CONCERNING THE CONVERGENCE OF SOME NETS

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THESIS

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CHAPTER I

AXIOMS

Let R^1 denote the set of real numbers and N denote the set of positive integers.

<u>Axiom 1.</u> $(R^1,+,\cdot)$ is a field ordered by the usual relation "less than".

<u>Axiom 2.</u> If S is a subset of N such that 1 is in S, and if k is in S, then k + 1 is in S, then S = N.

<u>Definition 1.1.</u> If M is a subset of R^1, then the statement that M is bounded above (below) means that there is a number b, called an upper (lower) bound of M, such that if a is in M, then $a \leq b$ $(b \leq a)$. The statement that M is bounded means that M is bounded above and below.

<u>Axiom 3.</u> If M is a subset of R^1 such that M is bounded above (below), then there is an upper (lower) bound of M, B, such that if b is an upper (lower) bound of M, then $B \leq b$ $(b \leq B)$. B will be called the l. u. b. (g. l. b.) of M.

<u>Axiom 4.</u> If c is a number and $0 < c$, then there is an n in N such that $1/n < c$.

<u>Axiom 5.</u> If each of a and b is a number and $a < b$, then there is a number c such that $a < c < b$.

<u>Definition 1.2.</u> The statement that $[a,b]$ is an interval means that each of a and b is a number, $a < b$, and x belongs

to $[a,b]$ iff x is a number and $a \leq x \leq b$. The statement that (a,b) is a segment means that each of a and b is a number, $a < b$, and x belongs to (a,b) iff x is a number and $a < x < b$.

Definition 1.3. If each of a and b is a number and $a < b$, then $(a,b]$ is the set to which x belongs iff x is a number and $a < x \leq b$. $[a,b)$ is the set to which x belongs iff x is a number and $a \leq x < b$.

Definition 1.4. If c is a number, then the statement that U is a neighborhood of c means that there is a positive number r such that $U = (c - r, c + r)$.

If A is a finite set and t is a function whose domain contains A and whose range is a subset of R^1, then

$$\sum_{a \in A} t(a) = \sum_{i=1}^{n} t(a_i)$$

where $A = a_1, a_2, \ldots, a_n$. If no conflict will result, then $\sum_{a \in A} t(a)$ will be written $\sum_A t(a)$.

Definition 1.5. If $A = \phi$ and t is a function whose range is a subset of R^1, then $\sum_{a \in A} t(a) = 0$.

CHAPTER II

THE CONVERGENCE OF NETS

Directed Sets

<u>Definition 2.1.</u> The statement that the ordered pair (M,R) is a directed set means that M is a set and R is a relation such that (1) if x is in M, then (x,x) is in R, (2) if (x,y) is in R and (y,z) is in R, then (x,z) is in R, and (3) if x and y are in M, then there is a z in M such that (x,z) and (y,z) are in R.

If there is a symbol "*" such that x*y means that (x,y) is in R, then (M,*) will be equivalent to (M,R).

Obviously $(N, \leq)$ is a directed set. Other examples will follow.

<u>Definition 2.2.</u> If [a,b] is an interval, then the statement that E is a subdivision of [a,b] means that E is a finite collection of intervals each of which is a subset of [a,b] such that if each of [p,q] and [r,s] is in E, then [p,q] and [r,s] have no more than one point in common, and if x is in [a,b], then x is in some element of E.

<u>Definition 2.3.</u> If [a,b] is an interval and E is a subdivision of [a,b], then the statement that G is a refinement of E means that G is a subdivision of [a,b] every element of which is a subset of some element of E.

If $[a,b]$ is an interval, let $\triangle[a,b]$ denote the set of ordered pairs to which (E,t) belongs iff E is a subdivision of $[a,b]$ and t is a function whose domain contains E such that if I is in E, then $t(I)$ is in I. Let $R[a,b]$ denote the set of ordered pairs to which $((E,t),(G,s))$ belongs iff each of (E,t) and (G,s) is in $\triangle[a,b]$ and G is a refinement of E. Let $(E,t) \lesssim (G,s)$ denote that $((E,t),(G,s))$ is in $R[a,b]$.

<u>Lemma 2.1.</u> If $[a,b]$ is an interval, then $(\triangle[a,b],\lesssim)$ is a directed set.

<u>Proof:</u> If (E,t) is in $\triangle[a,b]$, then E is a refinement of E. Thus $(E,t) \lesssim (E,t)$. Suppose each of (E,t), (E^*,t^*), and (E',t') is in $\triangle[a,b]$. If $(E,t) \lesssim (E^*,t^*)$ and $(E^*,t^*) \lesssim (E',t')$, then E^* is a refinement of E and E' is a refinement of E^*. Thus if $[p,q]$ is in E', then there is an $[r,s]$ in E^* such that $[p,q]$ is a subset of $[r,s]$. Furthermore, there is an $[u,v]$ in E such that $[r,s]$ is a subset of $[u,v]$. It follows that $[p,q]$ is a subset of $[u,v]$, and thus E' is a refinement of E. Therefore $(E,t) \lesssim (E',t')$. Now suppose that each of (E,t) and (G,s) is in $\triangle[a,b]$. Let D be the subdivision of $[a,b]$ such that the set of end points of the intervals of D is the union of the set of end points of intervals of E and the set of end points of the intervals of G. D is a refinement of both E and G. There is a function c whose domain contains D and such that (D,c) is in $\triangle[a,b]$. It follows that $(E,t) \lesssim (D,c)$ and $(G,s) \lesssim (D,c)$. Thus $(\triangle[a,b],\lesssim)$ is a directed set.

<u>Definition 2.4.</u> If $[a,b]$ is an interval and E is a subdivision of $[a,b]$, then $\|E\|$ is the maximum element of the set to which x belongs iff for some $[p,q]$ in E, $x = q - p$.

If $[a,b]$ is an interval, then let $M[a,b]$ denote the set of ordered pairs to which ((E,t),(G,s)) belongs iff each of (E,t) and (G,s) is in $\triangle[a,b]$ and $\|E\| \geq \|G\|$. Let $(E,t) \lesssim (G,s)$ denote that ((E,t),(G,s)) is in $M[a,b]$.

<u>Lemma 2.2.</u> If $[a,b]$ is an interval, then $(\triangle[a,b], \lesssim)$ is a directed set.

<u>Proof:</u> If (E,t) is in $\triangle[a,b]$, then $\|E\| = \|E\|$. Thus $(E,t) \lesssim (E,t)$. Suppose each of (E,t), (E*,t*), and (E',t') is in $\triangle[a,b]$, and $(E,t) \lesssim (E^*,t^*) \lesssim (E',t')$. Then $\|E\| \geq \|E^*\|$ and $\|E^*\| \geq \|E'\|$. It follows that $\|E\| \geq \|E'\|$ and that $(E,t) \lesssim (E',t')$. Suppose that each of (E,t) and (G,s) is in $\triangle[a,b]$. Let D be the common refinement of E and G as was obtained in Lemma 2.1. $\|E\| \geq \|D\|$ and $\|G\| \geq \|D\|$. Furthermore, there is a function c whose domain contains D and such that (D,c) is in $\triangle[a,b]$. Therefore $(E,t) \lesssim (D,c)$ and $(G,s) \lesssim (D,c)$. It follows that $(\triangle[a,b], \lesssim)$ is a directed set.

