A CORRECTION FACTOR FOR THE

FIRST BORN APPROXIMATION

APPROVED:

Robert E. Buduine

Major Professor

J R Sybert

Minor Professor

L. F. Connell Jr.

Director of the Department of Physics

Robert B. Toulouse

Dean of the Graduate School

A CORRECTION FACTOR FOR THE FIRST BORN APPROXIMATION

THESIS

Presented to the Graduate Council of the
North Texas State University in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

Jerry Brent Russell, B. S.

Denton, Texas

January, 1965

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

CHAPTER I

THE DEVELOPMENT OF THE GENERAL CASE

The problem of scattering of particles by a potential, V(r), has been treated, semi-successfully, by a method known as the Born approximation.

The scattering problem leading to the Born approximation is illustrated in Figure 1. Located at the origin is an arbitrary, fixed scattering center. A narrow beam of particles is incident upon the scattering potential. The beam of particles of reduced mass, μ , lies along the Z-axis.

The problem to be considered is the solution of the following Schroedinger equation

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi(r) = E\psi(r)\,. \tag{1}$$

This equation can be written as

$$\left[\nabla^2 + k^2\right]\psi(r) = U(r)\psi(r) \tag{2}$$

where

$$U(r) = \frac{2\mu}{\hbar^2}V(r) \tag{3}$$

and

$$k = \frac{\mu v}{\hbar^2} \tag{4}$$

The solution to equation 2 will have the form

$$\Psi(r) = e^{ikz} + \frac{1}{r} e^{ikr} f(\Theta) .$$

The quantity, $f(\Theta)$, is called the scattering amplitude of the process. The scattering amplitude, $f(\Theta)$, has the property

$$\frac{d\sigma}{d\omega} = |f(\Theta)| \tag{5}$$

where $d\sigma/d\omega$ is the differential cross section.

It is a well-known result that the scattering amplitude is given by (3, p. 147)

$$f(\Theta) = -\frac{1}{4\pi}\int e^{-i\bar{k}\cdot\bar{r}}\, U(r)\Psi(r)\,dr . \tag{6}$$

It is in this relation that the Born approximation is made. The $\Psi(r)$ appearing in the integrand of equation 6 is not explicitly known; the Born approximation is an assumption concerning this quantity. The assumption is that, for incident particles having sufficient energy, the $\Psi(r)$ in equation 6 is not appreciably distorted from a plane wave. If the incident particles are described by a plane wave, then, providing the particles have a large energy, the state of the particles will still be described by a plane wave after the scattering has begun. Thus, the Born approximation is

$$\Psi(r) = e^{i\bar{k}_0\cdot\bar{r}} . \tag{7}$$

With this approximation, equation 6 becomes

$$f(\Theta) = -\frac{1}{4\pi}\int e^{i(\bar{k}_0-\bar{k})\cdot\bar{r}}\, U(r)\,dr . \tag{8}$$

Letting

$$\bar{K} = \bar{k}_o - \bar{k} \tag{9}$$

then, if α is the angle between $\bar{K}$ and $\bar{r}$,

$$\bar{K}\cdot\bar{r} = Kr\cos\alpha \tag{10}$$

equation 8 becomes

$$f(\theta) = -\frac{1}{4\pi}\int e^{i\bar{K}\cdot\bar{r}\cos\alpha}\,U(r)\,dr\,. \tag{11}$$

Now, for a differential volume $r^2 \sin\alpha\, d\alpha\, d\beta\, dr$

$$f(\theta) = -\frac{1}{4\pi}\iiint e^{iKr\cos\alpha}\,U(r)r^2\sin\alpha\, d\alpha\, d\beta\, dr \tag{12}$$

or

$$f(\theta) = -\frac{1}{4\pi}\int_0^\infty U(r)r^2 dr\int_0^{2\pi} d\beta\int_0^{\pi} e^{iKr\cos\alpha}\sin\alpha\, d\alpha \tag{13}$$

which integrates to

$$f(\theta) = -\frac{1}{K}\int_0^\infty U(r)\,r\,\sin(Kr)\,dr\,. \tag{14}$$

Using equation 3, it is seen that

$$f(\theta) = -\frac{2\mu}{\hbar^2 K}\int_0^\infty V(r)\,r\,\sin(Kr)\,dr\,. \tag{15}$$

Equation 15 is the result given by the Born approximation for the scattering of the process described above.

A very important point to be noticed is that the Born approximation is valid only if the energy of the incident particles is greater than or equal to some lower energy bound. Qualitatively, the idea is that for low incident energies the particles are affected by the potential, V(r), to the extent that the approximation that $\Psi(r)$ is a plane wave at scattering is not valid. Thus, in order to make the approximation valid it is necessary that the incident energy be greater than some lower bounding energy corresponding to a momentum transfer K_B. An important observation about K_B is that it is not well defined. It varies from potential to potential and, for a given potential, it is more an order of magnitude than an explicit number.

For example, the Born approximation is valid, when the scattering center is hydrogen or any other light atom, for bombarding energies greater than about 100 ev. Again this is an order of magnitude and is used to find the momentum transfer. For heavier atoms, the bombarding energies must exceed approximately 1000 ev. For the scattering of protons by nuclei, the Born approximation is valid only for bombarding energies greater than 150 Mev. (3, p. 143).

Since the scattering amplitude given in equation 15 results in considerable error when the energy of the incident particles is below the range of validity, a method of correcting this discrepancy is desirable.

Since the scattering amplitude in the Born approximation, $f(\theta)$, can be evaluated for most important potentials and gives fairly accurate results, a reasonable approach to the correction problem is to assume a form involving a correction factor should be some function of the energy, since the range of validity of the Born approximation is energy dependent. Therefore, the correction factor can be taken to be

$$\Delta = \Delta(K). \tag{16}$$

In order to formulate a method for the evaluation of $\Delta(K)$, an equation must be arrived at that allows for the solution of $\Delta(K)$ as an explicit function of K.

The residue theorem on complex integrals provides a method of arriving at the necessary equation. The residue theorem is stated as: "Let C be a closed curve within and on which F(Z) is analytic except for a finite number of singularities Z_1, Z_2, ... , Z_n, inside of C. If K_1, K_2, ... , K_n, denote the residues of F(Z) at those points, then

$$\int_C F(Z)\,dZ = 2\pi i\,(K_1 + K_2 + \cdots + K_n)$$

where the integral is taken counterclockwise around C" (1, p. 188).

The next step is to determine exactly what the F(Z) should correspond to in the scattering problem. A reasonable form to assume for the corrected scattering amplitude is

$$f'(\theta) = f(\theta) + i\,\Delta\, f(\theta) \tag{17}$$

or

$$f'(\theta) = (1 + i\Delta) f(\theta). \tag{18}$$

This form for the corrected scattering amplitude, $f'(\theta)$, can be identified with the F(Z) in the statement of the residue theorem.

At this point, reconsider the meaning of $\bar{K}$, i.e.

$$\bar{K}^2 = (\bar{k}_0 - \bar{k})^2 = \bar{k}_0^2 + \bar{k}^2 - 2\bar{k}_0\bar{k} = 2k^2(1 - \cos\theta)$$

or

$$\bar{K}^2 = 4k^2 \sin^2 \theta/2$$

so that

$$K = 2k \sin \theta/2 . \tag{19}$$

From equation 19 it is seen that

$$f(\theta) \longrightarrow f(K)$$

Thus, in application to the scattering problem, $F(Z)$ in the residue theorem should contain as factors $[1 + i\Delta(K)]$, $f(K)$, and, in addition, a term to insure that a singularity will exist since it is not in any way guaranteed that $f(K)$ will contain the necessary pole. Therefore, it is necessary to include a factor of the form, $1/(K-K')$. Now the form of $F(Z)$ has been established as

$$F(z) = \frac{[1 + i\Delta(K)]\, f(K)}{K - K'} . \tag{20}$$

Using equation 20 as the form of $F(Z)$ in the residue theorem,

$$\int_C \frac{[1 + i\Delta(K)]\, f(K)}{K - K'}\, dK = 2\pi i \sum (\text{res.}) . \tag{21}$$

Equation 21 is the basis of the method for the evaluation of $\Delta(K)$.

It is now necessary to make one important assumption about $\Delta(K)$. In order to be able to write down the residues in equation 21, it is necessary to assume that $\Delta(K)$, itself, has no singularities. This condition is essential since there is no possible way of anticipating the actual location of any singularities $\Delta(K)$ might have in the complex plane.

Another important point is that a method of evaluating the integral in equation 21 must be devised and can be achieved by making the contour C enclose all the singularities in the upper half complex plane and those on the real axis. Figure 2 shows this situation. In Figure 2 the dots represent the singularities of $F(Z)$. The contour C encloses only those singularities which lie on the real axis or in the upper half plane. Notice that the singularity at K has been shown explicitly. Since K is the only singularity explicitly known, any other singularities which might arise must come from the scattering amplitude calculated in the Born approximation.

Now, with the assumptions made about $\Delta(K)$ and with an explicit form for the Born approximation scattering amplitude, it is possible to write down immediately the right-hand side of equation 21. However, the problem of solving an integral equation remains; that is, the integral on the left-hand side of equation 21 must be evaluated by some other means.

Consider the deformation of the contour C shown in Figure 3. That this deformation is valid follows from the fact that no singularities were crossed in the deformation (1, p. 118).

With the assumptions made about $\Delta(K)$, namely that it contains no singularities, by placing a restriction on $f(K)$ it is possible to arrive at a method of evaluating the integral using the contour shown in Figure 3. The restriction on $f(K)$ requires that the integrand in equation 21 tends to zero as $K \rightarrow \infty$. This is not a very strong restriction since most scattering amplitudes calculated in the Born approximation satisfy this condition.

