THE STRUCTURE OF A BOOLEAN ALGEBRA

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THE STRUCTURE OF A BOOLEAN ALGEBRA

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LIST OF ILLUSTRATIONS

CHAPTER I

FUNDAMENTAL POSTULATES AND THEOREMS

Let Σ be a set. The purpose of this chapter is to develop a form of a "free" Boolean algebra with Σ as a base, by imposing the usual Boolean operations on the set Σ and thus generating new elements freely within explicitly prescribed restrictions.

To this end let it be postulated that there exist a set Σ^g containing Σ as a subset, two binary operations $\cap$ and $\cup$, and a unary operation $'$, all closed on Σ^g and a relation $<$ on Σ^g subject only to the following restrictions:

<u>Postulate 1.1.</u> If A is an element of Σ^g, then $A < A$.

<u>Postulate 1.2.</u> If A, B, and C are elements of Σ^g, and if $A < B$ and $B < C$, then $A < C$.

<u>Postulate 1.3.</u> If A and B are elements of Σ^g, and if $A < B$ and $B < A$, then $A = B$.

<u>Postulate 1.4.</u> If A, B, and C are elements of Σ^g, then $A \cap (B \cup C) < (A \cap B) \cup (A \cap C)$.

<u>Postulate 1.5.</u> If A, B, and C are elements of Σ^g, then $A < (B \cap C)$ if and only if $A < B$ and $A < C$.

<u>Postulate 1.6.</u> If A, B, and C are elements of Σ^g, then $(A \cup B) < C$ if and only if $A < C$ and $B < C$.

Postulate 1.7. There exist two unique elements in Σ^g, namely O and I, such that if A is an element of Σ^g, then $O < A$ and $A < I$.

Postulate 1.8. If A and B are elements of Σ^g, then $A < B$ if and only if $B' \cap A < O$, and $A < B$ if and only if $I < A' \cup B$.

Postulate 1.9. If A and B are elements of Σ^g, and if $A < B$, then $B' < A'$.

Postulate 1.1 is the reflexive property, Postulate 1.2 is the transitive property, and Postulate 1.3 is the antisymmetric property.

For the following theorems assume A, B, and C are elements of Σ^g. Unless a symbol is defined, its commonly accepted meaning is assumed.

Theorem 1.1. $A \cup A < A$.

Proof: $(A \cup B) < C$ if and only if $A < C$ and $B < C$ by Postulate 1.6. Substituting A for B and A for C, $(A \cup A) < A$ if and only if $A < A$ and $A < A$. But $A < A$ by Postulate 1.1. Therefore, $(A \cup A) < A$.

Theorem 1.2. $A < A \cap A$.

Proof: $A < B \cap C$ if and only if $A < B$ and $A < C$, by Postulate 1.5. Substituting A for B and A for C, $A < A \cap A$ if and only if $A < A$ and $A < A$. But $A < A$ by Postulate 1.1. Therefore, $A < A \cap A$.

Theorem 1.3. $A \cap A < A$.

Proof: $A < A$ by Postulate 1.1. Substituting $(A \cap A)$ for A, $(A \cap A) < (A \cap A)$. But $(A \cap A) < A \cap A$ if and only if $(A \cap A) < A$ and $(A \cap A) < A$, by Postulate 1.5. Therefore $A \cap A < A$.

Theorem 1.4. $A < A \cup A$.

Proof: $A < A$ by Postulate 1.1. Substituting $(A \cup A)$ for A, $A \cup A < A \cup A$. But $A \cup A < (A \cup A)$ if and only if $A < A \cup A$ and $A < A \cup A$ by Postulate 1.6. Therefore, $A < A \cup A$.

Theorem 1.5. $A \cup A = A$.

Proof: $A \cup A < A$ by Theorem 1.1. $A < A \cup A$ by Theorem 1.4. Therefore, by Postulate 1.3 $A \cup A = A$.

Theorem 1.6. $A \cap A = A$.

Proof: $A < A \cap A$ by Theorem 1.2. $A \cap A < A$ by Theorem 1.3. Therefore, by Postulate 1.3, $A \cap A = A$.

Theorem 1.7. $A \cap A = A \cup A$.

Proof: $A \cap A = A$ by Theorem 1.6. $A \cup A = A$ by Theorem 1.5. Then by substitution $A \cap A = A \cup A$.

Theorem 1.8. $(A \cap B) < A$.

Proof: $A \cap B < A \cap B$ by Postulate 1.1. Therefore $(A \cap B) < A$ by Postulate 1.5.

Theorem 1.9. $(A \cap B) < B$.

Proof: $A \cap B < A \cap B$ by Postulate 1.1. Therefore $(A \cap B) < B$ by Postulate 1.5.

Theorem 1.10. $A \cap B < B \cap A$.

Proof: $A \cap B < B$ by Theorem 1.9. $A \cap B < A$ by Theorem 1.8. Therefore $A \cap B < B \cap A$ by Postulate 1.5.

Theorem 1.11. $A \cap B = B \cap A$.

Proof: $A \cap B < B \cap A$ by Theorem 1.10. Substituting A for B and B for A, $B \cap A < A \cap B$. Therefore by Postulate 1.3 $A \cap B = B \cap A$.

Theorem 1.12. $A < A \cup B$.

Proof: $A \cup B < A \cup B$ by Postulate 1.1. By Postulate 1.6 $A < A \cup B$.

Theorem 1.13. $B < A \cup B$.

Proof: $A \cup B < A \cup B$ by Postulate 1.1. By Postulate 1.6 $B < A \cup B$.

Theorem 1.14. $A \cup B < B \cup A$.

Proof: $A < B \cup A$ by Theorem 1.13. $B < B \cup A$ by Theorem 1.12. Therefore $A \cup B < B \cup A$ by Postulate 1.6.

Theorem 1.15. $A \cup B = B \cup A$.

Proof: $A \cup B < B \cup A$ by Theorem 1.14. Substituting A for B and B for A, $B \cup A < A \cup B$. Therefore by Postulate 1.3 $A \cup B = B \cup A$.

Theorem 1.16. $A \cap B < A \cup B$.

Proof: $A \cap B < A$ by Theorem 1.8. Also $A < A \cup B$ by Theorem 1.12. Therefore $A \cap B < A \cup B$ by Postulate 1.2.

Theorem 1.17. $A \cap (A \cup B) < A \cup (A \cap B)$.

Proof: $A \cap (B \cup C) < (A \cap B) \cup (A \cap C)$ by Postulate 1.4. Substituting A for B and B for C, $A \cap (A \cup B) < (A \cap A) \cup (A \cap B)$. Therefore by Theorem 1.6 $A \cap (A \cup B) < A \cup (A \cap B)$.

<u>Theorem 1.18.</u> If $(A \cup B) < A$, then $B < A$.

<u>Proof:</u> $(A \cup B) < A$ from the Hypothesis. Then by Postulate 1.6 $A < A$ and $B < A$. Therefore if $(A \cup B) < A$, then $B < A$.

<u>Theorem 1.19.</u> If $A < (A \cap B)$, then $A < B$.

<u>Proof:</u> $A < (A \cap B)$ from the Hypothesis. Then by Postulate 1.5 $A < A$ and $A < B$. Therefore if $A < (A \cap B)$, then $A < B$.

<u>Theorem 1.20.</u> $A \cup (A \cap B) < A$.

<u>Proof:</u> By Postulate 1.6 $A \cup B < C$ if and only if $A < C$ and $B < C$. Substituting $(A \cap B)$ for B and A for C, $A \cup (A \cap B) < A$ if and only if $A < A$ and $(A \cap B) < A$. Now $A < A$ by Postulate 1.1, and $(A \cap B) < A$ by Theorem 1.8. Therefore $A \cup (A \cap B) < A$.

<u>Theorem 1.21.</u> $A < A \cap (A \cup B)$.

<u>Proof:</u> $A < (B \cap C)$ if and only if $A < B$ and $A < C$ by Postulate 1.5. Substituting A for B and $(A \cup B)$ for C, $A < A \cap (A \cup B)$ if and only if $A < A$ and $A < (A \cup B)$. Now $A < A$ by Postulate 1.1, and $A < (A \cup B)$ by Theorem 1.12. Therefore $A < A \cap (A \cup B)$.

<u>Theorem 1.22.</u> $A \cup (A \cap B) < A \cap (A \cup B)$.

<u>Proof:</u> $A \cup (A \cap B) < A$ by Theorem 1.20. $A < A \cap (A \cup B)$ by Theorem 1.21. Therefore by Postulate 1.2,

$$A \cup (A \cap B) < A \cap (A \cup B).$$

<u>Theorem 1.23.</u> $A < A \cup (A \cap B)$.

<u>Proof:</u> $A \cap (B \cup C) < (A \cap B) \cup (A \cap C)$ by Postulate 1.4. Substituting A for B and B for C, $A \cap (A \cup B) < (A \cap A) \cup (A \cap B)$.

Then $A \cap (A \cup B) < A \cup (A \cap B)$ by Theorem 1.6. Also $A < A \cap (A \cup B)$ by Theorem 1.21. Therefore by Postulate 1.2 $A < A \cup (A \cap B)$.

<u>Theorem 1.24.</u> $A \cap (A \cup B) < A$.

<u>Proof:</u> $A \cap (B \cup C) < (A \cap B) \cup (A \cap C)$ by Postulate 1.4. Substituting A for B and B for C, $A \cap (A \cup B) < (A \cap A) \cup (A \cap B)$. Then $A \cap (A \cup B) < A \cup (A \cap B)$ by Theorem 1.6. But $A \cup (A \cap B) < A$ by Theorem 1.20. Therefore $A \cap (A \cup B) < A$ by Postulate 1.2.

<u>Theorem 1.25.</u> $A \cup (A \cap B) = A$.

<u>Proof:</u> $A \cup (A \cap B) < A$ by Theorem 1.20. Also $A < A \cup (A \cap B)$ by Theorem 1.23. Therefore by Postulate 1.3, $A \cup (A \cap B) = A$.

<u>Theorem 1.26.</u> $A \cap (A \cup B) = A$.

<u>Proof:</u> $A < A \cap (A \cup B)$ by Theorem 1.21. Also $A \cap (A \cup B) < A$ by Theorem 1.24. Therefore by Postulate 1.3, $A \cap (A \cup B) = A$.

<u>Theorem 1.27.</u> $(A \cap B) \cup (A \cap C) < A \cap (B \cup C)$.

<u>Proof:</u> $A \cap B < B$ by Theorem 1.9. And $B < B \cup C$ by Theorem 1.12. Then by Postulate 1.2, $A \cap B < B \cup C$. Also, $A \cap B < A$ by Theorem 1.8. Thus by Postulate 1.5,

$$(A \cap B) < A \cap (B \cup C).$$

Now to show that $(A \cap C) < A \cap (B \cup C)$: $A \cap C < C$ by Theorem 1.9. And $C < B \cup C$ by Theorem 1.13. Thus by Postulate 1.2, $A \cap C < B \cup C$. Also $A \cap C < A$ by Theorem 1.8. Then by Postulate 1.5, $(A \cap C) < A \cap (B \cup C)$. Therefore by Postulate 1.6, $(A \cap B) \cup (A \cap C) < A \cap (B \cup C)$.

<u>Theorem 1.28.</u> $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

<u>Proof:</u> $A \cap (B \cup C) < (A \cap B) \cup (A \cap C)$ by Postulate 1.4. Also $(A \cap B) \cup (A \cap C) < A \cap (B \cup C)$ by Theorem 1.27. Therefore by Postulate 1.3, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

<u>Theorem 1.29.</u> $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$.

<u>Proof:</u> $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by Theorem 1.28. Using Theorem 1.11 that $A \cap B = B \cap A$, then

$$(B \cup C) \cap A = (B \cap A) \cup (C \cap A).$$

<u>Theorem 1.30.</u> $A \cup (B \cap C) < (A \cup B) \cap (A \cup C)$.

<u>Proof:</u> $A \cup (B \cap C) < (A \cup B) \cap (A \cup C)$ is true if it can be proved that:

$A \cup (B \cap C) < (A \cup B)$ and $A \cup (B \cap C) < (A \cup C)$.

Now to prove $A \cup (B \cap C) < (A \cup B)$:

$(B \cap C) < B$ by Theorem 1.8. And $B < (A \cup B)$ by Theorem 1.13. Then by Postulate 1.2, $(B \cap C) < (A \cup B)$. Also, $A < (A \cup B)$ by Theorem 1.12. Therefore by Postulate 1.6, $A \cup (B \cap C) < (A \cup B)$.

Now to prove $A \cup (B \cap C) < (A \cup C)$:

$(B \cap C) < C$ by Theorem 1.9. And $C < (A \cup C)$ by Theorem 1.13. Then by Postulate 1.2, $(B \cap C) < (A \cup C)$. Also, $A < (A \cup C)$ by Theorem 1.12. Therefore by Postulate 1.6, $A \cup (B \cap C) < (A \cup C)$.

Thus since $A \cup (B \cap C) < (A \cup B)$ and $A \cup (B \cap C) < (A \cup C)$, by Postulate 1.5, then $A \cup (B \cap C) < (A \cup B) \cap (A \cup C)$.

Theorem 1.31. $(A \cup B) \cup C < A \cup (B \cup C)$.

Proof: $(A \cup B) \cup C < A \cup (B \cup C)$ is true if and only if $(A \cup B) < A \cup (B \cup C)$ and $C < A \cup (B \cup C)$.

To prove that $(A \cup B) < A \cup (B \cup C)$:
$B < (B \cup C)$ by Theorem 1.12. And $(B \cup C) < A \cup (B \cup C)$ by Theorem 1.13. Thus by Postulate 1.2, $B < A \cup (B \cup C)$. Also, $A < A \cup (B \cup C)$ by Theorem 1.12. Therefore by Postulate 1.6, $(A \cup B) < A \cup (B \cup C)$.

