IDEALS IN SEMIGROUPS

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IDEALS IN SEMIGROUPS

THESIS

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By

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CHAPTER I

GENERAL PROPERTIES OF IDEALS

A semigroup \((S, \cdot)\) is a non-empty set \(S\) together with an associative binary operation \(\cdot\), i.e. \((a \cdot b) \cdot c = a \cdot (b \cdot c)\) holds for all \(a, b, c\) in \(S\). If \(A\) and \(B\) are subsets of a semigroup \((S, \cdot)\), then \(A \cdot B = \{x \cdot y : x \in A\ \text{and} \ y \in B\}\). In the future \(A \cdot B\) will be denoted \(AB\), and a semigroup \((S, \cdot)\) will be referred to as a semigroup \(S\).

A left ideal is a non-empty subset \(A\) of a semigroup \(S\) with \(SA \subseteq A\). \(A\) is a right ideal if \(AS \subseteq A\) and \(A\) is called an ideal of \(S\) if \(A\) is both a left ideal and a right ideal. \(S\) is left (right) simple if \(S\) itself is the only left (right) ideal of \(S\), and \(S\) is also simple if \(S\) contains no proper ideal.

Now consider the following examples.

(1) Let \(S = \{a, b, c\}\), \(B = \{c\}\), \(A = \{b\}\), and

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Thus \(A\) and \(B\) are right ideals of \(S\) and they are not left ideals of \(S\).
(2) Let \( S = \{x, y, z\}, A = \{y, y\}, B = \{z, z\}, \) and

\[
\begin{array}{ccc}
  x & y & z \\
x & x & y & z \\
y & y & y & z \\
z & z & y & z \\
\end{array}
\]

Thus \( A \) and \( B \) are left ideals of \( S \) and they are not right ideals of \( S \).

(3) Let \( S = \{a, b, c\}, A = \{a, c\}, B = \{a\}, \) and

\[
\begin{array}{ccc}
a & b & c \\
a & a & a & a \\
b & a & b & c \\
c & a & c & a \\
\end{array}
\]

Thus both \( A \) and \( B \) are two-sided ideals of \( S \).

(4) Let \( G = \{x: x \text{ is a non-zero complex number}\} \). For \( a, b \in G \), \( ab = |a|b \), where if \( a = x+yi \) then \( |a| = \sqrt{x^2+y^2} \).

Illustration: \(|(3+i)(7-2i) = \sqrt{3^2+1^2} (7-2i) = 7\sqrt{10} - 2\sqrt{10}i \).

Obviously \( G \) is closed.

Let \( a, b, c \in G \) where \( a = u+ri, b = w+xi, \) and \( c = y+zi \). \( a(bc) = |a| (|b|c) = \sqrt{u^2+r^2 \sqrt{w^2+x^2}} (y+zi) \) =

\[
-\sqrt{(u^2+r^2)(w^2+x^2)} (y+zi)
\]

\( (ab)c = |(|a| \ b)|c = |\sqrt{u^2+r^2} (w+xi) (y+zi) =

\[
\sqrt{u^2+r^2} w + \sqrt{u^2+r^2} x i (y+zi) =
\]

\[
\sqrt{(u^2+r^2) w^2 + (u^2+r^2) x^2} (y+zi) =
\]

\[
\sqrt{(u^2+r^2)(w^2+x^2)} (y+zi).
\]
Therefore, since \(a(bc) = (ab)c\) and the operation is closed then \(G\) is a semigroup.

Let \(A = \{x \in G : x\) is a positive real number\}. Hence for any element \(a \in A\) and any element \(g \in G\), where \(g\) has the form \(g = x + yi\), then \(ga = \sqrt{x^2 + y^2} = a \sqrt{x^2 + y^2} = t\). Now \(t\) is a positive real number since both \(a\) and \(\sqrt{x^2 + y^2}\) are positive reals. Thus \(ga \in A\), and \(GA \subseteq A\), and hence \(A\) is a left ideal of \(G\). Now \((2 + i) \in G\), \(2 \in A\) and \(2(2 + i) = \sqrt{2^2 + 0^2} (2 + i) = 2(2 + i) = 4 + 2i \notin A\). Therefore, \(AG \nsubseteq A\) and hence \(A\) is not a right ideal of \(G\).

(5) Let \(S = \{x \in \text{reals} : \frac{1}{2} \leq x \leq 1\}\). If \(x, y \in S\), then \(xy \equiv \max(\frac{1}{2}, xy)\). Note: Since the real numbers are commutative, then every ideal is a two-sided ideal. There are infinitely many ideals contained in \(S\) for it follows that \(I_a = \{x : \frac{1}{2} \leq x \leq a, a \in \text{reals}\}\). Let \(b \in \text{reals}\) such that \(\frac{1}{2} \leq b \leq 1\). Thus \(I_b = \{x : \frac{1}{2} \leq x \leq b\}\) is an ideal of \(S\). Since \(b\) is the maximum element of \(I_b\) and \(1\) is the maximum element of \(S\), then clearly \(bl = b \in I_b\) and hence \(SI_b \subseteq I_b\) and \(I_bS \subseteq I_b\). Also, observe that \(1\) is the identity element of \(S\) and that \(\frac{1}{2}\) and \(1\) are the only idempotent elements of \(S\) since \(aa = \max(\frac{1}{2}, a^2)\). Now \((1)(1) = 1\) and \((\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}\) and it follows that there are no others since there are no other real numbers such that \(a^2 = a\).

(6) Perhaps the two most common semigroups \(S_1\) and \(S_2\) are \(S_1 = S_2 = \{x : x\) is a natural number\}\) and the operations are ordinary addition and multiplication, respectively. The set
of all even natural numbers is a two-sided ideal of the semigroup $S_2$ but not of the semigroup $S_1$.

An element $a$ of a semigroup $S$ is called a left (right) zero element of $S$ if $ax = a(xa = a)$ for every $x$ in $S$. An element $a$ of $S$ which is both a left and right zero element of $S$ is said to be a zero element of $S$, usually denoted by $0$.

The next three lemmas deal with certain properties of the zero element.

**Lemma 1.1.** A semigroup $S$ can contain at most one zero.
Proof: Assume $y$ and $z$ are both zeros of $S$, i.e. $xy = yx = y$ for every $x$ in $S$, and $xz = zx = z$ for every $x$ in $S$. Hence $zy = y$ and $zy = z$. Thus $y = zy = z$.

**Lemma 1.2.** The set consisting of only the zero element of a semigroup $S$ is a two-sided ideal and is contained in every left, right or two-sided ideal of $S$.
Proof: (1) Let $A = \{ 0 \}$. Then $SA = AS = A$ since $x0 = 0x = 0$ for every $x$ in $S$. Hence $SA \subseteq A$, $AS \subseteq A$ and, therefore, $A$ is a two-sided ideal of $S$.

(2) Assume $B$ is a left (right) ideal of $S$ such that $0 \notin B$. Since $B$ is a left (right) ideal of $S$, then $SB \subseteq B$ ($BS \subseteq B$). Let $a$ be any element of $B$. Now $0 \in S$, hence $0a = 0(a0 = 0)$. But $0 \notin B$ and hence $SB \not\subseteq B$ ($BS \not\subseteq B$), which is a contradiction of $B$ being a left (right) ideal of $S$. Thus $0$ is contained in every left (right) ideal of $S$ and obviously
from this fact 0 is also contained in every two-sided ideal of S.

**Lemma 1.3.** A left zero semigroup S is left simple and every element of S forms by itself a right ideal of S.

**Proof:** (1) Let A be any non-empty subset of S and A ≠ S. Now SA ⊆ A since SA = S. Thus SS ⊆ S and hence S itself is the only left ideal of S, and, therefore, S is left simple.

(2) Let a ∈ S. Then aS = a since a is a left zero element of S. Hence aS ⊆ a. Thus every element of S forms by itself a right ideal of S.

The next several theorems and two corollaries pertain to the simpler properties of ideals in semigroups.

**Theorem 1.1.** If S is a semigroup, then the intersection, if not empty, of any collection of left (right, two-sided) ideals of S is itself a left (right, two-sided) ideal of S.

**Proof:** Let \( \{A_\alpha\} \) be the intersection of a collection of left ideals of S. Now show S \( \{A_\alpha\} \subseteq \{A_\alpha\} \). Let \( x \in S \) and \( y \in \{A_\alpha\} \). Then \( y \) is contained in each \( A_\alpha \) and hence \( xy \) is contained in each \( A_\alpha \) and, therefore, \( xy \in \{A_\alpha\} \). Thus \( S \{A_\alpha\} \subseteq \{A_\alpha\} \).

Similar proofs hold for right and two-sided ideals of S.
A non-empty subset $B$ of a semigroup $(S, \cdot)$ is said to be a subsemigroup of $S$ if $B$ is also a semigroup under $\cdot$.

**Theorem 1.2.** If $S$ is a semigroup, $B$ a subsemigroup of $S$, $A$ a left (right, two-sided) ideal of $S$ and the intersection of $A$ and $B$ non-empty, then $A \cap B$ is a left (right, two-sided) ideal of $B$.

**Proof:** Now $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Thus $BA \subseteq A$ implies that $B(A \cap B) \subseteq A$. And $BB \subseteq B$ implies that $B(A \cap B) \subseteq B$. Therefore, $B(A \cap B) \subseteq (A \cap B)$.