Nets

<u>Definition 2.5.</u> The statement that the ordered triple (F,M,*) is a net means that (M,*) is a directed set and F is a function whose domain contains M.

<u>Definition 2.6.</u> If (F,M,*) is a net and K is a set, then the statement that (F,M,*) is in K means that if x is in M, then F(x) is in K.

<u>Definition 2.7.</u> If (F,M,*) is a net in R^1, then the statement that (F,M,*) converges to J means that J is a number and if U is a neighborhood of J, then there is an element x of M such that if y is in M and x*y, then F(y) is in U.

<u>Theorem 2.1.</u> If (F,M,*) is a net in R^1 and (F,M,*) converges to J and to K, then J = K.

<u>Proof:</u> Suppose that $J \neq K$. There is an x in M such that if y is in M and x*y, then $|F(y) - J| < (1/3)\,|J - K|$. There is a z in M such that if w is in M, and z*w, then $|F(w) - K| < (1/3)\,|J - K|$. Furthermore there is an x' in M such that x*x' and z*x'. Thus $|F(x') - J| < (1/3)\,|J - K|$ and $|F(x') - K| < (1/3)\,|J - K|$. It follows that

$$
\begin{aligned}
&|J - K| \\
&\leq |J - F(x')| + |F(x') - K| \\
&< (2/3)\,|J - K|.
\end{aligned}
$$

This implies that $1 < (2/3)$, a contradiction. Thus the only alternative is that J = K.

<u>Definition 2.8.</u> Suppose K is a set and R is a relation such that the union of the domain of R and the range of R contains K. If F is a function whose domain contains K and whose range is a subset of R^1, then the statement that F is non-decreasing with respect to R on K means that if each of x and y is in K and (x,y) is in R, then $F(x) \leq F(y)$. If

x*y denotes that (x,y) is in R, then F will be said to be non-decreasing with respect to * on K.

<u>Definition 2.9.</u> If (F,M,*) is a net in R^1, then the statement that (F,M,*) is a non-decreasing net means that F is non-decreasing with respect to * on M.

<u>Definition 2.10.</u> If M is a set and F is a function whose domain contains M and whose range is a subset of R^1, then the statement that F is bounded above (below) on M means that the set to which y belongs iff for some x in M, y = F(x), is bounded above (below). The statement that F is bounded on M means that the above set is bounded.

<u>Theorem 2.2.</u> If (F,M,*) is a non-decreasing net in R^1, then (F,M,*) converges iff F is bounded above on M.

<u>Proof:</u> Let U be the set to which y belongs iff for some x in M, y = F(x), and suppose that F is bounded above on M. Let b denote the l. u. b. of U. If c is a positive number, then there is an element y of U such that

$$\begin{aligned} &b \\ &\geq y \\ &> b-c. \end{aligned}$$

Furthermore, if x' is an element of M such that y = F(x'), and x'' is an element of M such that x'*x'', then

$$\begin{aligned} &b - c \\ &< F(x') \\ &\leq F(x'') \\ &< b. \end{aligned}$$

Thus (F,M,*) converges to b.

Conversely, suppose that $(F,M,*)$ converges to a number J. If F is not bounded above on M, then there is an element y of U such that $J < y$. Thus there is a positive number c such that $J + c = y$. There is an element x of M such that $y = F(x)$. There is an element z of M such that if w is in M and $z*w$, then $F(w)$ is in $(J - c, J + c)$. There is an element of M, w', such that $x*w'$ and $z*w'$. But then

$$F(w')$$
$$< J + c$$
$$= F(x)$$
$$\leq F(w'),$$

a contradiction. Thus U is bounded above by J. It follows that F is bounded above on M.

Consider the proof of Theorem 2.2. If there is an element x of M such that if y is in M, then $x*y$, then "bounded above" can be replaced by "bounded".

<u>Lemma 2.3.</u> If $(F,N,\leq)$ is a net in R^1 and for each positive number c there is an n in N such that if k and j are in N, $n \leq k$, and $n \leq j$, then $|F(k) - F(j)| < c$, then $(F,N,\leq)$ converges.

<u>Proof:</u> There is an element n of N such that if k and j are in N, $n \leq k$, and $n \leq j$, then $|F(k) - F(j)| < 1$. Thus if m is in N, and $n \leq m$, then

$$|F(m)| - |F(n)|$$
$$\leq |F(m) - F(n)|$$
$$< 1.$$

It follows that $|F(m)| \leq 1 + |F(n)|$. Let M be the set such that x is in M iff for some i in N, $1 \leq i < n$, $x = F(i)$, or $x = 1 + |F(n)|$. Let L denote the maximum element of M. If i is in N, then $|F(i)| \leq L$. Thus F is bounded on M.

Suppose the range of F is finite. There is a y in the range of F such that if n is in N, then there is an m in N such that $n < m$ and $F(m) = y$. If c is a positive number, then there is a K in N such that if k and j are in N, $K \leq k$, and $K \leq j$, then $|F(k) - F(j)| < c$. Let K' denote an element of N such that $K \leq K'$ and $F(K') = y$. If i is in N, and $K' \leq i$, then

$$|F(K') - F(i)|$$
$$= |y - F(i)|$$
$$< c.$$

Thus $(F, N, \leq)$ converges to y.

Let M be the set to which y belongs iff for some n in N, $y = F(n)$. Suppose M is infinite. Let U be the set to which x belongs iff there is an infinite subset, U_x, of M such that if y is in U_x, then $x < y$. $-L$ is in U, thus U is not empty. U is bounded above by L. Let p denote the l. u. b. of U. Suppose d is a positive number. If p is in U, then there is an element of M in $(p, p + d)$. If not, then there is an infinite subset, U', of M such that if y is in U', then $p + (d/2) < y$, a contradiction. If p is not in U, then there is an element of M in $(p - d, p)$. If not, then there is no infinite subset, U', of M such that if y is in U', then $p - (d/2) < y$. Thus $p - (d/2)$ is an upper bound of U, a contradiction.

Thus if c is a positive number, and if n is in N, then there is an m in N such that $n \leq m$, F(m) is in the segment $(p - (c/2), p + (c/2))$, and $F(m) \neq p$. There is an element K of N such that if j and k are in N, $K \leq j$, and $K \leq k$, then $|F(k) - F(j)| < (c/2)$. There is a K' in N such that $K \leq K'$ and

$$\begin{aligned} &0 \\ &< |F(K') - p| \\ &< (c/2). \end{aligned}$$

If i is in N and $K' \leq i$, then $|F(i) - F(K')| < (c/2)$. Thus

$$\begin{aligned} &|F(i) - p| \\ &\leq |F(K') - p| + |F(i) - F(K')| \\ &< c. \end{aligned}$$

It follows that $(F, N, \leq)$ converges to p.

