THE LAPLACE TRANSFORMATION

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THE LAPLACE TRANSFORMATION

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CHAPTER I

FUNCTIONS AND CONTINUITY

Definition 1-1: A function, denoted f, is a set of ordered pairs of real numbers, the first element of which is denoted by t, and the second element of which is denoted by f(t). The set of all t is called the domain of the function and is denoted by D(f); the set of all f(t) is called the range of the function and is denoted by R(f). Frequently, a function will be referred to as f(t).

Definition 1-2: f(t) is said to be continuous at a point a if and only if f(a) exists, and for each positive real number ε, there is a positive real number δ such that if $|t-a| < \delta$ and f(t) exists, then $|f(t)-f(a)| < \varepsilon$.

Definition 1-3: If f(t) is not continuous at a point a, it is said to be discontinuous at the point a.

It is seen that f(t) is discontinuous at a point a if and only if f(a) does not exist or f(a) exists and there is a positive real number ε, so that if δ is a positive real number, there is a number t so that $|t-a| < \delta$, f(t) exists and $|f(t)-f(a)| \geq \varepsilon$.

Example: If for all real numbers t, $f(t) = t$, and a is a real number, then $f(a) = a$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If t is any real number so that $|t-a| < \delta$ and $f(t)$ exists, then $f(t) = t$ and $|f(t)-f(a)| = |t-a| < \delta = \varepsilon$. Hence, $f(t)$ is continuous at any real number a.

In order to investigate the continuity of some functions, the following Lemmas will be proved.

Lemma: If k is a rational number and i is an irrational number, then $i+k$ is an irrational number.

Proof: Let k be a rational number and i be an irrational number. Assume $i+k$ is a rational number, say s. Since k is a rational number, it is expressible as p/q where each of p and q is an integer and $q \neq 0$. Since s is a rational number, it is expressible as m/n where each of m and n is an integer and $n \neq 0$. $i+k = s$ becomes $i+p/q = m/n$ and $i = m/n - p/q = \frac{m \cdot q - p \cdot n}{n \cdot q}$. The integers are closed with respect to addition and multiplication; hence $m \cdot q - p \cdot n$ is an integer and $n \cdot q$ is an integer. Thus i is expressible as the quotient of two integers, but this implies i is a rational number, a contradiction. Therefore, the sum of a rational number and an irrational number is an irrational number.

Lemma: If $k \neq 0$ is a rational number and i is an irrational number, then $i \cdot k$ is an irrational number.

Proof: Let $k \neq 0$ be a rational number and i be an irrational number. Assume $i \cdot k$ is a rational number, say s. Let $k = \frac{p}{q}$ and $s = \frac{m}{n}$, where each of p, q, m, and n are integers and

$p \neq 0$, $q \neq 0$, and $n \neq 0$. This can be done because k and s are rational numbers. Now, $i \cdot k = s$ implies $i \cdot \frac{p}{q} = \frac{m}{n}$ and $m \neq 0$, for if so then $i \cdot k = 0$ and either $i = 0$, a contradiction, or $k = 0$, a contradiction. Since $i \cdot \frac{p}{q} = \frac{m}{n}$, $i = \frac{m \cdot q}{n \cdot p}$. Again because of closure, $m \cdot q$ and $n \cdot p$ are integers and hence i is rational, a contradiction. Thus, the product of a non-zero rational number and an irrational number is an irrational number.

Example: Define $f(t) = 0$ if t is an irrational number and $f(t) = 1$ if t is a rational number. If a is a rational number let $\varepsilon = \frac{3}{4} > 0$ and δ be a positive real number. δ is either a rational or an irrational number.

Case I: If δ is a rational number, $\frac{\delta}{4}$ is a rational number and $\frac{\pi}{4} \cdot \delta$ is an irrational number. Now, $|\frac{\pi}{4}\delta| < \delta$ and $|a - \frac{\pi}{4}\delta - a| < \delta$. Let t be the irrational number $a - \frac{\pi}{4}\delta$, then $|t-a| < \delta$, and $f(t) = 0$. $|f(t)-f(a)| = |0-1| = 1 \nless \frac{3}{4} = \varepsilon$.

Case II: If δ is any irrational number, $\frac{3}{4}\delta$ is an irrational number. Now, $|\frac{3}{4}\delta| < \delta$ and $|a - \frac{3}{4}\delta - a| < \delta$. Let t be the irrational number $a - \frac{3}{4}\delta$, then $|t-a| < \delta$ and $f(t) = 0$. $|f(t)-f(a)| = |0-1| = 1 \nless \frac{3}{4} = \varepsilon$. Thus, f(t) is not continuous at any rational number a.

Definition 1-4: f(t) is said to be continuous on a set if and only if f(t) is continuous at each point of the set.

Definition 1-5: f(t) is said to be discontinuous on a set if it is not continuous on the set.

Remark: In the first example, $f(t)$ was defined on the set of all real numbers and was found to be continuous at any real number. Hence, $f(t) = t$ is continuous on the set of all real numbers. In the second example, $f(t)$ was defined on the set of all real numbers and was found to be discontinuous at all rational numbers. Hence, $f(t)$ is discontinuous on the set of all real numbers.

Example: Let a function f be defined in the following manner. If t is an irrational number, $f(t) = 0$. If t is a rational number, $t = \frac{p}{q}$ where p is zero or a positive integer, q is a positive integer and p and q are relatively prime, then $f(t) = \frac{1}{q}$. If $a \geq 1$, then there is a positive integer p and a number b so that $0 \leq b < 1$ and $a = p+b$. Now, $f(a) = f(p+b) = f(b)$. Hence, if f is continuous for $0 \leq t < 1$ and p is a positive integer, then f is continuous at $p+t$. Also, if f is not continuous at $0 \leq t < 1$, and p is a positive integer, then f is not continuous at $p+t$. Therefore, the investigation of the continuity of f will be limited to zero and numbers in the segment $(0,1)$. Let $0 \leq t < 1$ be rational, $t = \frac{p}{q}$, $f(t) = \frac{1}{q}$. Now, $0 < q < q+1$, so $\frac{1}{q} > \frac{1}{q+1} > 0$ and $\left(\frac{1}{q} - \frac{1}{q+1} \right) > 0$. Suppose $f(t)$ is continuous, then for the positive real number $\varepsilon = \left(\frac{1}{q} - \frac{1}{q+1}\right)$, there is a δ so that if a is a real number so that $|a-t| < \delta$ and $f(a)$ exists, then $|f(a)-f(t)| < \varepsilon$. Let a be an irrational number so that $|a-t| < \delta$ and $f(a)$ exists. Since in any interval there is at least one rational number and at least one irrational number, there is at least one such a. Now, $f(a) = 0$ and $|f(a)-f(t)| = |0-\frac{1}{q}| = \frac{1}{q} \not<$ $(\frac{1}{q} - \frac{1}{q+1}) = \varepsilon$. Hence, $f(t)$ is discontinuous at all rational numbers. Let $0 \leq t < 1$ be an irrational number, then $f(t) = 0$. Let $\varepsilon > 0$ and choose a positive integer N so that $N \geq 2$ and $\frac{1}{N} < \varepsilon$. Consider the set $S = \{x |\ x \varepsilon [0,1],$ x is rational, $x = \frac{p}{q}$ where each of p and q is a positive integer, $0 \leq p < q$, p and q are relatively prime, and $q \leq N\}$. There are N-1 positive integers which are less than N. There are $1 + \frac{(N-2)(N-1)}{2}$ rational expressions of the form $\frac{p}{q}$ where $0 < q < N$ and $0 \leq p < q$. Let y be an element of a set M if and only if y is one of these expressions or $y = 1$. Let z belong to the set K if and only if there is an element x in M so that $z = |t-x|$. K is a finite set of positive numbers, and hence has a smallest element d. Let $\delta = \frac{1}{2}$ d. If $|a-t| < \delta$ and $f(a)$ exists then $0 < a < 1$, also a is not an element of M. If a is irrational, $f(a) = 0$ and $|f(a)-f(t)| = |0-0| = 0 < \varepsilon$. If a is a rational number, $a = \frac{p}{q}$, $q \geq N$, $f(a) = \frac{1}{q} \leq \frac{1}{N} < \varepsilon$, and $|f(a)-f(t)| = |\frac{1}{q} - 0| < \varepsilon$. Therefore, $f(t)$ is discontinuous at each rational number and continuous at each irrational number.

CHAPTER II

INTEGRALS

Definition 2-1: If each of a and b is a real number and $a < b$, then $\{x| \ a \leq x \leq b\}$ is called an interval and is denoted by $[a,b]$. Also, $\{x| \ a < x < b\}$ is called a segment and is denoted by (a,b). The length of $[a,b]$ and (a,b) denoted by $\ell[a,b]$ and $\ell(a,b)$ respectively is $\ell[a,b] = \ell(a,b) = b - a$.

Definition 2-2: If $[a,b]$ is an interval, each of x_o, x_1, x_2,.... x_n is a real number, and $a = x_o < x_1 < x_2 < \cdots < x_n = b$, then if $\sigma = \{x_o, x_1, \cdots x_n\}$, σ is called a subdivision of $[a,b]$. The length of the i^{th} subinterval is $(x_i - x_{i-1})$.

Definition 2-3: If σ is a subdivision of the interval $[a,b]$, then the norm of σ is the max $\{(x_1-x_o),(x_2-x_1),\cdots (x_n-x_{n-1})\}$.

Definition 2-4: If σ is a subdivision of $[a,b]$ and each of c_1, c_2, c_3,.... c_n is a real number so that $x_o \leq c_1 \leq x_1$, $x_1 \leq c_2 \leq x_2, \cdots x_{n-1} \leq c_n \leq x_n$, then the ordered set $x_o, x_1, x_2 \cdots x_n$ together with $c_1, c_2, c_3, \cdots c_n$ is called an augmented subdivision of $[a,b]$.

Definition 2-5: If σ is an augmented subdivision of $[a,b]$, and for each positive integer p, $(c_p, f(c_p))$ is a pair in f, then define $S_\sigma = f(c_1)(x_1-x_o)+f(c_2)(x_2-x_1)+\cdots+f(c_n)(x_n-x_{n-1})$.

Definition 2-6: A function <u>f(x) has an integral</u> on the interval [a,b] if and only if f(x) is defined on [a,b], and if $\sigma_1, \sigma_2, \cdots$ is a sequence of augmented subdivisions of [a,b] with norms $\rho_1, \rho_2, \cdots$ respectively so that $\{\rho_n\}$ converges to 0, then $S_{\sigma_1}, S_{\sigma_2}, \cdots$ has a limit.

Definition 2-7: If K is a limit of one such sequence S_{σ_1}, $S_{\sigma_2}, \cdots$ then K will be called an <u>integral of f(x) on [a,b]</u>.

Theorem: If f(x) has an integral on [a,b], then f(x) is bounded on [a,b].

Proof: Let f(x) be a function defined on [a,b], such that f(x) is not bounded on [a,b]. Suppose f(x) is not bounded above on [a,b], then there is a number $\zeta \in [a,b]$, such that if I is any segment and $\zeta \in I$, f(x) is not bounded on $I \cap [a,b]$. Either $\zeta = a$, $\zeta = b$, or $a < \zeta < b$.

