v in $[0, 1]$. By Definitions 1.6 and 1.7 if $v \neq \underline{0}$, then $\frac{u}{v}$ is a function. Consider $\frac{u}{v}$ over the open intervals (x_{i-1}, x_i), $i = 1, 2, \ldots, n+1$, $x_0 = 0$, and $x_{n+1} = 1$. Since $u' \cdot v - u \cdot v' = \underline{0}$ by Theorem 1.7,

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2} = \underline{0}$$ over the open intervals (x_{i-1}, x_i).

Therefore by Theorem 1.9 there is a sequence of real numbers $\{c_i\}_{i=1}^{n+1}$ such that $u(x) = c_i \cdot v(x)$ for $x \in (x_{i-1}, x_i)$. Note that if 0 or 1 is a common root there will be a slight adjustment in the notation. Let ε be a positive number. Since v is differentiable at $(x_1, v(x_1))$ there is a positive number δ such that if $s \in [0, 1]$ and $0 < |s - x_1| < \delta$, then $\left|\frac{v(s) - v(x_1)}{s - x_1} - v'(x_1)\right| < \frac{\varepsilon}{|c_1 \cdot c_2| + 1}$. There is an $s \in [0, 1]$ such that $s < x_1$ and $0 < |s - x_1| < \delta$. Therefore $\left|\frac{u(s) - u(x_1)}{s - x_1} - c_1 \cdot v'(x_1)\right| = \left|\frac{c_1 \cdot v(s) - c_1 \cdot v(x_1)}{s - x_1} - c_1 \cdot v'(x_1)\right|$ $< |c_1| \frac{\varepsilon}{|c_1 \cdot c_2| + 1} < \varepsilon$. Therefore by Definition 1.12, $u'(x_1) = c_1 \cdot v'(x_1)$. There is a $t \in [0, 1]$ such that $t > x_1$ and $0 < |t - x_1| < \delta$. Therefore $\left|\frac{u(t) - u(x_1)}{t - x_1} - c_2 \cdot v'(x_1)\right| =$ $\left|\frac{c_2 \cdot v(t) - c_2 \cdot v(x_1)}{t - x_1} - c_2 \cdot v'(x_1)\right| < |c_2| \frac{\varepsilon}{|c_1 \cdot c_2| + 1} < \varepsilon$. Therefore by Definition 1.12, $u'(x_1) = c_2 \cdot v'(x_1)$. Since by Part A of Theorem 3.8, $v'(x_1) \neq 0$, then $c_1 = c_2$. It can be proved by a similar method that for i and j such that $0 < i \leq n + 1$ and $0 < j \leq n + 1$, $c_i = c_j$. Hence $u(x) = c_1 \cdot v(x)$ for $x \in [0, 1]$, and u and v are not linearly independent over [0, 1]. However u and v are linearly independent over [0, 1]; therefore u and v have no common root in [0, 1].

Proof of Part D. Suppose there is an $x_0 \in [0, 1]$ such that $v'(x_0) = u'(x_0) = 0$. Then by Theorem 3.5 and the fact that p is positive over [0, 1], $u' \cdot v - u \cdot v' = \underline{0}$ over [0, 1]. Suppose one of the functions, say u, has a root at $x_1 \in [0, 1]$. Since by Part A of Theorem 3.8, u is not tangent to the x-axis, then $u'(x_1) \neq 0$; hence $x_0 \neq x_1$. Since $u'(x_1) \cdot v(x_1) - u(x_1) \cdot v'(x_1) = 0$, then $v(x_1) = 0$. But u and v have no common root by Part C of Theorem 3.8; therefore $v(x_1) \neq 0$ and $u(x_1) \neq 0$. Hence either the derivatives of u and v do not have a common root at x_0, or neither u nor v has a root in [0, 1]. Suppose neither u nor v has a root in [0, 1]. Since $u' \cdot v - u \cdot v' = \underline{0}$ over [0, 1], by Definitions 1.6 and 1.7 and Theorem 1.7,

$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2} = \underline{0}$ over [0, 1]. By Theorem 1.9 there is a real number c such that $\frac{u}{v} = c$ over [0, 1]. Hence u and v are linearly dependent. But u and v are linearly independent. Therefore the derivatives of u and v have no common root. This completes the proof of Theorem 3.8.

Suppose p has continuous first and second derivatives over [0, 1].

Theorem 3.9. The substitution $y = \frac{z}{\sqrt{p}}$ transforms the differential equation $\mathfrak{L}(y) = \underline{0}$ into $M(z) = z'' + Q\cdot z = \underline{0}$, where Q is a continuous function of x over [0, 1].