Similar proofs hold for right and two-sided ideals of $S$.

**Theorem 1.3.** If $A$ is any non-empty subset of a semigroup $S$, the intersection of all left (right) ideals of $S$ containing $A$ is a left (right) ideal of $S$ containing $A$ and is contained in every other such left (right) ideal of $S$. It is called the left (right) ideal of $S$ generated by $A$.

**Proof:** Let $\{Z_\alpha\}$ denote the set of all left (right) ideals of $S$ containing $A$, where $A$ is any non-empty subset of $S$. Then $\bigcap_{\alpha \in \Lambda} Z_\alpha$ is the intersection of all left (right) ideals of $S$ containing $A$. Since $A$ is in each $Z_\alpha$ then clearly $A \subseteq \bigcap_{\alpha \in \Lambda} Z_\alpha$, and since $A \neq \emptyset$, then $\bigcap_{\alpha \in \Lambda} Z_\alpha \neq \emptyset$.

Now since $\bigcap_{\alpha \in \Lambda} Z_\alpha$ is contained in every left (right) ideal of $S$ containing $A$, then $S(\bigcap_{\alpha \in \Lambda} Z_\alpha) \subseteq (\bigcap_{\alpha \in \Lambda} Z_\alpha) S$ is contained in each $Z_\alpha$ for each $Z_\alpha$ to be a left (right) ideal. Hence,
since $S\left(\bigcap_{a \in \mathcal{A}} S\right)$ is in each $Z_{\alpha}$, then

$$S\left(\bigcap_{a \in \mathcal{A}} S\right) \subseteq \left(\bigcap_{a \in \mathcal{A}} S\right) = \left(\bigcap_{a \in \mathcal{A}} S\right) \subseteq \left(\bigcap_{a \in \mathcal{A}} S\right).$$

Thus $\bigcap_{a \in \mathcal{A}} S$ is a left (right) ideal and is clearly contained in every other left (right) ideal of $S$ containing $A$.

**Theorem 1.4.** If $S$ is a semigroup and $A \neq \emptyset$ with $A \subseteq S$, then the left (right) ideal of $S$ generated by $A$ is $A \cup (SA)$ [$A \cup (AS)$]. Also the ideal of $S$ generated by $A$ is $A \cup (SA) \cup (AS) \cup (SAS)$.

**Proof:** (1) Obviously $A \subseteq A \cup (SA)$. Now to indicate that $A \cup (SA)$ is a left ideal of $S$ it must be shown that $S(A \cup SA) \subseteq A \cup SA$. Clearly $SA \subseteq A \cup (SA)$. Also, since $S$ is a semigroup then $SS \subseteq S$ and thus $S(SA) \subseteq SA \subseteq A \cup (SA)$. Therefore, $S(A \cup SA) \subseteq A \cup SA$.

(2) Let $T$ be any left ideal of $S$ containing $A$. Because $T$ is a left ideal of $S$, then $ST \subseteq T$. Thus since $A \subseteq T$ and $ST \subseteq T$, then $SA \subseteq T$. Hence any left ideal of $S$ containing $A$ must contain $SA$.

Similar proof for $A \cup (AS)$ being the right ideal of $S$ generated by $A$.

(3) Let $B = A \cup (SA) \cup (AS) \cup (SAS)$. Clearly $A \subseteq B$ and $SA \subseteq B$. As shown above, $SS \subseteq S$ and hence $S(SA) \subseteq SA \subseteq B$. Obviously $S(AS) \subseteq B$ and $S(SAS) \subseteq SAS$ implies that $S(SAS) \subseteq B$. Therefore, $SB \subseteq B$.

Similar proof to show $BS \subseteq B$. Thus $B$ is a two-sided ideal of $S$ containing $A$. 
(4) Continue to let \( B = A \cup (SA) \cup (AS) \cup (SAS) \), and let \( T \) be any two-sided ideal of \( S \) containing \( A \). Now \( ST \subseteq T \) and \( TS \subseteq T \) since \( T \) is a two-sided ideal of \( S \). Hence \( SA \subseteq T \) and \( AS \subseteq T \) since \( A \subseteq T \) and \( T \) is a two-sided ideal of \( S \). Also this implies that \( SAS \subseteq T \). Therefore, \( B \subseteq T \) and the ideal of \( S \) generated by \( A \) is \( B \).

**Corollary 1.1.** If \( S \) is a semigroup and \( A \) is a non-empty left (right) ideal of \( S \), then \( A \cup (AS) \cup [A \cup (SA)] \) is a two-sided ideal of \( S \) containing \( A \).

**Proof:** By Theorem 1.4, \( A \cup (SA) \cup (AS) \cup (SAS) \) is the ideal of \( S \) generated by \( A \), where \( A \) is a left ideal of \( S \). Since \( A \) is a left ideal of \( S \) then \( SA \subseteq A \). Hence \( A \cup (SA) = A \), and also \( SAS \subseteq AS \). Thus \( A \cup (AS) \) is an ideal of \( S \) generated by \( A \) where \( A \) is a left ideal of \( S \).

Similar proof if \( A \) is a right ideal of \( S \).

**Theorem 1.5.** If \( S \) is a semigroup then the union of any collection of left (right, two-sided) ideals of \( S \) is itself a left (right, two-sided) ideal of \( S \).

**Proof:** Let \( A \) be the union of any collection of left ideals of \( S \). Now show \( SA \subseteq A \). Let \( x \in S \) and \( y \in A \). Since \( y \in A \) then \( y \) is contained in some left ideal \( I \) of \( S \) such that \( I \subseteq A \). Hence \( xy \) is some element, call one such element \( z \), contained in \( I \). And since \( I \subseteq A \) then \( z \in A \). Thus \( SA \subseteq A \).

Similar proofs hold for right and two-sided ideals of \( S \).
**Theorem 1.6.** If A is any non-empty subset of a semigroup S, then the union of any collection of left (right) ideals of S containing A is a left (right) ideal of S containing A.

Proof: Let \( \{Z_{\alpha}\}_{\alpha \in \Lambda} \) denote the set of all left (right) ideals of S containing A, where A is any non-empty subset of S. Then \( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \) is the union of all left (right) ideals of S containing A. Since A is in each \( Z_{\alpha} \), then clearly \( A \subseteq \bigcup_{\alpha \in \Lambda} Z_{\alpha} \) and since \( A \neq \emptyset \), then \( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \neq \emptyset \). Now show \( S \left( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) \subset \left( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) S \subset \left( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) \). Let \( a \) be any element of S and \( b \) be any element of \( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \). Now since \( b \in \bigcup_{\alpha \in \Lambda} Z_{\alpha} \), then \( b \) is contained in some \( Z_{\alpha} \), say \( Z_{i} \). Since \( Z_{i} \) is a left (right) ideal, then \( ab \in Z_{i} \) (\( ba \in Z_{i} \)). Hence \( ab \in \bigcup_{\alpha \in \Lambda} Z_{\alpha} \) \( \left( ba \in \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) \). Thus \( S \left( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) \subset \left( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) S \subset \left( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \right) \) and \( \bigcup_{\alpha \in \Lambda} Z_{\alpha} \) is a left (right) ideal of S containing A.

**Theorem 1.7.** Every non-empty subset of a semigroup S is a left (right) ideal of S if and only if every element of S is a right (left) zero of S.

Proof: (1) If every element of S is a right (left) zero of S, then every non-empty subset of S is a left (right) ideal of S. Now every element of S is a right (left) zero, i.e., for \( a \in S \) \( xa = a(ax=a) \), for every \( x \in S \). Let A be any non-empty subset of S. Let \( a \in A \) and \( x \in S \). Then
xa = a ∈ A (ax = a ∈ A). Thus SA ⊆ A (AS ⊆ A) and hence A is a left (right) ideal of S.

(2) If every non-empty subset of a semigroup S is a left (right) ideal of S, then every element of S is a right (left) zero of S. Assume there exists an element a ∈ S such that it is not true that xa = a (ax = a), for every x ∈ S. Hence for some element z ∈ S za = y (az = y), y ∈ S and y ≠ a.

Let A = {a} and hence A is a non-empty subset of S. Now for z ∈ S, za = y (az = y) y ≠ a. Thus SA ∉ A, (AS ∉ A) which contradicts the hypothesis. Therefore, every element of S is a right (left) zero of S.

If S is a semigroup and A is a subset of S, then the complement of A, denoted S \ A, is the set of all elements contained in S which are not contained in A.

Corollary 1.2. If every element of S is a right (left) zero of S, then the complement of every proper subset of S is a left (right) ideal of S.

Proof: Let A ⊆ S, A ≠ S, z ∈ S and a ∈ S \ A, then za = a ∈ S \ A (az = a ∈ S \ A). Therefore, S(S \ A) ⊆ S \ A[(S \ A)S ⊆ S \ A] and thus SA is a left (right) ideal of S.

Theorem 1.8. A semigroup S is simple if and only if SaS = S for every non-zero a ∈ S.

Proof: (1) If SaS = S for every non-zero a ∈ S then S is simple. Suppose A ≠ 0 is an ideal of S. Then SAS = (SA)S ⊆ AS ⊆ A. However, by hypothesis, for a ∈ A
and $a \neq 0$ $SaS = S \not\subseteq A$. This is a contradiction and, therefore, $S$ is simple.