To prove that $C < A \cup (B \cup C)$:
$C < (B \cup C)$ by Theorem 1.13. And $(B \cup C) < A \cup (B \cup C)$ by Theorem 1.13. Therefore by Postulate 1.2, $C < A \cup (B \cup C)$.

Now since $(A \cup B) < A \cup (B \cup C)$ and $C < A \cup (B \cup C)$, by Postulate 1.6, $(A \cup B) \cup C < A \cup (B \cup C)$.

Theorem 1.32. $A \cup (B \cup C) < (A \cup B) \cup C$.

Proof: $A \cup (B \cup C) < (A \cup B) \cup C$ is true if and only if $A < (A \cup B) \cup C$ and $(B \cup C) < (A \cup B) \cup C$.

To prove that $A < (A \cup B) \cup C$:
$A < (A \cup B)$ by Theorem 1.12. And $(A \cup B) < (A \cup B) \cup C$ by Theorem 1.12. Therefore by Postulate 1.2, $A < (A \cup B) \cup C$.

Now to prove that $(B \cup C) < (A \cup B) \cup C$:
$B < (A \cup B)$ by Theorem 1.13. And $(A \cup B) < (A \cup B) \cup C$ by Theorem 1.12. Therefore by Postulate 1.6,

$$(B \cup C) < (A \cup B) \cup C.$$

Now since $A < (A \cup B) \cup C$ and $(B \cup C) < (A \cup B) \cup C$, then by Postulate 1.6, $A \cup (B \cup C) < (A \cup B) \cup C$.

<u>Theorem 1.33.</u> $A \cup (B \cup C) = (A \cup B) \cup C$.

<u>Proof:</u> $(A \cup B) \cup C < A \cup (B \cup C)$ by Theorem 1.31. And $A \cup (B \cup C) < (A \cup B) \cup C$ by Theorem 1.32. Therefore by Postulate 1.3, $A \cup (B \cup C) = (A \cup B) \cup C$.

<u>Theorem 1.34.</u> $A \cap (B \cap C) < (A \cap B) \cap C$.

<u>Proof:</u> $A \cap (B \cap C) < (A \cap B) \cap C$ is true if and only if $A \cap (B \cap C) < (A \cap B)$ and $A \cap (B \cap C) < C$.

To prove that $A \cap (B \cap C) < (A \cap B)$:
$A \cap (B \cap C) < (B \cap C)$ by Theorem 1.9. By Theorem 1.8, $(B \cap C) < B$. Then by Postulate 1.2, $A \cap (B \cap C) < B$. Also $A \cap (B \cap C) < A$ by Theorem 1.8. Therefore $A \cap (B \cap C) < (A \cap B)$ by Postulate 1.5.

To prove that $A \cap (B \cap C) < C$:
$A \cap (B \cap C) < (B \cap C)$ by Theorem 1.9. And $(B \cap C) < C$ by Theorem 1.9. Therefore by Postulate 1.2, $A \cap (B \cap C) < C$.

Now since $A \cap (B \cap C) < (A \cap B)$ and $A \cap (B \cap C) < C$, then by Postulate 1.5, $A \cap (B \cap C) < (A \cap B) \cap C$.

<u>Theorem 1.35.</u> $(A \cap B) \cap C < A \cap (B \cap C)$.

<u>Proof:</u> $(A \cap B) \cap C < A \cap (B \cap C)$ is true if and only if $(A \cap B) \cap C < A$ and $(A \cap B) \cap C < (B \cap C)$.

To prove that $(A \cap B) \cap C < A$:
$(A \cap B) \cap C < (A \cap B)$ by Theorem 1.8. And $(A \cap B) < A$ by Theorem 1.8. Then by Postulate 1.2, $(A \cap B) \cap C < A$.

To prove that $(A \cap B) \cap C < (B \cap C)$:
$(A \cap B) \cap C < (A \cap B)$ by Theorem 1.8. And $(A \cap B) < B$ by

Theorem 1.9. Then by Postulate 1.2, $(A \cap B) \cap C < B$. Also, $(A \cap B) \cap C < C$ by Theorem 1.9. Therefore by Postulate 1.5, $(A \cap B) \cap C < (B \cap C)$.

Now since $(A \cap B) \cap C < A$ and $(A \cap B) \cap C < (B \cap C)$, then by Postulate 1.5, $(A \cap B) \cap C < A \cap (B \cap C)$.

Theorem 1.36. $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof: $A \cap (B \cap C) < (A \cap B) \cap C$ by Theorem 1.34. And $(A \cap B) \cap C < A \cap (B \cap C)$ by Theorem 1.35. Therefore by Postulate 1.3, $A \cap (B \cap C) = (A \cap B) \cap C$.

Theorem 1.37. $(A \cup B) \cap (A \cup C) < A \cup (B \cap C)$.

Proof: $B \cap (A \cup C) < (B \cap A) \cup (B \cap C)$ by Postulate 1.4. Now substituting $(A \cup B)$ for B,

$$(A \cup B) \cap (A \cup C) < [(A \cup B) \cap A] \cup [(A \cup B) \cap C].$$

Then $[(A \cup B) \cap (A \cup C)] < [A \cap (A \cup B)] \cup [(A \cup B) \cap C]$ by Theorem 1.11. Next by Theorem 1.26,

$$[(A \cup B) \cap (A \cup C)] < [A] \cup [(A \cup B) \cap C].$$

And then by Theorem 1.29,

$$[(A \cup B) \cap (A \cup C)] < A \cup [(A \cap C) \cup (B \cap C)].$$

Next by Theorem 1.33,

$$[(A \cup B) \cap (A \cup C)] < [A \cup (A \cap C)] \cup (B \cap C).$$

Therefore by Theorem 1.25,

$$(A \cup B) \cap (A \cup C) < A \cup (B \cap C).$$

Theorem 1.38. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof: $A \cup (B \cap C) < (A \cup B) \cap (A \cup C)$ by Theorem 1.30. Also, $(A \cup B) \cap (A \cup C) < A \cup (B \cap C)$ by Theorem 1.37. Then by Postulate 1.3, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 1.39. $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$.

Proof: By Theorem 1.38, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Then according to Theorem 1.15, $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$.

Theorem 1.40. $A \cap A' < 0$.

Proof: If $A < B$ then $B' \cap A < 0$ by Postulate 1.8. Substituting A for B, if $A < A$ then $A' \cap A < 0$. But $A < A$ by Postulate 1.1. Therefore by Postulate 1.8, $A \cap A' < 0$.

Theorem 1.41. $I < A \cup A'$.

Proof: If $A < B$ then $I < A' \cup B$ by Postulate 1.8. Substituting A for B, if $A < A$ then $I < A' \cup A$. But $A < A$ by Postulate 1.1. Therefore by Postulate 1.8, $I < A \cup A'$.

Theorem 1.42. $A \cup A' = I$.

Proof: $I < A' \cup A$ by Theorem 1.41. But $A \cup A' < I$ by Postulate 1.7. Therefore by Postulate 1.3, $A \cup A' = I$.

Theorem 1.43. $A \cap A' = 0$.

Proof: $A \cap A' < 0$ by Theorem 1.40. But $0 < A \cap A'$ by Postulate 1.7. Therefore by Postulate 1.3, $A \cap A' = 0$.

Theorem 1.44. $A < A \cap I$.

Proof: $A < I$ by Postulate 1.7. And $A < A$ by Postulate 1.1. Therefore by Postulate 1.5, $A < A \cap I$.

Theorem 1.45. $A \cup 0 < A$.

Proof: $0 < A$ by Postulate 1.7. And $A < A$ by Postulate 1.1. Therefore $A \cup 0 < A$ by Postulate 1.6.

Theorem 1.46. $0 < A \cap 0$.

Proof: $0 < A$ by Postulate 1.7. And $0 < 0$ by Postulate 1.1. Therefore $0 < A \cap 0$ by Postulate 1.5.

Theorem 1.47. $I < A \cup I$.

Proof: $B < A \cup B$ by Theorem 1.13. Now substituting I for B, $I < A \cup I$.

Theorem 1.48. $A \cup I = I$.

Proof: $A \cup I < I$ by Postulate 1.7. Also $I < A \cup I$ by Theorem 1.47. Thus by Postulate 1.3, $A \cup I = I$.

Theorem 1.49. $A \cap I < A$.

Proof: $A \cap B < A$ by Theorem 1.8. Now substituting I for B, $A \cap I < A$.

Theorem 1.50. $A \cap I = A$.

Proof: $A < A \cap I$ by Theorem 1.44. Also $A \cap I < A$ by Theorem 1.49. Therefore $A \cap I = A$ by Postulate 1.3.

Theorem 1.51. $A < A \cup 0$.

Proof: $A < A \cup B$ by Theorem 1.12. Now substituting 0 for B, $A < A \cup 0$.

Theorem 1.52. $A \cup 0 = A$.

Proof: $A \cup 0 < A$ by Theorem 1.45. And $A < A \cup 0$ by Theorem 1.51. Therefore $A \cup 0 = A$ by Postulate 1.3.

Theorem 1.53. $A \cap 0 < 0$.

Proof: $A \cap B < B$ by Theorem 1.9. Now substituting 0 for B, $A \cap 0 < 0$.

Theorem 1.54. $A \cap 0 = 0$.

Proof: $0 < A \cap 0$ by Theorem 1.46. Also $A \cap 0 < 0$ by Theorem 1.53. Therefore by Postulate 1.3, $A \cap 0 = 0$.

<u>Theorem 1.55.</u> $A' \cup B' < (A \cap B)'$.

<u>Proof:</u> $A \cap B < A$ by Theorem 1.8. Then by Postulate 1.9, $A' < (A \cap B)'$. $A \cap B < B$ by Theorem 1.9. Then by Postulate 1.9, $B' < (A \cap B)'$. Therefore by Postulate 1.6, $A' \cup B' < (A \cap B)'$.

<u>Theorem 1.56.</u> $(A \cup B)' < A' \cap B'$.

<u>Proof:</u> $A < A \cup B$ by Theorem 1.12. Then by Postulate 1.9, $(A \cup B)' < A'$. Also $B < A \cup B$ by Theorem 1.13. Then $(A \cup B)' < B'$ by Postulate 1.9. Therefore $(A \cup B)' < A' \cap B'$ by Postulate 1.5.

<u>Theorem 1.57.</u> $A < (A')'$.

<u>Proof:</u> $I < A \cup A'$ by Theorem 1.41. Now substituting A' for A, $I < A' \cup (A')'$. Then by Postulate 1.8, $A < (A')'$.

<u>Theorem 1.58.</u> $(A')' < A$.

<u>Proof:</u> $A \cap A' < 0$ by Theorem 1.40. Now substituting A' for A, $A' \cap (A')' < 0$. Then by Postulate 1.8, $(A')' < A$.

<u>Theorem 1.59.</u> $(A')' = A$.

<u>Proof:</u> $A < (A')'$ by Theorem 1.57. Also $(A')' < A$ by Theorem 1.58. Therefore $(A')' = A$ by Postulate 1.3.

<u>Theorem 1.60.</u> $(A' \cup B')' < A \cap B$.

<u>Proof:</u> $(A \cup B)' < A' \cap B'$ by Theorem 1.56. Now substituting A' for A and B' for B, $(A' \cup B')' < (A')' \cap (B')'$. Then by Theorem 1.59, $(A' \cup B')' < A \cap B$.

<u>Theorem 1.61.</u> $A \cup B < (A' \cap B')'$.

<u>Proof:</u> $A' \cup B' < (A \cap B)'$ by Theorem 1.55. Now substituting A' for A and B' for B, $(A')' \cup (B')' < (A' \cap B')'$. Then by Theorem 1.59, $A \cup B < (A' \cap B')'$.

<u>Theorem 1.62.</u> $(A \cap B)' < A' \cup B'$.

<u>Proof:</u> $(A' \cup B')' < (A \cap B)$ by Theorem 1.60. Then by Postulate 1.8, $(A \cap B)' \cap (A' \cup B')' < 0$. But by Theorem 1.10, $(A' \cup B')' \cap (A \cap B)' < 0$. Therefore $(A \cap B)' < A' \cup B'$ by Postulate 1.8.

<u>Theorem 1.63.</u> $(A \cap B)' = A' \cup B'$.

<u>Proof:</u> $(A \cap B)' < A' \cup B'$ by Theorem 1.62. And $A' \cup B' < (A \cap B)'$ by Theorem 1.55. Then by Postulate 1.3, $(A \cap B)' = A' \cup B'$.

<u>Theorem 1.64.</u> $A' \cap B' < (A \cup B)'$.

<u>Proof:</u> $A \cup B < (A' \cap B')'$ by Theorem 1.61. Then by Postulate 1.9, $[(A' \cap B')']' < (A \cup B)'$. Therefore by Theorem 1.59, $A' \cap B' < (A \cup B)'$.

<u>Theorem 1.65.</u> $(A \cup B)' = A' \cap B'$.

<u>Proof:</u> $(A \cup B)' < A' \cap B'$ by Theorem 1.56. Also by Theorem 1.64, $A' \cap B' < (A \cup B)'$. Then $(A \cup B)' = A' \cap B'$ by Postulate 1.3.

<u>Theorem 1.66.</u> $I < 0'$.

<u>Proof:</u> $0 < A'$ by Postulate 1.7. Then by Postulate 1.9 $(A')' < 0'$. Also $0 < A$ by Postulate 1.7. Then by Postulate 1.9, $A' < 0'$. Then by Postulate 1.6, $(A')' \cup A' < 0'$. Therefore by Theorem 1.42, $I < 0'$.

<u>Theorem 1.67.</u> $I' < 0$.

Proof: $I' < I'$ by Postulate 1.1. Also $I' < I$ by Postulate 1.7. Then by Postulate 1.5, $I' < I' \cap I$. Therefore $I' < 0$ by Theorem 1.43.