<u>Theorem 2.3.</u> If $(F, M, *)$ is a net in R^1, then $(F, M, *)$ converges iff for each positive number c there is an x in M such that if y is in M and $x * y$, then $|F(x) - F(y)| < c$.

<u>Proof:</u> Let f be a function whose domain is N and whose range is a subset of M such that f(1) is an element in M with the property that if y is in M and $f(1) * y$, then $|F(f(1)) - F(y)| < (1/2)$, and such that if n is in N and $1 < n$, then f(n) is in M, $f(n - 1) * f(n)$, and if y is in M and $f(n) * y$, then $|F(f(n)) - F(y)| < (1/2n)$. Let $(G, N, \leq)$ be the net such that if m is in N, then $G(m) = F(f(m))$.

Suppose d is a positive number. There is an element K of N such that $(1/K) < d$. If each of j and k is an element of N, $K < j$, and $K < k$, then $|F(f(K)) - F(f(k))| < (1/2K)$, and $|F(f(K)) - F(f(j))| < (1/2K)$. This implies that

$$\begin{aligned} &|F(f(k)) - F(f(j))| \\ &= |G(k) - G(j)| \\ &< (1/K) \\ &< d. \end{aligned}$$

Thus $(G,N,\leq)$ converges. Let J denote the number to which $(G,N,\leq)$ converges.

Suppose c is a positive number. There is a K in N such that $(1/K) < c$ and such that $|F(f(K)) - J| < (1/2)c$. If y is in M and $f(K)*y$, then

$$\begin{aligned} &|F(f(K)) - F(y)| \\ &< (1/2K) \\ &< (1/2)c. \end{aligned}$$

Thus it follows that $|F(y) - J| < c$, and that $(F,M,*)$ converges to J.

Conversely, if $(F,M,*)$ converges to J, then there is an element x of M such that if y is in M and $x*y$, then $|F(y) - J| < (1/2)c$. Thus

$$\begin{aligned} &|F(x) - F(y)| \\ &\leq |F(x) - J| + |J - F(y)| \\ &< c. \end{aligned}$$

CHAPTER III

AN EXAMPLE OF A NON-DECREASING NET

Suppose $[a,b]$ is an interval. Then $X([a,b])$ is the set to which g belongs iff g is a function whose domain contains $[a,b]$ and whose range is a subset of R^1.

If f is in $X([a,b])$, then let B_f denote the function whose domain contains $\triangle[a,b]$ and whose range is a subset of R^1 such that if (E,t) is in $\triangle[a,b]$, then

$$B_f(E,t) = \sum_{[p,q]\in E}|f(q) - f(p)| = \sum_E|\triangle f|.$$

Note that if each of (E,t) and (E,t') is in $\triangle[a,b]$, then $B_f(E,t) = B_f(E,t')$.

If $[a,b]$ is an interval, E is a subdivision of $[a,b]$, and each of r and s is a number such that $r < s$ and for some $[u,v]$ in E, $r = u$, and for some $[p,q]$ in E, $s = q$, then $E[r,s]$ is the subset of E which is a subdivision of $[r,s]$.

<u>Theorem 3.1.</u> If $[a,b]$ is an interval and f is in $X([a,b])$, then $(B_f, \triangle[a,b], \lesssim)$ is a non-decreasing net.

<u>Proof.</u> Suppose each of (E,t) and (E^*,t^*) is in $\triangle[a,b]$. Suppose $(E,t) \lesssim (E^*,t^*)$, $[p,q]$ is in E, and that F is a refinement of $E[p,q]$. If F has one more element than $E[p,q]$, then there is an x in (p,q) such that $[p,x]$ and $[x,q]$ are the

only elements of F, and

$$\sum_F |f(s) - f(r)|$$
$$= |f(x) - f(p)| + |f(q) - f(x)|$$
$$\geq |f(q) - f(p)|$$
$$= \sum_{E[p,q]} |f(s) - f(r)|.$$

Now, suppose that if F has k more intervals than $E[p,q]$, then $\sum_E |f(s) - f(r)| \leq \sum_F |f(s) - f(r)|$. If G is a refinement of $E[p,q]$ with k + 1 more intervals than $E[p,q]$, then there is an x and a y in (p,q) such that $[x,q]$ and $[y,x]$ are in $E[p,q]$. Let $G^* = (G - \{[y,x], [x,q]\}) \cup \{[y,q]\}$. Thus

$$\sum_G |f(s) - f(r)|$$
$$= (\sum_{G^*} |f(s) - f(r)|) - |f(q) - f(y)| + |f(x) - f(y)|$$
$$+ |f(q) - f(x)|$$
$$\geq (\sum_{G^*} |f(s) - f(r)|) - |f(q) - f(y)| + |f(q) - f(y)|$$
$$= \sum_{G^*} |f(s) - f(r)|$$
$$\geq \sum_{E[p,q]} |f(s) - f(r)|.$$

It follows from the above argument that $\sum_F |f(s) - f(r)|$ $\sum_{E[p,q]} |\Delta f|$. Let $E^*[p,q]$ be the subset of E^* which is a refinement of $E[p,q]$. $\sum_{E^*[p,q]} |\Delta f| \geq \sum_{E[p,q]} |\Delta f|$, thus it follows that

$$\sum_{[p,q] \in E} \sum_{E[p,q]} |\Delta f|$$
$$= \sum_E |\Delta f|$$
$$\leq \sum_{E^*} |\Delta f|$$
$$= \sum_{[p,q] \in E} \sum_{E^*[p,q]} |\Delta f|.$$

Thus $B_f(E,t) \leq B_f(E^*,t^*)$, and it follows that $(B_f, \Delta[a,b], \lesssim)$ is a non-decreasing net.

Theorem 3.2. If $[a,b]$ is an interval and f is in $X([a,b])$, then $(B_f, \Delta[a,b], \lesssim)$ converges iff B_f is bounded on $\Delta[a,b]$. Furthermore, if B_f is bounded, then $(B_f, \Delta[a,b], \lesssim)$ converges to the number J iff J is the l. u. b. of the set to which y belongs iff for some (E,t) in $\Delta[a,b]$, $y = B_f(E,t)$.

Proof: The first part is a direct application of Theorem 2.2. It was shown in the proof of Theorem 2.2 that $(B_f, \Delta[a,b], \lesssim)$ converges to the l. u. b. of the set to which y belongs iff for some (E,t) in $\Delta[a,b]$, $y = B_f(E,t)$.