Case I: $\zeta = a$. If n is a positive integer and $n > 1$, let $\sigma_n = \{x_o, x_1, \cdots x_n\}$ so that for each positive integer i, $i \leq n$, $x_i = a + \frac{i(b-a)}{n}$. If $2 \leq i \leq n$, let $c_i = a + \frac{i(b-a)}{n}$. Since f is not bounded on $[x_o, x_1]$, there is a c_1 so that $x_o \leq c_1 \leq x_1$, and $f(c_1) > -\sum_{p=2}^{n} f(c_p) + \frac{n^2}{b-a}$. Since $x_i - x_{i-1} = \frac{b-a}{n}$ then $S_{\sigma_n} = \sum_{p=1}^{n} f(c_p)(x_p - x_{p-1}) > n$. Also $\rho_n = \frac{b-a}{n}$, hence the sequence $\{\rho_i\}$ converges to 0, but $\{S_{\sigma_i}\}$ does not have a limit.

Case II: $\zeta = b$. A similar argument, choosing c_n so that $f(c_n) > -\sum_{p=1}^{n-1} f(c_p) + \frac{n^2}{b-a}$ will show that f does not have an integral on [a,b].

Case III: $a < \zeta < b$. If n is a positive integer greater than 1, let $z_i = a + \frac{i(\zeta-a)}{n}$ and $u_i = \zeta + \frac{i(b-\zeta)}{n}$, $i = 0, 1, 2, \cdots n$. Let $\sigma_n = \{x_o, x_1, \cdots x_{2n-1}\}$ so that if $0 \leq i \leq n-1$, $x_i = z_i$ and if $n \leq i \leq 2n-1$, $x_i = u_{i-n+1}$. $\rho_n = x_n - x_{n-1} = \frac{b-a}{n}$. If $1 \leq i \leq n-1$, let $c_i = x_i$. If $n+1 \leq i \leq 2n-1$, let $c_i = x_i$. Since f is not bounded on $[x_{n-1}, x_n]$, there is a number c_n so that $x_{n-1} \leq c_n \leq x_n$ and

$$f(c_n) > -\frac{n}{b-a}\left[\sum_{p=1}^{n-1} f(c_p)(x_p - x_{p-1}) + \sum_{p=n+1}^{2n-1} f(c_p)(x_p - x_{p-1}) - n\right].$$

Now $S_{\sigma_n} = \sum_{p=1}^{2n-1} f(c_p)(x_p - x_{p-1}) > n$. $\{\rho_n\}$ has a limit 0 but $\{S_{\sigma_n}\}$ does not converge so f does not have an integral on [a,b]. Therefore, <u>if f has an integral on [a,b], f is bounded on [a,b]</u>.

<u>Theorem</u>: If f(x) has an integral on [a,b] and $a < c < b$, then f(x) has an integral on [a,c] and on [c,b]. Furthermore, if A is an integral of f(x) on [a,c] and B is an integral of f(x) on [c,b], then A+B is an integral of f(x) on [a,b].

Proof: Let f(x) have an integral on [a,b] and $a < c < b$. Since f(x) has an integral on [a,b], it is bounded on [a,b], hence f(x) is bounded on [c,b]. Let J and j be the upper and lower bounds respectively of f(x) on [c,b]. Then, if $\zeta \in [c,b]$, $j \leq f(\zeta) \leq J$. If n is a positive integer, let $\sigma_n'' = \{x_o, x_1, \cdots x_n\}$ be a subdivision of [c,b] so that the length of each subinterval of σ_n'' is $\frac{b-c}{n}$. If σ_n'' is augmented, then for $1 \leq i \leq n$, $j \leq f(c_i'') \leq J$. Now $nj \leq f(c_1'') + f(c_2'') + \cdots + f(c_n'') \leq nJ$ and $j(b-c) \leq [f(c_1'') + f(c_2'') + \cdots + f(c_n'')] \frac{b-c}{n} \leq J(b-c)$. Hence $j(b-c) \leq S_{\sigma_n}'' \leq J(b-c)$. Thus, $\{S_{\sigma_n}''\}$ is a bounded sequence and by a previous theorem $\{S_{\sigma_n}''\}$ must contain a convergent subsequence $\{S_{\sigma_{n_p}}''\}$. Let $\sigma_1^* = \sigma_{n_1}''$; $\sigma_2^* = \sigma_{n_2}''$; $\cdots$. Now $0 \leq \rho_n^* \leq \rho_n'' = \frac{b-c}{n}$ and so $\{\rho_n^*\}$ converges to 0. Thus, $\{\sigma_n^*\}$ is a sequence of augmented subdivisions of $[c,b]$ with norms $\rho_1^*, \rho_2^*, \cdots$ such that $\{\rho_n^*\}$ converges to 0, and $\{S_{\sigma_n}^*\}$ converges. Let $\{S_{\sigma_n}^*\}$ converge to B. Let $\sigma_1', \sigma_2', \cdots$ be a sequence of augmented subdivisions of $[a,c]$ with norms $\rho_1', \rho_2', \cdots$ such that $\{\rho_n'\}$ converges to 0. Let σ_n be the union of σ_n' and σ_n^* with x_o^* deleted. Now $\rho_n = \max(\rho_n', \rho_n^*)$ and since both $\{\rho_n'\}$ and $\{\rho_n^*\}$ converge to 0, $\{\rho_n\}$ converges to 0. Hence $\{\sigma_n\}$ is a sequence of augmented subdivisions of $[a,b]$ with norms $\rho_1, \rho_2, \cdots$ so that $\{\rho_n\}$ converges to 0, and since $f(x)$ has an integral on $[a,b]$, $\{S_{\sigma_n}\}$ converges. Denote the limit of $\{S_{\sigma_n}\}$ by K. Now, for each positive integer s, $S_{\sigma_s} = S_{\sigma_s}' + S_{\sigma_s}^*$ so that $S_{\sigma_s}' = S_{\sigma_s} - S_{\sigma_s}^*$, but $\{S_{\sigma_n}\}$ converges to K and $\{S_{\sigma_n}^*\}$ converges to B, hence $\{S_{\sigma_n} - S_{\sigma_n}^*\}$ converges to (K-B), and $\{S_{\sigma_n}'\}$ converges. Therefore, $f(x)$ has an integral on $[a,c]$. Similarly, $f(x)$ has an integral on $[c,b]$.

Furthermore, if $\{S_{\sigma_n}'\}$ converges to A and $\{S_{\sigma_n}''\}$ converges to B, then $\{S_{\sigma_n}' + S_{\sigma_n}''\}$ converges to (A+B). But, for each positive integer m, $S_{\sigma_m} = S_{\sigma_m}' + S_{\sigma_m}''$, hence <u>$\{S_{\sigma_m}\}$ converges to (A+B).</u>

<u>Definition 2-8</u>: $\int_a^a f(x)\,dx = 0$.

<u>Definition 2-9</u>: If f(x) has an integral on [a,b] define a function ϕ so that $(t,K) \in \phi$ if and only if $t \in [a,b]$ and K is the integral of f(x) on [a,t].

<u>Theorem</u>: ϕ of definition 2-9 is continuous at each point of [a,b].

Proof: Let f(x) have an integral on [a,b]. Then if $c \in [a,b]$, f(x) has an integral on [a,c], denote the integral of f(x) on [a,c] by K, then $\phi(c) = K$ and $(c,K) \in \phi$. Since f(x) has an integral on [a,b], f(x) is bounded on [a,b]. Let m and M by the lower and upper bounds of f(x) on [a,b]. If $a \leq x \leq b$, then $m \leq f(x) \leq M$ and $|f(x)| \leq$ max. $(|m|,|M|)$. Let h = max. $(|m|,|M|)$. If $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{h+1}$. If $|x-c| < \delta$ and $\phi(x)$ exists, then $a \leq x \leq b$. If x = c, then $|\phi(x) - \phi(c)| = |K-K| < \varepsilon$. If $x \neq c$, then either $x > c$ or $x < c$. If $x > c$, then $|\phi(x) - \phi(c)| = |\int_a^x f(x)\,dx - \int_a^c f(x)\,dx| = |\int_c^x f(x)\,dx|$. If σ is an augmented subdivision of [c,x], then $m(x-c) = \sum_{p=1}^{n} m(x_p - x_{p-1}) \leq \sum_{p=1}^{n} f(c_p)(x_p - x_{p-1}) = S_\sigma \leq \sum_{p=1}^{n} M(x_p - x_{p-1}) = M(x-c)$. Hence, $|S_\sigma| \leq h(x-c)$ and $|\int_c^x f(x)\,dx| \leq h(x-c) < h\delta = h(\frac{\varepsilon}{h+1}) < \varepsilon$. If $x < c$, a similar argument will show that $|\phi(x) - \phi(c)| = |-\int_x^c f(x)\,dx| < \varepsilon$. Therefore, ϕ is continuous at each point of $[a,b]$.

Lemma: If $f(x)$ is continuous on $[a,b]$ and $\int_a^b f(x)\,dx$ exists, then there is a number ζ so that $\zeta \in [a,b]$ and $\int_a^b f(x)\,dx = f(\zeta)\cdot(b-a)$.

Proof: Let $f(x)$ be continuous on $[a,b]$ and $\int_a^b f(x)\,dx$ exist. Since $\int_a^b f(x)\,dx$ exists, $f(x)$ is bounded on $[a,b]$. Let m and M denote the greatest lower bound and least upper bound respectively of $f(x)$ on $[a,b]$. Since $f(x)$ is continuous there exists $c \in [a,b]$ and $d \in [a,b]$ such that $f(c) = m$ and $f(d) = M$. Furthermore, if $m \leq W \leq M$ there is a number y so that y is between c and d and $f(y) = W$. Let σ be an augmented subdivision of $[a,b]$, then

$$m(b-a) = \sum_{p=1}^{n} m(x_p - x_{p-1}) \leq \sum_{p=1}^{n} f(c_p)(x_p - x_{p-1}) = S_\sigma \leq$$

$$\sum_{p=1}^{n} M(x_p - x_{p-1}) = M(b-a)$$ and $m(b-a) \leq S_\sigma \leq M(b-a)$ so that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \text{ and } m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M.$$

Let $z = \dfrac{\int_a^b f(x)\,dx}{b-a}$ then $m \leq z \leq M$ and there is a number ζ between c and d so that $f(\zeta) = z$. Hence, there is a number ζ so that $\zeta \in [a,b]$ and $f(\zeta)\cdot(b-a) = \int_a^b f(x)\,dx$.

Definition 2-10: f(x) has a derivative at a point a if and only if there is a number K and a segment I containing a so that if $x \in I$, $f(x)$ exists and if $\varepsilon > 0$ there is a $\delta > 0$ so that if $0 < |x-a| < \delta$ and $f(x)$ exists, then $\left|\dfrac{f(x) - f(a)}{x-a} - K\right| < \varepsilon$. K will be denoted by $f'(a)$.

<u>Theorem</u>: If $f(x)$ is continuous on $[a,b]$, then $\phi(x)$ defined by definition 2-9 has a derivative on (a,b). Furthermore, if $c \in (a,b)$, then $\phi'(c) = f(c)$.

Proof: Let $f(x)$ be continuous on $[a,b]$, then $\phi(x)$ is defined for all $x \in [a,b]$, hence if $c \in (a,b)$, there is a segment containing c which is a subset of $D(\phi)$. If $\varepsilon > 0$, then since $f(x)$ is continuous there exists a $\delta > 0$ so that if $|x-c| < \delta$ and $a < x < b$, then $f(x)$ exists and $|f(x)-f(c)|<\varepsilon$. If $0 < |x-c| < \delta$ and $\phi(x)$ exists, then $a < x < b$. If $x > c$, then $\dfrac{\phi(x) - \phi(c)}{x-c} = \dfrac{\int_a^x f(x)\,dx - \int_a^c f(x)\,dx}{x - c} = \dfrac{\int_c^x f(x)\,dx}{x-c}$.