<u>Theorem 1.68.</u> $0 = I'$.

<u>Proof:</u> $0 < I'$ by Postulate 1.7. Also $I' < 0$ by Theorem 1.67. Then by Postulate 1.3, $0 = I'$.

<u>Theorem 1.69.</u> $0' < I$.

<u>Proof:</u> $A < I$ by Postulate 1.7. Substituting $0'$ for A, $0' < I$.

<u>Theorem 1.70.</u> $0' = I$.

<u>Proof:</u> $0' < I$ by Theorem 1.69. Also $I < 0'$ by Theorem 1.66. Therefore $0' = I$ by Postulate 1.3.

<u>Theorem 1.71.</u> $(A \cup B) \cup (A \cap C) = A \cup B$.

<u>Proof:</u> $(A \cup B) \cup (A \cap C) = [(A \cup B) \cup A] \cap [(A \cup B) \cup C]$ by Theorem 1.38 when $(A \cup B)$ is substituted for A, and A is substituted for B. Then

$$(A \cup B) \cup (A \cap C) = [(A \cup B) \cup A] \cap [(A \cup B) \cup C]$$

$$= [A \cup (B \cup A)] \cap [(A \cup B) \cup C]$$

by Theorem 1.33,

$$= [A \cup (A \cup B)] \cap [(A \cup B) \cup C]$$

by Theorem 1.15,

$$= [(A \cup A) \cup B] \cap [(A \cup B) \cup C]$$

by Theorem 1.33,

$$= [A \cup B] \cap [(A \cup B) \cup C]$$

by Theorem 1.5,

$$= [A \cup B]$$

by Theorem 1.26. Therefore the theorem is proved.

The associative laws, Theorems 1.36 and 1.33, guarantee that if A, B, C, and D are elements of Σ^{9}, then

$A \cap B \cap C \cap D$ and $A \cup B \cup C \cup D$ are elements just as $A \cap B$ and $A \cup B$ are elements. In other words, elements are formed which are the cap or cup of more than two elements at a time.

If $\sum$ is a finite set of n elements, $\sum_n$ denotes this set. $\sum_n^g$ denotes the set containing $\sum_n$ as a subset together with all elements generated from the operations $\cap$, $\cup$, and $'$ on elements of $\sum_n$.

Example 1.1. To demonstrate the generation of $\sum_n^g$, suppose $\sum_1 = \{A\}$. The elements of $\sum_n^g$ are generated as follows:

$$A$$

$$A'$$

$$A \cap A' = 0$$

$$A \cup A' = I$$

Now to examine the possibility of other elements generated from these, all possible cases are considered as follows:

$(A)' = A'$

$(A')' = A$ by Theorem 1.59.

$0' = I$ by Theorem 1.70.

$I' = 0$ by Theorem 1.68.

$A \cap A = A$ by Theorem 1.6.

$A \cap A' = 0$ by Theorem 1.43.

$A \cap 0 = 0$ by Theorem 1.54.

$A \cap I = A$ by Theorem 1.50.

$A' \cap A' = A'$ by Theorem 1.6.

$A' \cap 0 = 0$ by Theorem 1.54.

$A' \cap I = A'$ by Theorem 1.50.

$0 \cap 0 = 0$ by Theorem 1.6.

$I \cap 0 = 0$ by Theorem 1.54.

$I \cap I = I$ by Theorem 1.6.

$A \cup A = A$ by Theorem 1.5.

$A \cup A' = I$ by Theorem 1.42.

$A \cup 0 = A$ by Theorem 1.52.

$A \cup I = I$ by Theorem 1.48.

$A' \cup A' = A'$ by Theorem 1.5.

$A' \cup 0 = A'$ by Theorem 1.52.

$A' \cup I = I$ by Theorem 1.48.

$0 \cup 0 = 0$ by Theorem 1.5.

$0 \cup I = I$ by Theorem 1.48.

$I \cup I = I$ by Theorem 1.5.

Thus when the operations are applied to $\{A, A', 0, I\}$ the same set is generated. Thus $\sum_n^9$ is finite; moreover, if $\sum_1 = \{A\}$, then $\sum_1^9 = \{A, A', 0, I\}$.

<u>Example 1.2.</u> Now to demonstrate how the elements of are generated for $\sum_2$, suppose $\sum_2 = \{A, B\}$. The elements of $\sum_2^9$ are generated thus: (The elements are numbered to facilitate their further generation.)

1) A	7) $A \cap B'$	12) $A' \cup B'$
2) B	8) $A' \cap B'$	13) $(A \cap B') \cup (A' \cap B)$
3) A'	9) $A \cup B$	14) $(A \cap B) \cup (A' \cap B')$
4) B'	10) $A' \cup B$	15) $A \cap A' = 0$
5) $A \cap B$	11) $A \cup B'$	16) $A \cup A' = I$
6) $A' \cap B$		

Now to examine the possibility of other elements generated from these, four sample will be selected, for to examine all combinations of the sixteen elements under the three operations would be too extensive for its illustrative purpose.

Sample 1: $(2) \cap (2)$.

$B \cap B'$

$\simeq 0$ by Theorem 1.43. But 0 is element number 15.

Sample 2: $(4) \cup (9)$.

$B' \cup (A \cup B)$

$\simeq (A \cup B) \cup B'$ by Theorem 1.15,

$\simeq A \cup (B \cup B')$ by Theorem 1.33,

$\simeq A \cup I$ by Theorem 1.42,

$\simeq I$ by Theorem 1.48. But I is element number 16.

Sample 3: $(11)' \cap (13)$

$(A \cup B')' \cap [(A \cap B') \cup (A' \cap B)]$

$\simeq A' \cap (B')' \cap [(A \cap B') \cup (A' \cap B)]$ by Theorem 1.65,

$\simeq A' \cap B \cap [(A \cap B') \cup (A' \cap B)]$ by Theorem 1.59,

$\simeq A' \cap B \cap [\{(A \cap B') \cup A'\} \cap \{(A \cap B') \cup B\}]$ by Theorem 1.38,

$\simeq A' \cap B \cap [\{(A \cup A') \cap (B' \cup A')\} \cap \{(A \cup B) \cap (B' \cup B)\}]$
by Theorem 1.39,

$\simeq A' \cap B \cap [\{I \cap (B' \cup A')\} \cap \{(A \cup B) \cap I\}]$ by Theorem 1.42,

$\simeq A' \cap B \cap [\{(B' \cup A') \cap I\} \cap \{(A \cup B) \cap I\}]$ by Theorem 1.11,

$\simeq A' \cap B \cap [(B' \cup A') \cap (A \cup B)]$ by Theorem 1.50,

$\simeq A' \cap [B \cap (B' \cup A')] \cap (A \cup B)$ by Theorem 1.36,

$\simeq A' \cap [(B' \cup A') \cap B] \cap (A \cup B)$ by Theorem 1.11,

$\simeq [A' \cap (B' \cup A')] \cap [B \cap (A \cup B)]$ by Theorem 1.36,

$= [A' \cap (A' \cup B')] \cap [B \cap (B \cup A)]$ by Theorem 1.15,

$= A' \cap B$ by Theorem 1.26. But $(A' \cap B)$ is element number 6.

Sample 4: $(3) \cup (14)'$

$A' \cup [(A \cap B) \cup (A' \cap B')]'$

$= A' \cup [(A \cap B)' \cap (A' \cap B')']$ by Theorem 1.65,

$= A' \cup [(A' \cup B') \cap ((A')' \cup (B')')]$ by Theorem 1.63,

$= A' \cup [(A' \cup B') \cap (A \cup B)]$ by Theorem 1.59,

$= A' \cup [\{(A' \cup B') \cap A\} \cup \{(A' \cup B') \cap B\}]$ by Theorem 1.28,

$= A' \cup [\{(A' \cap A) \cup (B' \cap A)\} \cup \{(A' \cap B) \cup (B' \cap B)\}]$
by Theorem 1.29,

$= A' \cup [\{(A \cap A') \cup (A \cap B')\} \cup \{(A' \cap B) \cup (B \cap B')\}]$
by Theorem 1.11,

$= A' \cup [\{0 \cup (A \cap B')\} \cup \{(A' \cap B) \cup 0\}]$ by Theorem 1.43,

$= A' \cup [\{(A \cap B') \cup 0\} \cup \{(A' \cap B) \cup 0\}]$ by Theorem 1.15,

$= A' \cup [(A \cap B') \cup (A' \cap B)]$ by Theorem 1.52,

$= A' \cup [(A' \cap B) \cup (A \cap B')]$ by Theorem 1.15,

$= [A' \cup (A' \cap B)] \cup (A \cap B')$ by Theorem 1.33,

$= A' \cup (A \cap B')$ by Theorem 1.25,

$= (A' \cup A) \cap (A' \cup B')$ by Theorem 1.38,

$= (A \cup A') \cap (A' \cup B')$ by Theorem 1.15,

$= I \cap (A' \cup B')$ by Theorem 1.42,

$= (A' \cup B') \cap I$ by Theorem 1.11,

$= A' \cup B'$ by Theorem 1.50. But $(A' \cup B')$ is element number 12.

If the operations were performed on all possible combinations of the sixteen elements, the elements generated would be found to be the same as the above sixteen. Thus these sixteen elements comprise $\sum_2^3$.

These two examples illustrate a structure which will be developed in the following two chapters.

CHAPTER II

BOOLEAN POLYNOMIALS

Chapter One set forth some simple yet fundamental properties of the elements of Σ^9 under the three operations. Most of the commonly accepted Boolean postulates[1] were proved in Chapter One as theorems. These principle postulates are as follows:

1. Reflexive under $<$. $A < A$.--Theorem 1.16.

2. Anti-symmetric under $<$. If $A < B$ and $B < A$, then $A = B$.--Postulate 1.3.

3. Transitive under $<$. If $A < B$ and $B < C$, then $A < C$.--Postulate 1.2.

4. Idempotent. $A \cap A = A$.--Theorem 1.6.

   $A \cup A = A$.--Theorem 1.5.

5. Commutative. $A \cap B = B \cap A$.--Theorem 1.11.

   $A \cup B = B \cup A$.--Theorem 1.15.

6. Associative. $A \cap (B \cap C) = (A \cap B) \cap C$.--Theorem 1.36.

   $A \cup (B \cup C) = (A \cup B) \cup C$.--Theorem 1.33.

7. Distributive. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.--Theorem 1.28.

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.--Theorem 1.38.

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[1]Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra (New York, 1962), pp. 336-342.

8. Universal Bounds. There exist two unique elements, 0 and I, such that $0 < A$ and $A < I$ for all A.--Postulate 1.7.

9. Intersection. $A \cap 0 = 0$.--Theorem 1.54.

$A \cap I = A$.--Theorem 1.50.

10. Union. $A \cup 0 = A$.--Theorem 1.52.

$A \cup I = I$.--Theorem 1.48.

11. Complementarity. $A \cap A' = 0$.--Theorem 1.43.

$A \cup A' = I$.--Theorem 1.42.

12. Dualization or DeMorgan's Theorem.

$(A \cap B)' = A' \cup B'$.--Theorem 1.63.

$(A \cup B)' = A' \cap B'$.--Theorem 1.65.

13. Involution. $(A')' = A$.--Theorem 1.59.

14. Absorption. $A \cap (A \cup B) = A$.--Theorem 1.26.

$A \cup (A \cap B) = A$.--Theorem 1.25.

All of the above properties were proved in Chapter One with the exception of numbers one, two, three, and eight, which were postulated.

Definition 2.1. In any Boolean algebra, Σ^9, the operations of $\cap$, $\cup$, and $'$ will be called primary functions.

Definition 2.2. Suppose k is a positive integer. The statement that F is a polynomial in k variables means that there exists a finite composition of primary functions such that F maps each ordered k-tuple of elements of Σ^9 onto the element determined by that composition.

Example 2.1. $F(x) = (x \cap x') \cup [x' \cap (x \cup x)]$ where the replacement set for x is any element in Σ^9.

$F(x,y) = (x' \cup y)' \cap x'$ where the replacement set for x,y is any ordered pair (x,y) in $\sum^g$.

$F(x,y,z) = x' \cap (z \cup y')$ where x, y, and z is any ordered triple in $\sum^g$.

Definition 2.3. The statement that F^p is a primary polynomial means that F^p is a polynomial in one variable such that F^p maps each element of $\sum^g$ onto that same element. In other words, F^p is the identity polynomial.

Example 2.2. $F^p(x) = x$, where x is any element in $\sum^g$.

Definition 2.4. The statement that $F^{p'}$ is a primary prime polynomial means that $F^{p'}$ is a polynomial in one variable such that $F^{p'}$ maps each element of $\sum^g$ onto its prime.

Example 2.3. $F^{p'}(x) = x'$, where x is any element of $\sum^g$.

Definition 2.5. The symbol "F^o" will denote, in each occurence, either the primary polynomial or the primary prime polynomial.

Example 2.4. F_1^o is either F^p or $F^{p'}$, and F_2^o is also F^p or $F^{p'}$ independently of which one F_1^o is.

Definition 2.6. Suppose n is a positive integer. The statement that F^s is a simple polynomial in n variables means that there exist polynomials, F_1^o, $F_2^o, \ldots,$ F_n^o, such that F^s maps each ordered n-tuple, $(x_1, x_2, \ldots, x_n)$ onto the element $F_1^o(x_1) \cap F_2^o(x_2) \cap \ldots \cap F_n^o(x_n)$.

Example 2.5. The simple polynomials in three variables are:

$$F_1^s(x,y,z) = x \cap y \cap z$$
$$F_2^s(x,y,z) = x' \cap y \cap z$$
$$F_3^s(x,y,z) = x \cap y' \cap z$$
$$F_4^s(x,y,z) = x \cap y \cap z'$$
$$F_5^s(x,y,z) = x' \cap y' \cap z$$
$$F_6^s(x,y,z) = x' \cap y \cap z'$$
$$F_7^s(x,y,z) = x \cap y' \cap z'$$
$$F_8^s(x,y,z) = x' \cap y' \cap z'$$

Theorem 2.1. Let S_n be the set of simple polynomials in n variables. There are exactly 2^n simple polynomials in S_n.