Suppose $(B_f, \Delta[a,b], \lesssim)$ converges to J. If J' is the l. u. b. of the set to which y belongs iff for some (E,t) in $\Delta[a,b]$, $y = B_f(E,t)$, then $(B_f, \Delta[a,b], \lesssim)$ converges to J' by the above argument. Thus by Theorem 2.1, J = J'.

Definition 3.1. Suppose $[a,b]$ is an interval and F is a function such that if $[r,s]$ is a subinterval of $[a,b]$, then the domain of F contains $\Delta[r,s]$. Then the statement that F is additive on $\Delta[a,b]$ means that if (E,t) is in $\Delta[a,b]$ and E is a refinement of a subdivision G of $[a,b]$, then

$$F(E,t) = \sum_G F(E[p,q],t)$$

Theorem 3.3. Suppose $[a,b]$ is an interval and F is a function such that if $[r,s]$ is a subinterval of $[a,b]$, then the domain of F contains $\Delta[r,s]$. Then if F is additive on $\Delta[a,b]$, $[u,v]$ is a subset of $[a,b]$, and $(F, \Delta[a,b], \lesssim)$ converges, then $(F, \Delta[u,v], \lesssim)$ converges.

Proof: Suppose c is a positive number and $[u,v]$ is a subset of $[a,b]$. There is an (E,t) in $\triangle[a,b]$ such that E contains a subset which is a subdivision of $[u,v]$ and such that if (G,s) is in $\triangle[a,b]$ and $(E,t) \lesssim (G,s)$, then $|F(E,t) - F(G,s)| < c$. Suppose (E',t') is in $\triangle[u,v]$ and $(E[u,v],t) \lesssim (E',t')$. Let $E^* = (E - E[u,v]) \cup E'$. Let t^* be a function whose domain contains E^* such that if I is in E', then $t^*(I) = t'(I)$, and if I is in $E - E[u,v]$, then $t^*(I) = t(I)$. It follows that (E^*,t^*) is in $\triangle[a,b]$ and $(E,t) \lesssim (E^*,t^*)$. Thus

$$
\begin{aligned}
&|F(E,t) - F(E^*,t^*)| \\
&= |F(E[a,u] \cup E[u,v] \cup E[v,b], t) - F(E[a,u] \cup E' \cup E[v,b], t^*)| \\
&= |F(E[u,v], t) - F(E',t')| \\
&< c.
\end{aligned}
$$

It follows that $(F, \triangle[u,v], \lesssim)$ converges.

Corollary 3.3. If $[a,b]$ is an interval, f is in $X([a,b])$, $[u,v]$ is a subset of $[a,b]$, and $(B_f, \triangle[a,b], \lesssim)$ converges, then $(B_f, \triangle[u,v], \lesssim)$ converges.

Proof: If $[r,s]$ is a subinterval of $[a,b]$, then the domain of B_f contains $\triangle[r,s]$. Furthermore, it is obvious that B_f is additive on $\triangle[a,b]$. Thus it follows immediately from Theorem 3.3 that $(B_f, \triangle[u,v], \lesssim)$ converges.

Theorem 3.4. If $[a,b]$ is an interval, F is a function such that if $[r,s]$ is a subinterval of $[a,b]$ then the domain of F contains $\triangle[r,s]$, F is additive on $\triangle[a,b]$, $a \leq u < v < w \leq b$, $(F, \triangle[u,v], \lesssim)$ converges to $J([u,v])$, and $(F, \triangle[v,w], \lesssim)$

converges to $J([v,w])$, then $(F, \Delta[u,w], \lesssim)$ converges to $J([u,v]) + J([v,w])$.

<u>Proof:</u> If $a \leq u < v < w \leq b$ and c is a positive number, then there is an (E,t) in $\Delta[u,v]$ such that if (E',t') is in $\Delta[u,v]$, and $(E,t) \lesssim (E',t')$, then $|J([u,v]) - F(E',t')| < (c/2)$. Also there is an (G,s) in $\Delta[v,w]$ such that if (G',s') is in $\Delta[v,w]$ and $(G,s) \lesssim (G',s')$, then $|J([v,w]) - F(G',s')| < (c/2)$. If $E^* = E \cup G$ and if t^* is a function whose domain contains E^* such that if I is in E, then $t^*(I) = t(I)$, and if I is in G, then $t^*(I) = s(I)$, then (E^*,t^*) is in $\Delta u,w$. If (H,h) is in $\Delta[u,w]$ and $(E^*,t^*) \lesssim (H,h)$, then

$$|J([u,v]) + J([v,w]) - F(H,h)|$$
$$= |J([u,v]) + J([v,w]) - F(H[u,v] \cup H[v,w], h)|$$
$$\leq |J([u,v]) - F(H[u,v],h)| + |J([v,w]) - F(H[v,w],h)|.$$

But $(E,t) \lesssim (H[u,v],h)$ and $(G,s) \lesssim (H[v,w],h)$, thus $|J([u,v]) + J([v,w]) - F(H,h)| < c$. It follows that $(F, \Delta[u,v], \lesssim)$ converges to $J([u,v]) + J([v,w])$.

<u>Corollary 3.4.</u> If $[a,b]$ is an interval, F is a function such that if $[r,s]$ is a subinterval of $[a,b]$, then the domain of F contains $\Delta[r,s]$, F is additive on $\Delta[a,b]$, E is a subdivision of $[a,b]$ such that if $[p,q]$ is in E, then $(F, \Delta[p,q], \lesssim)$ converges to $J([p,q])$, then $(F, \Delta[a,b], \lesssim)$ converges to $\sum_E J([p,q])$.

<u>Proof:</u> Let S be the set to which n belongs iff n is in N and if G is a subdivision of $[a,b]$ containing n elements

such that if $[p,q]$ is in G, then $(F,\Delta[p,q],\lesssim)$ converges to $J([p,q])$, then $(F,\Delta[a,b],\lesssim)$ converges to $\sum_G J([p,q])$.

Certainly 1 is in S. Suppose G is a subdivision of $[a,b]$ containing only two elements, $[a,x]$ and $[x,b]$, such that $(F,\Delta[a,x],\lesssim)$ converges to $J([a,x])$ and $(F,\Delta[x,b],\lesssim)$ converges to $J([x,b])$. It follows from Theorem 3.4 that $(F,\Delta[a,b],\lesssim)$ converges to $J([a,x]) + J([x,b])$. Thus 2 is in S.