However, there is a number h such that $c < h < x$ and $f(h)\,(x-c) = \int_c^x f(x)\,dx$. Now $|h-c| < \delta$ and $\left|\dfrac{\phi(x)-\phi(c)}{x-c} - f(c)\right| = |f(h) - f(c)| < \varepsilon$. If $x < c$, a similar argument will show that $\left|\dfrac{\phi(x) - \phi(c)}{x-c} - f(c)\right| < \varepsilon$. Hence, <u>$\phi'(c) = f(c)$</u>.

<u>Definition 2-11</u>: If $f(x)$ is defined and bounded on $[a,b]$ and σ_n is a subdivision of $[a,b]$, then define $S_{\overline{\sigma_n}} = \sum_{\sigma_n} M_i(x_i-x_{i-1})$ where M_i is the least upper bound of $f(x)$ on $[x_{i-1},x_i]$. Also, define $S_{\underline{\sigma_n}} = \sum_{\sigma_n} m_i(x_i-x_{i-1})$ where m_i is the greatest lower bound of $f(x)$ on $[x_{i-1},x_i]$.

<u>Lemma</u>: Let $f(x)$ be defined and bounded on $[a,b]$ and let σ be a subdivision of $[a,b]$, then for each positive integer i such that there is $x_i \in \sigma$, $f(x)$ must be bounded on $[x_{i-1},x_i]$ because $f(x)$ is bounded on $[a,b]$. Let M_i nd m_i denote the least upper and greatest lower bounds respectively of $f(x)$ on $[x_{i-1},x_i]$. Then $m_i \leq M_i$, for if not then $m_i > M_i$ and $f(x)$ is a function

whose greatest lower bound is larger than its least upper bound, a contradiction. Now $(x_i - x_{i-1}) > 0$ so $m_i(x_i - x_{i-1}) \leq M_i(x_i - x_{i-1})$. Hence, $\sum_\sigma M_i(x_i - x_{i-1}) \geq \sum_\sigma m_i(x_i - x_{i-1})$ and $S_{\bar{\sigma}} \geq S_{\underline{\sigma}}$.

Lemma: If $f(x)$ is defined and bounded on $[a,b]$, σ and σ^* are subdivisions of $[a,b]$, and $\sigma \subset \sigma^*$, then $S_{\bar{\sigma}^*} \leq S_{\bar{\sigma}}$, also $S_{\underline{\sigma}} \leq S_{\underline{\sigma}^*}$.

Proof: Let $f(x)$ be defined and bounded on $[a,b]$, σ and σ^* be two subdivisions of $[a,b]$, and $\sigma \subset \sigma^*$. If $\sigma = \sigma^*$ then $S_{\bar{\sigma}} = S_{\bar{\sigma}^*}$. If $\sigma \neq \sigma^*$, then there is a positive integer j so that σ^* contains j elements not in σ. If $j = 1$, then there is a positive integer i such that $x_p = x_p^*$, $p = 0,1,2,\cdots, i-1$, $x_i^* \notin \sigma$, and $x_p = x_{p+1}^*$, $p = i, i+1, \cdots, n$. Now,

$$S_{\bar{\sigma}} = \sum_{p=1}^{i-1} M_p(x_p - x_{p-1}) + M_i(x_i - x_{i-1}) + \sum_{p=i+1}^{n} M_p(x_p - x_{p-1}) =$$

$$\sum_{p=1}^{i-1} M_p^*(x_p^* - x_{p-1}^*) + M_i(x_{i+1}^* - x_{i-1}^*) + \sum_{p=i+2}^{n+1} M_p^*(x_p^* - x_{p-1}^*) =$$

$$\sum_{p=1}^{i-1} M_p^*(x_p^* - x_{p-1}^*) + M_i(x_i^* - x_{i-1}^*) + M_i(x_{i+1}^* - x_i^*) + \sum_{p=i+2}^{n+1} M_p^*(x_p^* - x_{p-1}^*)$$

$$\geq \sum_{p=1}^{i-1} M_p^*(x_p^* - x_{p-1}^*) + M_i^*(x_i^* - x_{i-1}^*) + M_{i+1}^*(x_{i+1}^* - x_i^*) +$$

$$\sum_{p=i+2}^{n+1} M_p^*(x_p^* - x_{p-1}^*) = S_{\bar{\sigma}^*}.$$

Assume that for $\sigma \subset \sigma^*$ and σ^* having exactly k elements not in σ, then $S_{\bar{\sigma}} \geq S_{\bar{\sigma}^*}$. Let $\sigma \subset \sigma^*$ and σ^* have exactly $k+1$ elements not in σ. There is a positive integer i such that $x_i^* \notin \sigma$. Let σ' be σ^* with x_i^* deleted. Now $\sigma \subset \sigma'$ and σ' has exactly k elements not in σ. Therefore $S_{\bar{\sigma}} \geq S_{\bar{\sigma}'}$. Also $\sigma' \subset \sigma^*$ and σ^* has exactly one element not in σ'. Therefore $S_{\bar{\sigma}'} \geq S_{\bar{\sigma}}^*$ and $S_{\bar{\sigma}} \geq S_{\bar{\sigma}}^*$. By mathematical induction it follows that if $\sigma \subset \sigma^*$ then $S_{\bar{\sigma}} \geq S_{\bar{\sigma}}^*$. Also by a similar argument $S_{\underline{\sigma}} \leq S_{\underline{\sigma}}^*$.

Lemma: If f(x) is defined and bounded on [a,b], and σ and σ^* are two subdivisions of [a,b], then $S_{\bar{\sigma}} \geq S_{\underline{\sigma}}^*$.

Proof: Let f(x) be defined and bounded on [a,b] and σ and σ^* be two subdivisions of [a,b], then if $\sigma' = \sigma \cup \sigma^*$ by the previous lemma $S_{\bar{\sigma}} \geq S_{\bar{\sigma}}'$. Also by the same lemma $S_{\underline{\sigma}}' \geq S_{\underline{\sigma}}^*$. And by a previous lemma $S_{\bar{\sigma}}' \geq S_{\underline{\sigma}}'$ so that $S_{\bar{\sigma}} \geq S_{\bar{\sigma}}' \geq S_{\underline{\sigma}}' \geq S_{\underline{\sigma}}^*$. Hence $S_{\bar{\sigma}} \geq S_{\underline{\sigma}}^*$.

Theorem: A function f(x) defined on an interval [a,b] has an integral on [a,b] if the following are true: f(x) is bounded on [a,b]. Also if $\sigma_1, \sigma_2, \cdots$ is a sequence of subdivisions with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0 and $\varepsilon > 0$, there is a positive integer N so that if $n > N$, then $|S_{\bar{\sigma}_n} - S_{\underline{\sigma}_n}| < \varepsilon$.

Proof: Let f(x) be defined on [a,b], f(x) be bounded on [a,b] and if $\sigma_1, \sigma_2, \cdots$ is a sequence of subdivisions with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0 and $\varepsilon > 0$, there is a positive integer N so that if $n > N$, then $|S_{\bar{\sigma}_n} - S_{\underline{\sigma}_n}| < \varepsilon$.

Let $\sigma_1, \sigma_2, \cdots$ be a sequence of subdivisions of [a,b] with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0. Since f(x) is bounded on [a,b] it is bounded in each subinterval of [a,b]. Let M_i and m_i denote the least upper bound and greatest lower bound respectively of f(x) on $[x_{i-1}, x_i]$. Let M and m denote the least upper and greatest lower bounds respectively of f(x) on [a,b]. There is a set J such that $j \in J$ if and only if there is a subdivision σ_i such that $S_{\overline{\sigma}_i} = j$. This set is non-empty, for one such element is M(b-a). Also there is a set K such that $k \in K$ if and only if there is a subdivision σ_i such that $S_{\underline{\sigma}_i} = k$. Again this set is non-empty, for one such element is m(b-a). Now if $j \in J$, then $j = S_{\overline{\sigma}_n} = \sum_{\sigma_n} M_i(x_i - x_{i-1}) \geq \sum_{\sigma_n} m_i(x_i - x_{i-1}) \geq m(b-a)$, so that J is bounded below. Similarly, K is bounded above. Hence there are real numbers A and B such that A is the greatest lower bound of J and B is the least upper bound of K. Suppose A < B. Since A is the greatest lower bound of J, there is $j \in J$ so that $A \leq j < B$. If there is not, then all elements of J are greater than or equal to B or one element of J is less than A. This contradicts the fact that A is the greatest lower bound of J. Hence there is an $S_{\overline{\sigma}_i}$ such that $A \leq S_{\overline{\sigma}_i} < B$. For each positive integer n, $S_{\underline{\sigma}_n} \leq S_{\overline{\sigma}_i} < B$. Hence, B is not the least upper bound of K.

Thus, $A \nless B$. Suppose $A > B$, then $A-B > 0$ and by hypothesis there is a positive integer N so that if $n > N$ then $|S_{\overline{\sigma}_n} - S_{\underline{\sigma}}| < \varepsilon$. But if $j' = S_{\overline{\sigma}_n}$ and $k' = S_{\underline{\sigma}_n}$ then $|j' - k'| < A-B$ and by previous lemma $j' \geq k'$ so $j' - k' \geq 0$ and $0 \leq j' - k' < A-B$. Now $A \leq j'$ because A is the greatest lower bound of J and $k' \leq B$ because B is the least upper bound of K. Thus $A \leq j'$ and $-B \leq -k'$ and $A-B \leq j' - k'$, a contradiction of $j' - k' < A-B$. Therefore $A=B$. Let $\varepsilon > 0$, if $\sigma_1, \sigma_2, \cdots$ is a sequence of augmented subdivisions of $[a,b]$ with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0, then there is a positive integer N_1 so that if $n_1 > N_1$ then $A - \varepsilon < S_{\underline{\sigma}_{n_1}}$ if not then A is not the least upper bound of K. Also, there is a positive integer N_2 so that if $n_2 > N_2$ then $S_{\overline{\sigma}_{n_2}} < A + \varepsilon$. If not then A is not the greatest lower bound of J. Let $N = \max(N_1, N_2)$, then if $n > N$, $A - \varepsilon < S_{\underline{\sigma}_n}$ and $S_{\overline{\sigma}_n} < A + \varepsilon$. But $S_{\underline{\sigma}_n} \leq S_{\sigma_n} \leq S_{\overline{\sigma}_n}$ so that $A - \varepsilon < S_{\underline{\sigma}_n} \leq S_{\sigma_n} \leq S_{\overline{\sigma}_n} < A + \varepsilon$ and $A-\varepsilon < S_{\sigma_n} < A + \varepsilon$, hence $|S_{\sigma_n} - A| < \varepsilon$, and <u>f(x) has an integral on [a,b]</u>.

<u>Corollary</u>: If $f(x)$ has an integral on $[a,b]$ and if $\{\sigma_n\}$ is a sequence of subdivisions of $[a,b]$ with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0, then for each $\varepsilon > 0$ there is a positive integer N so that if $n > N$, $|S_{\overline{\sigma}_n} - S_{\underline{\sigma}_n}| < \varepsilon$.

