COMPACT CONVEX SETS IN
LINEAR TOPOLOGICAL
SPACES

APPROVED:

Major Professor

Minor Professor

Director of the Department of Mathematics

Dean of the Graduate School
COMPACT CONVEX SETS IN
LINEAR TOPOLOGICAL
SPACES

THESIS

Presented to the Graduate Council of the
North Texas State University in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

David E. Read, B. S.
Denton, Texas
May, 1964
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>List</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>iv</td>
</tr>
</tbody>
</table>

**Chapter**

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. CONVEX SETS IN LINEAR TOPOLOGICAL SPACES</td>
<td>22</td>
</tr>
<tr>
<td>III. COMPACT CONVEX SETS</td>
<td>36</td>
</tr>
</tbody>
</table>

**BIBLIOGRAPHY**  | 50   |
<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A Circled Symmetric Set Which Is Not Convex</td>
<td>14</td>
</tr>
<tr>
<td>2. An LTS Which Is Not An LCS</td>
<td>23</td>
</tr>
<tr>
<td>3. A Closed Set Whose Convex Hull Is Not Closed</td>
<td>25</td>
</tr>
<tr>
<td>4. Necessity of Compactness in Theorem 3.6</td>
<td>44</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

The purpose of this paper is to examine properties of convex sets in linear topological spaces with special emphasis on compact convex sets.

**Definition 1.1.** A set $L$ is called a linear space over a field $F$ if:

1. For every $a, b \in L$ a binary operation $+$ is defined on $L \times L \rightarrow L$ satisfying the following conditions:
   
   (a) $a + b = b + a$ for all $a, b \in L$,
   
   (b) $(a + b) + c = a + (b + c)$ for all $a, b, c \in L$,
   
   (c) There is an element $0 \in L$ such that $a + 0 = a$ for every $a \in L$,
   
   (d) Given $a \in L$, there exists an element $-a \in L$ such that $a + (-a) = 0$.

2. There is a mapping from $F \times L \rightarrow L$, symbolized by juxtaposition, such that for every $\lambda, \mu \in F$ and $a, b \in L$;
   
   (a) $\lambda(a + b) = \lambda a + \lambda b$,
   
   (b) $(\lambda + \mu)a = \lambda a + \mu a$,
   
   (c) $\lambda(\mu a) = (\lambda \mu)a$,
   
   (d) $1 \cdot a = a$ where $1$ is the unity of the field $F$.

For purposes of this paper the field $\mathbb{R}$, of real numbers will be used throughout even though many of the following
theorems are true for more general fields. The properties of the real number system will be assumed throughout this paper.

**Definition 1.2** A subset $M$ of a linear space $L$ is said to be flat if and only if for every $x, y \in M$ \{\(\mu x + (1-\mu)y; \mu \in \mathbb{R}\}\} \subset M$, i.e. $M$ contains the line through $x$ and $y$.

**Definition 1.3** A hyperplane $H$ in a linear space $L$ is a maximal flat proper subset of $L$, i.e. $H$ is flat and if $H'$ is a flat subset of $L$ such that $H \subseteq H'$, then either $H' = H$ or $H' = L$.

**Definition 1.4** If $L$ and $L'$ are linear spaces over $\mathbb{R}$, a function $f$, from $L$ into $L'$ is called additive if $f(x+y) = f(x)+f(y)$ for all $x, y \in L$; homogeneous if $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}$ and $x \in L$; linear if both additive and homogeneous.

**Definition 1.5** If $L$ is a linear space then $L^\#$, the conjugate space of $L$, is the set of all linear functions from $L$ into $\mathbb{R}$.

$L^\#$ is readily seen to be a linear space over $\mathbb{R}$ with addition and scalar multiplication defined in the usual way, i.e. \((f_1 + f_2)(x) = f_1(x) + f_2(x)\) and \((\lambda f)(x) = \lambda f(x)\).

A subset $L_o$ of a linear space $L$ is a subspace, i.e. is a linear space with operations as defined in $L$, if and only if for each $a, b \in L_o$ and each $\lambda \in \mathbb{R}$, $a + b \in L_o$ and $\lambda a \in L_o$. The proof of this follows directly from definition 1.1.
These definitions from set theory will be needed for many of the following theorems. Let $X$, $Y$ be subsets of a linear space $L$, and $r \in \mathbb{R}$.