Proof. Substitute $\frac{z}{\sqrt{p}}$ for y in $\mathfrak{L}(y) = \underline{0}$. Then $\left[p\cdot\left(\frac{z}{\sqrt{p}}\right)'\right]' = q\cdot\frac{z}{\sqrt{p}}$. Since p and y are differentiable, z is differentiable; therefore by Theorem 1.7,

$$\left[p\cdot\left(\frac{z}{\sqrt{p}}\right)'\right]' = \left[p\cdot\frac{z'\cdot\sqrt{p} - \frac{1}{2}\cdot z\cdot\frac{1}{\sqrt{p}}\, p'}{p}\right]' =$$

$$z''\cdot\sqrt{p} + \frac{z'\cdot p'}{2\cdot\sqrt{p}} - \frac{z'\cdot p'}{2\cdot\sqrt{p}} - \frac{z\cdot p''}{2\cdot\sqrt{p}} + \frac{z\cdot p'}{2\cdot p\cdot\sqrt{p}} =$$

$\sqrt{p}\cdot\left[z'' + \frac{z\cdot p'}{4\cdot p^2} - \frac{z\cdot p''}{2\cdot p}\right]$. Hence $\sqrt{p}\cdot\left[z'' + \frac{z\cdot p'}{4\cdot p^2} - \frac{z\cdot p''}{2\cdot p}\right] = q\cdot\frac{z}{\sqrt{p}}$. Thus $z'' + z\cdot\left[\frac{p'}{4\cdot p^2} - \frac{p''}{2\cdot p} - \frac{q}{p}\right] = \underline{0}$. Denote $\frac{p'}{4\cdot p^2} - \frac{p''}{2\cdot p} - \frac{q}{p}$ by Q. Note that Q is a continuous function of x over [0, 1]. Let $M(z) = z'' + Q\cdot z$. Then $M(z) = \underline{0}$.

Theorem 3.10. Let $M_1(z) = z'' + Q_1\cdot z$ and $M_2(z) = z'' + Q_2\cdot z$ where Q_1 and Q_2 are continuous over [0, 1]. Furthermore suppose that $Q_2(x) \geq Q_1(x)$ for all $x \in [0, 1]$ and there is at least one x in each subinterval of [0, 1] such that $Q_2(x) > Q_1(x)$. Suppose z_1 and z_2 are solutions of $M_1(z) = \underline{0}$ and $M_2(z) = \underline{0}$, respectively, and neither z_1 nor z_2

is identically equal to zero. Then there is at least one root of z_2 between any two roots of z_1.

Proof. Suppose $M_1(z_1) = \underline{0}$, $M_2(z_2) = \underline{0}$, and neither z_1 nor z_2 is identically zero over [0, 1]. Theorem 3.8 can be used to prove that any non-trivial solution of $\mathfrak{X}(f) = \underline{0}$ has at most a finite number of roots in [0, 1]. Since p has no root in [0, 1], y_1 has at most a finite number of roots in [0, 1], and $z_1 = \sqrt{p}\cdot y_1$, then z_1 has at most a finite number of roots in [0, 1]. Suppose s and t are consecutive roots of z_1. Since for any non-zero real number c, $M_1(c\cdot z_1) = \underline{0}$ and there is a non-zero real number c such that $c\cdot z_1 > 0$ over (s, t), it suffices to assume that $z_1 > 0$ over (s, t). Suppose z_2 has no root in (s, t). Since for any non-zero real number c, $M_2(c\cdot z_2) = 0$ and there is a non-zero real number c such that $c\cdot z_2 > 0$ over (s, t), it suffices to assume that $z_2 > 0$ over (s, t). Consider $g = z_1'\cdot z_2 - z_1\cdot z_2'$. By definition of M_1 and M_2, g' exists and by Theorem 1.7, $g' = z_1''\cdot z_2 - z_1\cdot z_2''$. Since $Q_1\cdot z_1 = z_1''$ and $Q_2\cdot z_2 = z_2''$, then $g' = z_2\cdot Q_2\cdot z_1 - z_1\cdot Q_1\cdot z_2$ over [0, 1]. Since z_1 and z_2 are continuous g' is continuous over (s, t) by Theorem 1.3. By Theorem 1.10, g' is integrable over [s, t]; therefore

$\int_s^t g'(r)dr = \int_s^t z_1(r)\cdot z_2(r)\cdot[Q_2(r) - Q_1(r)]dr$. By

Theorem 1.16, $\int_s^t g'(r)dr = g(t) - g(s) =$

$z_1'(t)\cdot z_2(t) - z_2'(t)\cdot z_1(t) - z_1'(s)\cdot z_2(s) + z_2'(s)\cdot z_1(s) =$

$z_1'(t)\cdot z_2(t) - z_1'(s)\cdot z_2(s)$. Since $z_1'(s) \geq 0$ and $z_2(s) \geq 0$, then $- z_1'(s)\cdot z_2(s) \leq 0$. And since $z_1'(t) \leq 0$ and $z_2(t) \geq 0$, then $z_1'(t)\cdot z_2(t) \leq 0$. Therefore $\int_s^t g'(r)dr \leq 0$. However since $z_1 > 0$ and $z_2 > 0$ over (s, t), and for each subinterval of (s, t), there is a number x such that $Q_2(x) - Q_1(x) > 0$, then by Theorem 1.15, $\int_s^t z_1(r)\cdot z_2(r)\cdot[Q_2(r) - Q_1(r)]dr > 0$.

Therefore, $\int_s^t g'(r)dr \neq \int_s^t z_1(r)\cdot z_2(r)\cdot[Q_2(r) - Q_1(r)]dr$.

Hence z_2 has a root between s and t. Therefore z_2 has a root between any two roots of z_1.

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