Proof: Let $f(x)$ have an integral K on $[a,b]$, and let $\{\sigma_n\}$ be a sequence of subdivisions of $[a,b]$ with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0. Let $\varepsilon > 0$ then $\frac{\varepsilon}{4(b-a)} > 0$. If $\sigma_k \varepsilon \{\sigma_n\}$ and σ_k has h subdivisions then let c_i, $i = 1,2,\cdots h$, be number so that $c_i \varepsilon [x_{i-1},x_i]$ and $f(c_i) > M_i - \frac{\varepsilon}{4(b-a)}$ Let $z_k = \{c_i | i = 1,2,\cdots h\}$. This is possible because M_i is the least upper bound of $f(x)$ on $[x_{i-1},x_i]$ and since $\frac{\varepsilon}{4(b-a)} > 0$ there must be at least one number $\zeta \varepsilon [x_{i-1},x_i]$ so that $f(\zeta) > M_i - \frac{\varepsilon}{4(b-a)}$. Let σ_k' be σ_k augmented with z_k. Then $\{\sigma_n'\}$ is a sequence of augmented subdivisions of $[a,b]$ with norms $\{\rho_n'\} = \{\rho_n\}$ which converges to 0 and hence $\{S_{\sigma_n}'\}$ converges to K. Thus, there is a positive integer P so that if $p > P$ then $|S_{\sigma_p}' - K| < \frac{\varepsilon}{4}$. If $\sigma_k \varepsilon \{\sigma_n\}$ and σ_k has h subdivisions then let d_i, $i = 1, 2,\cdots h$ be numbers so that $d_i \varepsilon [x_{i-1},x_i]$ and $f(d_i) < m_i + \frac{\varepsilon}{4(b-a)}$. Let $z_k' = \{d_i | i=1, 2,\cdots h\}$. This is possible because m_i is the greatest lower bound of $f(x)$ on $[x_{i-1},x_i]$ and since $\frac{\varepsilon}{4(b-a)} > 0$ there must be at least one number $\zeta \varepsilon [x_{i-1},x_i]$ so that $f(\zeta) < m_i - \frac{\varepsilon}{4(b-a)}$. Let σ_k'' be σ_k augmented with z_k', then $\{\sigma_n''\}$ is a sequence of augmented subdivisions of $[a,b]$ with norms $\{\rho_n''\} = \{\rho_n\}$ which converges to 0, and hence $\{S_{\sigma_n}''\}$ converges to K. Thus, there is a positive integer Q so that if $q > Q$ then $|S_{\sigma_q}'' - K| < \frac{\varepsilon}{4}$. Now, $f(c_i) > M_i - \frac{\varepsilon}{4(b-a)}$ so that $M_i - f(c_i) < \frac{\varepsilon}{4(b-a)}$ and $\sum_{r=1}^{m} [M_r - f(c_r)] [x_r - x_{r-1}] <$

$\sum_{p=1}^{m} \frac{\varepsilon}{4(b-a)} [x_p - x_{p-1}] = \frac{\varepsilon}{4}$. Hence, $|S_{\bar{\sigma}_k} - S_{\sigma_k}'| < \frac{\varepsilon}{4}$.

Also, $f(d_i) < m_i + \frac{\varepsilon}{4(b-a)}$ so that $f(d_i) - m_i < \frac{\varepsilon}{4(b-a)}$ and $[f(d_i) - m_i] [x_i - x_{i-1}] < \frac{\varepsilon}{4(b-a)} (x_i - x_{i-1})$ so that

$\sum_{r=1}^{m} [f(d_r) - m_r] [x_r - x_{r-1}] < \sum_{r=1}^{m} \frac{\varepsilon}{4(b-a)} (x_r - x_{r-1}) = \frac{\varepsilon}{4}$.

Hence $|S_{\sigma_k}'' - S_{\underline{\sigma_k}}| < \frac{\varepsilon}{4}$. Thus, if $N = \max(P,Q)$ and $n > N$, $|S_{\bar{\sigma}_n} - S_{\sigma_n}'| < \frac{\varepsilon}{4}$, $|S_{\sigma_n}'' - S_{\underline{\sigma_n}}| < \frac{\varepsilon}{4}$, $|S_{\sigma_n}' - K| < \frac{\varepsilon}{4}$ and $|S_{\sigma_n}'' - K| < \frac{\varepsilon}{4}$, and $|S_{\bar{\sigma}_n} - S_{\underline{\sigma_n}}| \leq |S_{\bar{\sigma}_n} - K| + |K - S_{\underline{\sigma_n}}| \leq |S_{\bar{\sigma}_n} - S_{\sigma_n}'| + |S_{\sigma_n}' - K| + |K - S_{\sigma_n}''| + |S_{\sigma_n}'' - S_{\underline{\sigma_n}}| < \varepsilon$.

By combining the last theorem and corollary, it is seen that a necessary and sufficient condition to insure the existence of an integral for $f(x)$ on $[a,b]$ is that for each sequence $\{\sigma_n\}$ of subdivisions of $[a,b]$ with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0, and for each $\varepsilon > 0$, there is a positive integer N so that if $n > N$ then $|S_{\bar{\sigma}_n} - S_{\underline{\sigma_n}}| < \varepsilon$.

Theorem: If $\int_a^b f(x)\,dx$ exists, then $\int_a^b |f(x)|\,dx$ exists.

Proof: Let $\int_a^b f(x)\,dx$ exist, then by a previous theorem $f(x)$ is bounded on $[a,b]$. Thus $|f(x)|$ is bounded on $[a,b]$. Let $\{\sigma_n\}$ be a sequence of subdivisions of $[a,b]$ with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0. Let M_i and $\overline{M}_i$

denote the least upper bound of $f(x)$ and $|f(x)|$ respectively on $[x_{i-1},x_i]$, also let m_i and $\overline{m}_i$ denote the greatest lower bound of $f(x)$ and $|f(x)|$ respectively on $[x_{i-1},x_i]$. For any subinterval $[x_{i-1},x_i]$ either;

(i) $f(x) \geq 0$ for all $x \in [x_{i-1},x_i]$. Then, $M_i = \overline{M}_i$ and $\underline{m}_i = m_i$ and $\overline{M}_i(x_i-x_{i-1}) - \underline{m}_i(x_i-x_{i-1}) = M_i(x_i-x_{i-1}) - m_i(x_i-x_{i-1})$.

(ii) $f(x) \leq 0$ for all $x \in [x_{i-1},x_i]$. Then, $\overline{M}_i = -m_i$, $\underline{m}_i = -M_i$, and $\overline{M}_i(x_i-x_{i-1}) - \underline{m}_i(x_i-x_{i-1}) = -m_i(x_i-x_{i-1}) + M_i(x_i-x_{i-1})$ so that $\overline{M}_i(x_i-x_{i-1}) - \underline{m}_i(x_i-x_{i-1}) = M_i(x_i-x_{i-1}) - m_i(x_i-x_{i-1})$, or

(iii) $f(x) > 0$ for some $x \in [x_{i-1},x_i]$ and $f(x) < 0$ for some $x \in [x_{i-1},x_i]$. Then $M_i > 0$ and $m_i < 0$, $\overline{M}_i$ is either M_i or $-m_i$, and $\underline{m}_i \geq 0$. If $\overline{M}_i = M_i$, then $\overline{M}_i - \underline{m}_i \leq \overline{M}_i = M_i < M_i - m_i$ so that $\overline{M}_i - \underline{m}_i < M_i - m_i$ and $\overline{M}_i(x_i-x_{i-1}) - \underline{m}_i(x_i-x_{i-1}) < M_i(x_i-x_{i-1}) - m_i(x_i-x_{i-1})$. If $\overline{M}_i = -m_i$, then $\overline{M}_i - \underline{m}_i \leq \overline{M}_i = -m_i < M_i - m_i$ and $\overline{M}_i(x_i-x_{i-1}) - \underline{m}_i(x_i-x_{i-1}) \leq M_i(x_i-x_{i-1}) - m_i(x_i-x_{i-1})$. Hence, by combining all three cases, $0 \leq \overline{M}_i(x_i-x_{i-1}) - \underline{m}_i(x_i-x_{i-1}) \leq M_i(x_i-x_{i-1}) - m_i(x_i-x_{i-1})$. If $S_{|\overline{\sigma}_n|}$ and $S_{|\underline{\sigma}_n|}$ denote the upper and lower sums respectively of $|f(x)|$ on $[a,b]$ for the subdivision σ_n, then $0 \leq S_{|\overline{\sigma}_n|} - S_{|\underline{\sigma}_n|} \leq S_{\overline{\sigma}_n} - S_{\underline{\sigma}_n}$. Let $\varepsilon > 0$ and choose N a positive integer so that if $n > N$ then $|S_{\overline{\sigma}_n} - S_{\underline{\sigma}_n}| < \varepsilon$. This can be done because $\int_a^b f(x)\,dx$ exists. But, $0 \leq S_{|\overline{\sigma}_n|} - S_{|\underline{\sigma}_n|} \leq S_{\overline{\sigma}_n} - S_{\underline{\sigma}_n} < \varepsilon$, so that $|S_{|\overline{\sigma}_n|} - S_{|\underline{\sigma}_n|}| < \varepsilon$. Hence, $\int_a^b |f(x)|\,dx$ exists.

Theorem: If $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, and $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\cdot g(x)\,dx$ exists.

Proof: Let $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, and $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in [a,b]$, then there are positive real numbers P and Q such that if $x \in [a,b]$ $|f(x)| \leq P$ and $|g(x)| \leq Q$. Let $\varepsilon > 0$, then $\frac{\varepsilon}{2P} > 0$ and $\frac{\varepsilon}{2Q} > 0$. If $\{\sigma_n\}$ is a sequence of subdivisions with norms $\rho_1, \rho_2, \cdots$ such that $\{\rho_n\}$ converges to 0, then since $\int_a^b f(x)\,dx$ exists, there is a positive integer N_1 so that if $r > N_1$, $|S_{\overline{\sigma}_r} f - S_{\underline{\sigma}_r} f| < \frac{\varepsilon}{2P}$, and there is a positive integer N_2 so that if $t > N_2$, then $|S_{\overline{\sigma}_t} g - S_{\underline{\sigma}_t} g| < \frac{\varepsilon}{2Q}$. Let $N = \max.(N_1, N_2)$ then if $n > N$, $|S_{\overline{\sigma}_n} f - S_{\underline{\sigma}_n} f| < \frac{\varepsilon}{2P}$ and $|S_{\overline{\sigma}_n} g - S_{\underline{\sigma}_n} g| < \frac{\varepsilon}{2Q}$. Since $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in [a,b]$, $M_i^{f\cdot g} \leq M_i^f \cdot M_i^g$ and $m_i^{f\cdot g} \geq m_i^f \cdot m_i^g$ for each i. Hence, $M_i^{f\cdot g} - m_i^{f\cdot g} \leq M_i^f \cdot M_i^g - m_i^f \cdot m_i^g$ and since $M_i^{f\cdot g} - m_i^{f\cdot g} \geq 0$, and $(x_i - x_{i-1}) > 0$, $|M_i^{f\cdot g}(x_i - x_{i-1}) - m_i^{f\cdot g}(x_i - x_{i-1})| \leq |(M_i^f \cdot M_i^g - m_i^f \cdot m_i^g)(x_i - x_{i-1})| = |(M_i^f M_i^g - m_i^f M_i^g + m_i^f M_i^g - m_i^f m_i^g)(x_i - x_{i-1})| = |M_i^g (M_i^f - m_i^f)(x_i - x_{i-1}) + m_i^f (M_i^g - m_i^g)(x_i - x_{i-1})| \leq |M_i^g| \cdot |M_i^f (x_i - x_{i-1}) - m_i^f (x_i - x_{i-1})| + |m_i^f| \cdot$

$|M_i^g(x_i-x_{i-1}) - m_i^g(x_i-x_{i-1})| \leq P \cdot |M_i^f(x_i-x_{i-1}) - m_i^f(x_i-x_{i-1})| + Q \cdot |M_i^g(x_i-x_{i-1}) - m_i^g(x_i-x_{i-1})|$ for each i. Therefore if $n > N$, $|S_{\bar{\sigma}_n} f \cdot g - S_{\underline{\sigma}_n} f \cdot g| \leq P \cdot |S_{\bar{\sigma}_n} f - S_{\underline{\sigma}_n} f| + Q|S_{\bar{\sigma}_n} g - S_{\underline{\sigma}_n} g| < P(\frac{\varepsilon}{2P}) + Q(\frac{\varepsilon}{2Q}) = \varepsilon$. Hence, if $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, and $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \varepsilon [a,b]$, then <u>$\int_a^b f(x)\, g(x)\,dx$ exists.</u>

<u>Corollary</u>: If $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, then $\int_a^b f(x)\cdot g(x)\,dx$ exists.