Suppose k is in S and that G is a subdivision of $[a,b]$ containing $k + 1$ elements such that if $[p,q]$ is in G, then $(F,\Delta[p,q],\lesssim)$ converges to $J([p,q])$. There are two numbers x and y, each in $(a,b]$, such that each of $[a,x]$ and $[x,y]$ is in G. $(F,\Delta[a,x],\lesssim)$ converges to $J([a,x])$ and $(F,\Delta[x,y],\lesssim)$ converges to $J([x,y])$, thus by Theorem 3.4, $(F,\Delta[a,y],\lesssim)$ converges to $J([a,x]) + J([x,y])$. Let $J([a,y]) = J([a,x]) + J([x,y])$. If $G^* = (G - \{[a,x],[x,y]\}) \cup \{[a,y]\}$, then G^* is a subdivision of $[a,b]$ containing k elements such that if $[p,q]$ is in G^*, then $(F,\Delta[p,q],\lesssim)$ converges to $J([p,q])$. Thus $(F,\Delta[a,b],\lesssim)$ converges to $\sum_{G^*} J([p,q])$. But

$$\begin{aligned}
&\textstyle\sum_{G^*} J([p,q]) \\
&= (\textstyle\sum_G J([p,q])) - J([a,x]) - J([x,y]) + J([a,y]) \\
&= \textstyle\sum_G J([p,q]).
\end{aligned}$$

Therefore $(F,\Delta[a,b],\lesssim)$ converges to $\sum_G J([p,q])$. Thus $k + 1$ is in S. It follows that $S = N$ and that the theorem is true.

<u>Theorem 3.5.</u> If $[a,b]$ is an interval and f is in $X([a,b])$, then $(B_f,\Delta[a,b],\lesssim)$ converges iff there are functions

f_1 and f_2 each in $X([a,b])$ such that $f = f_1 - f_2$ and each of f_1 and f_2 is non-decreasing with respect to $<$ on $[a,b]$.

<u>Proof:</u> Suppose there are functions f_1 and f_2 in $X([a,b])$ such that $f = f_1 - f_2$ and such that each of f_1 and f_2 is non-decreasing with respect to $<$ on $[a,b]$.

If i is 1 or 2 and (E,t) is in $\triangle[a,b]$, then for each $[p,q]$ in E, $f_i(q) - f_i(p) \geq 0$. Therefore

$$\begin{aligned} & 0 \\ \leq\ & B_{f_i}(E,t) \\ =\ & \textstyle\sum_E |\Delta f_i| \\ =\ & \textstyle\sum_E \Delta f_i \\ =\ & f_i(b) - f_i(a). \end{aligned}$$

Thus

$$\begin{aligned} & B_f(E,t) \\ =\ & \textstyle\sum_E |\Delta f| \\ =\ & \textstyle\sum_E |f_1(q) - f_2(q) - (f_1(p) - f_2(p))| \\ \leq\ & \textstyle\sum_E |f_1(q) - f_1(p)| + \sum_E |f_2(q) - f_2(p)| \\ =\ & B_{f_1}(E,t) + B_{f_2}(E,t) \\ =\ & f_1(b) - f_1(a) + f_2(b) - f_2(a). \end{aligned}$$

Thus B_f is bounded on $\triangle[a,b]$ and it follows from Theorem 3.2 that $(B_f, \triangle[a,b], \lesssim)$ converges.

If $(B_f, \triangle[a,b], \lesssim)$ converges, then for each subinterval $[r,s]$ of $[a,b]$, $(B_f, \triangle[r,s], \lesssim)$ converges. Thus if each of x and y is in $[a,b]$ and $x < y$, then let $J([x,y])$ denote the number to which $(B_f, \triangle[x,y], \lesssim)$ converges. Let $J([x,x]) = 0$.

If z is in [a,b], let $f_1(z) = (1/2)J([a,z]) + (1/2)f(z)$. If $a \leq x < y \leq b$, then

$$f_1(y) - f_1(x)$$
$$= (1/2)(J([a,y]) - J([a,x])) + (1/2)(f(y) - f(x))$$
$$= (1/2)(J([x,y])) + (1/2)(f(y) - f(x)).$$

If $f(y) \geq f(x)$, then $f_1(y) - f_1(x) \geq 0$. If $f(y) < f(x)$, then

$$f_1(y) - f_1(x)$$
$$= (1/2)J([x,y]) - (1/2)|f(y) - f(x)|$$
$$\geq 0.$$

It follows that f_1 is non-decreasing with respect to $<$ on [a,b].

If z is in [a,b], let $f_2(z) = (1/2)J([a,z]) - (1/2)f(z)$. If $a \leq x < y \leq b$, then

$$f_2(y) - f_2(x)$$
$$= (1/2)(J([a,y]) - J([a,x])) + (1/2)(f(x) - f(y))$$
$$= (1/2)J([x,y]) + (1/2)(f(x) - f(y)).$$

If $f(x) \geq f(y)$, then $f_2(y) - f_2(x) \geq 0$. If $f(x) < f(y)$, then

$$f_2(y) - f_2(x)$$
$$= (1/2)J([x,y]) - (1/2)|f(x) - f(y)|$$
$$\geq 0.$$

It follows that f_2 is non-decreasing with respect to $<$ on [a,b]. Furthermore, $f_1 - f_2 = f$.

<u>Definition 3.2.</u> Suppose [a,b] is an interval and f is in X([a,b]). If x is in (a,b] and (f, [a,x), $\leq$) converges,

then let $f(x-)$ denote the number to which $(f,[a,b),\leq)$ converges. If x is in $[a,b)$ and $(f,(x,b],\geq)$ converges, then let $f(x+)$ denote the number to which $(f,(x,b],\geq)$ converges. Furthermore, let $f(a-) = f(a)$ and $f(b+) = f(b)$.

The statement that f is in $X_1([a,b])$ means that if x is in $[a,b]$, then both $f(x+)$ and $f(x-)$ exist.

<u>Lemma 3.1.</u> If $[a,b]$ is an interval, f is in $X([a,b])$, and f is non-decreasing with respect to $<$ on $[a,b]$, then f is in $X_1([a,b])$.

<u>Proof:</u> Suppose $a < x \leq b$. Let M be the set such that z is in M iff for some x' such that $a \leq x' < x$, $z = f(x')$. M is bounded above by $f(x)$. Let m denote the l. u. b. of M.

Suppose c is a positive number. If m is in M, then there is a y such that $a \leq y < x$ and $m = f(y)$. If $y < w < x$, then $f(y) = f(w) = m$. It follows that $m - f(w) = 0 < c$. Thus $f(x-)$ exists and equals m. Suppose m is not in M. If there is no y such that $a \leq y < x$ and $m - f(y) < c$, then $m - c$ is an upper bound of M. Furthermore, $m - c < m$, a contradiction. Thus there is a y such that $a \leq y < x$ and $0 \leq m - f(y) < c$. If $a \leq y < w < x$, then $f(y) \leq f(w)$, and then $-f(w) \leq -f(y)$. It follows that $m - f(w) \leq m - f(y) < c$. Thus $f(x-)$ exists and is m. A similar argument will show that $f(x+)$ exists if $a \leq x < b$. Also $f(a-) = f(a)$ and $f(b+) = f(b)$. Thus f is in $X_1([a,b])$.

Theorem 3.6. If $[a,b]$ is an interval and f is in $X([a,b])$, and $(B_f, \triangle[a,b], \leq)$ converges, then f is in $X_1([a,b])$.

Proof: The proof follows immediately from the results of Theorem 3.5 and Lemma 3.1.