Proof by induction: Let $n = 1$. The simple polynomials are:

$$F_1^s(x) = F_1^o(x) = F^p(x) = x$$
$$F_2^s(x) = F_2^o(x) = F^{p'}(x) = x',$$

two of them, or $2^1 = 2$.

Now assume the theorem holds for $n = k$. Then

$$F^s(x_1,x_2,\ldots,x_k) = F_1^o(x_1) \cap F_2^o(x_2) \cap \ldots \cap F_k^o(x_k).$$

There are 2^k simple polynomials in S_k.

$$F_1^s(x_1,x_2,\ldots,x_k) = x_1 \cap x_2 \cap \ldots \cap x_k$$
$$F_2^s(x_1,x_2,\ldots,x_k) = x_1' \cap x_2 \cap \ldots \cap x_k$$
$$\vdots$$
$$F_{k+1}^s(x_1,x_2,\ldots,x_k) = x_1' \cap x_2' \cap \ldots \cap x_k$$
$$\vdots$$
$$F_{2^k}^s(x_1,x_2,\ldots,x_k) = x_1' \cap x_2' \cap \ldots \cap x_k'.$$

Now let the number of variables be $k+1$. The simple polynomials in $k+1$ variables are determined in the following manner.

$$F_1^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_1^s(x_1,x_2,\ldots,x_k) \cap F^p(x_{k+1})$$
$$= x_1 \cap x_2 \cap \ldots \cap x_k \cap x_{k+1}.$$
$$F_2^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_1^s(x_1,x_2,\ldots,x_k) \cap F^{p'}(x_{k+1})$$
$$= x_1 \cap x_2 \cap \ldots \cap x_k \cap x_{k+1}'.$$
$$F_3^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_2^s(x_1,x_2,\ldots,x_k) \cap F^p(x_{k+1})$$
$$= x_1' \cap x_2 \cap \ldots \cap x_k \cap x_{k+1}.$$
$$F_4^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_2^s(x_1,x_2,\ldots,x_k) \cap F^{p'}(x_{k+1})$$
$$= x_1' \cap x_2 \cap \ldots \cap x_k \cap x_{k+1}'.$$
$$\vdots$$
$$F_{2k+1}^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_{k+1}^s(x_1,x_2,\ldots,x_k) \cap F^p(x_{k+1})$$
$$= x_1' \cap x_2' \cap \ldots \cap x_k \cap x_{k+1}.$$
$$F_{2k+2}^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_{k+1}^s(x_1,x_2,\ldots,x_k) \cap F^{p'}(x_{k+1})$$
$$= x_1' \cap x_2' \cap \ldots \cap x_k \cap x_{k+1}'.$$
$$\vdots$$
$$F_{2\cdot 2^n-1}^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_{2^k}^s(x_1,x_2,\ldots,x_k) \cap F^p(x_{k+1})$$
$$= x_1' \cap x_2' \cap \ldots \cap x_k' \cap x_{k+1}.$$
$$F_{2\cdot 2^n}^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_{2^k}^s(x_1,x_2,\ldots,x_k) \cap F^{p'}(x_{k+1})$$
$$= x_1' \cap x_2' \cap \ldots \cap x_k' \cap x_{k+1}'.$$

Thus two simple polynomials of $k+1$ variables are determined from each polynomial F_i^s of k variables.

$F_{2i-1}^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_i^s(x_1,x_2,\ldots,x_k) \cap F_{k+1}^p(x_{k+1})$, and
$F_{2i}^s(x_1,x_2,\ldots,x_k,x_{k+1}) = F_i^s(x_1,x_2,\ldots,x_k) \cap F_{k+1}^{p'}(x_{k+1})$.

Therefore if there are 2^k simple polynomials in k variables, then there are $2^k \cdot 2$ simple polynomials in $k+1$ variables. But $2^k \cdot 2 = 2^k \cdot 2^1 = 2^{k+1}$.

Now since the theorem holds for $n=1$, and whenever it holds for $n=k$ it also holds for $n=k+1$, then the theorem is proved.

Example 2.6 (a). The simple polynomials in one variable are:

$$F_1^s(x) = x$$
$$F_2^s(x) = x',$$

two of them; $2^1 = 2$.

Example 2.6 (b). The simple polynomials in two variables are:

$$F_1^s(x,y) = x \cap y$$
$$F_2^s(x,y) = x' \cap y$$
$$F_3^s(x,y) = x \cap y'$$
$$F_4^s(x,y) = x' \cap y',$$

four of them; $2^2 = 4$.

Example 2.6 (c). The simple polynomials in three variables are:

$$F_1^s(x,y,z) = x \cap y \cap z$$
$$F_2^s(x,y,z) = x' \cap y \cap z$$
$$F_3^s(x,y,z) = x \cap y' \cap z$$
$$F_4^s(x,y,z) = x \cap y \cap z'$$
$$F_5^s(x,y,z) = x' \cap y' \cap z$$
$$F_6^s(x,y,z) = x' \cap y \cap z'$$

$$F_7^S(x,y,z) = x \cap y' \cap z'$$
$$F_8^S(x,y,z) = x' \cap y' \cap z',$$

eight of them; $2^3 = 8$.

Definition 2.7. Suppose n is a positive integer. The statement that F^M is a minimal polynomial in n variables means that there exists a finite collection of simple polynomials in n variables, F_1^S, $F_2^S, \ldots, F_k^S$, such that F^M maps each ordered n-tuple, $(x_1,x_2,\ldots,x_n)$, onto the element $F_1^S(x_1,x_2,\ldots,x_n) \cup F_2^S(x_1,x_2,\ldots,x_n) \cup \ldots \cup F_k^S(x_1,x_2,\ldots,x_n)$.

Example 2.7. Some minimal polynomials in three variables are:

$$F_1^M(x,y,z) = (x \cap y' \cap z) \cup (x' \cap y' \cap z)$$
$$F_2^M(x,y,z) = (x' \cap y \cap z') \cup (x \cap y \cap z) \cup (x' \cap y' \cap z')$$
$$F_3^M(x,y,z) = (x \cap y' \cap z)$$
$$F_4^M(x,y,z) = (x \cap y' \cap z') \cup (x' \cap y \cap z').$$

Definition 2.8. Suppose k and n are positive integers, $k < n$, F is a polynomial in n variables, and G is a polynomial in k variables. The statement that G is a reduction of F means that for each ordered n-tuple, $(x_1,x_2,\ldots,x_k,x_{k+1},\ldots,x_n)$, $F(x_1,x_2,\ldots,x_k,x_{k+1},\ldots,x_n) = G(x_1,x_2,\ldots,x_k)$.

The statement that F is reducible means there exists a polynomial, G, which is a reduction of F.

Example 2.8 (a). $F(x,y) = x \cup (y \cap x)$ is a reducible polynomial. Using Theorems 1.11 and 1.25, $x \cup (y \cap x) = x$. Therefore $G(x) = x$. Also, $F(x,y) = G(x)$.

Example 2.8 (b). $F(x,y,z) \asymp (x \cup y) \cup (x \cap z)$ is reducible. Theorem 1.71 states $(x \cup y) \cup (x \cap z) \asymp x \cup y$. In this case $G(x,y) \asymp x \cup y$, and $F(x,y,z) \asymp G(x,y)$.

Definition 2.9. An irreducible polynomial is a polynomial which is not reducible.

Lemma 2.1. If n is a positive integer, i is a positive integer, $i \leq n$, then there are exactly 2^{n-1} simple polynomials in n variables such that

$$F^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \asymp F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n).$$

Proof: There are 2^n simple polynomials in n variables, and each F^o is one of two distinct polynomials, F^p or $F^{p'}$. Thus if only one of these polynomials, F^p, is used for a specific x_i in determining simple polynomials in n variables, there will be $2^n/2 = 2^n/2^1 = 2^{n-1}$ simple polynomials of this type.

Theorem 2.2. Let i, n be positive integers such that $i \leq n$. Furthermore, let $F^M(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n)$

$$\asymp F_1^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup F_2^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup \ldots \cup F_{2^{n-1}}^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n)$$

where each F_j^s is a simple polynomial such that $F_i^o(x_i) \asymp F^p(x_i)$. In other words,

$$F_j^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \asymp F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n).$$

There exists a reduction of F^M, such that if G is that reduction, then $G^p(x_i) \asymp F^M(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n)$.

<u>Proof by induction:</u> Let $n = 1$. Then $x_i = x_1$ and $F^M(x_1) = F_1^s(x_1)$ since there is only one variable. Now $F_1^s(x_1) = x_1$ since the hypothesis specifies that $F_1^s(x_1)$ must contain x_1 only, and not x_1'. But there exists a primary polynomial, G^p, such that $G^p(x_1) = x_1$. Also, $F^M(x_1)$ is the cup of one simple polynomial; $2^{n-1} = 2^{1-1} = 2^0 = 1$.

Now assume the theorem is true for $n = k$.

$$F^M(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k) = F_1^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k) \cup F_2^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k) \cup \ldots \cup F_{2^{k-1}}^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k).$$

Then there exists a G^p such that

$$G^p(x_i) = F^M(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k).$$

Thus

$$G^p(x_i) = F_1^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k) \cup F_2^s(x_1,x_2,\ldots,x_i, x_{i+1},\ldots,x_k) \cup \ldots \cup F_{2^{k-1}}^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k).$$

Then

$$G^p(x_i) = \left[F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_k)\right] \cup \left[F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_k)\right] \cup \ldots \cup \left[F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_k)\right],$$

which is

$$x_i = (x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cup (x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cup \ldots \cup (x_1' \cap x_2' \cap \ldots \cap x_i \cap x_{i+1}' \cap \ldots \cap x_k').$$

And Lemma 2.1 guarantees that there are 2^{k-1} simple polynomials such that in each polynomial $F_i^o(x_i) = F_i^p(x_i)$.

Next prove the theorem is true for $n = k + 1$. Using theorems from Chapter One,

$$x_i = [(x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cup (x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cup \ldots \cup (x_1' \cap x_2' \cap \ldots \cap x_i \cap x_{i+1}' \cap \ldots \cap x_k')] \cap I$$

by Theorem 1.50,

$$= [(x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cup (x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cup \ldots \cup (x_1' \cap x_2' \cap \ldots \cap x_i \cap x_{i+1}' \cap \ldots \cap x_k')] \cap (x_{k+1} \cup x_{k+1}')$$

by Theorem 1.42,

$$= [(x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cap (x_{k+1} \cup x_{k+1}')] \cup [(x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cap (x_{k+1} \cup x_{k+1}')] \cup \ldots \cup [(x_1' \cap x_2' \cap \ldots \cap x_i \cap x_{i+1}' \cap \ldots \cap x_k') \cap (x_{k+1} \cup x_{k+1}')]$$

by Theorem 1.29 extended,

$$= [(x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cap x_{k+1}] \cup [(x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cap x_{k+1}'] \cup [(x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cap x_{k+1}] \cup [(x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k) \cap x_{k+1}'] \cup \ldots \cup [(x_1' \cap x_2' \cap \ldots \cap x_i \cap x_{i+1}' \cap \ldots \cap x_k') \cap x_{k+1}] \cup [(x_1' \cap x_2' \cap \ldots \cap x_i \cap x_{i+1}' \cap \ldots \cap x_k') \cap x_{k+1}']$$

by Theorem 1.28 extended,

$$= [x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k \cap x_{k+1}] \cup [x_1 \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k \cap x_{k+1}'] \cup [x_1' \cap x_2 \cap \ldots \cap x_i \cap x_{i+1} \cap \ldots \cap x_k \cap x_{k+1}] \cup [x_1' \cap x_2 \cap \ldots \cap x_1 \cap x_{i+1} \cap \ldots \cap x_k \cap x_{k+1}'] \cup$$

$$\cdots \cup [x_1' \cap x_2' \cap \cdots \cap x_i \cap x_{i+1}' \cap \cdots \cap x_k' \cap x_{k+1}]$$
$$\cup [x_1' \cap x_2' \cap \cdots \cap x_i \cap x_{i+1}' \cap \cdots \cap x_k' \cap x_{k+1}']$$

by Theorem 1.11, which is a cup of simple polynomials in $k+1$ variables of the type in Lemma 2.1. Now from each simple polynomial in k variables, two simple polynomials in $k+1$ variables are formed. Then $F^M(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k,x_{k+1})$ is a cup of $2^{k-1}\cdot 2 = 2^{k-1}\cdot 2^1 = 2^k = 2^{(k+1)-1}$ simple polynomials of the type under consideration. Then

$$G^p(x_i) = F^M(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_k,x_{k+1}).$$

Now since the theorem holds for $n = 1$ variables, and whenever it is true for k variables, it is also true for $k+1$ variables, the theorem is proved.

Example 2.9. To show an example of Theorem 2.2, let

$$F^M(x,y,z) = (x \cap y \cap z) \cup (x \cap y \cap z') \cup (x \cap y' \cap z)$$
$$\cup (x \cap y' \cap z').$$

Notice that F^M is the cup of $4 = 2^2 = 2^{3-1}$ simple polynomials such that in each simple polynomial $F^o(x) = F^p(x)$.

$(x \cap y \cap z) \cup (x \cap y \cap z') \cup (x \cap y' \cap z) \cup (x \cap y' \cap z')$

$= [(x \cap y) \cap (z \cup z')] \cup [(x \cap y') \cap (z \cup z')]$ by Theorem 1.28,

$= [(x \cap y) \cap I] \cup [(x \cap y') \cap I]$ by Theorem 1.42,

$= [x \cap y] \cup [x \cap y']$ by Theorem 1.50,

$= x \cap (y \cup y')$ by Theorem 1.28,

$= x \cap I$ by Theorem 1.42,

$= x$ by Theorem 1.50.