Proof: Let $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, then $f(x)$ and $g(x)$ are bounded on $[a,b]$ and there exists real numbers W and K so that $f(x) - W \geq 0$ and $g(x) - K \geq 0$ for all $x \varepsilon [a,b]$. Since $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, $\int_a^b K\, f(x)\,dx$ and $\int_a^b W\, g(x)\,dx$ exist, also, $\int_a^b - KW\,dx$, $\int_a^b f(x) - W$, and $\int_a^b g(x) - K$ exist. Hence, $\int_a^b (K\, f(x) + W\, g(x) - K\, W)\,dx$ exists. Now, since $f(x) - W \geq 0$ and $g(x) - K \geq 0$ for all $x \varepsilon [a,b]$, $\int_a^b (f(x) - W)\,(g(x) - W)\,dx$ exists. Thus, $\int_a^b [(f(x) - W)(g(x) - K) + (K\, f(x) + W\, g(x) - K\, W)]\,dx$ exists and $\int_a^b [(f(x) - W)\,(g(x) - K) + (K\, f(x) + W\, g(x) - K\, W)]\,dx = \int_a^b f(x)\, g(x)\,dx$. Hence <u>$\int_a^b f(x)\, g(x)\,dx$ exists.</u>

<u>Theorem</u>: If $g(x) > 0$ for all $x \varepsilon [a,b]$, $\int_a^b g(x)\,dx$ exists, and $\int_a^b f(x)\,dx$ exists, then there is a number H such that if $m \leq f(x) \leq M$ for all $x \varepsilon [a,b]$, $m \leq H \leq M$ and $\int_a^b f(x)\, g(x)\,dx = H \int_a^b g(x)\,dx$.

Proof: Let $g(x) > 0$ for all $x \in [a,b]$, $\int_a^b g(x)\,dx$ and $\int_a^b f(x)\,dx$ exist. Then $\int_a^b f(x)\,g(x)\,dx$ exists, also $f(x)$ is bounded on $[a,b]$, so that there exists real numbers m and M such that if $x \in [a,b]$, $m \leq f(x) \leq M$. Since $g(x) > 0$ for all $x \in [a,b]$, $mg(x) \leq f(x)\,g(x) \leq Mg(x)$. So that $\int_a^b mg(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \int_a^b Mg(x)\,dx$,

$$m \int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

Now, $g(x) > 0$ for all $x \in [a,b]$, hence $\int_a^b g(x)\,dx > 0$

so $m < \dfrac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} < M$. If $H = \dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)\,dx}$ then

$\underline{\int_a^b f(x)\,g(x)\,dx = H \int_a^b g(x)\,dx \text{ where } m \leq H \leq M.}$

Furthermore, if $f(x)$ is continuous, there is a number $\zeta \in [a,b]$ so that $f(\zeta) = H$ and $\int_a^b f(x)\,g(x)\,dx = f(\zeta)\int_a^b g(x)\,dx$.

CHAPTER III

THE LAPLACE TRANSFORMATION

Definition 3-1: Let $\{u_n\}$ be a sequence of positive real numbers such that if H is a real number, there is a positive integer N such that if n is a positive integer and $n > N$, then $u_n > H$. The sequence $\{u_n\}$ will be called an increasing unbounded sequence.

Definition 3-2: Let f(t) be a function such that if $w > 0$, $\int_0^w f(t)$ exists. If for each increasing unbounded sequence $\{u_n\}$, the sequence $\{\int_0^{u_n} f(t)dt\}$ has a limit, then $\int_0^\infty f(t)\,dt$ is the limit.

Definition 3-3: If there is a function and a real number k such that $\int_0^\infty f(t)e^{-kt}\,dt$ exists, then L will be the set of all ordered triplets so that $(x, y, z) \in L$ if and only if x is a function, f(t); y is a real number; $z = \int_0^\infty f(t)e^{-yt}dt$. z will be denoted by L[x,y], and z is called the Laplace Transformation of x and y.

Consider $f(t) = e^{at}$, then $L[f(t),k]=L[e^{at},k] = \lim_{n\to\infty}\{\int_0^{u_n} e^{at}\cdot e^{-kt}dt\}$

$= \lim_{n\to\infty} \{\int_0^{u_n} e^{-(k-a)t}dt\}$. Suppose $k > a$ then $k-a > 0$ and

$$\lim_{n\to\infty} \{\int_0^{u_n} e^{-(k-a)t}dt\} = \lim_{n\to\infty} [\frac{-1}{k-a} (e^{-(k-a)t})\Big|_0^{u_n}] =$$

$\frac{-1}{k-a} \lim_{n\to\infty}(e^{-(k-a)u_n} -1) = \frac{-1}{k-a} [\lim_{n\to\infty} \frac{1}{e^{(k-a)u_n}} -1] =$

$\frac{-1}{(k-a)} [0-1] = \frac{1}{(k-a)}$. Thus, L $[e^{at},k]$ exists for $k > a$.

Theorem: If $L[f(t), k]$ exists, for some $t_o \geq 0$ $f(t)e^{-kt}$ is bounded for all $t > t_o$, and $h > 0$, then $L[f(t),k+h]$ exists.

Proof: Since $f(t)e^{-kt}$ is bounded for all $t > t_o \geq 0$, there exist real numbers m and M so that if $t > t_o$, $m \leq f(t) e^{-kt} \leq M$. Let $H = \max.(|m|,|M|)$, then $H \geq 0$ and $-H \leq f(t)e^{-kt} \leq H$ for all $t > t_o$. Since $h > 0$, $\int_0^\infty e^{-ht}dt$ exists. Let $\varepsilon > 0$, $\frac{\varepsilon}{H+1} > 0$ and if $\{u_n\}$ is an increasing unbounded sequence, there is a positive integer N_1 so that if $p,q > N_1$ then $|\int_{u_q}^{u_p} e^{-ht} dt| < \frac{\varepsilon}{H+1}$. Let N_2 denote the smallest positive integer such that if $i > N_2$, then $u_i > t_o$. Let $N = \max.(N_1,N_2)$ then if $m,n > N$, $| \int_{u_n}^{u_m} e^{-ht}dt| < \frac{\varepsilon}{H+1}$, and $-H \leq f(t)e^{-kt} \leq H$ for all $t > t_o$, also $|f(t) e^{-ht}| \leq H$ for all t between u_n and u_m. Now, $L[f(t),k]$ exists so that $\int_{u_n}^{u_m} f(t) e^{-kt}dt$ exists and hence, $\int_{u_n}^{u_m} f(t) e^{-kt} (e^{-ht}) dt$ exists. Furthermore, since $e^{-ht} > 0$ for all $t \geq 0$, by a previous theorem, there is a real number K so that $|K| < H$ and $|\int_{u_n}^{u_m} f(t) e^{-kt} (e^{-ht}) dt| =$ $|K| \cdot |\int_{u_n}^{u_m} e^{-ht} dt| < H \cdot |\int_{u_n}^{u_m} e^{-ht} dt| < H (\frac{\varepsilon}{H+1}) < \varepsilon$.

Thus, $\left|\int_{u_n}^{u_m} f(t)\, e^{-(k+h)t}\, dt\right| < \varepsilon$ and <u>L[f(t),k+h] exists</u>.

<u>Theorem</u>: If L[f(t),k] oes not exist, h > 0, and $f(t)\, e^{-(k-h)t}$ is bounded for all t > $_o \geq$ 0, then L[f(t , h] does not exist.

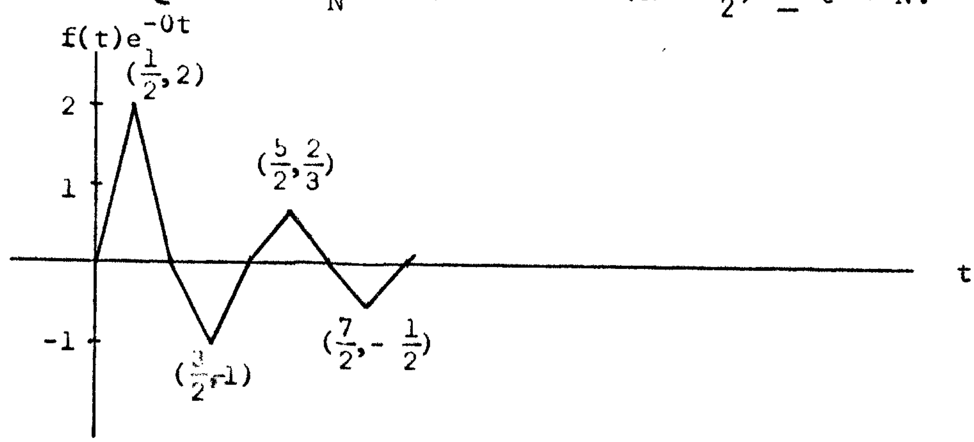
Proof: Le L[f(t),k] exist, h > 0, and $f(t)e^{-(k-h)t}$ be bounded for all t > $t_o \geq$ Suppose L[f(t),k-h] exists. Then since h > 0 and f(t) $^{-(k-h)t}$ is bounded for all t > t_o, by the previous theorem, L[f(t),(k-h)+h] = L[f(t),k] exists which contradicts the hypothesis. Hence, <u>L[f(t),k-h] does not exist</u>.

<u>Theorem:</u> There is a function so that if k $\geq$ 0, L[f(t),k] exists, but L[|f(t)|,0] does not exist.

Proof: Consider the following example.

For all t $\geq$ 0, define:

$$f(t) = \begin{cases} (-1)^N\left(-\frac{4}{N}t + \frac{4(N-1)}{N}\right) & \text{for } (N-1) \leq t \leq (N-\frac{1}{2}), \\ (-1)^N\left(\frac{4}{N}t - .4\right) & \text{for } (N-\frac{1}{2}) \leq t < N. \end{cases}$$



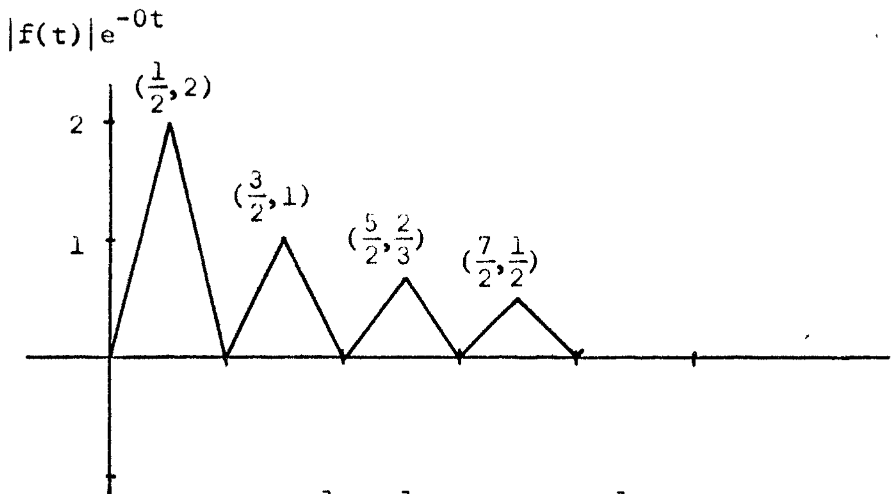
$$\int_0^\infty f(t)\, e^{-0t} dt = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots (-1)^{n-1}\frac{1}{n} + \cdots .$$

which converges. Hence, L[f(t),0] exists. Since $f(t)\, e^{-0t}$

is bounded for all $t \geq 0$, then by a previous theorem, if $k > 0$, $L[f(t),k]$ exists.