The integral in equation 21 can be written as

$$\int_C \frac{[1 + i\Delta(K)]f(K)}{(K-K')} dK = \int_R \frac{[1 + i\Delta(K)]f(K)}{(K-K')} dK$$

$$+ \int_{-R}^{K'-\rho} \frac{[1 + i\Delta(K)]f(K)}{(K-K')} dK \quad + \int_\rho \frac{[1 + i\Delta(K)]f(K)}{(K-K')} dK$$

$$+ \int_{K'+\rho}^{R} \frac{[1 + i\Delta(K)]f(K)}{(K-K')} dK.$$

Since the original contour C was assumed to contain all the singularities in the upper half plane, the following limits can be taken without crossing any singularities. Let $R \rightarrow \infty$ and $\rho \rightarrow 0$; then the contour integrals around R and ρ both tend to zero. The reason for these limits is that, in the case where $R \rightarrow \infty$, the line integration is performed at infinite values of K. Due to the previous assumptions about $f(K)$, the integrand is zero for these K-values. Thus,

$$\lim_{R\rightarrow\infty} \int_R \frac{[1 + i\Delta(K)]f(K)}{(K-K')} dK = 0.$$

As $\rho \rightarrow 0$, the path length goes to zero so that (2, pp. 529-530)

$$\lim_{\rho \to 0} \int_{\rho} \frac{[1 + i\Delta(K)] f(K)}{(K-K')} dK = 0.$$

It follows that

$$\int_C \frac{[1 + i\Delta(K)] f(K)}{(K-K')} dK = \lim_{\substack{\rho \to 0 \\ R \to \infty}} \int_{-R}^{K'-\rho} \frac{[1 + i\Delta(K)] f(K)}{(K-K')} + \lim_{\substack{\rho \to 0 \\ R \to \infty}} \int_{K'+\rho}^{R} \frac{[1 + i\Delta(K)] f(K)}{(K-K')} dK. \tag{22}$$

With the contour integral in this form, it is seen that

$$\int_C \frac{[1 + i\Delta(K)] f(K)}{(K-K')} dK = P\int_{-\infty}^{\infty} \frac{[1 + i\Delta(K)] f(K)}{(K-K')} dK \tag{23}$$

where the "P" indicates the Cauchy principal value of the integral.

Equation 23 is the second method of evaluating the integral in equation 21 making it possible to solve equation 21 for $\Delta(K)$. Using equations 23 and 21 gives

$$P\int_{-\infty}^{\infty} \frac{[1 + i\Delta(K)] f(K)}{(K-K')} dK = 2\pi i \sum(\text{res}). \tag{24}$$

Equation 24 then yields the solution which gives the form of the correction factor $\Delta(K)$. Thus, a corrected form for the Born approximation scattering amplitude has been chosen and a method for evaluating the correction factor has been developed.

CHAPTER BIBLIOGRAPHY

1. Churchill, Ruel V., Complex Variables, New York, McGraw-Hill, 1948.

2. Hildebrand, F. B., Advanced Calculus for Engineers, Englewood Cliffs, New Jersey, 1960.

3. Mandl, F., Quantum Mechanics, New York, Academic Press, Inc., 1957.

CHAPTER II

APPLICATION TO SCREENED COULOMB POTENTIAL

The method of correcting the Born approximation scattering amplitude, developed in Chapter I, should apply to any scattering amplitude calculated in the Born approximation. As an example of this method of correction, consider the screened Coulomb potential,

$$V(r) = \frac{-Ze^2 e^{-ar}}{r} \tag{25}$$

Using this potential in equation 17 gives

$$f(K) = -\frac{2\mu}{\hbar^2 K} \int_0^\infty \left(\frac{-Ze^2 e^{-ar}}{r} \right) r \sin Kr \, dr$$

or

$$f(K) = \frac{2Ze^2\mu}{\hbar^2 K} \left(\frac{K}{a^2 + K^2} \right) = \frac{2Ze^2\mu}{\hbar^2 (a^2 + K^2)} \; . \tag{26}$$

Equation 26 gives the Born-approximation scattering amplitude for the screened Coulomb potential.

Now,

$$f(K) \propto \frac{1}{(a^2 + K^2)}$$

so that, arising from the Born-approximation scattering amplitude, there are singularities at $K = \pm ia$ in the complex plane.

Then for the special case of the screened Coulomb potential the situation in Figure 3 becomes the situation shown in Figure 4.

For this case, equation 24 is

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}\,dK = 2\pi i\sum(\text{res.})\ .$$

Writing out the residues on the right gives

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}\,dK = \pi i\left[\frac{1+i\Delta(K')}{a^2+K'^2}\right]$$

$$+\,2\pi i\left[\frac{1+i\Delta(ia)}{(ia-K')(2ia)}\right]$$

Rationalizing the last term on the right gives

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}\,dK = \pi i\left[\frac{1+i\Delta(K')}{a^2+K'^2}\right]$$

$$-\,\pi\left[\frac{1+i\Delta(ia)}{(K'-ia)(a)}\right]\left[\frac{K'+ia}{K'+ia}\right]$$

or

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}\,dK = \pi i\left[\frac{1+i\Delta(K')}{a^2+K'^2}\right]-\pi\left[\frac{(1+i\Delta(ia))(K'+ia)}{a\,(K'^2+a^2)}\right].$$

Expanding the residues on the right side gives

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}\,dK = \frac{\pi i}{(a^2+K'^2)}-\frac{\pi\Delta(K')}{(a^2+K'^2)}-\frac{\pi K'}{a(a^2+K'^2)}$$

$$+\frac{\pi\Delta(ia)\,a}{a\,(a^2+K'^2)}-\frac{\pi i K'\Delta(ia)}{a(a^2+K'^2)}+\frac{\pi i a}{a(a^2+K'^2)}\ ,$$

or

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(K^2+a^2)(K-K')}dK = \frac{\pi i}{(a^2+K'^2)} - \frac{\pi\Delta(K')}{(a^2+K'^2)} - \frac{\pi K'}{a(a^2+K'^2)}$$

$$+\frac{\pi\Delta(ia)}{(a^2+K'^2)} - \frac{\pi i K'\Delta(ia)}{a(a^2+K'^2)} + \frac{\pi i}{(a^2+K'^2)}. \tag{27}$$

In general the most unrestricted assumption to make about the quantity, $\Delta(ia)$, is to assume that it is some complex number. Take

$$\Delta(ia) = V + iW$$

where V and W are two real numbers to be evaluated. Since $\Delta(ia)$ is a constant, it follows that V and W will both be constants. Then equation 27 can be rewritten as

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}dK = \frac{\pi i}{(a^2+K'^2)} - \frac{\pi\Delta(K')}{(a^2+K'^2)} - \frac{\pi K'}{a(a^2+K'^2)}$$

$$+\frac{\pi V}{(a^2+K'^2)} + \frac{\pi i W}{(a^2+K'^2)} - \frac{\pi i K' V}{a(a^2+K'^2)} + \frac{\pi K' W}{a(a^2+K'^2)} + \frac{\pi i}{(a^2+K'^2)}. \tag{28}$$

The problem now is to evaluate the Cauchy principal value on the left side of equation 28. The integral can be written

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')}dK = P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')}$$

$$+ P\int_{-\infty}^{\infty}\frac{i\Delta(K)\,dK}{(a^2+K^2)(K-K')}. \tag{29}$$

The first integral on the right is evaluated by the method of partial fractions,

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')} = P\int_{-\infty}^{\infty}\frac{1}{(a^2+K^2)(K-K')}\,dK\,;$$

then,

$$\frac{1}{(a^2+K^2)(K-K')} \equiv \frac{A}{K-K'} + \frac{BK+C}{a^2+K^2} \tag{30}$$

$$1 \equiv A(a^2+K^2) + (BK+C)(K-K')$$

$$1 \equiv A\,(a^2+K^2) + BK(K-K') + C\,(K-K')\,. \tag{31}$$

Now, for $K=K'$, equation 31 becomes

$$1 = A(a^2+K'^2)$$

so that

$$A = \frac{1}{a^2+K'^2}\,. \tag{32}$$

Also, for $K=0$, equation 31 becomes

$$1 = \frac{a^2}{a^2+K'^2} - CK'$$

or

$$C = \frac{-K'}{a^2+K'^2}\,. \tag{33}$$

Then, for $K=1$, equation 31 is

$$1 = \frac{a^2+1}{a^2+K'^2} + B(1-K') - \frac{K'(1-K')}{a^2+K'^2}$$

$$-B(1-K') = \frac{a^2+1-a^2-K'^2-K'+K'^2}{a^2+K'^2} = \frac{(1-K')}{a^2+K'^2}$$

$$B = \frac{-1}{a^2+K'^2} \quad . \tag{34}$$

Now, using equations 32, 33 and 34 in equation 30 gives the integral

$$P\int_{-\infty}^{\infty} \frac{dK}{(a^2+K^2)(K-K')} = P\int_{-\infty}^{\infty} \left\{ \frac{1}{K'^2+a^2}\left(\frac{dK}{K-K'}\right) - \frac{1}{a^2+K'^2}\left(\frac{K\,dK}{a^2+K^2}\right) - \frac{K'}{a^2+K'^2}\left(\frac{dK}{a^2+K^2}\right)\right\} \tag{35}$$

or

$$P\int_{-\infty}^{\infty} \frac{dK}{(a^2+K^2)(K-K')} = -\frac{1}{K'^2+a^2}\, P\int_{-\infty}^{\infty} \left\{\frac{dK}{K-K'} - \frac{K\,dK}{a^2+K^2} - \frac{K'dK}{a^2+K^2}\right\}. \tag{36}$$

Equation 36 gives a form of the integral in equation 29 which can be evaluated by using the definition of the Couchy principal value.