**Definitions 1.6**

\[
X + Y = \{x+y : x \in X \text{ and } y \in Y\}
\]
\[
x_r + Y = \{x + y : y \in Y\}
\]
\[
rX = \{rx : x \in X\}
\]
\[
L \setminus X = \{x \in L : x \notin X\}
\]

It is readily established that $r(L \setminus X) = \{rx : x \in L \text{ and } x \notin X\}$ and $L \setminus r(L \setminus X) = \{rx : x \in X\} = rX$.

**Theorem 1.1**

$E$ is flat if and only if for each $x \in E$, the set $E - x$ is a linear subspace of $L$.

*Proof* (only if) Let $E$ be a flat subset of $L$ and $x$ be an element of $E$. Let $a, b \in E - x$ and $\lambda \in \mathbb{R}$. There exist $y, y_1 \in E$ such that $a = y, -x$ and $b = y_1 - x$.

\[
a + b = (y, -x) + (y_1 - x) = (y, -x + y_1) - x.
\]

But $y, -x + y_1 \in E$ for let $\alpha \in \mathbb{R}$ such that $0 \neq \alpha \neq 1$. Then $C_y + (1 - \alpha) x \in E$. Let $\beta = \frac{-1}{\alpha - 1}$ and $\gamma = \frac{1}{\alpha}$. Since $\beta x + (1 - \beta)y_1 \in E$,

\[
y, -x + y_1 = y, + \gamma x - x - \gamma y_1 - y_2 + \gamma y_1
\]
\[
= y, + \gamma x - x - \gamma x + (1 - \gamma)y_2 + y_2
\]
\[
= \gamma(\alpha y, + (1 - \alpha)x) + (1 - \gamma)[\frac{-1}{\alpha - 1}x + (1 - \alpha) y_2]
\]
\[
= \gamma(\alpha y, + (1 - \alpha)x) + (1 - \gamma)[\beta x + (1 - \beta)y_2] \in E.\]

Therefore $a + b = (y, -x + y_1) - x \in E - x$.

\[
\lambda a = \lambda y, -\lambda x
\]
\[
= \lambda y, + x - x - \lambda x
\]
\[ \lambda \gamma, +x-x-\lambda x = (\lambda \gamma, +x-\lambda x)-x \]
\[ = (\lambda \gamma, +(1-\lambda)x)-x \text{ and since } E \text{ is flat and } x,y \in E, \lambda a \in E-x. \text{ Therefore } E-x \text{ is a linear subspace of } L. \]

(if) Suppose \( E-x \) is a linear subspace of \( L \) where \( x \in E \).
Let \( a,b \in E \) so \((a-x),(b-x)\in E-x \). Let \( \mu \in \mathbb{K} \).

\[ \mu(a-x)+(1-\mu)(b-x) \in E-x. \]
\[ \mu(a-x)+(1-\mu)(b-x) = \mu a-\mu x+b-x-\mu b+\mu x \]
\[ = \mu a+b-\mu b-x \]
\[ = \mu a+(1-\mu)b-x. \text{ Hence } \mu a+(1-\mu)b \in E \]

Therefore \( E \) is flat.

**Theorem 1.2** \( H \) is a hyperplane in \( L \) if and only if \( H = x+L_0 \) for some \( x \in L \) and some maximal proper linear subspace \( L_0 \) of \( L \).

**Proof:** (only if) Let \( H \) be a hyperplane in \( L \) and \( x_o \) be an element of \( H \). Since \( H \) is flat, \( L_o = H-x_o \) is a linear subspace of \( L \). Therefore \( H = x_o+L_o \) where \( x_o \in L \) and \( L_o \) is a linear subspace of \( L \).

Assume by way of contradiction that \( L_o \) is not a maximal proper linear subspace of \( L \). Then there exists an \( L_o > L_o \) such that \( L_o \) is a proper linear subspace of \( L \). Let \( H' = L_o+x_o \) with \( x_o \) as above. Since \( L_o \subset L \), then \( H \not\subset H' \subset L \). Let \( x \in H' \).