(2) If $S$ is simple then $SaS = S$ for every non-zero $a \in S$. Suppose that there exists an $x \in S$ such that $SxS \neq S$. Obviously $SxS \subseteq S$, and $S(SxS) = (SS)(xS) \subseteq S(xS) = SxS$. By a similar argument $(SxS)S \subseteq SxS$. Thus $SxS$ is an ideal of $S$, and this is a contradiction since $S$ is simple. Therefore, $SaS = S$ for every non-zero $a \in S$.

**Theorem 1.9.** A semigroup $S$ is right (left) simple if and only if $aS = S$ ($Sa = S$) for every $a \in S$.

**Proof:** (1) If $aS = S$ ($Sa = S$) for every $a \in S$, then $S$ is right (left) simple. Let $A$ be any non-empty proper subset of $S$. Hence $A$ contains at least one element. Let $x$ be any element contained in $A$. By hypothesis $xS = S$ ($S = Sx$). Hence $AS \subseteq A$ ($SA \subseteq A$) since $AS = S$ ($SA = S$) and $A \neq S$. Now $SS = S$ and hence $SS \subseteq S$. Thus $S$ is the only right (left) ideal of $S$, and $S$ is right (left) simple.

(2) If a semigroup $S$ is right (left) simple, then $AS = S$ ($Sa = S$) for every $a \in S$. Assume there is an $a \in S$ such that $aS \neq S$ ($Sa \neq S$). Hence $aS \subseteq S$ ($Sa \subseteq S$). Suppose $x \in (aS)S$ [$x \in S(Sa)$]. Then there exists $y, z \in S$ such that $x = (ay)z = a(yz) = aw$, $w = yz$. [$x = z(ya) = (zy)a = wa$, $w = yz$]. Hence $aw$ (wa) is an element of $aS$ ($Sa$). Thus $(aS)S \subseteq aS$ [$S(Sa) \subseteq Sa$] which is a contradiction since $S$ is right (left) simple. Therefore, $aS = S$ ($Sa = S$) for every $a \in S$. 
An element e of the semigroup S is called idempotent if 
\[ e^2 = e \] and nilpotent if \( e^n = 0 \) for some positive integer n.

An idempotent e of S is called primitive if \( e \neq 0 \) and if the 
only idempotents f of S such that \( ef = fe = f \) are \( f = e \) and 
f = 0. A non-empty subset T of S is called a nil subset if 
every element of T is nilpotent and is called nilpotent if 
\( T^n = 0 \) for some positive integer n; i.e. if \( t_1 t_2 \ldots t_n = 0 \) 
for all \( t_i \) in T.

A semigroup S is called completely simple if (i) S is 
simple, (ii) every non-zero idempotent of S is primitive;
(iii) to each x in S there corresponds at least one pair of 
non-zero idempotents e, f of S such that \( ex = xf = x \).

A left (right, two-sided) ideal A of the semigroup S is 
called minimal if \( A \neq \emptyset \) and if the only left (right, two-
sided) ideal of S contained in A is A itself.

Now consider the following example where \( S = \{a, b, c, d\} \) 
and

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Obviously the set \( T = \{a, b, c\} \) is both a nil and nilpotent 
subset of S.
Theorem 1.10. Let $S$ be a semigroup with zero. Then every nil (nilpotent) left (right) ideal of $S$ is contained in a nil (nilpotent) ideal of $S$.

Proof: Let $A$ be any nil left ideal of $S$. By Corollary 1.1, $A \cup (AS)$ is a two-sided ideal of $S$ containing $A$. Now it must be shown that $A \cup (AS)$ is a nil ideal of $S$. Let $a \in A \cup (AS)$. Hence $a \in A$ or $a \in AS$. If $a \in A$ then $a$ is nilpotent since $A$ is a nil left ideal of $S$. Suppose $a \in AS$.

Thus there exists $x \in A$ and $y \in S$ such that $a = xy$. Since $A$ is a left ideal of $S$, then $SA \subseteq A$. Hence $yx \in A$, and since $A$ is a nil left ideal of $S$, $(yx)^n = 0$, for some positive integer $n$.

Now by the associative law, $(xy)^{n+1} = x(yx)^n y = x0y = 0$.

Therefore, $A \cup (AS)$ is a nil ideal of $S$.

Now let $A$ be a nilpotent left ideal of $S$, and show $(A \cup AS)^n = 0$ for some positive integer $n$. First, let us show that $(A \cup AS)^n \subseteq A^n \cup A^n S$ for every positive integer $n$. For $n = 1$ it is obviously true that $A \cup AS \subseteq A \cup AS$. Now assume $(A \cup AS)^k \subseteq A^k \cup A^k S$ and show $(A \cup AS)^{k+1} \subseteq A^{k+1} \cup A^{k+1} S$ where $k$ is any positive integer. Let $y_1 y_2 \ldots y_k y_{k+1} = z \in (A \cup AS)^{k+1}$ and show $z \in A^{k+1} \cup A^{k+1} S$. By assumption $y_1 y_2 \ldots y_k \in A^k \cup A^k S$.

Case 1. Suppose $y_1 y_2 \ldots y_k \in A^k$.

Subcase 1. If $y_{k+1} \notin A$, then $z = y_1 y_2 \ldots y_k y_{k+1} \in A^k A = A^{k+1} \subseteq A^{k+1} \cup A^{k+1} S$.

Subcase 2. If $y_{k+1} \in AS$, where $y_{k+1} = ax$ such that $a \in A$ and $x \in S$ then $z = y_1 y_2 \ldots y_k y_{k+1} = y_1 y_2 \ldots y_k ax \in A^k AS = A^{k+1} S \subseteq A^{k+1} \cup A^{k+1} S$. 


Case 2. Suppose $y_1 y_2 \ldots y_k \in A^k S$.

Subcase 1. If $y_{k+1} \in A$ then $z = y_1 y_2 \ldots y_k y_{k+1} \in A^{k+1} S$. Since $A$ is a left ideal of $S$ then $A^k S A \subseteq A^k A$ and $A^k A = A^{k+1} \subseteq A^{k+1} S$.

Subcase 2. If $y_{k+1} \in AS$ then $z = y_1 y_2 \ldots y_k y_{k+1} \in (A^k S)(AS) = A^k (SA) S$. Since $A$ is a left ideal of $S$ then $A^k (SA) S \subseteq A^k (A) S = A^{k+1} S \subseteq A^{k+1} S$.

Thus $(A \cup AS)^n \subseteq A^n \cup A^n S$ for every positive integer $n$. Since $A$ is nilpotent then $A^n = 0$ for some positive integer $n$. Now $A^{n+1} S = 0$ for the same positive integer $n$ as above. Hence, since $(A \cup AS)^n \subseteq A^n \cup A^n S$ then $(A \cup AS)^n = 0$ for some positive integer $n$. Therefore, every nilpotent left ideal of $S$ is contained in a nilpotent ideal of $S$.

Similarly, for nil (nilpotent) right ideals of $S$.

The nil radical $N = N(S)$ is the set of all properly nilpotent elements of $S$. An element $x$ of $S$ is properly nilpotent if $xS$ is a nil subset. Since $x^2 \in xS$, $x$ itself is nilpotent.

The semigroup $S$ is said to satisfy the minimal condition for left ideals if every non-empty set $A$ of left ideals of $S$ contains a minimal element, say $M$, in the sense that $M$ is in $A$ and if $P$ is in $A$ and $P \subseteq M$ then $M \subseteq P$. Similarly for right ideals and two-sided ideals.

**Theorem 1.11.** Let $S$ be a semigroup with zero. Then every nilpotent subset of $S$ is a nil subset of $S$, and the nil radical, $N$, is itself a nil ideal of $S$. 
Proof: (1) Let \( A \) be any nilpotent subset of \( S \). Hence \( A^n = 0 \) for some positive integer \( n \). Thus if \( a \in A \) then \( a^n = 0 \), since \( A \) is nilpotent. Therefore, \( A \) is a nil subset of \( S \) since every element of \( A \) is nilpotent.

(2) Now show \( N \) is a nil ideal of \( S \). Let \( y \in N \) then \( yS \) is a nil subset of \( S \), and since \( y^2 \in yS \) then \( y \) itself is nilpotent. Hence \( N \) is a nil subset of \( S \). Let \( a \in N \) and \( x \in S \) and show \( ax \in N \). Let \( y \in S \). \((ax)y = a(xy) = az; \ z = xy \) and \( z \in S \). Since \( a \in N \) then \((az)^n = 0 \) for some positive integer \( n \). Hence \((ax)S \) is a nil subset of \( S \) and hence \( ax \) is properly nilpotent. Therefore, \( ax \in N \) and thus \( NS \subseteq N \). Now show \( SN \subseteq N \). Let \( x \in S \) and \( a \in N \) and show \( xa \) is a properly nilpotent element, i.e. that \((xa)S \) is a nil subset of \( S \). Let \( y \in S \). Now since \( x, y \in S \) then \( yx = z \in S \). Hence since \( a \in N \) then \((az)^n = 0 \) for some positive integer \( n \). Thus \((xay)^n+1 = x(axy)^nay = x(az)^nay = x0ay = 0 \) for some positive integer \( n \). Therefore, \((ax)S \) is a nil subset of \( S \) since every element is nilpotent; \( xa \) is properly nilpotent and thus \( xa \in N \). Hence \( SN \subseteq N \), and the nil radical \( N \) is itself a nil ideal of \( S \).