But there exists a G^p such that $G^p(x) = x$. Therefore $G^p(x) = F^M(x,y,z)$.

Lemma 2.2. If n is a positive integer, i is a positive integer, $i \leq n$, then there are exactly 2^{n-1} simple polynomials in n variables such that

$$F^{s}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) = F^{o}(x_1) \cap F^{o}(x_2) \cap \ldots \cap F^{p'}(x_i) \cap F^{o}(x_{i+1}) \cap \ldots \cap F^{o}(x_n).$$

Proof: There are 2^n simple polynomials in n variables, and each F^o is one of two distinct polynomials, F^p or $F^{p'}$. Thus if only one of these polynomials, $F^{p'}$, is used for a specific x_i in determining simple polynomials in n variables, there will be $2^n/2 = 2^n/2^1 = 2^{n-1}$ simple polynomials of this type.

Theorem 2.3. Let i, n be positive integers such that $i \leq n$. Furthermore, let

$$F^{M}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) = F_1^{s}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup F_2^{s}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup \ldots \cup F_{2^{n-1}}^{s}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n)$$

where each F_j^s is a simple polynomial such that $F_i^o(x_i) = F^{p'}(x_i)$. In other words,

$$F_j^{s}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) = F^{o}(x_1) \cap F^{o}(x_2) \cap \ldots \cap F^{p'}(x_i) \cap F^{o}(x_{i+1}) \cap \ldots \cap F^{o}(x_n).$$

There exists a reduction of F^M, such that if G is that reduction, then $G^{p'}(x_i) = F^{M}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n)$.

Proof: The proof of this theorem is similar to that of Theorem 2.2, with the exceptions that instead of the reduction being $G^p(x_i)$, it is $G^{p'}(x_i)$; and instead of each

$$F_j^s(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) = F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n)$$

as in Theorem 2.2, in Theorem 2.3,

$$F_j^s(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) = F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^{p'}(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n).$$

In other words, in each F_j^s, $F_i^o(x_i) = F^{p'}(x_i)$. Also, instead of Lemma 2.1, Lemma 2.2 must be used.

Example 2.10. To show an example of Theorem 2.3, let

$$F^M(w,x,y,z) = (w \cap x' \cap y \cap z) \cup (w \cap x' \cap y \cap z') \cup (w \cap x' \cap y' \cap z) \cup (w \cap x' \cap y' \cap z') \cup (w' \cap x' \cap y \cap z) \cup (w' \cap x' \cap y \cap z') \cup (w' \cap x' \cap y' \cap z) \cup (w' \cap x' \cap y' \cap z').$$

Notice that F^M is the cup of $8 = 2^3 = 2^{4-1}$ simple polynomials in each of which $F^o(x) = F^{p'}(x)$.

$$F^M(w,x,y,z) = [(w \cap x' \cap y \cap z) \cup (w \cap x' \cap y \cap z')] \cup [(w \cap x' \cap y' \cap z) \cup (w \cap x' \cap y' \cap z')] \cup [(w' \cap x' \cap y \cap z) \cup (w' \cap x' \cap y \cap z')] \cup [(w' \cap x' \cap y' \cap z) \cup (w' \cap x' \cap y' \cap z')]$$

by Theorem 1.33,

$$= [(w \cap x' \cap y) \cap (z \cup z')] \cup [(w \cap x' \cap y') \cap (z \cup z')] \cup [(w' \cap x' \cap y) \cap (z \cup z')] \cup [(w' \cap x' \cap y') \cap (z \cup z')]$$

by Theorem 1.28,

$$= [(w \cap x' \cap y) \cap I] \cup [(w \cap x' \cap y') \cap I] \cup [(w' \cap x' \cap y) \cap I] \cup [(w' \cap x' \cap y') \cap I]$$

by Theorem 1.42,

$= [W \cap x' \cap y] \cup [W \cap x' \cap y'] \cup [w' \cap x' \cap y] \cup [w' \cap x' \cap y']$

by Theorem 1.50,

$= \{[w \cap x' \cap y] \cup [w \cap x' \cap y']\} \cup \{[w' \cap x' \cap y]$
$\cup [w' \cap x' \cap y']\}$

by Theorem 1.33,

$= \{(w \cap x') \cap (y \cup y')\} \cup \{(w' \cap x') \cap (y \cup y')\}$

by Theorem 1.28,

$= \{(w \cap x') \cap I\} \cup \{(w' \cap x') \cap I\}$

by Theorem 1.42,

$= \{w \cap x'\} \cup \{w' \cap x'\}$

by Theorem 1.50,

$= (w \cup w') \cap x'$

by Theorem 1.29,

$= I \cap x'$

by Theorem 1.42,

$= x' \cap I$

by Theorem 1.11,

$= x'$

by Theorem 1.50.

But there exists a $G^{p'}$ such that $G^{p'}(x) = x'$. Therefore $G^{p'}(x) = F^M(w,x,y,z)$.

Theorem 2.4. If $(x_1, x_2, \ldots, x_n)$ is an ordered n-tuple, and if F^M is a minimal polynomial in n variables such that $F^M = F_1^s \cup F_2^s \cup \ldots \cup F_{2^n}^s$ (in other words F^M is the cup of all simple polynomials in n variables), then $F^M(x_1, x_2, \ldots, x_n) = I$.

Proof: Let i be a positive integer, n be a positive integer, $i \leq n$. Let F^{Mi} be a minimal polynomial such that

$$F^{Mi}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \simeq F_1^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup F_2^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup \ldots \cup F_{2^n-1}^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n),$$

where each

$$F_j^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \simeq F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n).$$

Let $F^{Mi'}$ be a minimal polynomial such that

$$F^{Mi'}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \simeq F_1^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup F_2^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup \ldots \cup F_{2^n-1}^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n),$$

where each

$$F_k^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \simeq F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^{p'}(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n).$$

Then

$$F^M(x_1,x_2,\ldots,x_n) \simeq F^{Mi}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n) \cup F^{Mi'}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n).$$

But Theorem 2.2 states that there exists a G^p such that

$$G^p(x_i) \simeq F^{Mi}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n),$$

and Theorem 2.3 states that there exists a $G^{p'}$ such that

$$G^{p'}(x_i) \simeq F^{Mi'}(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n).$$

Then $F^M(x_1,x_2,\ldots,x_n) \simeq G^p(x_i) \cup G^{p'}(x_i)$

$\simeq x_i \cup x_i'$ by Definitions 2.3 and 2.4,

$\simeq I$ by Theorem 1.42.

Theorem 2.5. Let n be a positive integer, and P_n^M be the set of all minimal polynomials in n variables. There are exactly $2^{2^n}-1$ minimal polynomials in P_n^M.

Proof: There are 2^n distinct simple polynomials in n variables by Theorem 2.1. Let $k = 2^n$, and let r be any positive integer, $r \leq k$. From Definition 2.7, the set of all minimal polynomials in n variables, P_n^M, is the set of all F^M where F^M is the cup of r simple polynomials. The order in which the simple polynomials appear in any one minimal polynomial is immaterial by Theorem 1.15.

Let the symbol $\binom{k}{r}$ denote the number of combinations of k things taken r at a time. From the binomial expansion,

$$(a+b)^k = \binom{k}{0}a^kb^0 + \binom{k}{1}a^{k-1}b^1 + \binom{k}{2}a^{k-2}b^2 + \binom{k}{3}a^{k-3}b^3 + \ldots + \binom{k}{k-3}a^3b^{k-3} + \binom{k}{k-2}a^2b^{k-2} + \binom{k}{k-1}a^1b^{k-1} + \binom{k}{k}a^0b^k,$$

where $\binom{k}{0}$ is defined to be 1.[1]

Now let $a = b = 1$. The expansion becomes

$$(1+1)^k = \binom{k}{0}1^k1^0 + \binom{k}{1}1^{k-1}1^1 + \binom{k}{2}1^{k-2}1^2 + \binom{k}{3}1^{k-3}1^3 + \ldots + \binom{k}{k-3}1^31^{k-3} + \binom{k}{k-2}1^21^{k-2} + \binom{k}{k-1}1^11^{k-1} + \binom{k}{k}1^01^k,$$

which is

$$2^k = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \binom{k}{3} + \ldots + \binom{k}{k-3} + \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k}.$$

Now $\binom{k}{0}$ is the number of k things taken none at a time, and is defined to be 1, therefore,

$$2^k-1 = \binom{k}{1} + \binom{k}{2} + \binom{k}{3} + \ldots + \binom{k}{k-3} + \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k}.$$

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[1] Richard E. Johnson, Lona Lee Lendsey, William E. Slesnick, Grace E. Bates, Modern Algebra, Second Course (Reading, Massachusetts, 1962), p. 409.

The right side of the last equation is the sum of all possible combinations of k things, in this theorem simple polynomials, taken one at a time, two at a time, and so on to k at a time; and this sum is 2^k-1. Therefore there are 2^k-1, or $2^{2^n}-1$ distinct minimal polynomials in n variables, or $2^{2^n}-1$ minimal polynomials in P_n^M.

<u>Example 2.11.</u> To show an example of Theorem 2.5, suppose (x,y) is an ordered pair such that $F^M(x,y)$ is a minimal polynomial. Notice that $n = 2$. The minimal polynomials in 2 variables are:

$$\left.\begin{array}{l} 1)\ x \cap y \\ 2)\ x' \cap y \\ 3)\ x \cap y' \\ 4)\ x' \cap y' \end{array}\right\} \binom{4}{1} = 4$$

$$\left.\begin{array}{l} 5)\ (x \cap y) \cup (x' \cap y) \\ 6)\ (x \cap y) \cup (x \cap y') \\ 7)\ (x \cap y) \cup (x' \cap y') \\ 8)\ (x' \cap y) \cup (x \cap y') \\ 9)\ (x' \cap y) \cup (x' \cap y') \\ 10)\ (x \cap y') \cup (x' \cap y') \end{array}\right\} \binom{4}{2} = 6$$

$$\left.\begin{array}{l} 11)\ (x \cap y) \cup (x' \cap y) \cup (x \cap y') \\ 12)\ (x \cap y) \cup (x' \cap y) \cup (x' \cap y') \\ 13)\ (x \cap y) \cup (x \cap y') \cup (x' \cap y') \\ 14)\ (x' \cap y) \cup (x \cap y') \cup (x' \cap y') \end{array}\right\} \binom{4}{3} = 4$$

$$15)\ (x \cap y) \cup (x' \cap y) \cup (x \cap y') \cup (x' \cap y') = I \} \ \binom{4}{4} = 1$$

There are $2^{2^n}-1 = 2^{2^2}-1 = 2^4-1 = 16-1 = 15$ minimal polynomials.

CHAPTER III

THE SET $\sum_n^q$

In this chapter the properties developed in Chapter One and the development of simple polynomials in Chapter Two will be used to investigate the structure of $\sum_n^q$.

Definition 3.1. Let n be a positive integer and F^p a primary polynomial. The statement that x is a primary element of $\sum_n^q$ means that x is the element onto which $F^p(x)$ maps any element of $\sum_n$. Thus $F^p(x)$ maps each element of $\sum_n$ onto itself.

The order of $\sum_n$ is n. If $F^p(A) \in \sum_n$ then $F^{p'}(A) \notin \sum_n$.

The set of primary elements, $\sum_n$, is not closed under the three primary functions $\cap$, $\cup$, and $'$. $\sum_n$ is a proper subset of $\sum_n^q$.

Definition 3.2. Let n be a positive integer and $F^{p'}$ a primary prime polynomial. The set which consists of all $F^{p'}(x)$, where x is any element of $\sum_n$, is denoted by $\sum_{n'}$. Elements of $\sum_{n'}$ are called primary prime elements.

If $F^{p'}(A) \in \sum_{n'}$ then $F^{p'}(A) \notin \sum_n$. In other words, $\sum_n$ and $\sum_{n'}$ are mutually exclusive sets. $\sum_{n'}$ is a proper subset of $\sum_n^q$.

Example 3.1. Suppose $\sum_4 = \{A,B,C,D\}$. Then the set $\sum_4$ is $\{F^p(x) \mid x = A,B,C,\text{or}D\}$ or simply $\sum_4 = \{A,B,C,D\}$.

Also, the set $\Sigma_{4'} = \{F^{p'}(x) \mid x = A,B,C,\text{or} D\}$ or simply $\Sigma_{4'} = \{A',B',C',D'\}$. Furthermore, $\Sigma_4 \subset \Sigma_4^g$ and $\Sigma_{4'} \subset \Sigma_4^g$.

Definition 3.3. If $A \in \Sigma_n$, then the set which consists of the two elements, A and A', is denoted by A*, and is called the "pair set of A". Thus if one element of A* is in Σ_n, namely A, then the other element of A* is in $\Sigma_{n'}$, namely A'.

Example 3.2. Suppose $\Sigma_5 = \{A,B,C,D,E\}$. Then

$$A^* = \{A,A'\}$$
$$B^* = \{B,B'\}$$
$$C^* = \{C,C'\}$$
$$D^* = \{D,D'\}$$
$$E^* = \{E,E'\}$$

Definition 3.4. Let n be a positive integer. The set which is the union of all pair sets of Σ_n is denoted by Σ_{n^*}. Thus if $\Sigma_n = \{A_1,A_2,\ldots,A_n\}$ then $\Sigma_{n^*} = \{A_1,A_1',A_2,A_2',\ldots,A_n,A_n'\}$ and therefore has exactly 2n elements.

If $A,B \in \Sigma_{n^*}$, then $(A \cap B) \notin \Sigma_{n^*}$, and $(A \cup B) \notin \Sigma_{n^*}$. Thus Σ_{n^*} is a proper subset of Σ_n^g.