There exists an \( x', \in L \), such that \( x = x'+x_o \). For any \( z \in H' -x \) there exists a \( y, \in L \), such that \( z = (y, +x_o)-x = y, +x_o-x'-x_o \) and \( y, +x_o-x'-x_o = y, -x' \). Hence if \( a,b \in H' -x \) there exist \( y, y, \in L \), such that
\[ a = y_i - x_i', b = y_i - x_i, \text{ and} \]
\[ a + b = y_i - x_i + y_i - x_i' \]
\[ = (y_i - x + y_i') - x_i', \text{ and since } y_i, x_i, y_i' \in L, \text{ then} \]
\[ a + b \in H - x. \]
\[ \lambda a = \lambda(y_i - x_i) \]
\[ = \lambda y_i - \lambda x_i. \text{ Let } \alpha = 1 - \lambda. \text{ Then } \lambda = 1 - \alpha. \]

Hence \[ \lambda a = (1 - \alpha)y_i, -(1 - \alpha)x_i' = (1 - \alpha)y_i + \alpha x_i - x_i' \] and since
\[ \alpha x_i \in L, \text{ then } \lambda a \in H - x. \]

Therefore \( H - x \) is a linear subspace of \( L \) for any \( x \in H' \).

Hence \( H' \) is flat. This, however, contradicts the assumption that \( H \) is a hyperplane in \( L \). Therefore \( L_0 \) is a maximal proper linear subspace of \( L \).

(if) Let \( H = x + L_0 \) for some \( x \in L \) and some maximal proper linear subspace \( L_0 \) of \( L \). Now \( L_0 = H - z \) for any \( z \in H \) so, by theorem 1.2, \( H \) is flat. \( H \) is a maximal flat proper subset of \( L \) for suppose by way of contradiction that \( H \) is not maximal. Then there exists a flat proper subset \( H' \) of \( L \) such that \( H \subset H' \). Let \( x \in H \). \( L_1 = H - x \) is a proper linear subspace of \( L \). \( L_0 = H - x, c \in H - x = L_1. \) This contradicts the assumption that \( L_0 \) is a maximal proper linear subspace of \( L \). Hence \( H \) is maximal and is therefore a hyperplane in \( L \).

**Theorem 1.3** If \( f \in L' \), \( f \neq 0 \) and if \( \lambda \in \mathbb{R} \) then \( \{x : f(x) = \lambda\} \) is a hyperplane in \( L \).

Proof: Let \( H = \{x : f(x) = \lambda\} \), \( x, y \in H \) and \( \mu \in \mathbb{R} \). Suppose without loss of generality that \( \lambda > 0 \) (otherwise let \( f_1(x) = -f(x) \)).
\[ f(\mu x + (1-\mu)y) = f(\mu x) + f((1-\mu)y) \]
\[ = \mu f(x) + (1-\mu)f(y) \]
\[ = \mu \lambda + (1-\mu)\lambda \]
\[ = \lambda. \]

Therefore \( \mu x + (1-\mu)y \in H \). Hence \( H \) is flat.

Suppose by way of contradiction that \( H \) is not maximal. Then there exists a flat proper subset \( H' \) of \( L \) such that \( H < H' \) and \( H' \neq H \). Let \( z \in H' \) such that \( z \notin H \). Then \( f(z) \notin \lambda \).

1. Suppose \( f(z) > \lambda \). There exists an \( \epsilon > 0 \) such that \( f(z) = \lambda + \epsilon \).

(a) If \( y \in L \) such that \( f(y) < \lambda \), then \( y \in H' \). For suppose \( y \in L \) such that \( f(y) < \lambda \) \((z \) is one such element\). There exists an \( \epsilon_1 > 0 \), such that \( f(y) = \lambda - \epsilon_1 \). Let \( \beta = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \).

\[ f(\beta z + (1-\beta)y) = f(\beta z) + f((1-\beta)y) \]
\[ = \beta f(z) + (1-\beta)f(y) \]
\[ = \beta(\lambda + \epsilon_1) + (1-\beta)(\lambda - \epsilon_2) \]
\[ = \lambda. \]

Therefore \( x = \beta z + (1-\beta)y \in H \). Since \( H' \) is flat, \( x \in H \subset H' \), and \( y \) is on the line between \( x \) and \( z \), \( y \in H' \). Hence if \( y \in L \) and \( f(y) < \lambda \), then \( y \notin H' \).