Using this definition,

$$P\int_{-\infty}^{\infty} \frac{dK}{(a^2+K^2)(K-K')} = \frac{1}{a^2+K'^2}\left\{\lim_{\epsilon\to 0}\int_{-\infty}^{K'-\epsilon}\left[\left(\frac{dK}{K-K'}\right) - \left(\frac{K\,dK}{a^2+K^2}\right) - K'\left(\frac{dK}{a^2+K^2}\right)\right]\right.$$

$$\left. + \lim_{\epsilon\to 0}\int_{K'+\epsilon}^{\infty}\left[\left(\frac{dK}{K-K'}\right) - \left(\frac{K\,dK}{a^2+K^2}\right) - K'\left(\frac{dK}{a^2+K^2}\right)\right]\right\}. \tag{37}$$

Integrating the factors in equation 37 gives

$$P\int_{-\infty}^{\infty}\frac{dK}{(K^2+a^2)(K-K')} = \frac{1}{a^2+K'^2}\left\{\lim_{\epsilon\to 0}\left[\ln(K-K') - \tfrac{1}{2}\ln(a^2+K^2) - \frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]_{-\infty}^{K'-\epsilon}\right. \tag{38}$$

$$\left. + \lim_{\epsilon\to 0}\left[\ln(K-K') - \tfrac{1}{2}\ln(K^2+a^2) - \frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]_{K'+\epsilon}^{\infty}\right\}.$$

Equation 38 can be rewritten as

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')} = \frac{1}{a^2+K'^2}\left\{\lim_{\epsilon\to 0}\left[\ln\frac{(K-K')}{(a^2+K^2)^{1/2}} - \frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]_{-\infty}^{K'-\epsilon}\right.$$

$$\left. + \lim_{\epsilon\to 0}\left[\ln\frac{(K-K')}{(a^2+K^2)^{1/2}} - \frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]_{K'+\epsilon}^{\infty}\right\}. \tag{39}$$

At this point it is necessary to evaluate the limits given equation 39 by first inserting the integration limits and then taking the limit as ϵ approaches zero. Equation 39 is then written

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')} = \frac{1}{K'^2+a^2}\left\{\lim_{\epsilon\to 0}\left[\ln\frac{(K'-\epsilon)-K'}{[(K'-\epsilon)^2+a^2]^{1/2}} - \frac{K'}{a}\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right.\right.$$

$$- \left[\ln\frac{(1-K'/K)}{[1-(a/K)^2]^{1/2}} - \frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{K\to-\infty}$$

$$+ \left[\ln\frac{(1-K'/K)}{[1-(a/K)^2]^{1/2}} - \frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{K\to\infty} \tag{40}$$

$$\left. - \lim_{\epsilon\to 0}\left[\ln\frac{(K'+\epsilon)-K'}{[(K'+\epsilon)^2+a^2]^{1/2}} - \frac{K'}{a}\tan^{-1}\left(\frac{K'+\epsilon}{a}\right)\right]\right\}.$$

Now,

$$-\left[\ln\frac{(1-K'/K)}{[1-(a/K)^2]^{1/2}}-\frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{K\to-\infty}=\frac{K'}{a}\tan^{-1}(-\infty)$$

and

$$\left[\ln\frac{(1-K'/K)}{[1-(a/K)^2]^{1/2}}-\frac{K'}{a}\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{K\to\infty}=\frac{K'}{a}\tan^{-1}(-\infty)\ ,$$

so that

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')}=\frac{1}{a^2+K'^2}\Bigg\{\lim_{\epsilon\to 0}\left[\ln\frac{-\epsilon}{[(K'-\epsilon)^2+a^2]^{1/2}}-\frac{K'}{a}\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right]$$
$$+\frac{K'}{a}\tan^{-1}(-\infty)-\frac{K'}{a}\tan^{-1}(\infty)-\lim_{\epsilon\to 0}\left[\ln\frac{\epsilon}{[(K'+\epsilon)^2+a^2]^{1/2}}-\frac{K'}{a}\tan\left(\frac{K'+\epsilon}{a}\right)\right]\Bigg\}\ . \tag{41}$$

The two arctangent terms can be taken as

$$\left(\frac{K'}{a}\right)\tan^{-1}(-\infty)=-\left(\frac{K'}{a}\right)\frac{\pi}{2}$$

and

$$-\left(\frac{K'}{a}\right)\tan^{-1}(\infty)=-\left(\frac{K'}{a}\right)\frac{\pi}{2}\ ,$$

so that equation 41 becomes

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')}=\frac{1}{K'^2+a^2}\Bigg\{\lim_{\epsilon\to 0}\left[\ln\frac{-\epsilon}{[(K'-\epsilon)^2+a^2]^{1/2}}-\frac{K}{a}\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right]$$
$$-\lim_{\epsilon\to 0}\left[\ln\frac{-\epsilon}{[(K'+\epsilon)^2+a^2]^{1/2}}-\frac{K'}{a}\tan^{-1}\left(\frac{K'+\epsilon}{a}\right)\right]-\frac{K'\pi}{a}\Bigg\}\ . \tag{42}$$

Now, since

$$\lim_{\epsilon\to 0}\left[-\frac{K'}{a}\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right]=-\frac{K}{a}\tan^{-1}\left(\frac{K'}{a}\right)$$

and

$$-\lim_{\epsilon\to 0}\left[-\frac{K'}{a}\tan^{-1}\left(\frac{K'+\epsilon}{a}\right)\right]=\frac{K'}{a}\tan^{-1}\left(\frac{K'}{a}\right) ,$$

it is seen that these terms cancel. Then equation 42 becomes

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')}=\frac{1}{a^2+K'^2}\Bigg\{-\frac{K'\pi}{a}+\lim_{\epsilon\to 0}\left[\ln\frac{-\epsilon}{[(K'-\epsilon)^2+a^2]^{1/2}}\right] \qquad (43)$$

$$-\lim_{\epsilon\to 0}\left[\ln\frac{\epsilon}{[(K'+\epsilon)^2+a^2]^{1/2}}\right]\Bigg\} .$$

The two remaining limits can be evaluated by noticing that

$$\lim_{\epsilon\to 0}\left\{\ln\frac{-\epsilon}{[(K'-\epsilon)^2+a^2]^{1/2}}-\ln\frac{\epsilon}{[(K'+\epsilon)^2+a^2]^{1/2}}\right\}$$

$$=\lim_{\epsilon\to 0}\left\{\ln\frac{\dfrac{-\epsilon}{\pm[(K'-\epsilon)^2+a^2]^{1/2}}}{\dfrac{\epsilon}{\pm[(K'+\epsilon)^2+a^2]^{1/2}}}\right\} . \qquad (44)$$

Since $\Delta(K)$, itself, is a real quantity, the following choice of signs on the radicals is made:

$$\lim_{\epsilon\to 0}\left\{\ln\frac{\dfrac{-\epsilon}{-[(K'-\epsilon)^2+a^2]^{1/2}}}{\dfrac{\epsilon}{+[(K'+\epsilon)^2+a^2]^{1/2}}}\right\}=\lim_{\epsilon\to 0}\left\{\ln\frac{[(K'+\epsilon)^2+a^2]^{1/2}}{[(K'-\epsilon)^2+a^2]^{1/2}}\right\}$$

or

$$\lim_{\epsilon\to 0}\left\{\ln\frac{[(K'+\epsilon)^2+a^2]^{1/2}}{[(K'-\epsilon)^2+a^2]^{1/2}}\right\}=\ln(1)=0 . \qquad (45)$$

Finally, the integral is

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')}=\frac{1}{a^2+K'^2}\left\{-\frac{K'\pi}{a}\right\}$$

or

$$P\int_{-\infty}^{\infty}\frac{dK}{(a^2+K^2)(K-K')} = -\frac{K'\pi}{a(K'^2+a^2)} \quad . \tag{46}$$

Using equation 46 in equation 29 gives

$$P\int_{-\infty}^{\infty}\frac{1+i\Delta(K)}{(a^2+K^2)(K-K')} = -\frac{K'\pi}{a(a^2+K'^2)} + P\int_{-\infty}^{\infty}\frac{i\Delta(K)}{(a^2+K^2)(K-K')}dK \quad . \tag{47}$$

Now, returning to equation 28 and substituting equation 47 for the integral on the left,

$$\frac{-K'\pi}{a(a^2+K'^2)} + P\int_{-\infty}^{\infty}\frac{i\Delta(K)}{(a^2+K^2)(K-K')}dK = \frac{\pi i}{(a^2+K'^2)} - \frac{\pi\Delta(K')}{(a^2+K'^2)} - \frac{\pi K'}{a(a^2+K'^2)} + \frac{\pi i W}{(a^2+K'^2)} - \frac{\pi i K' V}{a(a^2+K'^2)} + \frac{\pi K' W}{a(a^2+K'^2)} + \frac{\pi i}{(a^2+K'^2)} . \tag{48}$$

Since the only way two complex numbers can be equal is for the real parts of the numbers to be equal and the imaginary parts to be equal, two equations can be obtained from equation 48. Equating the real parts of equation 48 gives

$$-\frac{\pi K'}{a(a^2+K'^2)} = -\frac{\pi\Delta(K')}{(a^2+K'^2)} - \frac{\pi K'}{a(a^2+K'^2)} + \frac{\pi V}{(a^2+K'^2)} + \frac{\pi K' W}{a(a^2+K'^2)} \quad . \tag{49}$$

Equation 49 gives

$$-\frac{K'}{a} = -\Delta(K') - \frac{K'}{a} + V + \frac{K'W}{a}$$

or

$$\Delta(K') = V + \frac{K'W}{a} \; . \tag{50}$$

The Born approximation gives a scattering amplitude that is valid providing the incident energies of the bombarding particles are sufficiently high. As the incident energy of the bombarding particles is increased to the range where the scattering amplitude calculated in the Born approximation is valid, it is reasonable to expect the correction factor given in equation 50 to approach zero as the energy approaches the sufficiently high Born energy. Letting K_B represent the momentum transfer associated with the lowest energy for a given scattering center to which the Born approximation will give a valid scattering amplitude,

$$\Delta(K_B') = 0 \; .$$

Using this idea as a boundary condition in equation 50 gives

$$\Delta(K_B') = 0 = V + \frac{K_B' W}{a} \; ,$$

so that

$$V = -\frac{K_B' W}{a} \; . \tag{51}$$

Thus, one of the two unknown quantities, V and W, has been evaluated. Using equation 51 in equation 50

$$\Delta(K') = -\frac{K_B' W}{a} + \frac{K' W}{a}$$

or

$$\Delta(K') = (K' - K_B')\frac{W}{a} \; . \qquad (52)$$

The unknown W can be evaluated by means of the relation obtained by equating the imaginary parts of equation 48;

$$P\int_{-\infty}^{\infty} \frac{i\,\Delta(K)}{(K^2+a^2)(K-K')}\,dK = \frac{\pi i}{(a^2+K'^2)}$$
$$+ \frac{\pi i W}{(a^2+K'^2)} - \frac{\pi i K' V}{a(a^2+K'^2)} + \frac{\pi i}{(a^2+K'^2)} \; . \qquad (53)$$