Definition 3.5. Let $(A_1,A_2,\ldots,A_n)$ be an ordered n-tuple of distinct elements from Σ_n, and F^s be any simple polynomial in n variables. Then $F^s(A_1,A_2,\ldots,A_n)$ will be called a "minterm". The set of all minterms of Σ_n^g is denoted by Σ_n^M.

Example 3.3. Let $\Sigma_2 = \{A,B\}$. Then

$$\Sigma_2^M = \{A \cap B,\ A' \cap B,\ A \cap B',\ A' \cap B'\}.$$

Keeping the same $\Sigma_2 = \{A, B\}$, it is clearly seen that $A \cap A'$ is not a minterm. A similar argument could be proposed for any n. Not only $\Sigma_n^M \subset \Sigma_n^q$, but Σ_n^M is a proper subset of Σ_n^q .

Theorem 3.1. Let $\Sigma_n = \{A_1, A_2, \ldots, A_n\}$. There are 2^n elements in Σ_n^M.

Proof: Let the ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of the simple polynomials in n variables be the set Σ_n such that $x_1 = A_1$; $x_2 = A_2$; ...; $x_n = A_n$. Then

$$F_1^s(x_1, x_2, \ldots, x_n) = F^p(x_1) \cap F^p(x_2) \cap \ldots \cap F^p(x_n)$$
$$= A_1 \cap A_2 \cap \ldots \cap A_n.$$
$$F_2^s(x_1, x_2, \ldots, x_n) = F^{p'}(x_1) \cap F^p(x_2) \cap \ldots \cap F^p(x_n)$$
$$= A_1' \cap A_2 \cap \ldots \cap A_n.$$
$$\vdots$$
$$F_{2^n}^s(x_1, x_2, \ldots, x_n) = F^{p'}(x_1) \cap F^{p'}(x_2) \cap \ldots \cap F^{p'}(x_n)$$
$$= A_1' \cap A_2' \cap \ldots \cap A_n'.$$

Now by Theorem 2.1, there are exactly 2^n simple polynomials in n variables. And since each x_i is the exact element $A_i \in \Sigma_n$, then there are 2^n minterms of Σ_n^q, or 2^n elements in Σ_n^M.

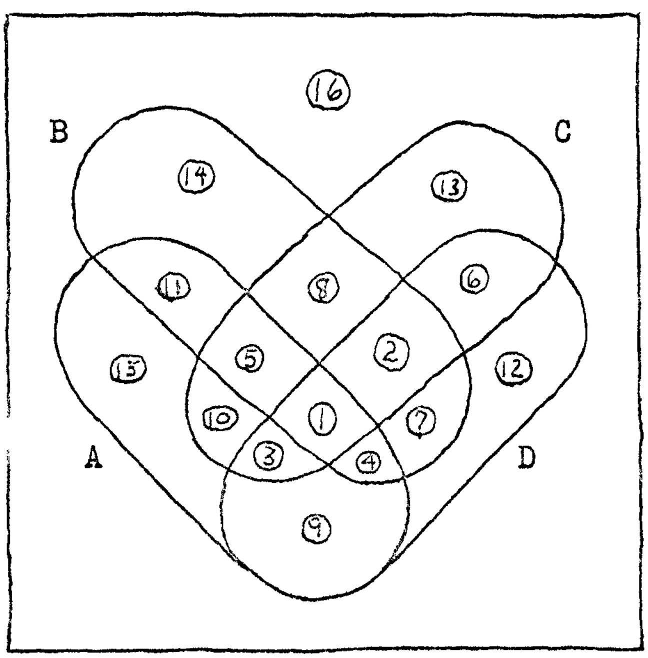
When restricting the variables of the simple polynomials in the manner prescribed in Theorem 3.1, an interesting feature is evolved. The minterms of Σ_n^q may be illustrated by Venn diagrams. If each element of Σ_n is represented by a circle or ellipse, the prime of each element $A_i \in \Sigma_n$ represented as the area not in the area A_i, and the primary function $\cap$

is interpreted in the sense of the class algebra connective, then each minterm is represented as a distinct area in the figure, and no two minterm areas overlap.

<u>Illustration 3.1.</u> Suppose $\Sigma_4 = \{A,B,C,D\}$. Then the elements of Σ_4^M are:

1) $A \cap B \cap C \cap D$
2) $A' \cap B \cap C \cap D$
3) $A \cap B' \cap C \cap D$
4) $A \cap B \cap C' \cap D$
5) $A \cap B \cap C \cap D'$
6) $A' \cap B' \cap C \cap D$
7) $A' \cap B \cap C' \cap D$
8) $A' \cap B \cap C \cap D'$
9) $A \cap B' \cap C' \cap D$
10) $A \cap B' \cap C \cap D'$
11) $A \cap B \cap C' \cap D'$
12) $A' \cap B' \cap C' \cap D$
13) $A' \cap B' \cap C \cap D'$
14) $A' \cap B \cap C' \cap D'$
15) $A \cap B' \cap C' \cap D'$
16) $A' \cap B' \cap C' \cap D'$.

Now suppose each element of Σ_4 is represented as an ellipse in a Venn diagram. Then the small sixteen areas represent the sixteen minterms of Σ_4^9.



Definition 3.6. Let $(A_1, A_2, \ldots, A_n)$ be an ordered n-tuple of distinct elements from Σ_n, and F^M be any minimal polynomial in n variables. Then elements $F^M(A_1, A_2, \ldots, A_n)$ and the element 0 are called polyterms. The set of all polyterms of Σ_n^g is denoted by Σ_n^U.

Example 3.4. Suppose $\Sigma_2 = \{A, B\}$. Then

$$\Sigma_2^U = \{[0],\ [A \cap B],\ [A' \cap B],\ [A \cap B'],\ [A' \cap B'],$$
$$[(A \cap B) \cup (A' \cap B)],\ [(A \cap B) \cup (A \cap B')],$$
$$[(A \cap B) \cup (A' \cap B')],\ [(A \cap B') \cup (A' \cap B')],$$
$$[(A' \cap B) \cup (A \cap B')],\ [(A' \cap B) \cup (A' \cap B')],$$
$$[(A \cap B) \cup (A \cap B') \cup (A' \cap B)],$$
$$[(A \cap B) \cup (A' \cap B) \cup (A' \cap B')],$$
$$[(A \cap B) \cup (A \cap B') \cup (A' \cap B')],$$
$$[(A' \cap B) \cup (A \cap B') \cup (A' \cap B')],$$
$$[(A \cap B) \cup (A' \cap B) \cup (A \cap B') \cup (A' \cap B')]\}.$$

Not only $\Sigma_n^M \subset \Sigma_n^U$, but Σ_n^M is a proper subset of Σ_n^U.

Clearly $\Sigma_n^U \subset \Sigma_n^g$, but the question "Is Σ_n^U a proper subset of Σ_n^g?" is yet unanswered. The solution to this query is one of the basic concepts in establishing the structure of Σ_n^g.

Theorem 3.2. If $A \in \Sigma_n$, then $A \in \Sigma_n^U$.

Proof: Let $\Sigma_n = \{A_1, A_2, \ldots, A_i, A_{i+1}, \ldots, A_n\}$. For the n-tuple $(x_1, x_2, \ldots, x_n)$,

$$F^M(x_1, x_2, \ldots, x_n) = F_1^S(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) \cup F_2^S(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) \cup \ldots \cup F_{2^n-1}^S(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n),$$

where each

$$F_j^s(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_n)$$
$$= F^o(x_1) \cap F^o(x_2) \cap \ldots \cap F^p(x_i) \cap F^o(x_{i+1}) \cap \ldots \cap F^o(x_n).$$

Then by Theorem 2.2, there exists a G^p such that

$$G^p(A_i) = F^M(A_1,A_2,\ldots,A_i,A_{i+1},\ldots,A_n) = F_1^s(A_1,A_2,\ldots,A_i,$$
$$A_{i+1},\ldots,A_n) \cup F_2^s(A_1,A_2,\ldots,A_i,A_{i+1},\ldots,A_n) \cup$$
$$\ldots \cup F^s_{2^n-1}(A_1,A_2,\ldots,A_i,A_{i+1},\ldots,A_n).$$

But $F^M(A_1,A_2,\ldots,A_i,A_{i\ 1},\ldots,A_n)$ is a polyterm of $\sum_n^g$, in other words, an element of $\sum_n^\cup$, by Definition 3.6. And $G^p(A_i)$ is an element of $\sum_n$, by Definition 3.1. Therefore, if $A_i \in \sum_n$, then $A_i \in \sum_n^\cup$.

Example 3.5. Suppose $\sum_2 = \{A,B\}$. $B = (A \cap B) \cup (A \cap B')$. Obviously from Theorem 3.2, $\sum_n \subset \sum_n^\cup$.

Theorem 3.3. If $A \in \sum_n$, then $A' \in \sum_n^\cup$.

Proof: The same procedure is used in proving this theorem that was employed in the proof of Theorem 3.2, except this theorem follows as a result of Theorem 2.3.

Example 3.6. Suppose $\sum_2 = \{A,B\}$. $B' = (A \cap B') \cup (A' \cap B')$. From Theorem 3.3 it is seen that $\sum_{n'} \subset \sum_n^\cup$. Also, it follows that $\sum_{n^*} \subset \sum_n^\cup$.

Theorem 3.4. The cup of all minterms of $\sum_n^g$ is I.

Proof: Let $\sum_n = \{A_1,A_2,\ldots,A_n\}$. For the ordered n-tuple $(x_1,x_2,\ldots,x_n)$,

$$F^M(x_1,x_2,\ldots,x_n) = F_1^s(x_1,x_2,\ldots,x_n) \cup F_2^s(x_1,x_2,\ldots,x_n) \cup$$
$$\ldots \cup F^s_{2^n}(x_1,x_2,\ldots,x_n).$$

Then

$$F^M(A_1,A_2,\ldots,A_n) = F_1^s(A_1,A_2,\ldots,A_n) \cup F_2^s(A_1,A_2,\ldots,A_n) \cup \ldots \cup F_{2^n}^s(A_1,A_2,\ldots,A_n).$$

Then by Theorem 2.4,

$$F^M(A_1,A_2,\ldots,A_n) = I.$$

But $F_1^s(A_1,A_2,\ldots,A_n) \cup F_2^s(A_1,A_2,\ldots,A_n) \cup \ldots \cup F_{2^n}^s(A_1,A_2,\ldots,A_n)$ is the cup of all minterms of $\sum_n^q$. Therefore the theorem is proved.

Again referring to the analogy in a Venn diagram, with the class algebra interpretation of $\cup$, the element I is represented as the total area in the illustration--the union of all minterm areas.

<u>Illustration 3.2.</u> Suppose $\sum_2 = \{A,B\}$. Then

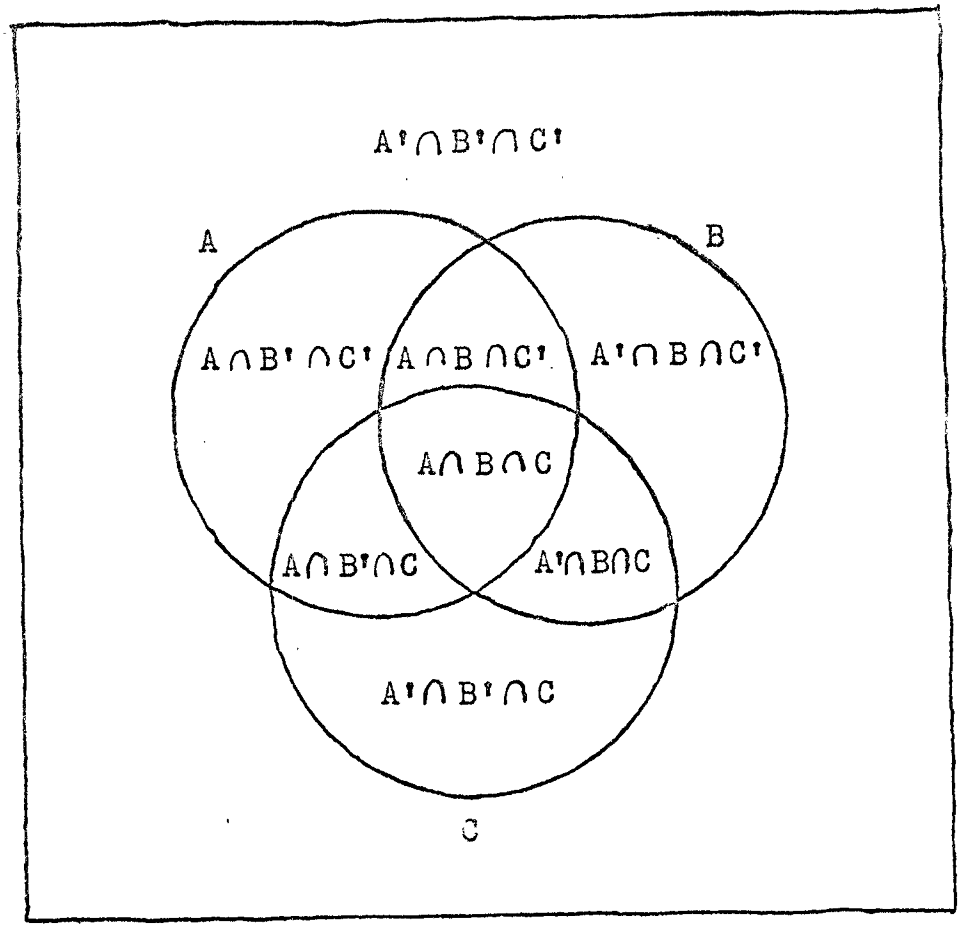
$$I = (A \cap B \cap C) \cup (A' \cap B \cap C) \cup (A \cap B' \cap C) \cup (A \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A' \cap B \cap C') \cup (A \cap B' \cap C') \cup (A' \cap B' \cap C').$$

Also it is shown that

$$A = (A \cap B' \cap C) \cup (A \cap B \cap C') \cup (A \cap B \cap C) \cup (A \cap B' \cap C').$$

$$A' = (A' \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B \cap C) \cup (A' \cap B' \cap C).$$

It is noted that A and A' are both the union of $2^{n-1} = 2^{3-1} = 2^2 = 4$ minterms; $A \in \sum_3^{\cup}$, $A' \in \sum_3^{\cup}$.