(b) If \( y' \in L \) such that \( f(y') > \lambda \) \((2z \) is one such element\) then \( y' \in H \). Since there exists an \( \epsilon' > 0 \) such that \( f(y') = \lambda + \epsilon' \), and letting \( \beta' = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \),

\[ f(\beta' y' + (1-\beta')y) = f(\beta' y') + f((1-\beta')y) \]

Therefore \( \beta' = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \).
\[ f(\rho y') + f((1-\rho)z) = \rho f(y') + (1-\rho)f(z) = \rho(\lambda + \epsilon') + (1-\rho)(\lambda - \epsilon) = \lambda. \]

Therefore \( x' = \rho y' + (1-\rho)z \in H'. \) Since \( x' \in H' \cap H' \), \( y' \) is on the line between \( x' \) and \( y', \) and \( H' \) is flat, \( y' \in H'. \) Hence if \( y' \in L \) and \( f(y') > \lambda \) then \( y' \in H'. \)

(2) In like manner if \( f(x) \) is supposed to be greater than \( \lambda \) then \( x \in H' \) whenever \( f(x) > \lambda \) or \( f(x) < \lambda \). Further, since \( H \subset H' \), \( x \in H' \) whenever \( f(x) = \lambda \).

Therefore \( H' = L. \) But this contradicts the assumption that \( H' \) is a proper subset of \( L. \) Hence \( H \) is maximal so \( H \) is a hyperplane in \( L. \)

**Definition 1.7** If \( A \subset L \), then a point \( x \in L \) is said to be a finite linear combination of elements of \( A \) if there exist \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), and \( x_1, \ldots, x_n \in A \) such that \( x = \lambda_1 x_1 + \ldots + \lambda_n x_n. \)

**Definition 1.8** A subset \( X \) of \( L \) is called linearly independent if for \( x_1, \ldots, x_n \) distinct elements of \( X \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) then \( \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0 \) implies that \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0. \)

**Definition 1.9** A vector basis (or basis) for \( L \) is a maximal linearly independent set.

If \( X \) is a basis for \( L \) then every point of \( L \) is a finite linear combination of elements of \( L. \) Further, "if \( B \) is a linearly independent set in \( L \), then there is a vector basis \( B \) of \( L \) such that \( B \supseteq E. \)" (1, p. 2).
Theorem 1.4 If \( H \) is a hyperplane, there exists an \( f \in L^\# \), \( f \neq 0 \), and a \( \lambda \in \mathbb{R} \) such that \( H = \{ x : f(x) = \lambda \} \). \( H \) is linear if and only if \( \lambda = 0 \).

Proof: Let \( H \) be a hyperplane in \( L \). There is an \( x_0 \in L \) and a maximal proper linear subspace \( L_o \) of \( L \) such that \( H = x_0 + L_o \). Let \( x_0 \) be a basis for \( L_o \) and \( y \) be an element of \( L \setminus L_o \) such that \( x_0 \cup \{ y \} \) forms a basis for \( L \). There exist a positive integer \( n \), elements \( x_1, x_2, \ldots, x_n \in x_0 \), and \( \lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) such that \( x_0 = \lambda x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n + \lambda y \).

Define \( f : L \rightarrow \mathbb{R} \) as follows:

\[
f(x) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \in L_o \\
\alpha f(x_i') + \alpha_2 f(x_2') + \ldots + \alpha_m f(x_m') + \alpha f(y) = \alpha & \text{if } x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + \alpha y \text{ where } \alpha, \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R} \text{ and } x_1, x_2, \ldots, x_m \in x_0.
\end{cases}
\]

Let \( x_1', x_2' \in L \). There exist \( \alpha, \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) and \( x_1', x_2', \ldots, x_m' \in x_0 \) such that \( x_1' = \alpha x_1' + \ldots + \alpha_m x_m' + \alpha y \) and there exist \( \beta, \beta_1, \ldots, \beta_m \in \mathbb{R} \) and \( x_1^2, x_2^2, \ldots, x_m^2 \in x_0 \) such that \( x_1^2 = \beta x_1^2 + \beta_1 x_2^2 + \ldots + \beta_m x_m^2 + \beta y \).