The Cauchy principal value on the left can be evaluated by substituting equation 52 into the integral, giving

$$P\int_{-\infty}^{\infty} \frac{\Delta(K)}{(a^2+K^2)(K-K')}\,dK = P\int_{-\infty}^{\infty} \frac{(K-K_B')\frac{W}{a}}{(a^2+K^2)(K-K')}\,dK$$

or

$$P\int_{-\infty}^{\infty} \frac{\Delta(K)}{(a^2+K^2)(K-K')}\,dK = \frac{W}{a}\,P\int_{-\infty}^{\infty} \frac{K\,dK}{(a^2+K^2)(K-K')}$$
$$- \frac{K_B' W}{a}\,P\int_{-\infty}^{\infty} \frac{dK}{(a^2+K^2)(K-K')} \; . \qquad (54)$$

In this equation, the second principal value on the right-hand side is exactly the same as that previously calculated except for the coefficient; thus

$$-\frac{K_B' W}{a} P\int_{-\infty}^{\infty} \frac{dK}{(a^2+K'^2)(K-K')} = \frac{-K_B' W}{a}\left\{-\frac{K'\pi}{a(K'^2+a^2)}\right\} = \frac{\pi K' K_B' W}{a^2(K'^2+a^2)} \, . \quad (55)$$

The first integral on the right side of equation 54 is again evaluated by the method of partial fractions. The integrand of this integral can then be rewritten

$$\frac{K}{(a^2+K^2)(K-K')} \equiv \frac{A}{K-K'} + \frac{BK+C}{a^2+K^2} \quad (56)$$

or

$$K \equiv A(a^2+K^2) + BK(K-K') + C(K-K') \, . \quad (57)$$

For $K=K'$, equation 57 becomes

$$K' = A(a^2+K'^2)$$

or

$$A = \frac{K'}{a^2+K'^2} \, . \quad (58)$$

For $K=0$, equation 57 is

$$0 = \frac{K'a^2}{K'^2+a^2} - CK'$$

or

$$C = \frac{a^2}{K'^2+a^2} \, . \quad (59)$$

For $K=1$, equation 57 gives

$$1 = \frac{K'(a^2+1)}{a^2+K'^2} + B(1-K') + \frac{a^2(1-K')}{a^2+K'^2}$$

or

$$B = -\frac{K'}{a^2+K'^2} \, . \quad (60)$$

Then

$$\frac{W}{a}\,P\int_{-\infty}^{\infty}\frac{K\,dK}{(a^2+K^2)(K-K')} = \frac{W}{a(K'^2+a^2)}\,P\int_{-\infty}^{\infty}\left\{\frac{K'}{K-K'} - \frac{K'K}{a^2+K^2} + \frac{a^2}{a^2+K^2}\right\}dK.$$

Using the definition of the Cauchy principal value,

$$\frac{W}{a}\,P\int_{-\infty}^{\infty}\frac{K\,dK}{(a^2+K^2)(K-K')} = \left\{\lim_{\epsilon\to 0}\int_{-\infty}^{K'-\epsilon}\left[\frac{K'dK}{K-K'} - \frac{K'K\,dK}{a^2+K^2} + \frac{a^2 dK}{a^2+K^2}\right]\right.$$
$$\left. + \lim_{\epsilon\to 0}\int_{K'+\epsilon}^{\infty}\left[\frac{K'dK}{K-K'} - \frac{K'K\,dK}{a^2+K^2} + \frac{a^2 dK}{a^2+K^2}\right]\right\}\frac{W}{a(a^2+K'^2)}. \qquad (61)$$

Integrating equation 61 gives

$$\frac{W}{a}\,P\int_{-\infty}^{\infty}\frac{K\,dK}{(a^2+K^2)(K-K')} = \frac{W}{a(a+K'^2)}\left\{\lim_{\epsilon\to 0}\left[K'\ln(K-K') - \frac{K'}{2}\ln(a^2+K^2) + a\tan^{-1}\left(\frac{K}{a}\right)\right]_{-\infty}^{K'-\epsilon}\right.$$
$$\left. + \lim_{\epsilon\to 0}\left[K'\ln(K-K') - \frac{K'}{2}\ln(K^2+a^2) + a\tan^{-1}\left(\frac{K}{a}\right)\right]_{K'+\epsilon}^{\infty}\right\}$$

or

$$\frac{W}{a}\,P\int_{-\infty}^{\infty}\frac{K\,dK}{(a^2+K^2)(K-K')} = \frac{W}{a(a^2+K'^2)}\left\{\lim_{\epsilon\to 0}\left[\ln\left(\frac{(K-K')}{(a^2+K^2)^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K}{a}\right)\right]_{-\infty}^{K'-\epsilon}\right.$$
$$\left. + \lim_{\epsilon\to 0}\left[\ln\left(\frac{(K-K')}{(K^2+a^2)^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K}{a}\right)\right]_{K'+\epsilon}^{\infty}\right\}. \qquad (62)$$

Now inserting the integration limits in equation 62 gives

$$\frac{W}{a} P\int_{-\infty}^{\infty} \frac{K\,dK}{(a^2+K^2)(K-K')} = \frac{W}{a(a^2+K'^2)}\Bigg\{\lim_{\epsilon\to 0}\left[\ln\left(\frac{(K'-\epsilon)-K'}{[(K'-\epsilon)^2+a^2]^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right]$$

$$-\left[\ln\left(\frac{(1-K'/K)}{[1-a^2/K^2]^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{K\to-\infty}$$

$$+\left[\ln\left(\frac{(1-K'/K)}{[1-a^2/K^2]^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{K\to\infty} \tag{63}$$

$$-\lim_{\epsilon\to 0}\left[\ln\left(\frac{(K'+\epsilon)-K'}{[(K'+\epsilon)^2+a^2]^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K'+\epsilon}{a}\right)\right]\Bigg\} .$$

Then,

$$-\left[\ln\left(\frac{(1-K'/K)}{[1-a^2/K^2]^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{-\infty} = -a\tan^{-1}(-\infty),$$

and

$$\left[\ln\left(\frac{(1-K'/K)}{[1-a^2/K^2]^{1/2}}\right)^{K'} + a\tan^{-1}\left(\frac{K}{a}\right)\right]\Bigg|_{\infty} = a\tan^{-1}(\infty) .$$

It is also seen that

$$-\left[\ln\left(\frac{(1-K'/K)}{[1-a^2/K^2]^{1/2}}\right)^{K'}\right]\Bigg|_{-\infty} = -\ln(1) = 0,$$

and

$$\left[\ln\left(\frac{(1-K'/K)}{[1-a^2/K^2]^{1/2}}\right)^{K'}\right]\Bigg|_{\infty} = \ln(1) = 0.$$

The remaining arctangent terms in equation 63 cancel each other, since

$$\lim_{\epsilon\to 0}\left\{a\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right\} = a\tan^{-1}\left(\frac{K'}{a}\right)$$

and

$$\lim_{\epsilon\to 0}\left\{-a\tan^{-1}\left(\frac{K'-\epsilon}{a}\right)\right\} = -a\tan^{-1}\left(\frac{K'}{a}\right).$$

Using these results in equation 63 gives for the Cauchy principal value

$$\frac{W}{a}\,P\int_{-\infty}^{\infty}\frac{K\,dK}{(a^2+K^2)(K-K')} = -\frac{\pi W}{a^2+K'^2}\,. \tag{64}$$

Finally, equation 54 becomes, by the use of equations 55 and 64,

$$P\int_{-\infty}^{\infty}\frac{(K-K_B)\frac{W}{a}\,dK}{(a^2+K^2)(K-K')} = -\frac{\pi W}{a^2+K'^2}+\frac{\pi K'K'_B W}{a^2(a^2+K'^2)}\,. \tag{65}$$

Substituting equation 65 in equation 53 gives

$$\begin{aligned}-\frac{\pi W}{(a^2+K'^2)}-\frac{\pi K'K'_B W}{a^2(a^2+K'^2)} &= \frac{\pi}{(a^2+K'^2)}\\ &+\frac{\pi W}{(a^2+K'^2)}-\frac{\pi K'V}{a(a^2+K'^2)}+\frac{\pi}{(a^2+K'^2)}\end{aligned} \tag{66}$$

or

$$-W+\frac{K'K'_B W}{a^2} = 1+W-\frac{K'V}{a}+1. \tag{67}$$

Using the value found for V in equation 51 gives

$$-W+\frac{K'K'_B W}{a^2} = 2+W+\frac{K'K'_B W}{a^2}$$

or

$$W = -1. \tag{68}$$

Now that W has been evaluated, equation 52 can be written as

$$\Delta(K') = (K'_B - K')\frac{1}{a}. \tag{69}$$

Equation 69 gives the correction factor for the screened Coulomb potential in terms of known quantities. The factor K' corresponds to the incident energy of the bombarding particles. The factor K'_B corresponds to the lower energy at which the Born approximation ceases to give a valid expression for the scattering amplitude. The quantity, a, is determined by equation 25.

The corrected scattering amplitude was assumed to have the form,

$$f(K') = [1 + i\Delta(K')]\, f_B(K'). \tag{70}$$

The scattering amplitude calculated in the Born approximation is given in equation 26, so that

$$f(K') = [1 + i\Delta(K')]\left\{\frac{2Ze^2\mu}{\hbar^2(a^2+K'^2)}\right\}. \tag{71}$$

Using equation 69 in equation 71 gives, for the corrected scattering amplitude,

$$f(K) = \frac{2Ze^2\mu}{\hbar^2}\left\{\frac{1 + i\left[\frac{(K_B-K)}{a}\right]}{a^2+K^2}\right\}, \tag{72}$$

where the primed notation has been dropped. Equation 72 is the relation which has been sought.