Now,

$$|f(t)|\ e^{-0t} = \begin{cases} \left|-\frac{4}{N}t + \frac{4(N-1)}{N}\right| & \text{for } (N-1) \leq t \leq (N - \frac{1}{2}), \\ \left|\frac{4}{N}t - 4\right| & \text{for } (N - \frac{1}{2}) \leq t < N. \end{cases}$$



$\int_0^\infty |f(t)|\ e^{-0t} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n} + \cdots\cdots$ which is the divergent harmonic series. Hence, $L[|f(t)|,0]$ does not exist.

Similarly, if $F(t) = f(t)\ e^{ht}$, then <u>$L[F(t),k]$ exists for $k \geq h$, and $L[|f(t)|, h]$ does not exist</u>.

<u>Theorem</u>: There is a function so that if $k > 0$, $L[|f(t)|,k]$ exists, but $L[f(t),k]$ does not exist.

Proof: Consider the following example. Let $k > 0$, then for all $t \geq 0$, define;

$$f(t) = \begin{cases} 1 \text{ if } t \text{ is rational} \\ -1 \text{ if } t \text{ is irrational} \end{cases}$$

Then, $|f(t)| = 1$ for all $t \geq 0$. Now $L[|f(t)|, k] = \int_0^\infty |f(t)| e^{-kt} dt = \int_0^\infty e^{-kt} dt$ which exists since $k > 0$. Hence, $L[|f(t)|, k]$ exists. Between any tw numbers $0 \leq t_1 \quad t_2$ there is at least one rational and one irrational number. Hence, if $0 \leq a < b$ and σ is any subdivision of $[a,b]$, $S_{\overline{\sigma}} = b-a$ and $S_{\underline{\sigma}} = -(b-a)$ so that $S_{\overline{\sigma}} - S_{\underline{\sigma}} = 2(b-a)$ and $f(t)$ does not have an integral on $[a,b]$. $\frac{1}{e^{-kt}}$ has an integral on $[a,b]$, suppose $f(t)\, e^{-kt}$ has an integral on $[a,b]$, then $[f(t)\, e^{-kt} (\frac{1}{e^{-kt}})]$ has an integral on $[a,b]$ and $\int_a^b f(t)\, e^{-kt}(\frac{1}{e^{-kt}})\, dt$ exists. Hence $\int_a^b f(t)\, dt$ exists. This is a contradiction. Thus, $\int_a^b f(t)\, e^{-kt}\, dt$ does not exist. Hence, $L[f(t),k]$ does not exist. Thus, there is a function so that if $k > 0$, <u>$L[|f(t)|, k]$ exists and $L[f(t),k]$ does not exist</u>.

<u>Theorem</u>: If $L[f(t),k]$ exists and K is a real number, then $L[Kf(t),k]$ exists and $K \cdot L[f(t),k] = L[K \cdot f(t),k]$.

Proof: Let $L[f(t),k]$ exist, $K \neq 0$ be a real number and $\varepsilon > 0$. $\frac{\varepsilon}{|K|} > 0$ and there is a positive integer N so that if $m,n > N$, $|\int_{u_n}^{u_m} f(t)\, e^{-kt}\, dt| < \frac{\varepsilon}{|K|}$. So that $|K| \cdot |\int_{u_n}^{u_m} f(t)\, e^{-kt}\, dt| < \varepsilon$. Now $|\int_{u_n}^{u_m} K\, f(t)\, e^{-kt} dt| < \varepsilon$. Thus, for $K \neq 0$, $L[K\, f(t),k]$ exists. If $L[f(t),k]$ exists and $K = 0$, then $K\, f(t) = 0$ and for any two positive integers s,t, $|\int_{u_s}^{u_t} K\, f(t)\, e^{-kt}\, dt| = 0 < \varepsilon$. Hence if $L[f(t),k]$ exists and K is a real number, $L[K \cdot f(t),k]$ exists.

Furthermore, $L[K \cdot f(t),k] = \int_0^\infty K \cdot f(t)\ e^{-kt} dt$ $=$ $K \int_0^\infty f(t)\ e^{-kt}\ dt$ $=$ $K \cdot L[f(t),k]$. Therefore, <u>$L[K \cdot f(t),k]$ exists and $L[K \cdot f(t),k] = K \cdot L[f(t),k]$</u>.

<u>Theorem</u>: If $L[f(t),k]$ exists, and there is a number z and a number M so that if $t \geq z$, then $|f(t)e^{-kt}| < M$, then if $j > 0$, $L[|f(t)|, k+j]$ exists.

Proof: Let $L[f(t),k]$ exist and let $f(t)\ e^{-kt}$ be such that there are real numbers M and z so that if $t > z$, $|f(t)e^{-kt}| < M$. Since $L[f(t),k]$ exists, if $a > 0$, $\int_0^a f(t)\ e^{-kt}\ dt$ exists and $\int_0^a |f(t)|\ e^{-kt}\ dt$ exists. Furthermore, let $j > 0$, then $\int_0^a e^{-jt}\ dt$ exists also $\int_0^a e^{-jt}\ dt > 0$ and $\int_0^a (|f(t)|e^{-kt})(e^{-jt})\ dt$ exists. Let $\{u_n\}$ be an increasing unbounded sequence and $\epsilon > 0$. There is a positive number W so that if $h \geq W$, then $e^{-h} < \frac{\epsilon}{M}$. There is a positive integer N so that if $n > N$, then $u_n > \max.(\frac{W}{j}, z)$ which implies $u_n > \frac{W}{j}$ and $ju_n > W$ and $e^{-ju_n} < e^{-W}$ also $|f(u_n)\ e^{-ku_n}| < M$. If $m > n > N$, let $p = \min.(u_n, u_m)$ and $q = \max.(u_n, u_m)$ then $|\int_0^{u_m} |f(t)|\ e^{-(k+j)t}\ dt - \int_0^{u_n} |f(t)|\ e^{-(k+j)t} dt|$ $=$ $|\int_p^q |f(t)|\ e^{-(k+j)t} dt|$ $=$ $\int_p^q |f(t)|\ e^{-(k+j)t}\ dt$. If $p = q$ $\int_p^q |f(t)|\ e^{-(k+j)t} dt = 0 < \epsilon$. If $p \neq q$, then there is a real number H so that $0 \leq H \leq M$ and $0 \leq \int_p^q |f(t)|\ e^{-(k+j)t} dt$ $=$

$\int_p^q (|f(t)|\, e^{-kt}) \cdot (e^{-jt})\, dt = H \cdot \int_p^q e^{-jt} dt =$
$H(e^{-jp} - e^{-jq}) \leq H\, e^{-jp} \leq M\, e^{-jp}$. Since $p > \frac{W}{j}$, $jp > W$ and $e^{-jp} < \frac{\varepsilon}{M}$ so that $M\, e^{-jp} < M\, (\frac{\varepsilon}{M}) = \varepsilon$. Therefore, $L[|f(t)|, k+j]$ exists.

Theorem: If $L[f(t),k]$ exists and $L[g(t),k]$ exists, then $L[f(t) + g(t),k]$ exists. Furthermore, $L[f(t),k] + L[g(t),k] = L[f(t) + g(t),k]$.

Proof: Let $L[f(t),k]$ exist and $L[g(t),k]$ exist; then if $a > 0$, $\int_0^a f(t)\, e^{-kt} dt$ and $\int_0^a g(t)\, e^{-kt}\, dt$ exist, so that $\int_0^a [f(t) + g(t)]\, e^{-kt}\, dt$ exists and $\int_0^a [f(t) + g(t)]\, e^{-kt}\, dt = \int_0^a f(t)\, e^{-kt}\, dt + \int_0^a g(t) e^{-kt} dt$. Let $\varepsilon > 0$ and $\{u_n\}$ be an increasing unbounded sequence then for $\frac{\varepsilon}{2} > 0$ there is a positive integer S so that if $s > S$, $|\int_0^{u_s} f(t)\, e^{-kt} - L[f(t),k]| < \frac{\varepsilon}{2}$, and there is a positive integer V so that if $v > V$, $|\int_0^{u_v} g(t)\, e^{-kt} dt - L[g(t),k]| < \frac{\varepsilon}{2}$. Let $N = \max.(S,V)$ then if $n > N$, $|\int_0^{u_n} f(t)\, e^{-kt}\, dt - L[f(t),k]| < \frac{\varepsilon}{2}$ and $|\int_0^{u_n} g(t)\, e^{-kt}\, dt - L[g(t),k]| < \frac{\varepsilon}{2}$. Consider,

$|\int_0^{u_n} [f(t) + g(t)]\, e^{-kt} dt - \{L[f(t),k] + L[g(t),k]\}| =$
$|\int_0^{u_n} f(t) e^{-kt} dt - L[f(t),k] + \int_0^{u_n} g(t) e^{-kt} dt - L[g(t),k]| \leq$

$|\int_0^{u_n} f(t)\, e^{-kt}\, dt - L[f(t),k]| + |\int_0^{u_n} g(t)e^{-kt}dt - L[g(t),k]| <$

$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, $|\int_0^{u_n} [f(t) + g(t)]\, e^{-kt} - \{L[f(t),k] +$ $L[g(t),k]\}| < \varepsilon$. Therefore, $L[f(t)+g(t),k]$ exists.

Furthermore, since $\{\int_0^{u_n} [f(t) + g(t)]\, e^{-kt}dt\}$ converges to $L[f(t),k] + L[g(t),k]$ and to $L[f(t) + g(t),k]$ it is seen $L[f(t),k] + L[g(t),k] = L[f(t) + g(t),k]$.

Theorem: If $f(t) > 0$ for all $t \geq 0$, $k > 0$, $a > 0$, and $L[f(t),k]$ exists, then $L[f(t), k+a] < L[f(t),k]$.