**Theorem 1.12.** Let \( S \) be a semigroup with zero which satisfies the minimal condition for nil ideals. Then the nil radical \( N \) of \( S \) is nilpotent.

Proof: First, let us consider the set \( \{N^n; \ n = 1, 2, \ldots \} \). Now \( N^{n+1} \subseteq N^n \) for every positive integer \( n \); \( N^2 \subseteq N \) since \( N \)
is an ideal of S. Assume $N^{k+1} \subseteq N^k$ and show $N^{k+2} \subseteq N^{k+1}$. Thus $N^{k+2} = N^{k+1}N \subseteq N^kN = N^{k+1}$. Therefore, the above set consists of nil ideals such that $N^{n+1} \subseteq N^n$ for every positive integer $n$.

Hence \( \{N^n; n = 1, 2, \ldots\} \) has a minimal element, say $M$ where $M = N^M$ for some positive integer $m$. It is also noted here that $M = M^2 = M^3 = \ldots$. This follows directly from the above argument that $N^{n+1} \subseteq N^n$ together with $M$ being the minimal element. Assume that $M \neq 0$ for if $M = 0$ then the proof is complete. Now consider the set $T$ of all ideals $A$ of $S$ such that $A \subseteq M$ and $MAM \neq 0$. Obviously $T$ is not empty since $MMM = M \neq 0$ and hence $M \in T$. Hence $T$ has a minimal element, call it $B$. Thus $MBM \neq 0$ and there exists $b \in B$ such that $MBM \neq 0$. Now $M(MBM)M = MBM \neq 0$ and $MBM \subseteq M$ so $MBM \in T$. However, $MBM \subseteq B$ so $B = MBM$ since $B$ is the minimal element of $T$. Thus $b = abc$ for some $a, c \in M$. Now show $b = a^nbc^n$ for every positive integer $n$. For $n = 1$, $b = abc$. Assume $b = a^kbc^k$ and show that $b = a^{k+1}bc^{k+1}$. $a^{k+1}bc^{k+1} = a(a^kbc^k)c = abc = b$. Therefore, $b = a^nbc^n$ for every positive integer $n$. Since $M$ is a nil ideal of $S$ then $b = 0$, which is a contradiction of $MBM \neq 0$. Therefore, $N^M = M = 0$ for some positive integer $m$.

**Theorem 1.13.** Let $M$ be a non-zero ideal of the semigroup $S$ with zero. If $M$ is a completely simple semigroup then $M$ is simple and contains a primitive idempotent of $S$. 
Proof: Since \( M \) is completely simple then by definition \( M \) is simple. We are given that \( M \) is an ideal of \( S \), and thus by Lemma 1.2 \( 0 \in M \), where \( 0 \) is the zero of \( S \). Now there corresponds to \( 0 \) at least one pair of non-zero idempotents \( p, q \) of \( M \) such that \( p0 = 0q = 0 \). Also, it is noted that \( p \) and \( q \) are primitive with respect to \( M \). Since \( p \) and \( q \) are non-zero idempotents of \( M \) then they are also non-zero idempotents of \( S \). Suppose there exists a non-zero idempotent \( e \) of \( S \) and \( e \neq p \) such that \( pe = ep = e \). Since \( M \) is an ideal of \( S \) and \( p \in M \) then \( pe = e \in M \) and hence the above contradicts the fact that \( p \) is primitive with respect to \( M \). Therefore, \( M \) contains a primitive idempotent of \( S \).

**Theorem 1.14.** Let \( M \) be a non-zero ideal of the semi-group \( S \) with zero. If \( M \) contains a minimal left ideal \( L \) and a minimal right ideal \( R \) of \( S \) such that \( LR = M \) and \( RL \neq 0 \) then \( M^2 \neq 0 \), \( M \) is a minimal ideal of \( S \) and \( M \) is both a union of minimal left ideals of \( S \) and a union of minimal right ideals of \( S \).

Proof: Now \( R \subseteq M \) and \( L \subseteq M \) and hence \( 0 \neq RL \subseteq MM = M^2 \) and thus \( M^2 \neq 0 \).

Let \( x \) be a non-zero element of \( L \), then \( Sx \subseteq L \) since \( L \) is a left ideal of \( S \). Now \( S(Sx) = (SS)x \subseteq Sx \) and thus \( Sx \) is a left ideal of \( S \). Hence \( L = Sx \) since \( L \) is a minimal left ideal of \( S \). By a similar argument \( yS = R \) for every non-zero element \( y \in R \). Now let \( z \) be any non-zero element of \( M \),
then \( z = ab \) where \( a \in L \) and \( b \in R \) and \( a \neq 0 \) and \( b \neq 0 \) since \( LR = M \). Thus \( SzS = (Sa)(bS) = LR = M \). Therefore, \( M \) is a minimal ideal of \( S \).

Let \( a \) be any non-zero element of \( R \) and show that \( L_a \) is a minimal left ideal of \( S \). \( S(L_a) = (SL)a \subseteq L_a \) and thus \( L_a \) is a left ideal of \( S \). Suppose \( A \neq 0 \) is a left ideal of \( S \) and \( A \subseteq L_a \). Thus \( A = Ba \) where \( B \subseteq L \) and \( B \) is a left ideal of \( S \) for if \( B \) is not a left ideal of \( S \) then \( SA = S(Ba) = (SB)a \not\subseteq Ba = A \) which contradicts the fact that \( A \) is a left ideal of \( S \). Now \( B \neq 0 \) since \( A \neq 0 \) and thus \( B = L \) since \( L \) is a minimal left ideal of \( S \). Therefore, \( A = L_a \) and hence \( L_a \) is a minimal left ideal of \( S \).

By a similar argument it can be shown that for any non-zero element \( b \in L \) that \( bR \) is a minimal right ideal of \( S \).

Thus \( M \) is both a union of minimal left ideals and a union of minimal right ideals of \( S \).
CHAPTER II

MINIMAL IDEALS

A left (right, two-sided) ideal $A$ of the semigroup $S$ is called minimal if $A \neq \emptyset$ and if the only left (right, two-sided) ideal of $S$ contained in $A$ is $A$ itself. The last theorem in the previous chapter was concerned with, to some extent, minimal ideals. The content of the present chapter will be devoted to the consideration of that specific type of (right, left, two-sided) ideal.

If the intersection $K$ of all the two-sided ideals of a semigroup $S$ is not empty, then $K$ shall be called the kernel of $S$. The statement that a semigroup has a kernel means that $K$ is not empty.

Now consider the following examples of minimal ideals.

(1) Let $S = \{a, b, c, d\}$ and

$$
\begin{array}{cccc}
  & a & b & c & d \\
\hline
  a & a & a & a & a \\
  b & a & b & a & d \\
  c & a & c & a & d \\
  d & a & a & a & d \\
\end{array}
$$

Thus both $A = \{a\}$ and $B = \{d\}$ are minimal right ideals of $S$ and $T = \{a, d\}$ is a minimal two-sided ideal of $S$. 
(2) Let $S = \{a, b, c, d\}$ and

\[
\begin{array}{c|cccc}
& a & b & c & d \\
\hline
a & a & a & a & a \\
b & b & b & b & b \\
c & c & c & c & c \\
d & c & c & c & d \\
\end{array}
\]

Thus $A = \{a\}$, $B = \{b\}$, and $T = \{c\}$ are all minimal left ideals of $S$ and $D = \{a, b, c\}$ is a minimal two-sided ideal of $S$.

(3) In example (5) of Chapter I the set $A = \{\frac{1}{2}\}$ is a minimal two-sided ideal of the semigroup $S$ since $\frac{1}{2}$ is the zero element of $S$. In fact, for any semigroup $S$ the set consisting of only the zero element will be a minimal ideal.

If a semigroup $S$ has a kernel $K$, then $K$ is a two-sided ideal of $S$. This fact is a consequence of Theorem 1.1. Also it is obviously true, by definition, that $K$ is a subset of every two-sided ideal of $S$.

**Lemma 2.1.** If a semigroup $S$ contains a minimal ideal $M$, then $M$ is the kernel of $S$.

**Proof:** Let $A$ be any ideal of $S$. Then $AM \subseteq A$ and $AM \subseteq M$. Obviously $AM$ is an ideal of $S$ and thus by the minimality of $M$, $AM = M$. Hence $M \subseteq A$. Thus $M$ is contained in every ideal of $S$ and it follows that $M$ is the kernel of $S$.

**Theorem 2.1.** If a semigroup $S$ has a non-degenerate kernel $K$, then $K$ is a simple subsemigroup of $S$ without zero, i.e. $K$ contains no zero element.
Proof: Let \( a, b \in K \) then \( ab \in K \) since \( K \) is an ideal of \( S \), and \( K \) is associative since \( K \subseteq S \). Thus \( K \) is a subsemigroup of \( S \). In fact, every ideal of \( S \) is a subsemigroup of \( S \).

Suppose \( A \) is an ideal of \( S \) and \( A \subseteq K \). From the above proof we have \( K \subseteq A \). Hence \( A = K \). Thus \( K \) is a simple subsemigroup of \( S \) without zero.

If \( S \) contains a zero element then \( K \) consists of only one element, the zero element of \( S \).