The differential cross section is calculated from equation 72 by taking

$$\frac{d\sigma}{d\omega} = f(K)\, f^*(K).$$

Then, the differential cross section is

$$\frac{d\sigma}{d\omega} = \frac{4Z^2 e^4 \mu^2}{\hbar^2} \left\{ \frac{1 + \left[\frac{K_B - K}{a}\right]^2}{(a^2 + K^2)^2} \right\}. \qquad (73)$$

CHAPTER III

DISCUSSIONS AND CONCLUSIONS

The expression for the differential cross section of a screened Coulomb scattering center, given in equation 73, is seen to be

$$\left(\frac{d\sigma}{d\omega}\right) = \left(\frac{d\sigma}{d\omega}\right)_{Born} + (\text{correction factor}). \qquad (74)$$

The correction factor in equation 74 is

$$\left[\frac{K_B - K}{a}\right]^2 \left(\frac{d\sigma}{d\omega}\right)_{Born}$$

where $\left(\frac{d\sigma}{d\omega}\right)_{Born}$ is the differential cross section calculated in the Born approximation.

Recalling that

$$K = 2k \sin\theta/2 \qquad (75)$$

it is necessary to examine closely the meaning of K_B in order to see the significance of the factor $(K_B - K)$. First of all, K and K_B are actually momentum transfers; K_B is a constant momentum transfer, characteristic of the scattering problem under consideration.

The constant K_B is defined as

$$K_B = 2k_B \qquad (76)$$

where k_B is the order of magnitude of the incident energy of the bombarding particles at which the Born approximation fails to give satisfactory results. Since k_B is not an explicit number, but rather a range of energy, the choice of k_B, and thus K_B, is to a limited extent arbitrary. The reason for the choice of K_B in equation 76 stems from the fact that the Born approximation's failure is a function of incident energy. The quantity arising in the calculations, most closely corresponding to the incident energy, is the momentum vector, k. The subscript, B, has been used to indicate quantities referring to this "Born energy" and following this reasoning the momentum vector corresponding to this energy is denoted by k_B.

Since

$$\bar{K} = \bar{k}_o - \bar{k} \quad , \tag{77}$$

it follows that the magnitude of the $\bar{K}$ vector varies with the energies involved and with the "impact parameter" involved. Obviously, as the calculations have been made with no reference to an impact parameter, it is impossible to relate K_B to the Born energy through an impact parameter. Thus, a choice must be made such that no impact parameter is involved. The most obvious way to bypass the problem of the impact parameter is to consider a "head-on" collision, so that the effects of the impact parameter will be zero.

For a given k_B the choices for K_B, in view of the foregoing restrictions, must lie in the range

$$0 \leq K_B \leq 2k_B \quad . \tag{78}$$

In this range, only two explicitly known relations exist; i.e.

$$K_B = 0 \qquad \text{or} \qquad K_B = 2k_B \; .$$

The first choice corresponds to zero momentum transfer, that is, no scattering. The second corresponds to a maximum momentum transfer of $2k_B$. The first choice is obviously not desirable, leaving the second for the defining relation of K_B.

Now that K_B has been defined as given by equation 76, the factor (K_B-K) can be examined. It is very important that the correction apply only in the case where

$$0 \leq K \leq K_B \ .$$

Thus, the quantity (K_B-K) will always be greater than or equal to zero. It is then seen that the factor will be large in two cases. First, the factor will be large when the momentum vector, k, is small, corresponding to low incident energies. This idea is in agreement with experimental results. The magnitude of the error in using the Born approximation to calculate differential cross sections increases as incident energies are taken lower and lower.

Second, the correction factor is large when the scattering angle is small, also in agreement with experimental results. Thus, the correction factor in equation 74 accomplishes the desired results. That is, the corrective effect of the factor upon the differential cross section calculated in the Born approximation is greatest where the error in the original is greatest.

Figure 5 shows the scattering of electrons from hydrogen atoms for bombarding energies of 30 ev. The experimental data in Figures 5 and 6 were taken from an article by Webb (4, p. 386). The figure illustrates the effect of the correction factor upon the differential cross section calculated in the Born approximation.

For angles greater than approximately 30° the "corrected" curve gives a good fit to the experimental points. However, at angles close to 0° scattering, the correction, though largest in this region, is still smaller than the experimental values. Over all, in the range 0° to 180°, the corrected differential cross section is an improvement over the Born differential cross section.

As an illustration of the situation which occurs when the bombarding energies are equal to the assumed "Born" energy, Figure 6 shows the differential cross sections for bombarding energies of 100 ev. Figure 6 shows that there is an overcorrection resulting from the application of the correction factor. An important point to notice is that the correction factor is zero at a scattering angle of 180°, which is consistent with the definition of K_B. That is, even though the incident energy is 100 ev., equal to the assumed Born energy, the correction factor is zero only when $K=K_B$. The fact that the correction factor is zero only when $k=k_B$ and $\theta=180°$, explains the overcorrective effect for scattering angles less than 180°.

Thus the corrected scattering amplitude gives better results than the Born scattering amplitude, in the energy range up to 100 ev. At energies of 100 ev. the overcorrection is to be expected since, for hydrogen, 100 ev. was assumed to be the lowest energy at which the Born approximation gave acceptable results. Thus making a correction for 100 ev. bombarding energies would be expected to alter an already correct result.

Figure 7 shows the differential cross section for electron scattering by helium atoms for an incident energy of 100 ev. The experimental data in this figure are from a book by Mott and Massey (3, p. 122). Figure 7 serves to illustrate that the "Born energy" is a range of energy rather than an explicit number. For purposes of calculating the corrected differential cross section in this figure a Born energy of 110 ev. was assumed. The figure illustrates that the choice of the Born energy was too low since the corrected differential cross section fell below the experimental points at all angles. The Born energy should be chosen to give the best fit when this energy is not well defined.

Now referring to equation 72 and equation 26, an important and basic difference between the Born scattering amplitude and the corrected scattering amplitude is seen. The Born scattering amplitude is a real quantity, whereas the corrected scattering amplitude is a complex quantity. The significant difference between the two becomes apparent when the "optical theorem" is applied (2, p. 144). According to this theorem

$$Q = \frac{4\pi}{k} \operatorname{Im} f(0) \tag{79}$$

where Q is the total cross section and $\operatorname{Im} f(0)$ is the imaginary part of the forward scattering amplitude.

In the case of the scattering amplitude calculated in the Born approximation there is no imaginary part so that

$$\operatorname{Im} f(0) = 0 \tag{80}$$

in the Born approximation. Thus

$$Q_{Born} = 0$$

according to the optical theorem.

In the corrected scattering amplitude, however, there is an imaginary part, namely

$$\mathrm{Im}\, f(\theta) = \frac{2Ze^2\mu}{\hbar^2(K^2+a^2)}\left[\frac{(K_B-K)}{a}\right] . \tag{81}$$

Using equation 75 in equation 81 gives

$$\mathrm{Im}\, f(\theta) = \frac{2Ze^2\mu}{\hbar^2[(2k\sin\theta/2)^2+a^2]}\left\{\frac{K_B-2k\sin\theta/2}{a}\right\} . \tag{82}$$

Thus

$$\mathrm{Im}\, f(0) = \frac{2Ze^2\mu K_B}{\hbar^2 a^3} . \tag{83}$$

Now the total cross section is seen to be

$$Q = \frac{2Ze^2\mu K_B 4\pi}{\hbar^2 a^3 k} . \tag{84}$$

Using equation 76 in equation 84 gives

$$Q = \frac{16\pi Ze^2\mu}{\hbar^2 a^3}\left(\frac{k_B}{k}\right) . \tag{85}$$

Now, although application of the optical theorem to the scattering amplitude calculated in the first Born approximation yields zero for the total cross section, a method known as second Born approximation gives a scattering amplitude which does not yield zero when the optical theorem is applied. The scattering amplitude calculated in second Born approximation is given by (2, p. 156)

$$f^2(\theta) = \frac{\left(\frac{2mze^2}{\hbar^2}\right)^2}{K^2 + a^2}$$

$$+ \frac{\left(\frac{2mze^2}{\hbar^2}\right)^4}{2kA\sin\theta/2}\left\{\tan^{-1}\left(\frac{ka\sin\theta/2}{A}\right) + \frac{i}{2}\ln\left(\frac{A+2k^2\sin\theta/2}{A-2k^2\sin\theta/2}\right)\right\} \tag{86}$$

where

$$A^2 = a^4 + 4a^2k^2 + 4k^4\sin^2\theta/2 .$$

Application of the optical theorem to the scattering amplitude in equation 86 gives

$$Q_{2nd\ Born} = \frac{4\pi\left(\frac{2mze^2}{\hbar^2 a}\right)^4}{4k^2/a^2 + 1} \tag{87}$$

Comparing equations 85 and 86, it is seen that

$$Q_{corrected} \propto \frac{1}{k} \propto \frac{1}{v}$$

whereas

$$Q_{2nd\ Born} \propto \frac{1}{k^2} \propto \frac{1}{v^2} .$$

Therefore, the total cross section calculated from the corrected scattering amplitude leads to a different velocity dependence from that calculated in the second Born approximation.

There are two limitations to the method described in Chapter I. The first limitation can be seen from equation 24. If the method of Chapter I is to be of value, it is necessary that $\Delta(K)$ can be solved for as an explicit function of K. The preceding condition requires that the $f(K)$ in equation 24, the scattering amplitude calculated in the first Born approximation, be such that the left side of equation 24 can be explicitly integrated. For this reason numerical integrations of the left side of equation 24 are of no value.

The second limitation of the method described in Chapter I is that the scattering amplitude calculated in the Born approximation cannot contain poles of order greater than one. To understand this restriction it is necessary to recall the method for writing the residue of a pole of order greater than one.