Proof: Let $f(t) > 0$ for all $t \geq 0$, $k > 0$, $a > 0$, and $L[f(t),k]$ exist. Clearly, $L[f(t),k+a]$ exists. Also since $k > 0$, and $a > 0$, $k+a > k$ so that $0 < e^{-(k+a)} < e^{-k}$ and since $f(t) > 0$, $0 < f(t)\, e^{-(k+a)t} < f(t)\, e^{-kt}$ for all $t > 0$ and $0 < f(t)\, e^{-(k+a)t} \leq f(t)\, e^{-kt}$ for all $t \geq 0$. If $h > 0$ and $0 < \delta < h$, $\int_\delta^h f(t)e^{-(k+a)t}dt < \int_\delta^h f(t)e^{-kt}dt$ and $\int_0^h f(t)e^{-(k+a)t}dt \leq \int_0^h f(t)e^{-kt}dt$ thus, $\int_\delta^h f(t)e^{-kt}dt - \int_\delta^h f(t)\, e^{-(k+a)t}dt > 0$ and $\int_0^h f(t)e^{-kt}dt - \int_0^h f(t)e^{-(k+a)t}dt \geq 0$. Hence, $\int_\delta^h f(t)e^{-kt}[1-e^{-at}]dt > 0$ and $\int_0^h f(t)\, e^{-kt}[1-e^{-at}]dt \geq 0$. Furthermore, if $h > 2$ and $0 < \delta < 1$, from the above equations

$\int_0^h f(t)e^{-kt}[1-e^{-at}]\, dt = \int_0^\delta f(t)\, e^{-kt}\, [1-e^{-at}]\, dt + \int_\delta^1 f(t)\, e^{-kt}[1-e^{-at}]\, dt + \int_1^2 f(t)\, e^{-kt}[1-e^{-at}]\, dt + \int_2^h f(t)\, e^{-kt}\, [1-e^{-at}]\, dt$, where $\int_0^\delta f(t)e^{-kt}[1-e^{-at}]\, dt \geq 0$, $\int_\delta^1 f(t)\, e^{-kt}[1-e^{-at}]\, dt > 0$, $\int_1^2 f(t)e^{-kt}\, [1-e^{-at}]\, dt > 0$, and $\int_2^h f(t)\, e^{-kt}[1-e^{-at}]\, dt > 0$. Hence, $\int_0^h f(t)e^{-kt}[1-e^{-at}]\, dt > \int_1^2 f(t)e^{-kt}[1-e^{-at}]\, dt$. Let $\int_1^2 f(t)e^{-kt}[1-e^{-at}]\, dt = K$, and $\{u_n\}$ be an increasing unbounded sequence. Choose N so that if $n > N$, $u_n > 2$, then $\int_0^{u_n} f(t)\, e^{-kt}\, [1-e^{-at}]\, dt > \int_1^2 f(t)\, e^{-kt}[1-e^{-at}]\, dt = K$. Hence, $\int_0^{u_n} f(t)e^{-kt}\, dt - \int_0^{u_n} f(t)\, e^{-(k+a)t}\, dt > K$ and $\int_0^{u_n} f(t)e^{-kt}dt > \int_0^{u_n} f(t)e^{-(k+a)t}\, dt + K$. Thus, except for at most a finite number of positive integers $\{\int_0^{u_n} f(t)e^{-kt}dt\}$ and $\{\int_0^{u_n} f(t)e^{-(k+a)t}dt\}$ differ term by term by at least K. Thus, $\text{Lim}\{\int_0^{u_n} f(t)\, e^{-kt}dt\} \neq \text{Lim}\{\int_0^{u_n} f(t)e^{-(k+a)t}dt\}$. Furthermore, since $\{\int_0^{u_n} f(t)e^{-kt}dt\}$ is term by term greater than $\{\int_0^{u_n} f(t)\, e^{-(k+a)t}dt + K\}$, hence $\underline{L[f(t),k] \geq L[f(t),k+a] + K}$.

Theorem: If $f(t) < 0$ for all $t \geq 0$, $k > 0$, $a > 0$, and $L[f(t),k]$ exists, then $L[f(t),k] < L[f(t),k+a]$.

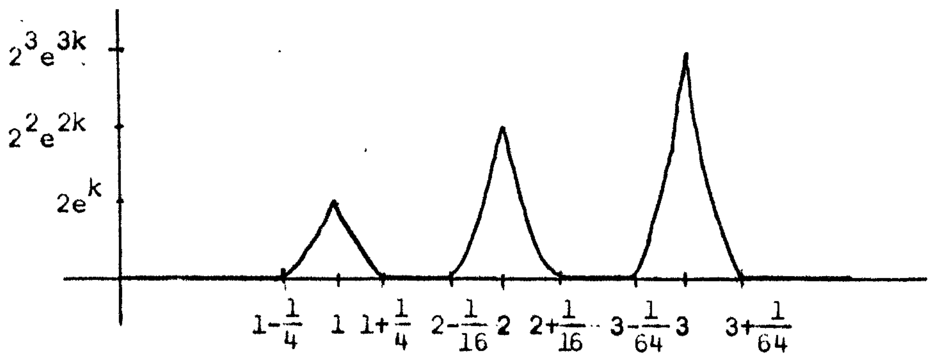
Proof: Let $f(t) < 0$ for all $t \geq 0$, $k > 0$, $a > 0$, and $L[f(t),k]$ exist. Clearly, $L[f(t),k+a]$ exists. Also since $k > 0$ and $a > 0$, $k+a > k$ and $0 < e^{-(k+a)} < e^{-k}$ and since $f(t) < 0$, $0 > f(t)e^{-(k+a)t} > f(t)e^{-kt}$ for all $t > 0$ and $0 > f(t)e^{-(k+a)t} \geq f(t)\, e^{-kt}$ for all $t \geq 0$. If $h > 0$ and $0 ? \delta > h$, $\int_\delta^h f(t)e^{-(k+a)t}dt > \int_\delta^h f(t)e^{-kt}dt$ and $\int_0^h f(t)e^{-(k+a)t}dt \geq \int_0^h f(t)e^{-kt}dt$ so that $\int_\delta^h f(t)e^{-(k+a)}dt - \int_\delta^h f(t)e^{-kt}dt > 0$ and $\int_0^h f(t)e^{-(k+a)t}dt -$

$\int_0^h f(t)e^{-kt}dt \geq 0$, hence $\int_\delta^h f(t)e^{-kt}[e^{-at}-1]\,dt > 0$ and $\int_0^h f(t)e^{-kt}[e^{-at}-1]dt \geq 0$. Furthermore, if $h > 2$ and $0 < \delta < 1$, from the above equation $\int_0^h f(t)e^{-kt}[e^{-at}-1]\,dt =$ $\int_0^\delta f(t)e^{-kt}[e^{-at}-1]dt + \int_\delta^1 f(t)e^{-kt}[e^{-at}-1]dt + \int_1^2 f(t)e^{-kt}[e^{-at}-1]dt +$ $\int_2^h f(t)e^{-kt}[e^{-at}-1]dt$, where $\int_0^\delta f(t)e^{-kt}[e^{-at}-1]\,dt \geq 0$, $\int_\delta^1 f(t)e^{-kt}[e^{-at}-1]dt > 0$, $\int_1^2 f(t)e^{-kt}[e^{-at}-1]dt > 0$, $\int_2^h f(t)\,e^{-kt}[e^{-at}-1]dt > 0$. Hence, $\int_0^h f(t)e^{-kt}[e^{-at}-1]dt > \int_1^2 f(t)e^{-kt}[e^{-at}-1]dt$. Let $\int_1^2 f(t)e^{-kt}[e^{-at}-1]\,dt = K$, and $\{u_n\}$ be an increasing unbounded sequence. Choose N so that if $n > N$, $u_n > 2$, then $\int_0^{u_n} f(t)e^{-kt}[e^{-at}-1]dt > \int_1^2 f(t)e^{-kt}[e^{-at}-1]dt = K$. Thus, $\int_0^{u_n} f(t)e^{-(k+a)t}dt - \int_0^{u_n} f(t)e^{-kt}dt > K$, and $\int_0^{u_n} f(t)e^{-(k+a)t}dt >$ $\int_0^{u_n} f(t)e^{-kt}dt + K$. Thus, except for at most a finite number of positive integers $\{\int_0^{u_n} f(t)e^{-(k+a)t}dt\}$ and $\{\int_0^{u_n} f(t)e^{-kt}dt\}$ differ term by term by at least K, hence $\text{Lim}\,\{\int_0^{u_n} f(t)e^{-(k+a)t}dt\} \neq \text{Lim}\,\{\int_0^{u_n} f(t)e^{-kt}dt\}$. Furthermore, since $\{\int_0^{u_n} f(t)e^{-(k+a)t}dt\}$ is term by term greater than $\{\int_0^{u_n} f(t)e^{-kt}dt + K\}$, <u>$L[f(t),k+a] \geq L[f(t),k]+K$.</u>

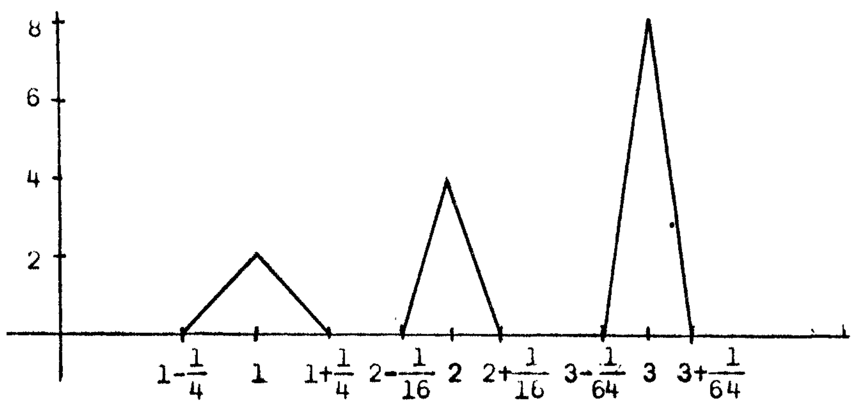
<u>Example</u>: The following is an example of a continuous, unbounded function which has a laplace transformation. Define, for $k > 0$;

$$f(t) = \begin{cases} 2^{3n}\left[t-n+\frac{1}{2^{2n}}\right]e^{kt} & \text{for } n - \frac{1}{2^{2n}} \leq t \leq n \\ -2^{3n}\left[t-n-\frac{1}{2^{2n}}\right]e^{kt} & \text{for } n < t \leq n + \frac{1}{2^{2n}} \\ 0 \text{ elsewhere} & \end{cases}$$

Consider the following graph of f(t)



It is seen that f(t) is unbounded. Now consider the graph of $f(t)\ e^{-kt}$:



From the graph $\int_0^\infty f(t)e^{-kt}dt = \frac{1}{2}\left(\frac{2}{4}\right)(2) + \frac{1}{2}\left(\frac{2}{16}\right)(4) + \frac{1}{2}\left(\frac{2}{64}\right)(8) + \cdots\cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots = 1.$

Thus, <u>L[f(t),k] exists</u>.

Furthermore, the above example can easily be made to be differentiable everywhere.

<u>Theorem</u>: If $L[f(t),k]$ exists, $f(t)e^{-kt}$ is bounded for all $t > t_o \geq 0$, and $a > 0$, then $L[e^{-at}f(t),k]$ exists and $L[e^{-at}f(t),k] = L[f(t),k+a]$.

Proof: Let $\varepsilon > 0$ and $\{u_n\}$ be an increasing unbounded sequence. Since $a > 0$ and $L[f(t),k]$ exists, $L[f(t),k+a]$ exists. Hence, there is a positive integer N so that if $m,n > N$, $\left|\int_{u_n}^{u_m} f(t)e^{-(k+a)t}dt\right| < \varepsilon$. And $\left|\int_{u_n}^{u_m} (e^{-at}f(t))e^{-kt}dt\right| = \left|\int_{u_n}^{u_m} f(t)\, e^{-(k+a)t}dt\right| < \varepsilon$. Thus, $L[e^{-at}f(t),k]$ exists.

Also $L[e^{-at}f(t),k] = \int_0^\infty e^{-at}f(t)\, e^{-kt}dt = \int_0^\infty f(t)e^{-(k+a)}dt = L[f(t),k+a]$. Therefore, <u>$L[e^{-at}f(t),k]$ exists and $L[e^{-at}f(t),k] = L[f(t),k+a]$</u>.