As has been stated in the previous chapter, a semigroup is said to be simple if it contains no proper two-sided ideal. A semigroup \( S \) is left (right) simple without zero if and only if the equation \( ya = b \) (\( ay = b \)) is always solvable in \( y, y \in S \), for any given pair of elements \( a, b \) of \( S \).

**Lemma 2.2.** If \( L \) is a minimal left ideal of a semigroup \( S \), and \( c \) is any element of \( S \), then \( Lc \) is also a minimal left ideal of \( S \).

Proof: \( SLc = (SL)c \subseteq Lc \) since \( L \) is a left ideal of \( S \) and thus \( Lc \) is a left ideal of \( S \). Suppose \( A \) is a left ideal of \( S \) and \( A \subseteq Lc \). Then \( A \) must have the form \( Bc \) where \( B \) is a left ideal of \( S \) and \( B \subseteq L \). However, since \( L \) is a minimal left ideal of \( S \), then \( B = L \) and hence \( A = Lc \). Thus \( Lc \) is also a minimal left ideal of \( S \).

**Lemma 2.3.** A two-sided ideal \( A \) of \( S \) contains every minimal left ideal \( L \).
Proof: \( SAL = (SA)L \subseteq AL \) since \( A \) is an ideal of \( S \). Thus \( AL \) is a left ideal of \( S \). Also \( AL \subseteq L \) since \( L \) is a left ideal of \( S \). Hence \( AL = L \) since \( L \) is minimal. Therefore, \( L = AL \subseteq A \) since \( A \) is a two-sided ideal of \( S \).

**Theorem 2.2.** If a semigroup \( S \) contains at least one minimal left ideal, then it has a kernel \( K \). \( K \) is the union of all the minimal left ideals of \( S \).

Proof: Let \( A \) be the union of all the minimal left ideals of \( S \). Obviously \( A \) is not empty since by hypothesis \( S \) contains at least one minimal left ideal. Let \( a \in S \) and \( b \in A \). Then \( b \) is contained in some minimal left ideal \( D \) and since \( ab \in D \) then \( ab \in A \). Hence \( A \) is a left ideal of \( S \). Also for \( a \in S \) and \( b \in A \) we have \( ba \in Da \). By Lemma 2.2, \( Da \) is a minimal left ideal and hence \( Da \subseteq A \). Thus \( ba \in A \) and \( AS \subseteq A \). Therefore, \( A \) is a two-sided ideal of \( S \). Now by Lemma 2.3 \( A \) is contained in every two-sided ideal of \( S \) and hence \( S \) has a kernel \( K \) and \( K = A \).

Note that in example (1) of this chapter, \( K = T \) where \( T \) is the union of the minimal right ideals \( A \) and \( B \). Also in example (2) of this chapter, \( K = D \) when \( D \) is the union of the minimal left ideals \( A, B, \) and \( T \).

**Lemma 2.4.** Let \( S \) be a semigroup with minimal left ideals. Then distinct minimal left ideals of \( S \) have no common elements.
Proof: Let \( A \) and \( B \) be minimal left ideals of \( S \) and suppose \( A \) and \( B \) have an element in common. As was noted in Chapter I, \( A \cap B \) is a left ideal of \( S \) and obviously \( (A \cap B) \subseteq A \) and \( (A \cap B) \subseteq B \). However, by the minimality of \( A \) and \( B \) \( A \cap B = A \) and \( A \cap B = B \). Thus \( A = B \).

**Theorem 2.3.** If a semigroup \( S \) contains at least one minimal left ideal, then every left ideal of \( K \), \( K \) the kernel of \( S \), is a left ideal of \( S \).

Proof: \( K \) exists by Theorem 2.2. Let \( A \) be a left ideal of \( K \). Then \( KA \subseteq A \). To show \( A \subseteq KA \), let \( x \in A \). By Theorem 2.2, \( x \) is contained in some minimal left ideal \( B \) of \( S \) and \( B \subseteq K \).

By Lemma 2.2, \( Bx \) is also a minimal left ideal of \( S \). Obviously, \( Bx \subseteq B \) but since \( B \) is a minimal left ideal of \( S \) then \( Bx = B \). Hence for some \( y \in B \subseteq K \), \( x = yx \). Thus \( x \in KA \) and \( A \subseteq KA \). Therefore, \( KA = A \). Clearly, \( KA \) is a left ideal of \( S \) and thus every left ideal of \( K \) is a left ideal of \( S \).

Observe at this point, that if \( S \) is a semigroup and \( A \) is a left ideal of \( S \) and \( B \) is a left ideal of \( A \), then it is not necessarily true that \( B \) is also a left ideal of \( S \). As an example, let \( S = \{a, b, c, 0\} \), where \( 0 \) denotes the zero element of \( S \). Define the product of any \( x, y \in S \) to be 0 except for \( a, b \in S \) and let \( ab = c \). Thus \( A = \{b, c, 0\} \) is a left ideal of \( S \) since \( SA = \{c, 0\} \subseteq A \). Also \( B = \{b, 0\} \) is a left ideal of \( A \) since \( AB = \{0\} \subseteq B \). However, \( B \) is not a left ideal of \( S \) since \( SB = \{c, 0\} \not\subseteq B \).
Theorem 2.4. If a semigroup S contains at least one minimal left ideal, then every left ideal of S contains at least one minimal left ideal of S.

Proof: Let A be a left ideal of S and L be a minimal left ideal of S. Let $a \in A$ and hence $La$ is a minimal left ideal of S. Now $La \subseteq A$ and thus A contains at least one minimal left ideal of S.

A semigroup S which is both left and right simple and without zero is a group.

Theorem 2.5. A minimal left ideal L of a semigroup S is a left simple subsemigroup of S without zero.

Proof: Since L is a left ideal of S then L is a subsemigroup of S. Let $a, b \in L$. Then $La$ is a minimal left ideal of S and $La \subseteq L$. Thus by the minimality of L, $La = L$. Hence for $b \in L$ there exists some $y \in L$ such that $ya = b$. Therefore, L is a left simple subsemigroup of S without zero.

It is quite clear that the preceding lemmas and theorems apply equally to semigroups containing minimal right ideals. The following theorem is a direct result of Theorem 2.2 and its application to minimal right ideals, together with Theorem 2.1.

Theorem 2.6. If a semigroup S contains at least one non-zero minimal left ideal and at least one non-zero minimal right ideal, then the union of all the minimal left ideals of S coincides with that of all its minimal right
ideals and constitutes the kernel $K$ of $S$. $K$ is a simple subsemigroup of $S$ without zero.

In the following lemmas, $R$ denotes a minimal right ideal and $L$ denotes a minimal left ideal of a semigroup $S$.

**Lemma 2.5.** If $a \in R$ and $b \in RL$, the equation $ax = b$ has a solution $x$ in $RL$.

*Proof:* Now $aR$ is a minimal right ideal of $S$ and $aR \subseteq R$. Thus by the minimality of $R$, $aR = R$. Hence $aRL = RL$, and $ax = b$ has a solution $x$ in $RL$.

**Lemma 2.6.** If $a \in L$ and $b \in RL$, the equation $ya = b$ has a solution $y$ in $RL$.

*Proof:* Now $La$ is a minimal left ideal of $S$ and $La \subseteq L$. Thus by the minimality of $L$, $La = L$. Hence $RLa = RL$, and $ya = b$ has a solution $y$ in $RL$.

**Lemma 2.7.** $RL$ is a group.

*Proof:* Obviously $RL \subseteq S$ and thus $RL$ is associative. $(RL)(RL) \subseteq (RL)L = R(LL) \subseteq RL$ and hence $RL$ is closed. Therefore, $RL$ is a semigroup.

Now $RL \subseteq R$ and $RL \subseteq L$. Let $a, b \in RL$, then $a \in R$ and hence by Lemma 2.5 $RL$ is a right simple semigroup without zero.

Similarly $a, b \in RL$ implies $a \in L$ and thus by Lemma 2.6 $RL$ is a left simple semigroup without zero. Therefore, $RL$ is a group.

An identity element of a semigroup $S$ is an element $e \in S$ such that $ex = xe = x$ for every $x \in S$. 
Let $S$ be a semigroup. An element $a \in S$ is right (left) cancellable if for any $x, y \in S$, if $ax = ay$ ($xa = ya$) then $x = y$. $S$ is left (right) cancellative if every element in $S$ is left (right) cancellable. $S$ is cancellative if it is both left and right cancellative.

**Lemma 2.8.** A group $G$ with an identity element $e$ is cancellative.

**Proof:** Let $a, b, c \in G$ and $ab = ac$. The equation $ya = e$ is solvable for $y$ in $G$. Hence $y(ab) = y(ac)$. $(ya)b = (ya)c$. $eb = ec$. $b = c$. Therefore, $G$ is left cancellative.

By a similar proof $G$ can be shown to be right cancellative and thus $G$ is cancellative.

**Lemma 2.9.** If $G$ is a group, then $G$ has a unique two-sided identity $e$.

**Proof:** Since $G$ is a group, then $G$ is both left and right simple and without zero. For $a \in G$ the equation $ya = a$ has a solution $y$ in $G$; call one such solution $e$. Hence $ea = a$. Also the equation $e = ax$ has a solution $x$ in $G$; call one such solution $b$. Hence $e = ab$. Thus $ee = e(ab) = (ea)b = ab = e$ and, therefore, $e$ is an idempotent element of $G$.