If a function $f(z)$ has a pole of order m at Z=Z , then the function

$$\phi(z) = (z-z_0)^m f(z) \tag{88}$$

has a removable singularity at Z . The residue at Z is given by (1, p. 121)

$$\text{res}\,(z_0) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \tag{89}$$

where $\phi^{(m-1)}(z_0)$ is the $(m-1)$th derivative of the function given in equation 88 evaluated at $Z=Z_0$. Referring to equation 83, it is seen that in order to write the residues in equation 24 it is necessary to take at least one derivative of $\Delta(K)$ for a second-order pole. Writing the residues requires a knowledge of the explicit form of $\Delta(K)$ before $\Delta(K)$ is actually evaluated. An example of a potential which leads to this kind of difficulty is

$$V(r) = U_0 e^{-ar} \tag{90}$$

Applying equation 17 to this potential yields a scattering amplitude which has a second-order pole at $K = \pm ia$,

$$f(K) \propto \frac{1}{(K^2+a^2)^2} \quad .$$

Thus the method described in Chapter I does not apply to the potential in equation 90.

There is no apparent means of predicting which potentials will lead to second-order (or higher) poles without performing the integration indicated in equation 17.

In conclusion, the correction factor succeeds in lowering the "Born energy" for the case of the screened Coulomb potential. The method described in Chapter I should provide a solution for $\Delta(K)$ for any potential which leads to an analytic integrand in equation 24 and which leads to a Born scattering amplitude containing no poles of order greater than one. The correction factor found in Chapter I should then lower the Born energy for any potential satisfying the preceding conditions. Thus the method should extend the energy range where the Born approximation is valid.

CHAPTER BIBLIOGRAPHY

1. Churchill, Ruel V., _Complex Variables_, New York, McGraw-Hill, 1948.

2. Moisiewitsch, B. L., _Lectures in Theoretical Physics_, New York, Interscience Publishers, 1956.

3. Mott, N. F. and H. S. W. Massey, _The Theory of Atomic Collisions_, Great Britain, Oxford University Press, 1933.

4. Webb, Glenn M., "The Elastic Scattering of Electrons in Moleculer Hydrogen," _Physical Review_, XLVII, Series 2 (January 7, 1935), 384-388.