CHAPTER IV

A COMPARISON OF FOUR SIMILAR NETS

Suppose that each of f and g is in $X([a,b])$. $R_{f,g}$ will denote the function whose domain contains $\Delta[a,b]$ and whose range is a subset of R^1 such that if (E,t) is in $\Delta[a,b]$, then $R_{f,g}(E,t) = \sum_E f(t([p,q]))(g(q) - g(p))$.

The proper choice of g will insure the convergence of the net $(R_{f,g}, \Delta[a,b], \lesssim)$. For example, if $[a,b]$ is an interval, f is in $X([a,b])$, and g is an element of $X([a,b])$ such that if x is in $[a,b]$, then $g(x) = d$, where d is in R^1, then $\sum_E f(t(I)) \Delta g = 0$ for all (E,t) in $\Delta[a,b]$. Thus if c is a positive number, $E = \{[a,b]\}$, $t([a,b]) = a$, (E',t') is in $\Delta[a,b]$, and $(E,t) \lesssim (E',t')$, then $|R_{f,g}(E',t') - 0| < c$. Thus $(R_{f,g}, \Delta[a,b], \lesssim)$ converges to 0.

<u>Example 4.1.</u> Suppose f is an element of $X([0,2])$ such that if x is in $[0,(1/2))$, then $f(x) = 1$, and if x is in $[(1/2),2]$, then $f(x) = 2$; and suppose g is in $X([0,2])$ such that if x is in $[0,1)$, then $g(x) = 0$, and if x is in $[1,2]$, then $g(x) = 1$. Now suppose that (E,t) is in $\Delta[0,2]$ such that $E = \{[0,(1/2)], [(1/2),1], [1,(3/2)], [(3/2),2]\}$ and if $[p,q]$ is in E, then $t([p,q]) = p$. Thus

$$\begin{aligned} &R_{f,g}(E,t) \\ &= \textstyle\sum_E f(t(I)) \Delta g \end{aligned}$$

$= f(t([(1/2),1]))(g(1) - g(1/2))$

$= 2.$

If (E',t') is in $\triangle[0,2]$ and $(E,t) \lesssim (E',t')$, then there is a $[p,q]$ in E' such that $(1/2) \leq p < 1 \leq q$. Thus

$$
\begin{aligned}
& R_{f,g}(E',t') \\
&= \textstyle\sum_{E'} f(t'(I))\Delta g \\
&= f(t'([p,q]))(g(q) - g(p)) \\
&= 2.
\end{aligned}
$$

Thus if c is a positive number, then

$$
\begin{aligned}
& 0 \\
&= |R_{f,g}(E,t) - R_{f,g}(E',t')| \\
&< c.
\end{aligned}
$$

It follows from Theorem 2.3 that $(R_{f,g}, \triangle[a,b], \lesssim)$ converges.

<u>Definition 4.1.</u> If $[a,b]$ is an interval and f is in $X([a,b])$, then $C_f([a,b])$ is the set to which x belongs iff x is in $[a,b]$, $f(x+)$ exists, $f(x-)$ exists, and each of $f(x+)$ and $f(x-)$ is equal to $f(x)$. Denote $[a,b] - C_f([a,b])$ by $D_f([a,b])$.

<u>Theorem 4.1.</u> If $[a,b]$ is an interval, each of f and g is in $X([a,b])$, and $(R_{f,g}, \triangle[a,b], \lesssim)$ converges, then $D_f([a,b])$ and $D_g([a,b])$ have no point in common.

<u>Proof:</u> Suppose x is in $D_g([a,b])$. If $g(x+)$ exists, $x \neq b$, and $g(x+) = g(x)$, then there is a positive number c' such that if d is a positive number, then there is a y in $(x - d,x) \cap [a,b]$ such that $|g(y) - g(x)| > c'$. Let T_d be the set of all such y. Furthermore there is a positive number

d' such that if z is in $(x, x + d')$, then $|g(z) - g(x)| < c'/2$. Thus if y is in $T_{d'}$, then

$$
\begin{aligned}
&|g(y) - g(z)| \\
&\geq |g(y) - g(x)| - |g(x) - g(z)| \\
&> c'/2.
\end{aligned}
$$

If c is a positive number, then there is an (E,t) in $\triangle[a,b]$ such that (1) if $[p,q]$ is in E, then p is not x and q is not x, (2) if $[r,s]$ is in E and x is in $[r,s]$, then $s - r < d'$ and r is in $T_{d'}$, and (3) if (E',t') is in $\triangle[a,b]$ and $(E,t) \lessapprox (E',t')$, then $|R_{f,g}(E,t) - R_{f,g}(E',t')| < cc'/4$. Suppose (E,t_1) is in $\triangle[a,b]$ such that if I is in $E - \{[r,s]\}$, then $t_1(I) = t(I)$, and $t_1([r,s]) = x$. If y is in (r,s), then there is an (E,t_2) in $\triangle[a,b]$ such that if I is in $E - \{[r,s]\}$, $t_2(I) = t(I)$, and $t([r,s]) = y$. $(E,t) \lessapprox (E,t_1)$ and $(E,t) \lessapprox (E,t_2)$, thus it follows that $|R_{f,g}(E,t) - R_{f,g}(E,t_1)| < cc'/4$ and that $|R_{f,g}(E,t) - R_{f,g}(E,t_2)| < cc'/4$. Thus

$$
\begin{aligned}
&|R_{f,g}(E,t_1) - R_{f,g}(E,t_2)| \\
&= |(f(x) - f(y))(g(s) - g(r))| \\
&= |f(x) - f(y)||g(s) - g(r)| \\
&< cc'/2.
\end{aligned}
$$

But

$$
\begin{aligned}
&0 \\
&< c'/2 \\
&< |g(s) - g(r)|,
\end{aligned}
$$

therefore

$$
0
$$

$< 1/|g(s) - g(r)|$

$< 2/c'$.

Thus

$$(|f(x) - f(y)||g(s) - g(r)|)/|g(s) - g(r)|$$

$$= |f(x) - f(y)|$$

$$< (cc'/2)(2/c')$$

$$= c.$$

It follows that x is not in $D_f([a,b])$.

If g(x-) exists, $x \neq a$, and g(x-) = g(x), then a similar argument will show that x is not in $D_f([a,b])$.