Now let $t$ be any element of $G$. The equation $ex = t$ has a solution $x$ in $G$; call one such solution $u$. Hence $eu = t$. $e(eu) = et$. $(ee)u = et$. $eu = et$. $u = t$. Thus $et = t$. 
By similar proof \( te = t \). Thus \( e \) is a two-sided identity for \( G \).

Assume that \( f \) is also a two-sided identity for \( G \). Hence \( e = ef = f \). Therefore, \( e \) is the unique two-sided identity for \( G \).

**Lemma 2.10.** Let \( e \) be the identity element of the group \( RL \). Then \( R = eS, L = Se, \) and \( R \cap L = eSe \).

**Proof:** Again, note that \( RL \subseteq R \) and \( RL \subseteq L \). Hence \( e \in RL \) implies \( e \in R \) and \( e \in L \). Obviously \( eS \) is a right ideal of \( S \) and \( eS \subseteq R \). Thus by the minimality of \( R \), \( R = eS \). By similar argument \( L = Se \).

Let \( x \in R \cap L \). Then \( x \in R = eS \) and \( x \in L = Se \). Thus \( x = ey \) where \( y \in S \) and \( x = ze \) where \( z \in S \). Hence \( ey = ze \), and \( eye = zee = ze = x \). Therefore, \( x \in eSe \) and \( (R \cap L) \subseteq eSe \). Now let \( x \in eSe \). Then \( x = ete \) where \( t \in S \). Thus \( x = ete = e \in eS = R \), and \( x = ete = te \in Se = L \). Therefore, \( x \in R \cap L, eSe \subseteq (R \cap L) \), and hence \( R \cap L = eSe \).

**Lemma 2.11.** \( R \cap L = RL \).

**Proof:** From the above lemma, \( R \cap L = eSe, R = eS, \) and \( L = Se \) where \( e \) is the identity element of \( RL \). Now \( R \cap L = eSe = (eS)e = Re \subseteq RL \) and \( RL = (eS)(Se) \subseteq eSe = R \cap L \). Thus \( R \cap L = RL \).

**Lemma 2.12.** The identity element \( e \) of the group \( RL \) is a primitive idempotent.
Proof: Assume that $f$ is an idempotent of $RL$ such that $ef = fe = f$. Thus $ef = f = ff$, and $e = f$ since $RL$ is cancellative. Hence $e$ is a primitive idempotent of $RL$.

Theorem 2.7. Under the hypothesis of Theorem 2.6, the kernel $K$ of $S$ is a completely simple semigroup without zero.

Proof: By hypothesis $K$ is a simple semigroup without zero.

Let $f$ be any non-zero idempotent of $K$. Then $f$ is contained in exactly one minimal left ideal $L$ of $S$ and $f$ is contained in exactly one minimal right ideal $R$ of $S$. Now $RL$ is a group and $f \in RL$. Thus by Lemma 2.12, $f$ is a primitive idempotent.

Let $x \in K$. Then by a similar argument to the one above, $x \in RL$ where $R$ is a minimal right ideal of $S$ and $L$ is a minimal left ideal of $S$. Hence $RL$ has an identity element; call it $e$. Thus $xe = ex = x$. Therefore, $K$ is a completely simple semigroup without zero.

An element $1$ of a semigroup $S$ is a left zeroid of $S$ if, for any $a \in S$, the equation $xa = 1$ has a solution $x$ in $S$.

Theorem 2.8. A semigroup $S$ contains a left zeroid element if and only if it contains exactly one minimal left ideal $L$. $L$ consists of all the left zeroids of $S$ and is contained in every left ideal of $S$. $L$ is a right as well as a left ideal, being in fact the kernel of $S$, and is a left simple subsemigroup of $S$ without zero.
Proof: (1) Show that if $S$ contains exactly one minimal left ideal $L$, then $S$ contains a left zeroid element. By Lemma 2.2, $La$ is a minimal left ideal of $S$ for every $a \in S$. Since $L$ is the only minimal left ideal of $S$ then $L = La$ for every $a \in S$. Hence every element of $L$ is a left zeroid of $S$.

(2) Show that if a semigroup $S$ contains a left zeroid element then it contains exactly one minimal left ideal $L$. Let $L$ equal the set of all left zeroid elements of $S$. Now show $S \subseteq L$. Let $a \in S$ and $1 \in L$ and show $al$ is a left zeroid element of $S$. Let $z$ be any element in $S$. Hence there exists some $y \in S$ such that $yz = 1$. Thus $(ay)z = al$ and the equation $xz = al$ has a solution $x$ in $S$. Thus $al$ is a left zeroid element of $S$ and hence $S \subseteq L$. Let $A$ be any left ideal of $S$, let $1 \in L$, and $a \in A$. Since $1$ is a left zeroid of $S$ then there exists some $y \in S$ such that $ya = 1$. Hence $1 \in A$. Therefore, $L \subseteq A$ and $L$ is the only minimal left ideal contained in $S$.

As shown in an earlier part of the proof, $La = L$ for every $a \in S$. Thus it is true that $LS \subseteq L$. Hence $L$ is a right as well as a left ideal of $S$. Since $L$ is the only minimal left ideal of $S$ then by Theorem 2.2, $L$ is the kernel of $S$. The fact that $L$ is a left simple subsemigroup of $S$ without zero follows from Theorem 2.5.

In the following example the semigroup $S$ contains minimal left ideals, but $S$ does not contain a minimal left
ideal $L$ as described in Theorem 2.8 where $L$ is contained in every left ideal of $S$. Let $S = \{a_{11}, a_{12}, a_{21}, a_{22}\}$ and the operation on $S$ be defined as follows: $a_{ij}a_{kl} = a_{il}$. The set $A = \{a_{11}, a_{21}\}$ is a minimal left ideal of $S$ and the set $B = \{a_{12}, a_{22}\}$ is a minimal left ideal of $S$. However, since the intersection of $A$ and $B$ is empty there exists no minimal left ideal $L$ of $S$ such that $L$ is contained in every left ideal of $S$.

The following theorem is an immediate consequence of Theorem 2.8 with its dual for right minimal ideals and right zeroid elements together with the definition of a group in the earlier part of this chapter.

Theorem 2.9. If a semigroup $S$ contains exactly one minimal left ideal $L$ and exactly one minimal right ideal $R$, then the two coincide and constitute the kernel $K$ of $S$. $K$ is a group and consists of all the (left and right) zeroid elements of $S$.

A two-sided (left, right) ideal $M$ of a semigroup $S$ with zero is called 0-minimal if (i) $M \neq 0$ and (ii) 0 is the only two-sided (left, right) ideal of $S$ properly contained in $M$. Also a semigroup $S$ with zero is called 0-simple (left 0-simple, right 0-simple) if (i) $S^2 \neq 0$ and (ii) 0 is the only proper two-sided (left, right) ideal of $S$.

A one-to-one correspondence exists between two sets $A$ and $B$ if it is possible to associate the elements of $A$ with
the elements of $B$ in such a way that each element of each set is associated with exactly one element of the other.

If $S$ is a semigroup with zero, then all of the ideals of $S \setminus 0$ can be placed in a one-to-one correspondence with all of the non-zero ideals of $S$. Consequently, any theorem concerning minimal ideals implies an evident corollary concerning 0-minimal ideals in a semigroup with zero. Similar statements can be made concerning simple and 0-simple semigroups.

We shall now present a few theorems dealing with 0-minimal ideals and 0-simple semigroups.

The following example illustrates the necessity of part (i) in the definition of a 0-simple semigroup $S$. Let $S = \{a, b, c\}$ and

\[
\begin{array}{ccc}
  & a & b & c \\
a & a & a & a \\
b & a & a & a \\
c & a & a & a \\
\end{array}
\]

Thus the set consisting of the zero element $a$ is not the only proper two-sided ideal of $S$ since $B = \{a, c\}$ is also a proper two-sided ideal of $S$.

**Theorem 2.10.** If $S$ is a right (left) 0-simple semigroup, then $S \setminus 0$ is a right (left) simple subsemigroup of $S$.

**Proof:** Obviously, since $S \setminus 0$ is a subset of $S$, then the elements of $S \setminus 0$ are associative. Assume there exists
Let $a, b \in S \setminus 0$ such that $ab = 0$. Let $A = \{ x \in S : ax = 0 \}$. Now $A$ is non-empty since $0, b \in A$. To show $AS \subseteq A$, let $r \in A$ and $t \in S$. Then $a(rt) = (ar)t = 0t = 0$ and thus $rt \in A$ and hence $AS \subseteq A$. Also $A \neq \{0\}$ since $b \neq 0$ and $b \in A$. Thus by the 0-simplicity of $S$, $A = S$.

Now consider the set $B = \{0, a\}$, where $a \in S \setminus 0$ as indicated above. Since $A = S$ then $aS = \{0\}$ and thus $BS \subseteq B$. Again by the 0-simplicity of $S$, $S = B$. But $S^2 = b^2 = 0$ which contradicts the definition of 0-simplicity and hence there exist no $a, b \in S \setminus 0$ such that $ab = 0$. Therefore, $S \setminus 0$ is a subsemigroup of $S$.

Assume that $T$ is a proper right ideal of $S \setminus 0$. Then obviously $T \cup \{0\}$ is a proper right ideal of $S$ and $T \cup \{0\} \neq 0$ since $T \neq 0$. This contradicts the 0-simplicity of $S$ and, therefore, $S \setminus 0$ is a right simple subsemigroup of $S$.