APPENDIX

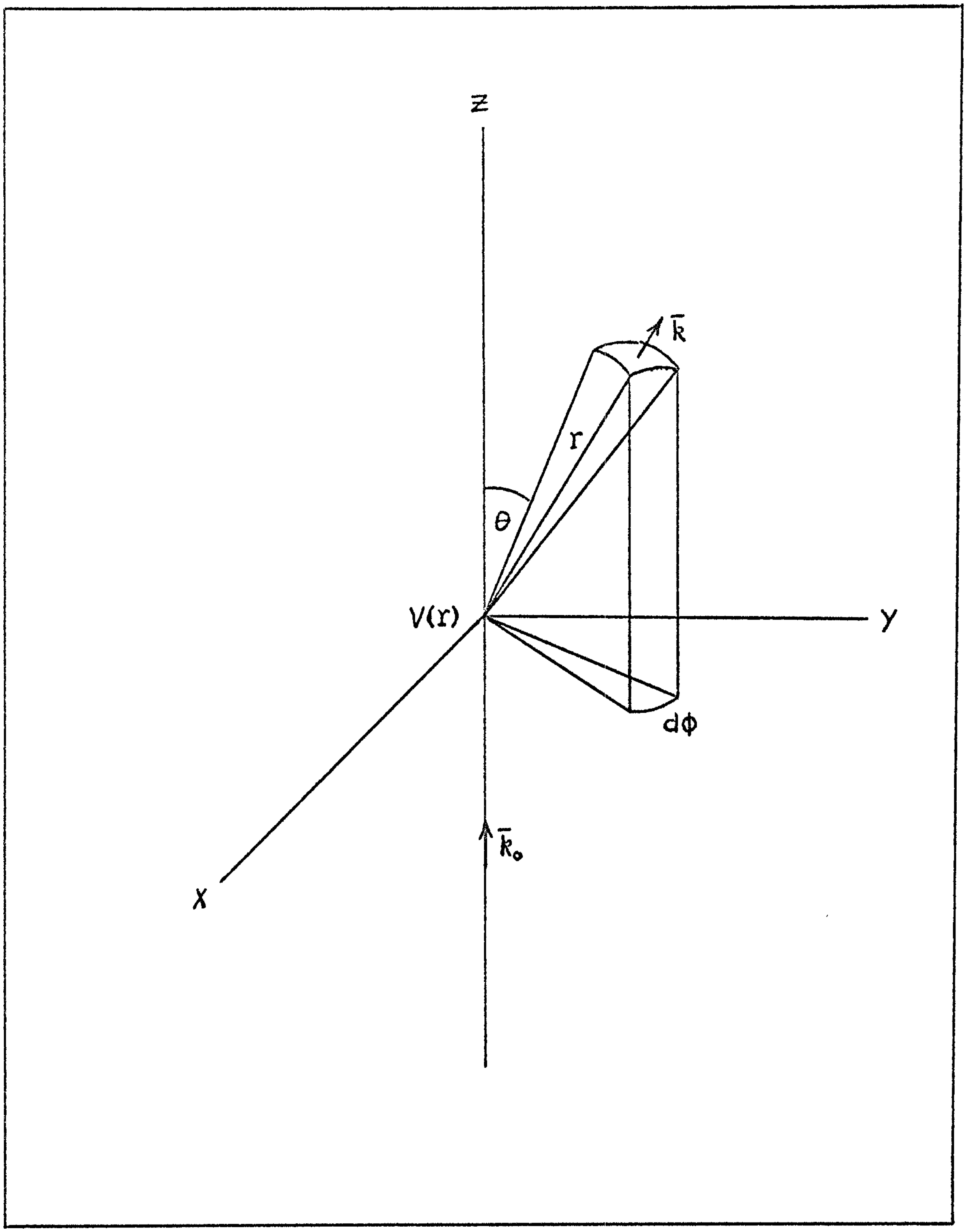

Fig. 1--Illustration of the scattering problem

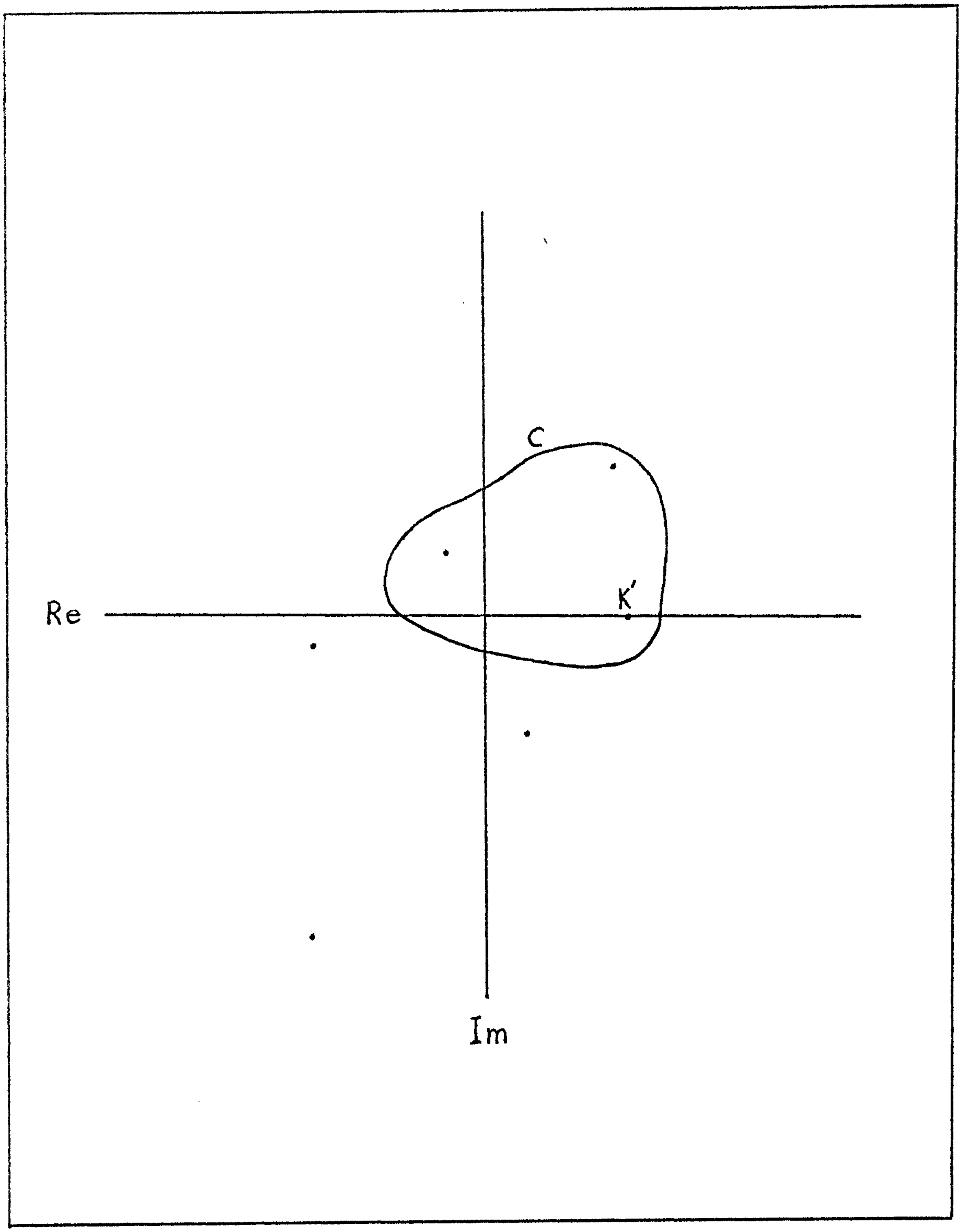

Fig. 2--Arbitrary contour and poles in the complex plane

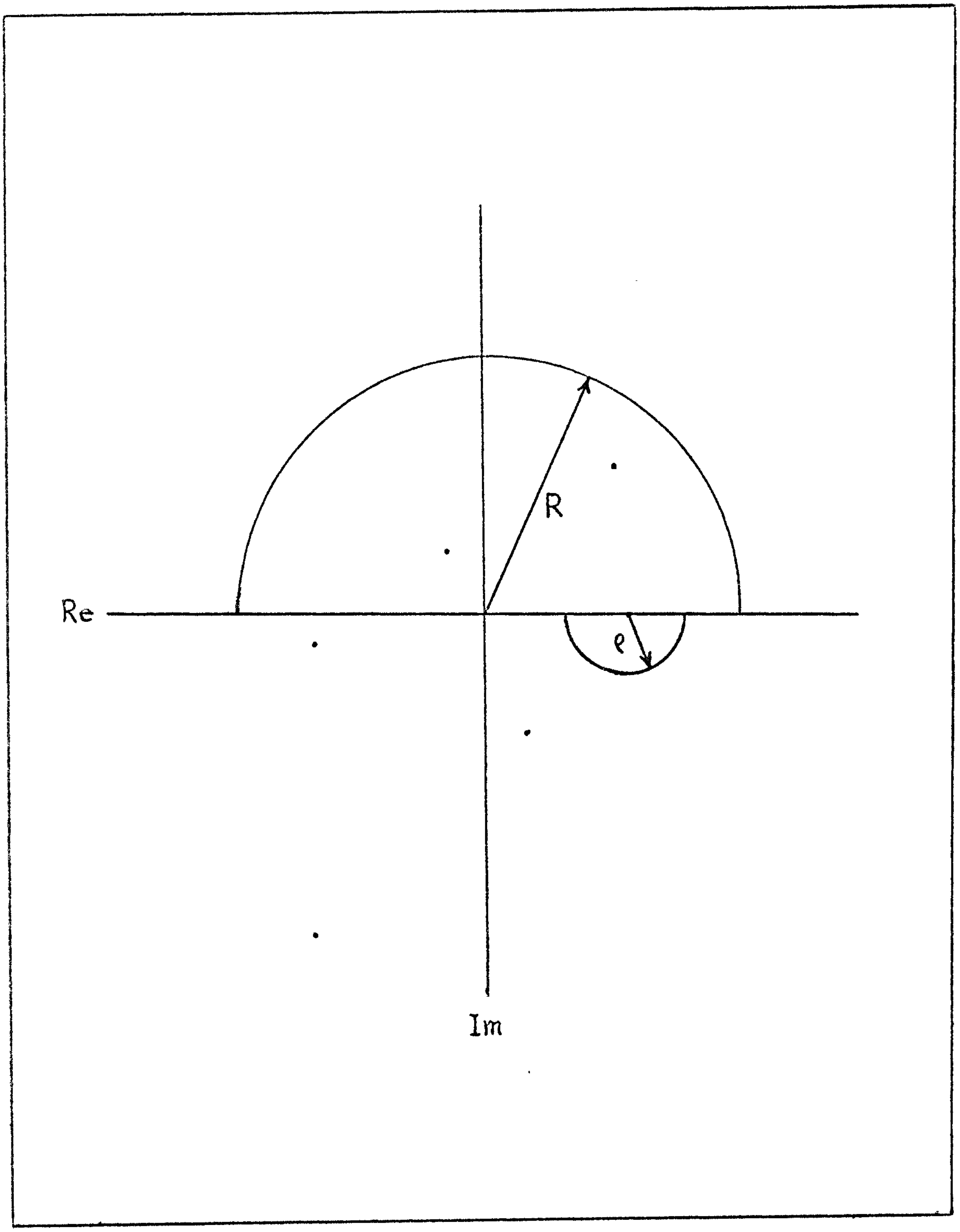

Fig. 3--Deformed contour

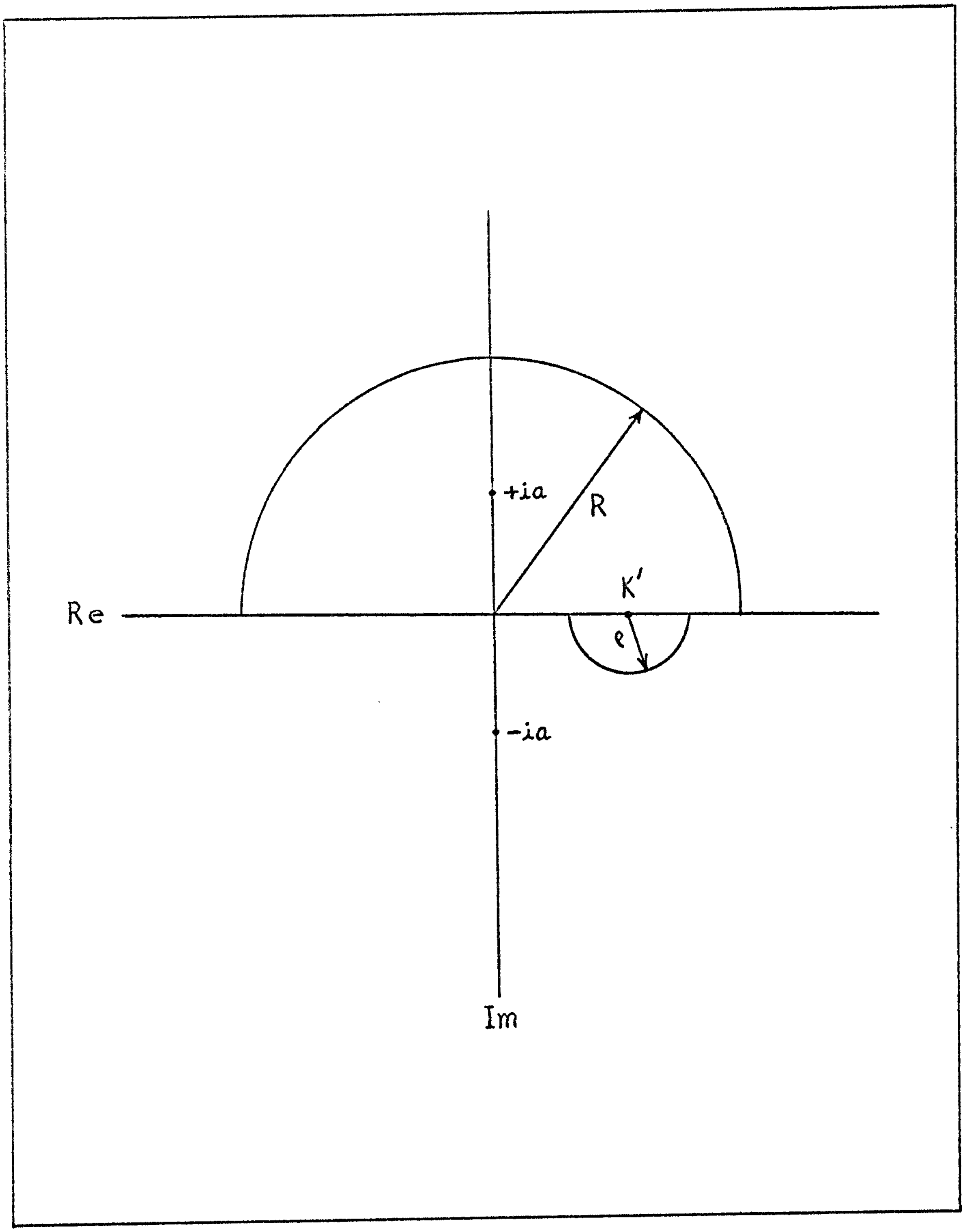

Fig. 4--Deformed contour and poles for application to the screened Coulomb potential.

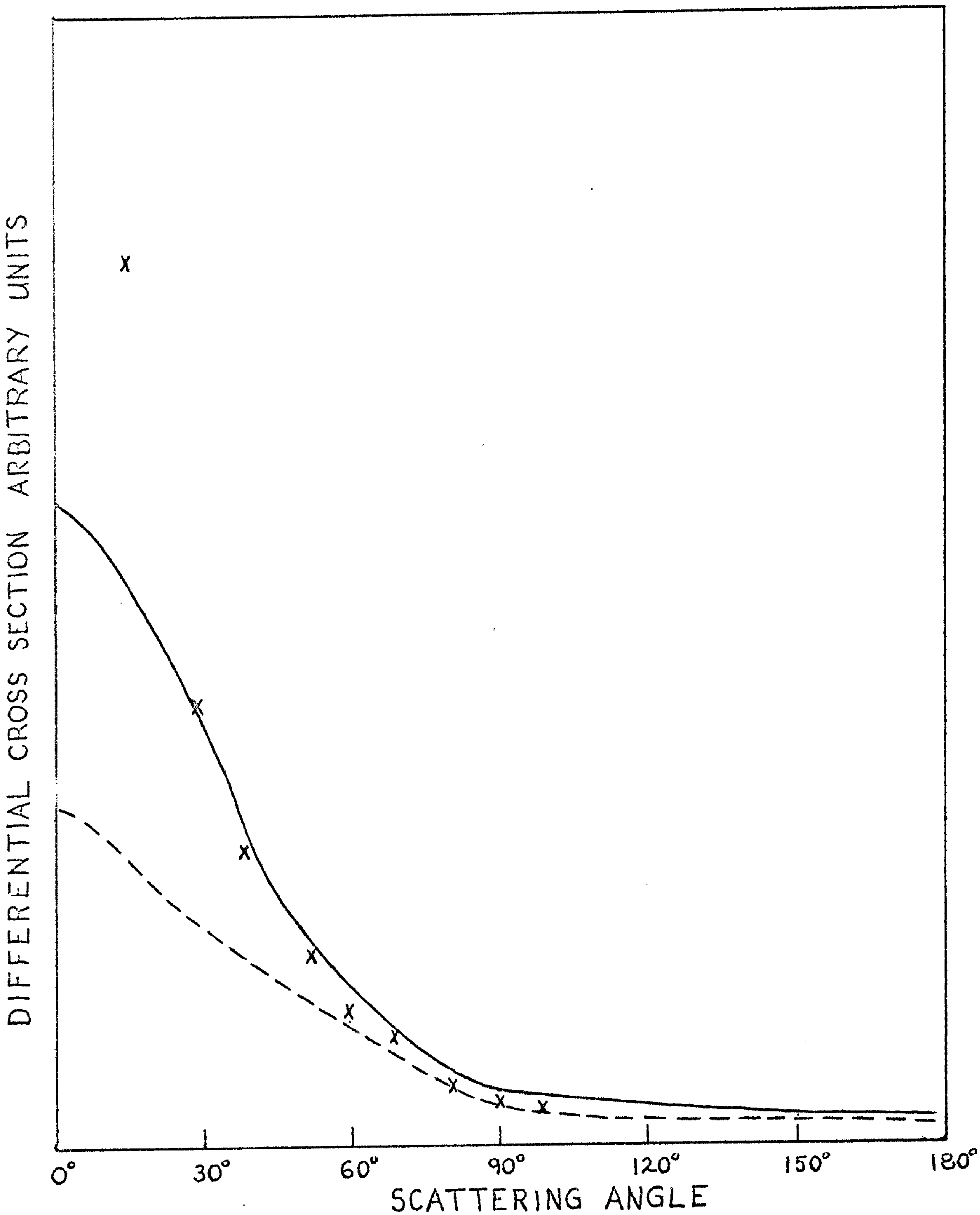

Fig. 5--Electron scattering by hydrogen for incident energies of 30 ev (assumed Born energy 100 ev).