Theorem 3.5. There are 2^{2^n} elements in $\Sigma_n^{\cup}$.

Proof: Let $\Sigma_n = \{A_1, A_2, \ldots, A_n\}$, P_n^M be the set of all minimal polynomials in n variables, and let r be any positive integer, $r \leq 2^n$. For the ordered n-tuple $(x_1, x_2, \ldots, x_n)$, P_n^M consists of all F^M such that for each F_j^M,

$$F_j^M(x_1, x_2, \ldots, x_n) = F_1^S(x_1, x_2, \ldots, x_n) \cup F_2^S(x_1, x_2, \ldots, x_n) \cup \ldots \cup F_r^S(x_1, x_2, \ldots, x_n).$$

By Theorem 2.5, there are $2^{2^n}-1$ elements of P_n^M. But by Definition 3.6, each $F_j^M(A_1, A_2, \ldots, A_n)$ is an element of $\Sigma_n^{\cup}$. Now

$\sum_n^{\cup}$ consists of all $F_j^M(A_1, A_2, \ldots, A_n)$ together with the element 0. Thus there are $(2^{2^n}-1)+1=2^{2^n}$ elements in $\sum_n^{\cup}$.

Example 3.7. Suppose $\sum_2 = \{A, B\}$. There are $2^{2^n} = 2^{2^2} = 2^4 = 16$ elements in $\sum_2^{\cup}$, namely

1) 0
2) $A \cap B$
3) $A' \cap B$
4) $A \cap B'$
5) $A' \cap B'$
6) $(A \cap B) \cup (A' \cap B)$
7) $(A \cap B) \cup (A \cap B')$
8) $(A \cap B) \cup (A' \cap B')$
9) $(A' \cap B) \cup (A \cap B')$
10) $(A' \cap B) \cup (A' \cap B')$
11) $(A \cap B') \cup (A' \cap B')$
12) $(A \cap B) \cup (A' \cap B) \cup (A \cap B')$
13) $(A \cap B) \cup (A' \cap B) \cup (A' \cap B')$
14) $(A \cap B) \cup (A \cap B') \cup (A' \cap B')$
15) $(A' \cap B) \cup (A \cap B') \cup (A' \cap B')$
16) $(A \cap B) \cup (A' \cap B) \cup (A \cap B') \cup (A' \cap B') = I.$

Theorems 1.28, 1.29, 1.38, and 1.39 may be extended to apply to the cap or cup of any number of elements.

$A \cup (B_1 \cap B_2 \cap \ldots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_n).$

Likewise,

$A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n).$

Theorems 1.63 and 1.65 may also be extended to apply to the cap or cup of any number of elements.

$$(A_1 \cap A_2 \cap \ldots \cap A_n)' = A_1' \cup A_2' \cup \ldots \cup A_n',$$

and

$$(A_1 \cup A_2 \cup \ldots \cup A_n)' = A_1' \cap A_2' \cap \ldots \cap A_n'.$$

The process of showing this extension is the same for any n. To illustrate, suppose $A, B, C, D \in \sum_n^g$. Then

$(A \cup B \cup C \cup D)' = \{[(A \cup B) \cup C] \cup D\}'$ by Theorem 1.33,

$= [(A \cup B) \cup C]' \cap D'$ by Theorem 1.65,

$= (A \cup B)' \cap C' \cap D'$ by Theorem 1.65,

$= A' \cap B' \cap C' \cap D'$ by Theorem 1.65.

A similar procedure is used in showing

$$(A \cap B \cap C \cap D)' = A' \cup B' \cup C' \cup D'.$$

Considering that the cup of all minterms of $\sum_n^g$ is I, it is an interesting procedure to show that $I' = 0$. In demonstrating this, assume $\sum_2 = \{A, B\}$. Thus

$I = (A \cap B) \cup (A' \cap B) \cup (A \cap B') \cup (A' \cap B')$ by Theorem 3.4.

Then

$I' = [(A \cap B) \cup (A' \cap B) \cup (A \cap B') \cup (A' \cap B')]'$

$= (A \cap B)' \cap (A' \cap B)' \cap (A \cap B')' \cap (A' \cap B')'$ by Theorem 1.65,

$= (A' \cup B') \cap (A \cup B') \cap (A' \cup B) \cap (A \cup B)$ by Theorems 1.59,63,

$= \{(A' \cup B') \cap (A \cup B')\} \cap \{(A' \cup B) \cap (A \cup B)\}$ by Theorem 1.36,

$= \{(A' \cap A) \cup B'\} \cap \{(A' \cap A) \cup B\}$ by Theorem 1.39,

$= (A' \cap A) \cup (B' \cap B)$ by Theorem 1.38,

$= (A \cap A') \cup (B \cap B')$ by Theorem 1.11,

$= 0 \cup 0$ by Theorem 1.43,

$= 0$ by Theorem 1.5.

Lemma 3.1. If M_n^i and M_n^j are two minterms of $\sum_n^g$, then either $M_n^i \cap M_n^j = 0$, or $M_n^i \cap M_n^j = M_n^i = M_n^j$.

Proof: Assume $M_n^i = M_n^j$. Then $M_n^i \cap M_n^j = M_n^i = M_n^j$ follows immediately by Theorem 1.6.

Now assume $M_n^i \neq M_n^j$. Let $\sum_n = \{A_1, A_2, \ldots, A_n\}$. Then each minterm is $F^s(A_1, A_2, \ldots, A_n)$, where

$$F^s(A_1, A_2, \ldots, A_n) = F^o(A_1) \cap F^o(A_2) \cap \ldots \cap F^o(A_n).$$

Now since these two minterms are not the same, there exists at least one A_i such that $F_1^o(A_i) = F^p(A_i)$ in one of the minterms, while $F_2^o(A_i) = F^{p'}(A_i)$ in the other minterm. Thus $M_n^i \cap M_n^j$ is the cap of certain elements, S, of $\sum_{n*}$, two of which are the two elements of A_i*. But $A_i \cap A_i' = 0$ by Theorem 1.43, and $B \cap 0 = 0$ by Theorem 1.54 where B is the element in $\sum_n^g$ generated by the caps of the rest of the elements of S. Therefore, if $M_n^i \neq M_n^j$, then $M_n^i \cap M_n^j = 0$.

Example 3.8. Suppose $\sum_n = \{A, B, C\}$, $M_3^i = A \cap B' \cap C'$, and $M_3^j = A \cap B \cap C'$. Then

$M_3^i \cap M_3^j = (A \cap B' \cap C') \cap (A \cap B \cap C')$

$= (A \cap A) \cap (B \cap B') \cap (C' \cap C')$ by Theorems 1.36 and 1.11,

$= A \cap (B \cap B') \cap C'$ by Theorem 1.6,

$= A \cap 0 \cap C'$ by Theorem 1.43,

$= A \cap C' \cap 0$ by Theorem 1.11,

$= (A \cap C') \cap 0$ by Theorem 1.36,

$= 0$ by Theorem 1.54.

<u>Theorem 3.6.</u> $\sum_n^{\cup}$ is closed under $\cap$.

<u>Proof:</u> Let k and p be two positive integers, q be a non-negative integer such that k, (p + q) are less then or equal 2^n. Also, let $M_n^{\cup i}$, $M_n^{\cup j}$ be two polyterms of $\sum_n^{g}$ such that

$$M_n^{\cup i} = M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k,$$
$$M_n^{\cup j} = M_n^p \cup M_n^{p+1} \cup \ldots \cup M_n^{p+q},$$

where each M_n^m is a minterm of $\sum_n^{g}$. Then

$$M_n^{\cup i} \cap M_n^{\cup j} = (M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k) \cap (M_n^p \cup M_n^{p+1} \cup \ldots \cup M_n^{p+q})$$
$$= \left[(M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k) \cap M_n^p\right] \cup \left[(M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k) \cap M_n^{p+1}\right] \cup \ldots \cup \left[(M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k) \cap M_n^{p+q}\right]$$

by Theorem 1.28 extended,

$$= \left[(M_n^1 \cap M_n^p) \cup (M_n^2 \cap M_n^p) \cup \ldots \cup (M_n^k \cap M_n^p)\right] \cup \left[(M_n^1 \cap M_n^{p+1}) \cup (M_n^2 \cap M_n^{p+1}) \cup \ldots \cup (M_n^k \cap M_n^{p+1})\right] \cup \ldots \cup \left[(M_n^1 \cap M_n^{p+q}) \cup (M_n^2 \cap M_n^{p+q}) \cup \ldots \cup (M_n^k \cap M_n^{p+q})\right]$$

by Theorem 1.29 extended. Now by Lemma 3.1, the cap of any two minterms M_n^x, M_n^y of $\sum_n^{g}$ is either 0 or M_n^x ($M_n^x = M_n^y$). Then the right side of the above equation is the cup of elements $(M_n^x \cap M_n^y)$, each of which is either 0 or a minterm.

Case I: If each $(M_n^x \cap M_n^y)$ is a minterm, then $M_n^{\cup i} \cap M_n^{\cup j}$ is the cup of minterms. Then by Definition 3.6, $(M_n^{\cup i} \cap M_n^{\cup j}) \in \sum_n^{\cup}$.

Case II: If there exists at least one $(M_n^x \cap M_n^y)$ such that $M_n^x \cap M_n^y = 0$, and one $(M_n^x \cap M_n^y)$ such that $M_n^x \cap M_n^y \neq 0$, then $M_n^{\cup i} \cap M_n^{\cup j}$ is the cup of 0 and a polyterm, $M_n^{\cup a}$. But by Theorem 1.52, $M_n^{\cup a} \cup 0 = M_n^{\cup a}$. Thus $(M_n^{\cup i} \cap M_n^{\cup j}) \in \sum_n^{\cup}$.

Case III: If every $(M_n^x \cap M_n^y) = 0$, then $M_n^{\cup i} \cap M_n^{\cup j}$ is the cup of 0's. Then by Theorem 1.5, $M_n^{\cup i} \cap M_n^{\cup j} = 0$. But by Definition 3.6, $0 \in \sum_n^{\cup}$. Thus $(M_n^{\cup i} \cap M_n^{\cup j}) \in \sum_n^{\cup}$.

Since for all possible cases $(M_n^{\cup i} \cap M_n^{\cup j}) \in \sum_n^{\cup}$, $\sum_n^{\cup}$ is closed under $\cap$.

<u>Theorem 3.7.</u> $\sum_n^{\cup}$ is closed under $\cup$.

<u>Proof:</u> By Definition 3.6, each element of $\sum_n^{\cup}$ is either 0 or the cup of one or more minterms of $\sum_n^{g}$. Let $M_n^{\cup i}$ and $M_n^{\cup j}$ be two polyterms of $\sum_n^{g}$.

Case I: If $M_n^{\cup i} = 0$ and $M_n^{\cup j} \neq 0$, then $M_n^{\cup i} \cup M_n^{\cup j} = M_n^{\cup j}$ by Theorem 1.52. But $M_n^{\cup j} \in \sum_n^{\cup}$. Therefore $(M_n^{\cup i} \cup M_n^{\cup j}) \in \sum_n^{\cup}$.

Case II: If $M_n^{\cup i} = 0$ and $M_n^{\cup j} = 0$, then $M_n^{\cup i} \cup M_n^{\cup j} = 0$ by Theorem 1.5. But $0 \in \sum_n^{\cup}$. Therefore $(M_n^{\cup i} \cup M_n^{\cup j}) \in \sum_n^{\cup}$.

Case III: If $M_n^{\cup i} \neq 0$ and $M_n^{\cup j} \neq 0$, then $M_n^{\cup i} \cup M_n^{\cup j}$ is the cup of all minterms which are either in $M_n^{\cup i}$ or $M_n^{\cup j}$. But the cup of minterms of $\sum_n^{g}$ is a polyterm of $\sum_n^{g}$. Thus $(M_n^{\cup i} \cup M_n^{\cup j}) \in \sum_n^{\cup}$.

Now since $(M_n^{\cup i} \cup M_n^{\cup j}) \in \sum_n^{\cup}$ for all possible cases, then $\sum_n^{\cup}$ is closed under $\cup$.

<u>Lemma 3.2.</u> If $M_n^i \in \sum_n^{M}$, then $(M_n^i)' \in \sum_n^{\cup}$.

<u>Proof:</u> Let $\sum_n = \{A_1, A_2, \ldots, A_n\}$. Then there exists a minimal polynomial, F^{si}, such that $F^{si}(A_1, A_2, \ldots, A_n) = M_n^i$. Now

$$F^{si}(A_1, A_2, \ldots, A_n) = F^o(A_1) \cap F^o(A_2) \cap \ldots \cap F^o(A_n)$$

where each $F^o(A_j)$ is either $F^p(A_j)$ or $F^{p'}(A_j)$. Therefore

$$(M_n^i)' = [F^o(A_1) \cap F^o(A_2) \cap \ldots \cap F^o(A_n)]'$$
$$= [F^o(A_1)]' \cup [F^o(A_2)]' \cup \ldots \cup [F^o(A_n)]'$$

by Theorem 1.63 extended. Now since $F^o(A_j)$ is either A_j or A_j', then $[F^o(A_j)]'$ is either A_j' or A_j. Thus $(M_n^i)'$ is the cup of elements of $\sum_{n*}$. But by Theorems 3.2 and 3.3, A_j' and A_j are polyterms. Then $(M_n^i)'$ is the cup of polyterms of $\sum_n^g$. But by Theorem 3.7, the cup of polyterms of $\sum_n^g$ is an element of $\sum_n^{\cup}$. Therefore, $(M_n^i)' \in \sum_n^{\cup}$.