If neither of the above cases is true and x is in (a,b), then there is a positive number c' such that if d is a positive number, then there is a y in $(x - d,x) \cap [a,b]$ and a z in $(x,x + d) \cap [a,b]$ such that $|g(y) - g(x)| > c'$ and $|g(z) - g(x)| > c'$. Let T_d denote the set of all such y, and S_d denote the set of all such z. If c is a positive number, then there is an (E,t) in $\Delta[a,b]$ such that (1) for some [p,q] in E, x is p or x is q, (2) for some positive number e if [r,x] is in E, then r is in T_e, and if [x,s] is in E, then s is in S_e, and (3) if (E',t') is in $\Delta[a,b]$ and if $(E,t) \lesseqgtr (E',t')$, then $|R_{f,g}(E,t) - R_{f,g}(E',t')| < cc'/2$. If w is in (r,x], then there is an (E,t_1) in $\Delta[a,b]$ such that if I is in $E - \{[r,x]\}$, then $t_1(I) = t(I)$, and $t_1([r,x]) = w$. There is also an (E,t_2) in $\Delta[a,b]$ such that if I is in $E - \{[r,x]\}$, then $t_2(I) = t(I)$ and $t_2([r,x]) = x$. Thus it follows that $|R_{f,g}(E,t) - R_{f,g}(E,t_1)| < cc'/2$ and that

$|R_{f,g}(E,t) - R_{f,g}(E,t_2)| < cc'/2$. This implies that

$$
\begin{aligned}
&|R_{f,g}(E,t_2) - R_{f,g}(E,t_1)| \\
&= |f(x) - f(w)||g(x) - g(r)| \\
&< cc'.
\end{aligned}
$$

But

$$
\begin{aligned}
&0 \\
&< c' \\
&< |g(x) - g(r)|,
\end{aligned}
$$

thus

$$
\begin{aligned}
&(|f(x) - f(w)||g(x) - g(r)|)/|g(x) - g(r)| \\
&= |f(x) - f(w)| \\
&< cc'/c' \\
&= c.
\end{aligned}
$$

If w is in $[x,s)$ a similar argument will show that x is not in $D_f([a,b])$. Also a similar argument will hold for $x = a$ and $x = b$.

Thus $D_f([a,b])$ and $D_g([a,b])$ have no point in common.

Example 4.2. Let f be the element of $X([0,2])$ such that if x is in $[0,1)$, then $f(x) = 1$, and if x is in $[1,2]$, then $f(x) = 2$. Let g be the element of $X([0,2])$ such that if x is in $[0,1]$, $g(x) = 0$, and if x is in $(1,2]$, then $g(x) = 1$. It follows from Theorem 4.1 that $(R_{f,g}, \triangle[0,2], \lesssim)$ does not converge.

If $[a,b]$ is an interval and each of f and g is in $X([a,b])$, then it is obvious that $R_{f,g}$ is additive on $\triangle[a,b]$.

<u>Example 4.3.</u> Let each of f and g be defined as in Example 4.2. Consider the net $(R_{f,g}, \triangle[0,2], \lesssim)$. If (E,t) is in $\triangle[0,1]$, then $R_{f,g}(E,t) = \sum_E f(t(I))\,\Delta g = 0$, thus $(R_{f,g}, \triangle[0,1], \lesssim)$ converges to 0. If (E,t) is in $\triangle[1,2]$, then $R_{f,g}(E,t) = \sum_E f(t(I))\,\Delta g = 2$. Thus $(R_{f,g}, \triangle[0,1], \lesssim)$ converges to 2. It follows from Theorem 3.3 that $(R_{f,g}, \triangle[0,2], \lesssim)$ converges to 2.

<u>Theorem 4.2.</u> If $[a,b]$ is an interval, each of f and g is in $X([a,b])$, and $(R_{f,g}, \triangle[a,b], \lessapprox)$ converges to J, then $(R_{f,g}, \triangle[a,b], \lesssim)$ converges to J.

<u>Proof:</u> If U is a neighborhood of J, then there is an (E,t) in $\triangle[a,b]$ such that if (E',t') is in $\triangle[a,b]$ and $(E,t) \lessapprox (E',t')$, then $R_{f,g}(E',t')$ is in U. If G is a subdivision of $[a,b]$ and G' is a refinement of G, then $\|G'\| \leq \|G\|$. Thus if (G,s) is in $\triangle[a,b]$ and (G',s') is in $\triangle[a,b]$ such that $(G,s) \lesssim (G',s')$, then $(G,s) \lessapprox (G',s')$. It follows that if (E^*,t^*) is in $\triangle[a,b]$ such that $(E,t) \lesssim (E^*,t^*)$, then $(E,t) \lessapprox (E^*,t^*)$ and thus $R_{f,g}(E^*,t^*)$ is in U. Thus $(R_{f,g}, \triangle[a,b], \lesssim)$ converges to J.

The results of Example 4.2 and Example 4.3 show that the converse of Theorem 4.2 is not true.

In Theorem 4.2, $R_{f,g}$ could have been replaced with any function whose domain contains $\triangle[a,b]$ and whose range is a subset of R^1.

<u>Example 4.4.</u> Let f be the element of $X([0,2])$ such that if x is in $[0,1]$, then $f(x) = 1$, and if x is in $(1,2]$,

then $f(x) = 2$. Let g be the element of $X([0,2])$ such that if x is in $[0,1]$, $g(x) = 0$, and if x is in $(1,2]$, then $g(x) = 1$.

Suppose there is an (E,t) in $\triangle[0,2]$ such that if (E',t') is in $\triangle[0,2]$ and $(E,t) \lesssim (E',t')$, then

$$|R_{f,g}(E,t) - R_{f,g}(E',t')| < 1/2.$$

There is an (G,s) in $\triangle[0,2]$ such that for some $[p,q]$ in G, p is 1, such that if $[1,q]$ is in G, then $s([1,q]) = 1$, and such that $(E,t) \lesssim (G,s)$. Also there is an (G',s') in $\triangle[0,2]$ such that for some $[p,q]$ in G', p is 1, such that if $[1,q]$ is in G', then $s'([1,q]) = q$, and such that $(E,t) \lesssim (G',s')$. It follows that

$$|R_{f,g}(E,t) - R_{f,g}(G,s)| < 1/2$$

and

$$|R_{f,g}(E,t) - R_{f,g}(G',s')| < 1/2.$$

Thus

$$\begin{aligned} &1 \\ &> |R_{f,g}(G,s) - R_{f,g}(G',s')| \\ &= |1 - 2| \\ &= 1, \end{aligned}$$

a contradiction. Thus it follows that $(R_{f,g}, \triangle[0,2], \lesssim)$ does not converge.

Suppose $[a,b]$ is an interval, and each of f and g is in $X([a,b])$. $S_{f,g}$ will denote the function whose domain contains $\triangle[a,b]$ and whose range is a subset of R^1 such that if (E,t) is in $\triangle[a,b]$, then

$$S_{f,g}(E,t)$$
$$= \sum_E (1/2)(f(q) + f(p))(g(q) - g(p))$$
$$= \sum_E (1/2)(f(q) + f(p))\Delta g.$$

Example 4.5. Let f and g be the functions of Example 4.4. If (E,t) is in $\triangle[0,2]$, then

$$S_{f,g}(E,t)$$
$$= \sum_E (1/2)(f(q) + f(p))\Delta g$$
$$= (1/2)(1+2)(1-0)$$
$$= 3/2.$$

It follows that $(S_{f,g}, \triangle[0,2], \lesssim)$ converges to 3/2.