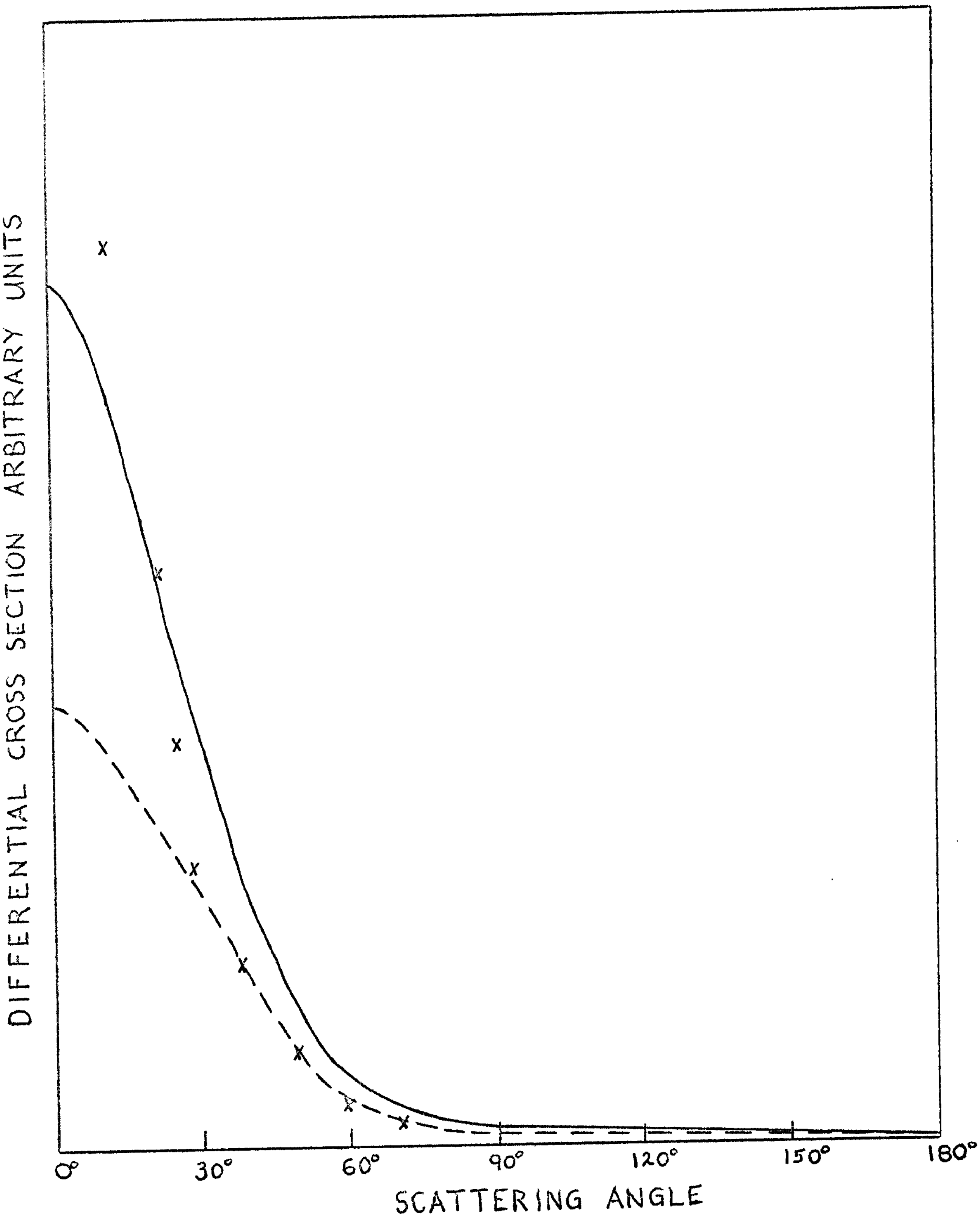

Fig. 6--Electron scattering by hydrogen for incident energies of 100 ev (assumed Born energy 100 ev).

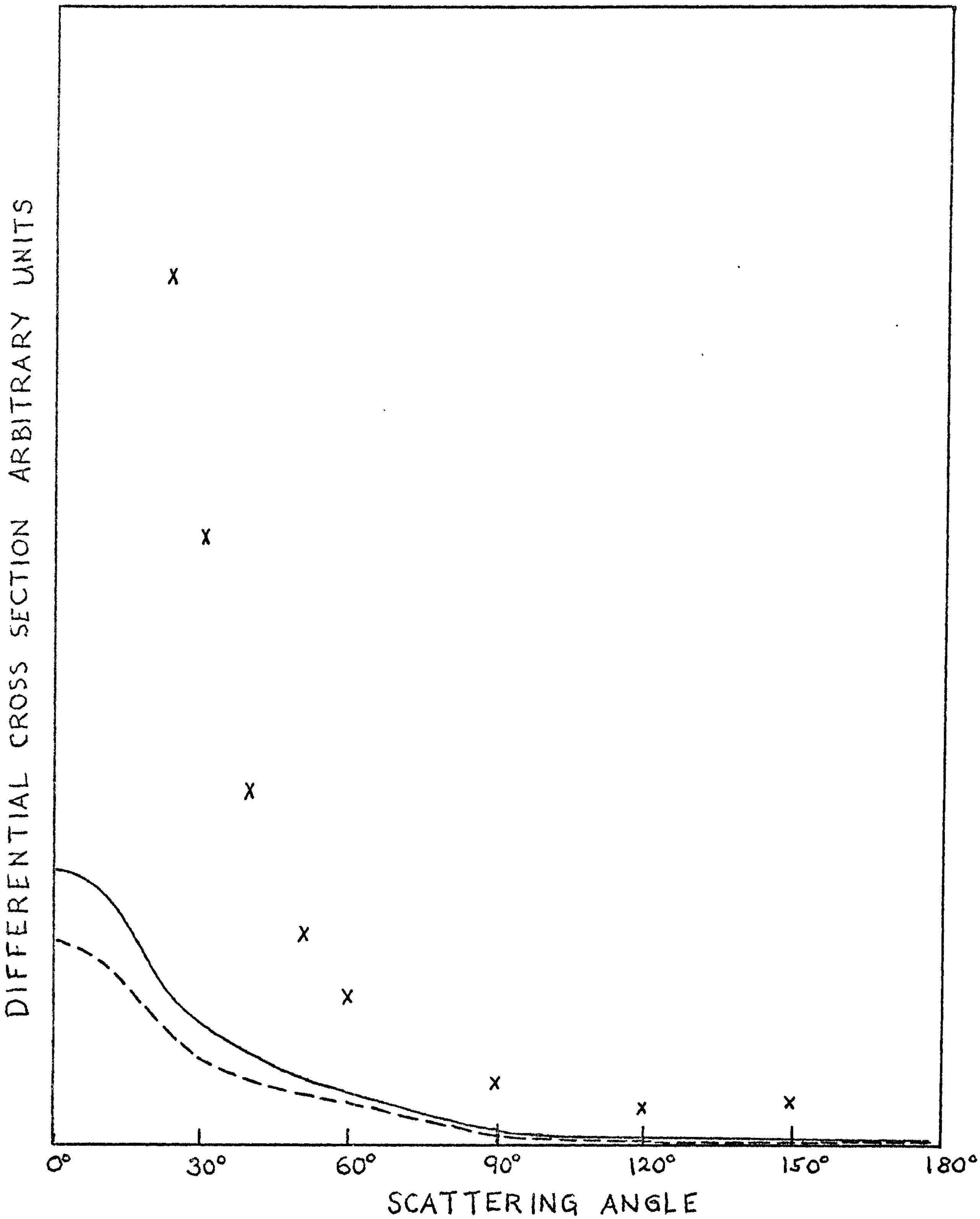

Fig. 7--Electron scattering by helium for incident energies of 100 ev (assumed Born energy 110 ev).

BIBLIOGRAPHY

Books

Churchill, Ruel V., Complex Variables, New York, McGraw-Hill, 1948.

Hildebrand, F. B., Advanced Calculus for Engineers, Englewood Cliffs, New Jersey, 1960.

Mandl, F., Quantum Mechanics, New York, Academic Press, Inc., 1957.

Moisiewitsch, B. L., Lectures in Theoretical Physics, New York, Interscience Publishers, 1956.

Mott, N. F. and H. S. W. Massey, The Theory of Atomic Collisions, Great Britain, Oxford University Press, 1933.

Articles

Webb, Glenn M., "The Elastic Scattering of Electrons in Moleculer Hydrogen," Physical Review, XLVII, Series 2 (January 7, 1935), 384-388.