By a similar argument, if $S$ is a left 0-simple semigroup then $S \setminus 0$ is a left simple subsemigroup of $S$.

**Theorem 2.11.** If $L$ is a 0-minimal left ideal of a semigroup $S$ with 0 such that $L^2 \neq 0$, then $L = Sa$ for any element $a \neq 0$ of $L$.

Proof: Let $a \in L$ and $a \neq 0$. Now $Sa \subseteq L$ and also $Sa$ is a left ideal of $S$. Hence $L = Sa$ or $0 = Sa$. Assume $0 = Sa$ for if $L = Sa$ the lemma is complete. Thus $A = \{0, a\}$ is a left ideal of $S$ since $SA = 0$ implies that $SA \subseteq A$. Also $A \subseteq L$ and $A \neq 0$ since $a \neq 0$. Therefore, $L = A$ and $LL = L^2 = 0$.
which contradicts the hypothesis. Hence $L = Sa$ for any element $a \neq 0$ of $L$.

**Theorem 2.12.** If $M$ is a 0-minimal ideal of a semigroup $S$ with 0 such that $M^2 \neq 0$, and if $L$ is a non-zero left ideal of $S$ contained in $M$, then $L^2 \neq 0$.

**Proof:** Consider the set $LS$ and observe that $S(LS) \subset LS$, and $(LS)S = L(SS) \subset LS$. Thus $LS$ is an ideal of $S$. Hence $LS = 0$ or $LS = M$. Assume $LS = 0$. Then $L$ is an ideal of $S$ and since by hypothesis $LCM$ and $L \neq 0$ then $L = M$. Therefore, $M^2 = IM \subset LS = 0$ and this contradicts the hypothesis $M^2 \neq 0$. Thus $LS = M$.

Assume $L^2 = 0$. Now $M^2 = (LS)(LS) = L(SL)S \subset LSS = L^2S = OS = 0$, which contradicts the hypothesis. Hence $L^2 \neq 0$.

In concluding this chapter, consider the following theorem which pertains to right zero elements of a semigroup $S$.

**Theorem 2.13.** If $S$ is a semigroup having a right zero element, then the set $T$ of right zero elements of $S$ is a right zero subsemigroup of $S$ and is a two-sided ideal of $S$ contained in every two-sided ideal of $S$.

**Proof:** (1) $T$ is associative since $T \subset S$ and by hypothesis $T$ contains at least one element. Let $x, y \in T$ then $xy = y \in T$. Thus $T$ is closed and since $T$ consists of right zero elements of $S$, then all the elements of $T$ are right
zero elements of $T$. Therefore, $T$ is a right zero subsemigroup of $S$.

(2) Clearly $ST \subseteq T$ since all the elements of $T$ are right zero elements. Let $x \in TS$. Then $x = ab$ where $a \in T$ and $b \in S$. Hence for $c \in S$, then $c(ab) = (ca)b = ab$ and thus $ab$ is a right zero element of $S$. Therefore, $ab \in T$, $TS \subseteq T$, and $T$ is a two-sided ideal of $S$.

Assume $A$ is a two-sided ideal of $S$ and $T \nsubseteq A$. Then there exists a $f \in S$ such that $f \in T$ and $f \notin A$. Let $g \in A$. Then $gf = f$ since $f \in T$. This implies $AS \nsubseteq A$ which contradicts the fact that $A$ is a two-sided ideal of $S$. Thus $T$ is contained in every two-sided ideal of $S$. 
CHAPTER III

MAXIMAL IDEALS

Now the opposite extreme of the previous chapter will be considered, i.e. maximal ideals.

A right (left, two-sided) proper ideal $A$ of a semigroup $S$ is said to be a maximal right (left, two-sided) ideal of $S$ if $A$ is not properly contained in any proper right (left, two-sided) ideal of $S$.

It is trivial to speak of a maximal ideal of a semigroup $S$ which is not a proper ideal of $S$, since otherwise $S$ would be the only maximal ideal of $S$. Thus in the future when referring to a maximal ideal of a semigroup $S$, it will be understood to be a proper ideal. Also note that neither the union nor the intersection of two maximal ideals is a maximal ideal. As examples of maximal ideals, consider the following: Let $S = \{0, 1, 2, 3, 4, 5\}$ and define the binary operation of $S$ as follows:

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Thus it is evident that $A = \{0, 2, 3, 4\}$ is the only maximal ideal of $S$.

As a last example here, let $S = \{a, b, c, d\}$ and define the binary operation of $S$ in the following manner.

\[
\begin{array}{cccc}
| & a & b & c & d \\
\hline
a & a & a & c & d \\
b & b & b & c & d \\
c & c & c & c & c \\
d & c & c & c & c \\
\end{array}
\]

Thus $\{a, c, d\}$ and $\{b, c, d\}$ are maximal right ideals of $S$, $\{a, b, c\}$ is a maximal left ideal of $S$, and $\{c, d\}$ is a maximal two-sided ideal of $S$.

If $S$ is a semigroup and $x, y \in S$, then $x$ is said to be a left divisor of $y$ if $y \in xS$, a right divisor of $y$ if $y \in Sx$, and an internal divisor of $y$ if $y \in SxS$. An element which is not a left (right, internal) divisor of $y$ is called a left (right, internal) non-divisor of $y$. A left (right, two-sided) ideal $A$ of $S$ is called universally maximal if $A \neq S$ and $A$ contains every proper left (right, two-sided) ideal of $S$.

**Lemma 3.1.** If $S$ is a semigroup without identity, then $S^1 = S \cup \{e\}$, where $e$ is the identity element of $S^1$, has a universally maximal ideal, namely $S$.

**Proof:** Show that $S$ is a universally maximal ideal of $S^1$. Let $A$ be any proper ideal of $S^1$ and let $x$ be any
element of \( A \). Now either \( x \in S \) or \( x = e \). Assume \( x = e \).

Then \( S^1A = S^1 \neq A \) which contradicts \( A \) being a proper ideal of \( S^1 \). Thus \( x \in S \) and \( S \) is a universally maximal ideal of \( S^1 \).

**Theorem 3.1.** If the set of all right (left) non-divisors of an arbitrary element \( b \) is non-empty, then it is a left (right) ideal of \( S \); and if the set of all internal non-divisors of \( b \) is non-empty, then it is a two-sided ideal of \( S \).

**Proof:** Let \( x \) be any right non-divisor of \( b \) and let \( y \) be any element of \( S \). Assume that \( yx \) is a right divisor of \( b \). Then there exists an element; call one such element \( z \), of \( S \) such that \( zyx = b \). Thus \( zyx = (zy)x = b \), which implies that \( x \) is a right divisor of \( b \). This fact contradicts \( x \) being a right non-divisor of \( b \) and, therefore, \( yx \) is right non-divisor of \( b \). Hence the set of all right non-divisors of \( b \) is a left ideal of \( S \).

By a similar proof, the set of all left non-divisors of \( b \) is a right ideal of \( S \).

Now let \( x \) be an internal non-divisor of \( b \) and \( y \in S \). Assume that \( yx \) is an internal divisor of \( b \). Then there exists \( r, t \in S \) such that \( r(yx)t = b \). But \( r(yx)t = ry(x)t = b \), which implies \( x \) is an internal divisor of \( b \) contrary to \( x \) being an internal non-divisor of \( b \). Thus \( yx \) is an internal non-divisor of \( b \).
By a similar argument xy is an internal non-divisor of b, and, therefore, the set of all internal non-divisors of b is a two-sided ideal of S.

As a matter of convenience, the following notation will be introduced. If S is a semigroup with a fixed left identity element e, let U be the set of all left divisors of e, V be the set of all right divisors of e, and W be the set of all internal divisors of e.

**Theorem 3.2.** If S is a semigroup with a left identity element e and U \( \neq \) S, then S \( \setminus \) U is a universally maximal right ideal of S.

**Proof:** By Theorem 3.1, S \( \setminus \) U is a right ideal of S. Let A be any proper right ideal of S, and assume there exists an element \( r \in A \) such that \( r \in U \). Thus there exists some element \( z \in S \) such that \( rz = e \). Since A is a right ideal of S and \( rz = e \), then \( e \in A \). Hence AS = S which contradicts A being a proper right ideal of S. Therefore, every element of A is an element of S \( \setminus \) U.

It may be remarked that the right (left) non-divisors, when they exist, of an arbitrary element x of a semigroup S need not be a right (left) ideal of S; for example, let S = \( \{ a, b, c \} \) and define the binary operation of S as follows:
V = \{a\} is the set of right divisors of the element \(a \in S\), and hence \(S \setminus V = \{b, c\}\). It is obvious that \(S \setminus V\) is a left ideal of \(S\) but not a right ideal of \(S\). Dually, for left non-divisors of some element \(x \in S\) not necessarily a left ideal of \(S\).

**Corollary 3.1.** If a semigroup \(S\) contains a left identity element \(e\) and contains a proper right ideal \(A\), then \(S\) contains a universally maximal right ideal.

**Proof:** Assume \(A\) contains an element; call one such element \(t\), such that \(t \in U\). Hence there exists some element \(z \in S\) such that \(tz = e\), and since \(A\) is a right ideal of \(S\) then \(e \in A\). But then \(AS = S\) since \(eS = S\), which contradicts the hypothesis that \(A\) is a proper right ideal of \(S\). Hence \(U \neq S\) and by Theorem 3.2, \(S\) contains a universally maximal right ideal.