<u>Notation</u>: $f^{(n)}_{(t)}$ denotes the n^{th} derivative of $f(t)$. If $n=0$, $f^{(0)}_{(t)} = f(t)$.

<u>Notation</u>: $f^{(n)}_{(0+)} = \lim_{t\to 0^+} f^{(n)}_{(t)}$.

<u>Theorem</u>: Let $f(t)$ be defined for all $t \geq 0$, $f^{(n)}_{(t)}$ exists for all $t > 0$, $k > 0$, $L[f(t),k]$ exists, $L[f^{(n)}_{(t)},k]$ exists, and $\lim_{t\to\infty} f(t)e^{-kt} = \lim_{t\to\infty} f^{(1)}(t)e^{-kt} = \cdots = \lim_{t\to\infty} f^{(n-1)}(t) = 0$, then $L[f^{(n)}_{(t)},k] = -\sum_{p=1}^{n} k^{p-1} f^{(n-p)}_{(0+)} + k^n L[f(t),k]$.

Proof: Let $\varepsilon > 0$ and $\{u_n\}$be an increasing unbounded sequence. Since $\lim_{t\to\infty} f(t)\, e^{-kt} = 0$, there is a positive integer N_1 such that if $n_1 > N_1$, $|f(u_{n_1})e^{-ku_{n_1}}| < \frac{\varepsilon}{2}$. Also, since $L[f(t),k]$ exists, for $\frac{\varepsilon}{2k} > 0$, there is a positive integer N_2 such that if $n_2 > N_2$, $|\int_0^{u_{n_2}} f(t)e^{-kt}dt$ $-$ $L[f(t),k]| < \frac{\varepsilon}{2k}$. Choose $N = \max.(N_1,N_2)$, then if $n > N$, $|f(u_n)e^{-ku_n}| < \frac{\varepsilon}{2}$ and $|\int_0^{u_n} f(t)e^{-kt}dt - L[f(t),k]| < \frac{\varepsilon}{2k}$.

Since $L[f^{(1)}_{(t)}, k]$ exists, integration by parts is applied and $|\int_0^{u_n} f(t)\, e^{-kt}dt - L[f(t),k]| = |[-\frac{1}{k} f(t)e^{-kt} + \frac{1}{k}\int f^{(1)}_{(t)} e^{-kt}dt]_0^{u_n}$ $- L[f(t),k]| < \frac{\varepsilon}{2k}$ so $\frac{1}{k}\,|-f(u_n)e^{-ku_n} + f(0+) + \int_0^{u_n} f^{(1)}(t)e^{-kt}dt -$ $k\cdot L[f(t),k]| < \frac{\varepsilon}{2k}$, so that $|\int_0^{u_n} f^{(1)}(t)e^{-kt}dt - (-f(0+) +$ $k\, L[f(t),k])| < \frac{\varepsilon}{2}$ $+$ $|f(u_n)e^{-ku_n}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus,

$L[f'(t),k] = -f(0+) + k\, L[f(t),k]$.

Assume for $1 \leq m < n$, $L[f^{(m)}(t),k] = -\sum_{p=1}^{m} k^{p-1} f^{(m-p)}(0+)$ $+$ $k^m L[f(t),k]$. Then, since $\lim_{t\to\infty} f^{(m)}(t)e^{-kt} = 0$, there is a positive integer N_1 such that if $n_1 > N_1$, $|f(u_{n_1})\, e^{-ku_{n_1}}| < \frac{\varepsilon}{2}$. Also, since $L[f^{(m)}(t),k]$ exists, for $\frac{\varepsilon}{2k} > 0$, there is a positive integer N_2 so that if $n_2 > N_2$, $|\int_0^{u_{n_2}} f^{(m)}(t)\, e^{-kt}dt - L[f^{(m)}(t),k]|$

$\leq \frac{\varepsilon}{2k}$. Choose $N = \max.(N_1, N_2)$, then if $n > N$, $|f(u_n)e^{-ku_n}| < \frac{\varepsilon}{2}$ and $|\int_0^{u_n} f^{(m)}(t)e^{-kt}dt - L[f^{(m)}(t),k]| < \frac{\varepsilon}{2k}$. Since $m+1 \leq n$, $L[f^{(m)}(t),k]$ exists and integration by parts is applied so that $|\int_0^{u_n} f^{(m)}(t)e^{-kt}dt - L[f^{(m)}(t),k]| =$ $|[-\frac{1}{k} f^{(m)}(t)e^{-kt} + \frac{1}{k}\int f^{(m+1)}(t)e^{-kt}\,dt]_0^{u_n} - L[f(t),k]| <$

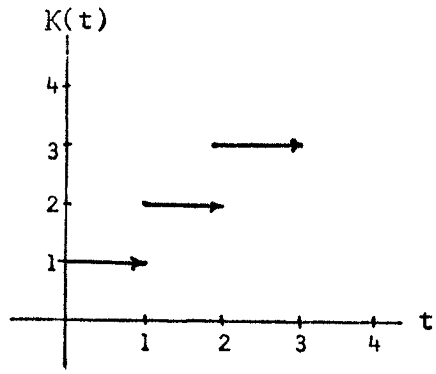
$\frac{\varepsilon}{2k}$. So $\frac{1}{k} \cdot |\int_0^{u_n} f^{(m+1)}(t)e^{-kt}dt - (-f^{(m)}(0+) + kL[f^{(m)}(t),k]) - f(u_n)e^{-ku_n}| < \frac{\varepsilon}{2k}$. So $|\int_0^{u_n} f^{(m+1)}(t)\,e^{-kt}dt - (-f^{(m)}(0+) + kL[f^{(m)}(t),k])| < \frac{\varepsilon}{2k} + |f(u_n)\,e^{-ku_n}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, $L[f^{(m)}(t),k] = -f^{(m)}(0+) + kL[f^{(m)}(t),k]$. But, $L[f^{(m)}(t),k] = -\sum_{p=1}^{m} k^{p-1}f^{(m-p)}(0+) + k^m L[f(t),k]$. Hence, $L[f^{(m+1)}(t),k] =$ $-f^{(m)}(0+) - k\sum_{p=1}^{m} k^{p-1}f^{(m-p)}(0+) + k^{m+1}L[f(t),k] = -f^{(m)}(0+) - \sum_{p=1}^{m} k^p f^{(m-p)}(0+) + k^{m+1}L[f(t),k] = -f^{(m)}(0+) - \sum_{p=2}^{m+1} k^{p-1} f^{(m-(p-1))}(0+) + k^{m+1}L[f(t),k] = -[k^{1-1}f^{m}(0+) + \sum_{p=2}^{m+1} k^{p-1} f^{((m+1)-p)}(0+)] + k^{m+1}L[f(t),k] = -[k^{1-1}f^{(m+1)-1}(0+) + \sum_{p=2}^{m+1} k^{p-1}f^{((m+1)-p)}(0+)] + k^{m+1}L[f(t),k] = -\sum_{p=1}^{m+1} k^{p-1}f^{((m+1)-p)}(0+) + k^{m+1}L[f(t),k]$. Hence, $\underline{L[f^{(m+1)}(t),k] = -\sum_{p=1}^{m+1} k^{p-1}f^{((m+1)-p)}(0+) + k^{m+1}L[f(t),k]}$.

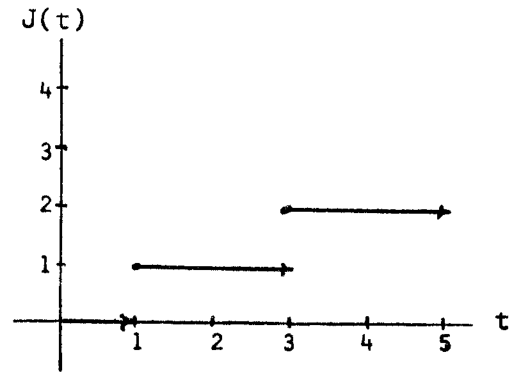
Examples: Define $H(t) = \begin{Bmatrix} 0 \text{ if } t < 0 \\ 1 \text{ if } t \geq 0 \end{Bmatrix}$

If $k > 0$, $L[H(t),k] = \int_0^\infty H(t)\, e^{-kt}\, dt = \int_0^\infty e^{-kt} dt = \frac{1}{k}$.

Thus, $L[H(t),k] = \frac{1}{k}$.

Now, define $K(t) = H(t) + H(t-1) + H(t-2) + \cdots$; and $J(t) = H(t-1) + H(t-3) + H(t-5) + \cdots$.





K(t) and J(t) are called staircase functions. If $k > 0$,

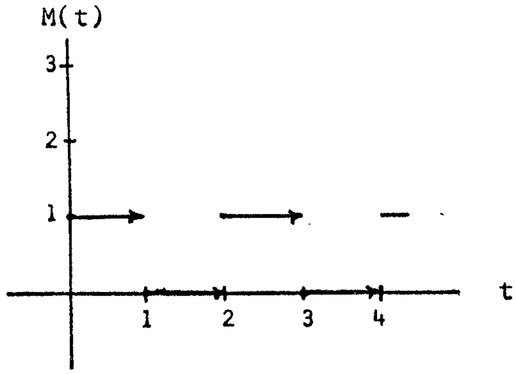
$$L[K(t),k] = \int_0^\infty K(t)e^{-kt}dt = \int_0^1 e^{-kt}dt + 2\int_1^2 e^{-kt}dt + 3\int_2^3 e^{-kt}dt + \cdots = \frac{1}{k}(1+e^{-k}+e^{-2k}+e^{-3k}+\cdots) = \frac{1}{K(1-e^{-K})}.$$

Similarly,

$$L[J(t),k] = \int_0^\infty J(t)e^{-kt}dt = \int_1^3 e^{-kt}dt + 2\int_3^5 e^{-kt}dt + 3\int_5^7 e^{-kt}dt + \cdots = \frac{1}{k}(e^{-kt}+e^{-3k}+e^{-5k}+\cdots) = \frac{e^{-kt}}{k}[1 + e^{-2k} + e^{-4k} + e^{-6k}+\cdots] = \frac{e^{-k}}{k(1-e^{-2k})}.$$

Thus, $L[K(t),k]$ and $L[J(t),k]$ exist, and $L[K(t),k] = \frac{1}{k(1-e^{-k})}$ and $L[J(t),k] = \frac{e^{-k}}{k(1-e^{-2k})}$.

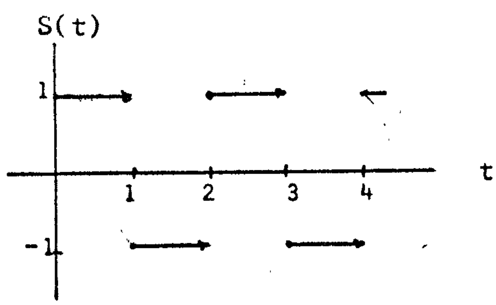
Now, define $M(t) = K(t) - 2J(t)$. M(t) is called the meander function.



$$L[M(t),k] = L[K(t),k] - 2L[J(t),k] = \frac{1}{k(1-e^{-k})} - \frac{2e^{-k}}{k(1-e^{-2k})}.$$

Hence, $\underline{L[M(t),k] = \frac{1}{k(1+e^{-k})}}$.

Define, $S(t) = 2M(t) - H(t)$. $S(t)$ is called the square-wave function.



$$L[S(t),k] = 2M(t) - H(t) = \frac{2}{k\ (1+e^{-k})} - \frac{1}{k}.$$ Hence,

$L[S(t),k]$ exists, and $\underline{L[S(t),k] = \frac{1-e^{-k}}{k\ (1+e^{-k})}}$.