<u>Theorem 3.8.</u> $\sum_n^{\cup}$ is closed under $'$.

<u>Proof:</u> Let $M_n^{\cup i}$ be an element of $\sum_n^{\cup}$. Then by Definition 3.6, $M_n^{\cup i}$ is either 0 or the cup of one or more minterms of $\sum_n^g$.

Case I: Suppose $M_n^{\cup i} = 0$. Then

$$(M_n^{\cup i})' = 0'$$
$$= I$$

by Theorem 1.70. But $I \in \sum_n^{\cup}$ by Theorem 3.4. Therefore $(M_n^{\cup i})' \in \sum_n^{\cup}$.

Case II: Suppose $M_n^{\cup i} \neq 0$. Let k be a positive integer, $k \leq 2^n$. Then

$$M_n^{\cup i} = M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k$$

where each M_n^q is a minterm of $\sum_n^g$. Then

$$(M_n^{\cup i})' = (M_n^1 \cup M_n^2 \cup \ldots \cup M_n^k)'$$
$$= (M_n^1)' \cap (M_n^2)' \cap \ldots \cap (M_n^k)'$$

by Theorem 1.65 extended. But each $(M_n^q)'$ is a polyterm by Lemma 3.2. Thus $(M_n^{\cup i})'$ is the cap of polyterms. Now by

Theorem 3.6, the cap of polyterms of Σ_n^g is an element of $\Sigma_n^\cup$. Therefore $(M_n^{\cup i})' \in \Sigma_n^\cup$.

Now since $(M_n^{\cup i})' \in \Sigma_n^\cup$ for all possible cases, then $\Sigma_n^\cup$ is closed under $'$.

Theorem 3.9. Fundamental Theorem of the Algebra of Σ_n^g. $\Sigma_n^\cup = \Sigma_n^g$.

Proof: Basic in the concept of the Boolean algebra developed is the establishment that Σ_n^g is the set generated by the three primary functions on the elements of Σ_n. Now it has been shown, by Theorem 3.2, that each element of Σ_n is an element of $\Sigma_n^\cup$. So to inspect Σ_n^g further, it is necessary to examine the elements generated under the three primary functions on the elements of $\Sigma_n^\cup$.

1) Theorem 3.6 states $\Sigma_n^\cup$ is closed under $\cap$.

2) Theorem 3.7 states $\Sigma_n^\cup$ is closed under $\cup$.

3) Theorem 3.8 states $\Sigma_n^\cup$ is closed under $'$.

0 and I are in $\Sigma_n^\cup$; $0 \in \Sigma_n^\cup$ by Definition 3.6, and $I \in \Sigma_n^\cup$ by Theorem 3.4. Therefore, $\Sigma_n^\cup = \Sigma_n^g$.

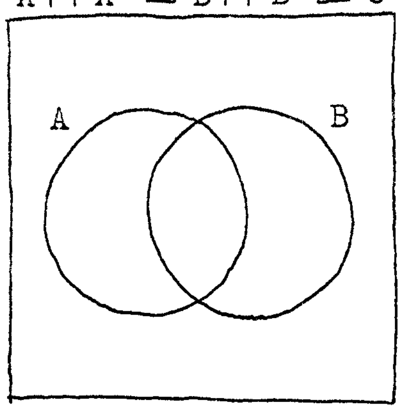
Corollary 3.1. There are 2^{2^n} elements in Σ_n^g.

Proof: By Theorem 3.5, there are 2^{2^n} elements in $\Sigma_n^\cup$. Then since $\Sigma_n^\cup = \Sigma_n^g$, there are 2^{2^n} elements in Σ_n^g.

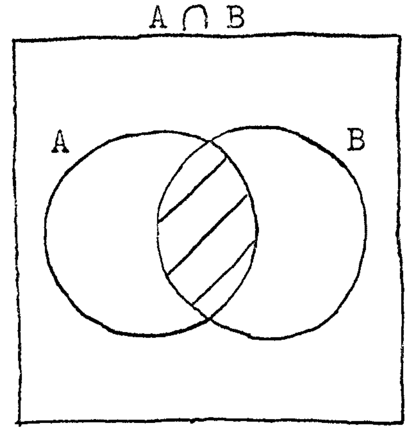
Illustration 3.3. In showing how the set $\Sigma_n^\cup$ may be applied to Venn diagrams, let $\Sigma_2 = \{A,B\}$. Example 3.4 lists the elements of $\Sigma_n^\cup$. As in illustration 3.2, each small area is a minterm of Σ_2^g. Each of the $2^{2^n} = 2^{2^2} = 16$ different elements of $\sum_{n}^{\cup}$ is illustrated as the striped portion in a separate square.

It is easily seen that the very same set of squares represents $\sum_{2}^{9}$. Notice that element number six is B, and that element number seven is A. Each of the elements generated under the primary functions on A and B is represented by one of the sixteen squares.

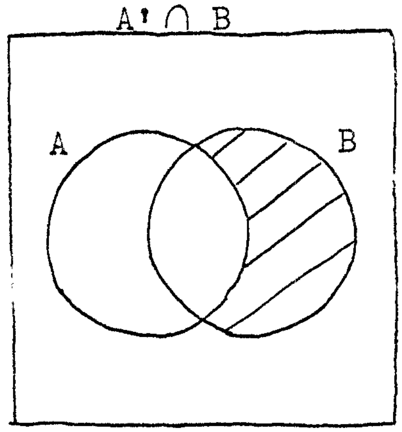
1) $A \cap A' = B \cap B' = 0$



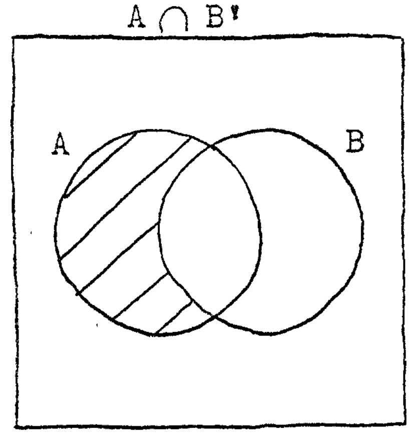
2) $A \cap B$



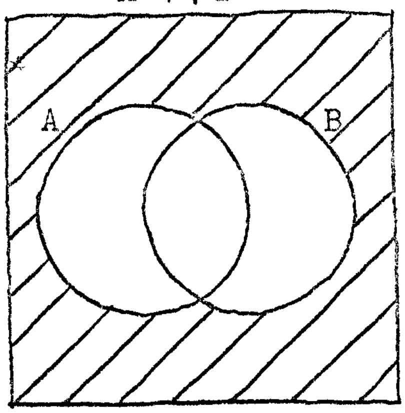
3) $A' \cap B$



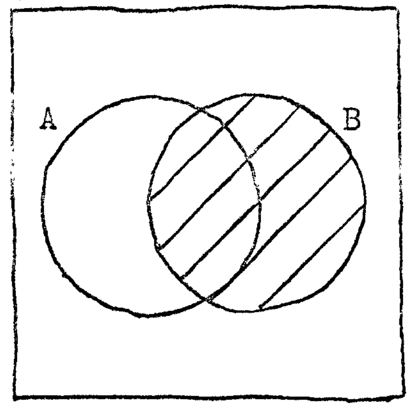
4) $A \cap B'$



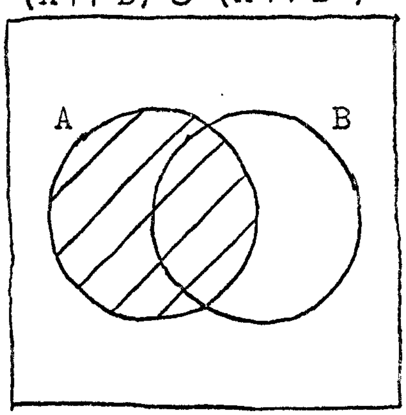
5) $A' \cap B'$



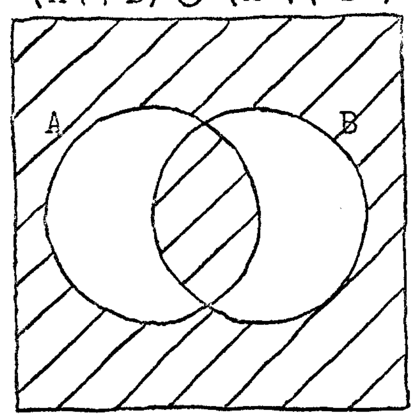
6) $(A \cap B) \cup (A' \cap B)$



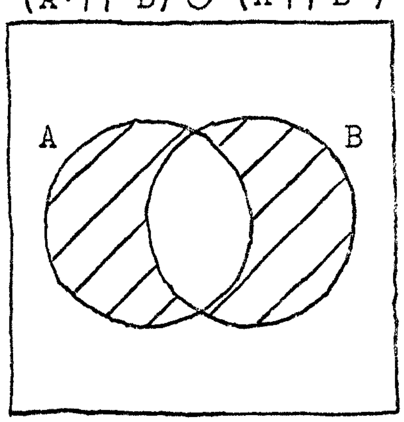
7) $(A \cap B) \cup (A \cap B')$



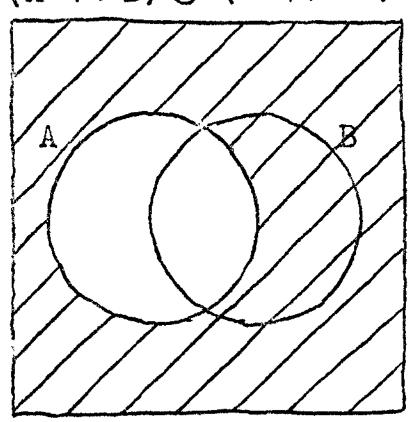
8) $(A \cap B) \cup (A' \cap B')$



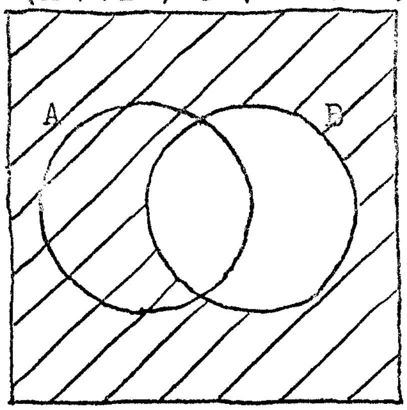
9) $(A' \cap B) \cup (A \cap B')$



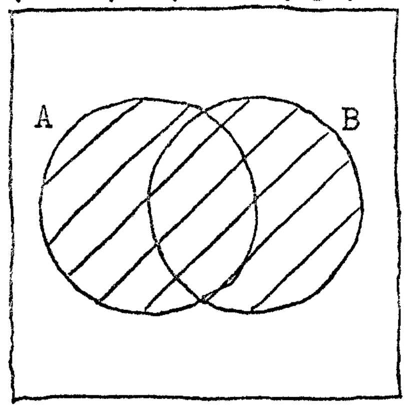
10) $(A' \cap B) \cup (A' \cap B')$



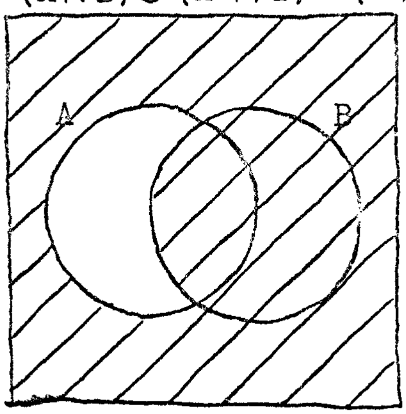
11) $(A \cap B') \cup (A' \cap B')$



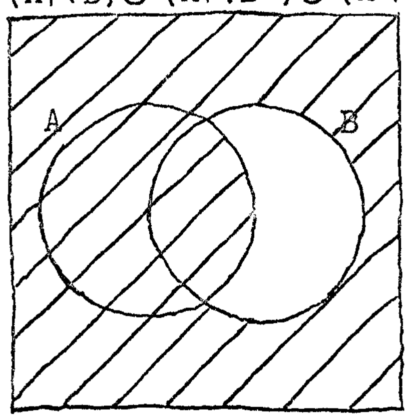
12) $(A \cap B) \cup (A' \cap B) \cup (A \cap B')$



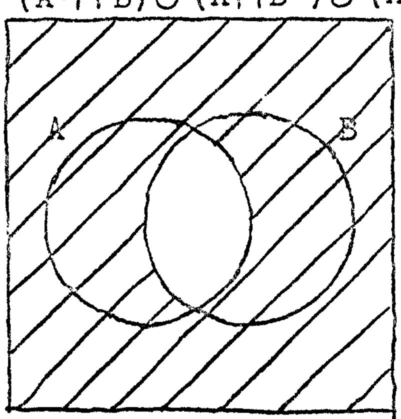
13) $(A \cap B) \cup (A' \cap B) \cup (A' \cap B')$



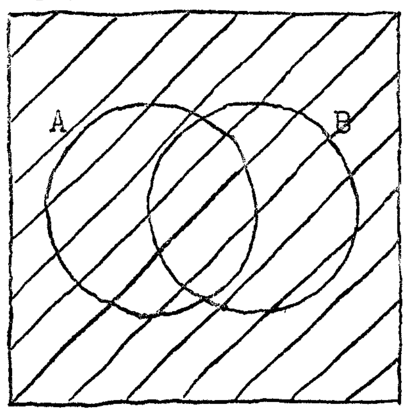
14) $(A \cap B) \cup (A \cap B') \cup (A' \cap B')$



15) $(A' \cap B) \cup (A \cap B') \cup (A' \cap B')$



16) $(A \cap B) \cup (A' \cap B)$
$\cup (A \cap B') \cup (A' \cap B') = I.$



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