If $[a,b]$ is an interval and each of f and g is in $X([a,b])$, then the convergence of $(S_{f,g}, \triangle[a,b], \lesssim)$ does not necessarily imply that $(R_{f,g}, \triangle[a,b], \lesssim)$ converges as is shown by Example 4.4 and Example 4.5. Likewise, the convergence of $(R_{f,g}, \triangle[a,b], \lesssim)$ does not imply that $(S_{f,g}, \triangle[a,b], \lesssim)$ converges. Consider the following example.

Example 4.6. Let each of f and g be defined as in Example 4.2. It was shown in Example 4.3 that $(R_{f,g}, \triangle[0,2], \lesssim)$ converges. Now suppose there is an (E,t) in $\triangle[0,2]$ such that if (E',t') is in $\triangle[0,2]$ and $(E,t) \lesssim (E',t')$, then

$$|S_{f,g}(E,t) - S_{f,g}(E',t')|$$
$$< 1/4.$$

There are elements (G,s) and (G',s') of $\triangle[0,2]$ such that $[1,q]$ is in G for some q, $1 < q \leq 2$, such that for some $[r,s]$ in

G', $r < 1 < s$, and such that $(E,t) \lesssim (G,s)$ and $(E,t) \lesssim (G',s')$. Thus

$$|S_{f,g}(G,s) - S_{f,g}(G',s')|$$
$$= 1/2$$
$$\leq |S_{f,g}(E,t) - S_{f,g}(G,s)| + |S_{f,g}(G',s') - S_{f,g}(E,t)|$$
$$< 1/2,$$

a contradiction. It follows that $(S_{f,g}, \triangle[0,2], \lesssim)$ does not converge.

But it is true that the convergence of $(R_{f,g}, \triangle[a,b], \lesssim)$ implies that $(S_{f,g}, \triangle[a,b], \lesssim)$ converges.

<u>Theorem 4.3.</u> If $[a,b]$ is an interval and each of f and g is in $X([a,b])$, and $(R_{f,g}, \triangle[a,b], \lesssim)$ converges to J, then $(S_{f,g}, \triangle[a,b], \lesssim)$ converges to J.

<u>Proof:</u> Suppose c is a positive number. There is an (E,t) in $\triangle[a,b]$ such that if (E',t') is in $\triangle[a,b]$ and $(E,t) \lesssim (E',t')$, then $|R_{f,g}(E',t') - J| < c$. Thus if (G,m) is in $\triangle[a,b]$ such that $(E,t) \lesssim (G,m)$, and for each $[p,q]$ in G, $s([p,q]) = p$ and $s'([p,q]) = q$, then each of (G,s) and (G,s') is in $\triangle[a,b]$, $(E,t) \lesssim (G,s)$ and $(E,t) \lesssim (G,s')$. Thus

$$|R_{f,g}(G,s) + R_{f,g}(G,s') - 2J|$$
$$\leq |R_{f,g}(G,s) - J| + |R_{f,g}(G,s') - J|$$
$$< 2c.$$

It follows that

$$|S_{f,g}(G,m) - J|$$

$= |(1/2)(R_{f,g}(G,s) + R_{f,g}(G,s')) - J|$

$< c.$

Thus $(S_{f,g}, \Delta[a,b], \lessapprox)$ converges to J.

<u>Example 4.7.</u> Let each of f and g be defined as in Example 4.4. It was shown in Example 4.5 that $(S_{f,g}, \Delta[0,2], \lessapprox)$ converges. However, $D_f([0,2]) \cap D_g([0,2])$ is not empty; thus by Theorem 4.1, $(R_{f,g}, \Delta[0,2], \lessapprox)$ does not converge. It follows that the converse of Theorem 4.3 is not true.

Suppose [a,b] is an interval and each of f and g is in X([a,b]). Consider the net $(S_{f,g}, \Delta[a,b], \lesssim)$. The convergence of any of the nets discussed in this chapter implies the convergence of this net. Thus for each ordered pair of functions (f,g) described in Example 4.1 through Example 4.7, $(S_{f,g}, \Delta[0,2], \lesssim)$ converges. These statements will be verified in Chapter V.

<u>Theorem 4.4.</u> If [a,b] is an interval and each of f and g is in X([a,b]), and any one of the following is true,

(1) $(R_{f,g}, \Delta[a,b], \lessapprox)$ converges to the number J,

(2) $(S_{f,g}, \Delta[a,b], \lessapprox)$ converges to the number J,

(3) $(R_{f,g}, \Delta[a,b], \lesssim)$ converges to the number J,

then $(S_{f,g}, \Delta[a,b], \lesssim)$ converges to the number J.

<u>Proof:</u> The proof of part (2) is similar to the proof of Theorem 4.2. Part (1) follows from part (2), and the proof of part (3) is similar to the proof of Theorem 4.3.

CHAPTER V

SOME THEOREMS CONCERNING THE CONVERGENCE OF $(S_{f,g},\triangle[a,b],\leq)$

Miscellaneous Convergence Theorems

Suppose $[a,b]$ is an interval. $L[a,b]$ will denote a set of ordered pairs to which (f,g) belongs iff each of f and g is a function whose domain contains $[a,b]$ and whose range is a subset of R^1 and $(S_{f,g},\triangle[a,b],\leq)$ converges.

<u>Theorem 5.1.</u> If $[a,b]$ is an interval and (f,g) is in $L[a,b]$, then (g,f) is in $L[a,b]$. Furthermore, if $(S_{f,g},\triangle[a,b],\leq)$ converges to J, then $(S_{g,f},\triangle[a,b],\leq)$ converges to $f(b)g(b) - f(a)g(a) - J$.

<u>Proof:</u> Suppose (E,t) is in $\triangle[a,b]$.

$$
\begin{aligned}
&2S_{f,g}(E,t)\\
&= \textstyle\sum_E (f(q) + f(p))(g(q) - g(p))\\
&= -\textstyle\sum_E f(q)g(p) + \sum_E f(q)g(q) - \sum_E f(p)g(p)\\
&\quad + \textstyle\sum_E f(p)g(q)\\
&= -\textstyle\sum_E f(q)g(p) + (\sum_E f(p)g(p)) - f(a)g(a) + f(b)g(b)\\
&\quad - (\textstyle\sum_E f(q)g(q)) - f(a)g(a) + f(b)g(b) + \sum_E f(p)g(q)\\
&= 2f(b)g(b) - 2f(a)g(a) - \textstyle\sum_E g(p)(f(q) - f(p))\\
&\quad - \textstyle\sum_E g(q)(f(q) - f(p))\\
&= 2f(b)g(b) - 2f(a)g(a) - \textstyle\sum_E (g(q) + g(p))(f(q) - f(p))\\
&= 2f(b)g(b) - 2f(a)g(a) - 2S_{g,f}(E,t).
\end{aligned}
$$