**Theorem 3.3.** If a semigroup \(S\) contains a left identity element \(e\) and \(V \neq S\), then \(S \setminus V\) is a maximal left ideal of \(S\).

**Proof:** By Theorem 3.1, \(S \setminus V\) is a left ideal of \(S\). Assume \(A\) is a left ideal of \(S\), \(A \neq S \setminus V\) and \((S \setminus V) \subseteq A\). Then there exists an element, call one such element \(r\), contained in \(A\) such that \(r \in V\). Hence there exists some element \(t \in S\)
such that \( tr = e \). Since \( A \) is a left ideal of \( S \) and \( r \in A \), then \( e \in A \). Obviously \( e \in V \) since \( ee = e \). Now show \( e \) is an identity element for \( V \). Let \( a \) be any element contained in \( V \) and suppose \( ax = ay \) where \( x, y \in V \). Since \( a \in V \) then there exists some element, call one such element \( d \), contained in \( S \) such that \( da = e \). Hence \( ax = ay, d(ax) = d(ay), (da)x = (da)y, ex = ey, x = y \). Thus \( V \) is left cancellative. Now show \( ae = a \) where \( a \) is as indicated above. Thus \( da = e = ee = dae \). Hence \( dae = da \) and \( ae = a \) since \( V \) is left cancellative.

Obviously, \( ea = a \) and, therefore, \( e \) is an identity element for \( V \). Thus \( V = Ve \subseteq VA \subseteq A \). But \( V \cup (S \setminus V) = S \), and since \( (S \setminus V) \subseteq A \) and \( V \subseteq A \) then \( A = S \). Hence \( S \setminus V \) is a maximal left ideal of \( S \).

Note that although \( S \setminus V \) is a maximal left ideal, unlike \( S \setminus U \), it is not necessarily a universal maximal left ideal. For example, let \( S = \{a, b, c, d\} \) and

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
a & a & a & a & a \\
b & a & b & c & d \\
c & a & c & b & d \\
d & a & a & a & d
\end{array}
\]

Thus \( V = \{b, c\} \) and \( S \setminus V = \{a, d\} \). However, \( A = \{a, b, c\} \) is also a maximal left ideal of \( S \). Thus \( S \setminus V \) is not a universally maximal left ideal of \( S \).
The above example also implies that the set of all right non-divisors of an arbitrary element \( b \) of a semigroup \( S \) is not necessarily a maximal left ideal. Let \( T \) be the set of all right non-divisors of the element \( a \in S \). Thus \( T = \{d\} \). But \( T \subseteq (S\backslash W) \) and, therefore, \( T \) is not a maximal left ideal of \( S \).

It is likewise true that the set of all left non-divisors of an arbitrary element \( b \) of a semigroup \( S \) is not necessarily a maximal right ideal. Let \( S = \{a, b, c, d\} \) and

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Let \( R \) be the set of all left non-divisors of the element \( b \in S \). Thus \( R = \{a\} \). However, \( B = \{a, b\} \) is also a right ideal of \( S \) and \( R \subseteq B \).

**Theorem 3.4.** If \( S \) is a semigroup containing a left identity element \( e \) and \( W \neq S \), then \( S\backslash W \) is a universally maximal two-sided ideal of \( S \).

**Proof:** By Theorem 3.1, \( S\backslash W \) is a two-sided ideal of \( S \). Let \( A \) be a proper two-sided ideal of \( S \) such that \( A \neq (S\backslash W) \). Assume that there exists some element \( z \in A \) such that \( z \in W \). Thus there exists \( x, y \in S \) such that \( xzy = e \). Since \( A \) is a two-sided ideal of \( S \), then \( e \in A \). Hence \( AS = S \) which
contradicts $AS \subseteq A \neq S$. Therefore, $S \setminus W$ is a universally maximal two-sided ideal of $S$.

In the following example is illustrated the fact that the set of all internal non-divisors of an arbitrary element $b$ is not necessarily a maximal two-sided ideal. Let $S = \{a, b, c\}$ and

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The set $T$ of internal non-divisors of the element $b \in S$ is $T = \{a\}$. However, $A = \{a, b\}$ is also a two-sided ideal of $S$ and $T \subseteq A$.

**Corollary 3.2.** If a semigroup $S$ contains a left identity element $e$ and a proper two-sided ideal $A$, then $S$ contains a universally maximal two-sided ideal.

**Proof:** Assume $A$ contains an element, call one such element $t$, such that $t \in W$. Hence there exist some elements $x, y \in S$ such that $xty = e$, and since $A$ is a two-sided ideal of $S$ then $e \in A$. Thus, $AS = S$ since $eS = S$, which contradicts $AS \subseteq A \neq S$. Hence $W \neq S$ and by Theorem 3.4, $S$ contains a universally maximal two-sided ideal.

If a semigroup $S$ with a left identity element $e$ is a group, then $S \setminus U = S \setminus V = S \setminus W = \emptyset$. This follows since $S$ is both left and right simple without zero.
Now by combining the previous statement together with the foregoing theorems and corollaries the following theorem is obtained.

Theorem 3.5. If a semigroup S containing a left identity element is not a group, then it contains a maximal left ideal, a universally maximal right ideal, and a universally maximal two-sided ideal.

For semigroups with two-sided identity elements, the foregoing theorems may be combined with their left-right duals to obtain the following theorem.

Theorem 3.6. In a semigroup S having a two-sided identity element, if S is not a group then S contains universally maximal left, right, and two-sided ideals.

If S is a semigroup, A is any non-empty subset of S and \( x \in A \) then \( xA \) will be defined as \( \{xa : a \in A\} \). Also for \( x, y \in S \) \( x \sim y \) if and only if \( xA = yA \).

Theorem 3.7. \( \sim \) is reflective, symmetric, and transitive.

Proof: (1) Obviously \( x \sim x \) since \( xA = xA, x \in S \).

(2) Show if \( x \sim y \) then \( y \sim x \) for \( x, y \in S \). Since \( x \sim y \) then \( xA = yA \) and thus \( yA = xA \). Hence \( y \sim x \).

(3) Show if \( x \sim y \) and \( y \sim z \) then \( x \sim z \) for \( x, y, z \in S \). Since \( x \sim y \) and \( y \sim z \) then \( xA = yA \) and \( yA = zA \), respectively. Thus \( xA = yA = zA \) and hence \( x \sim z \). Therefore, \( \sim \) is reflective, symmetric, and transitive.
If $S$ is a semigroup, $A$ is any non-empty subset of $S$, and $x \in S$ then define $[x]$ as $\{y \in S : x \cap y\}$, and $S/A$ as $\{[x] : x \in S\}$. If $A$ is an ideal of $S$ such that $xA = Ax$ for every $x$ in $S$ then for $[x], [y] \in S/A$, $[x] \cdot [y] = [xy]$.

**Theorem 3.8.** $\cdot$ is well defined, i.e. the operation on the sets does not depend upon the particular elements used to designate these sets, but only upon the sets themselves.

Proof: Suppose $[a], [b], [c], [d] \in S/A$ where $[a] = [c]$ and $[b] = [d]$, and show $[a] \cdot [b] = [c] \cdot [d]$. Hence $[a] \cdot [b] = [ab]$ and $[ab] = \{y \in S : ab \cap y\}$. Thus $yA = (ab)A = a(bA) = a(dA) = a(Ad) = (aA)d = (cA)d = c(Ad) = c(dA) = (cd)A$. Therefore, $[ab] = [cd]$ and $\cdot$ is well defined.

**Theorem 3.9.** If $S$ is a semigroup and $A$ is any ideal of $S$ such that $xA = Ax$ for all $x \in S$, then $S/A$ is a subsemigroup of $S$; and if $S$ contains an identity $e$, then $S/A$ contains an identity element, namely $[e]$.

Proof: Let $[x], [y]$ and $[z]$ be any elements of $S/A$. Thus $([x] \cdot [y]) \cdot [z] = ([xy]) \cdot [z] = [xyz] = [x] \cdot ([yz]) = [x] \cdot ([y] \cdot [z])$. Hence $S/A$ is associative.

To show $S/A$ is closed with respect to $\cdot$, let $[x]$ and $[y]$ be any elements of $S/A$. $[x] \cdot [y] = [xy] = [z]$ where $z = xy$ and thus $S/A$ is closed.

If $S$ contains an identity $e$ then $[e]$ is contained in $S/A$. Thus for any $[x] \in S/A$, $[x] \cdot [e] = [xe] = [x] = [ex] = [e] \cdot [x]$ and hence $[e]$ is an identity element of $S/A$. 
Maximal ideals are especially useful in the study of group and ring theory, particularly in polynomial rings. Consider the theorem from group theory that says "a normal subgroup \( K \) of a group \( G \) is maximal if and only if \( G/K \) is a non-degenerate simple group." A simple group is a group that has no proper normal subgroups. In the theory developed in this chapter, a normal subgroup would correspond to an ideal \( A \) such that \( xA = Ax \) for every \( x \) contained in a semigroup \( S \). Hence it would be interesting to know whether the above theorem from group theory would hold true if \( G \) were a semigroup and if \( K \) were an ideal of \( G \) such that \( xK = Kx \) for every \( x \) contained in \( G \).
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