\[
f(x_1' + x_2^2) = f(\alpha x_1' + \ldots + \alpha_m x_m' + \alpha y + \beta x_1^2 + \ldots + \beta_m x_m^2 + \beta y) \\
= f(\alpha x_1' + \ldots + \alpha_m x_m' + \beta x_1^2 + \ldots + \beta_m x_m^2 + (\alpha + \beta) y) \\
= \alpha + \beta \\
= f(x_1') + f(x_2^2).
\]

Let \( \lambda \in \mathbb{R} \). Then

\[
f(\lambda x_1') = f(\lambda \alpha x_1' + \ldots + \lambda \alpha_m x_m' + \lambda \alpha y) \\
= \lambda \alpha = \lambda f(x_1').
\]
Hence $f$ is a linear function, so $f \in \mathbb{L}$. Since $f(y) = 1$, $f \neq 0$.

Let $x' \in H$. There exist $\mu_1, \ldots, \mu_s \in \mathbb{R}$ and $x_1', x_2', \ldots, x_s' \in X_o$ such that $x' = x_1' + \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_s x_s$.

\[ f(x') = f(x_1' + \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_s x_s) \]
\[ = f(x_1') + f(\mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_s x_s) \]
\[ = \lambda. \]

Therefore $H \subseteq \{ x : f(x) = \lambda \}$. Let $x'' \in L$ such that $x'' \notin H$. There exist $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_s \in \mathbb{R}$, $\gamma \neq \lambda$ and $x_1'', x_2'', \ldots, x_s'' \in X_o$ such that $x'' = \gamma x_1'' + \gamma_1 x_2'' + \ldots + \gamma_s x_s'' + \gamma y$.

\[ f(x'') = f(\gamma x_1'' + \gamma_1 x_2'' + \ldots + \gamma_s x_s'' + \gamma y) \]
\[ = \gamma. \]

Therefore $x'' \notin \{ x : f(x) = \lambda \}$. Hence $\{ x : f(x) = \lambda \} \subset H$ so

\[ H = \{ x : f(x) = \lambda \}. \]

If $\lambda = 0$ then $x_0 \in L$. Therefore $H = L$ so $H$ is linear.

If $H$ is linear then $x_0 \in L$ so $\lambda = 0$. Therefore $H$ is linear if and only if $\lambda = 0$.

**Theorem 1.5** If the hyperplane $H = \{ x : f_1(x) = \lambda_1 \} = \{ x : f_2(x) = \lambda_2 \}$, then there exists $\mu \in \mathbb{R}$, $\mu \neq 0$ such that $f_1 = \mu f_2$ and $\lambda_1 = \mu \lambda_2$.

**Proof:** Let $H$ be a hyperplane in a linear space $L$ such that $H = \{ x : f_1(x) = \lambda_1 \} = \{ x : f_2(x) = \lambda_2 \}$. As follows from the previous theorem $\lambda_2 = 0$ if and only if $\lambda_1 = 0$. In such case $f_1 = 1 \cdot f_2$, $\lambda_1 = 1 \cdot \lambda_2$. Suppose $\lambda_2 \neq 0$. $H = x + L_o$ for some $x \in L$ and a maximal proper linear subspace $L_o$ of $L$. Let
$X_0$ be a basis for $L$. Since $\lambda_2 \neq 0$ in is not linear, so $x \notin L$.

$X_0 \cup \{x\}$ forms a basis for $L$. Since $x \in H$, $f_1(x) = \lambda_1$, $f_2(x) = \lambda_2$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and $x_1, x_2, \ldots, x_n \in X_0$. Then

$$x' = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + x \in H.$$ Therefore

$$f_1(x') = \alpha_1 f_1(x_1) + \alpha_2 f_1(x_2) + \ldots + \alpha_n f_1(x_n) + \lambda_1 = \lambda_1,$$

and

$$f_2(x') = \alpha_1 f_2(x_1) + \alpha_2 f_2(x_2) + \ldots + \alpha_n f_2(x_n) + \lambda_2 = \lambda_2.$$

Hence for any values of $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$

$$\alpha_1 f_1(x_1) + \alpha_2 f_1(x_2) + \ldots + \alpha_n f_1(x_n) = 0$$ and

$$\alpha_1 f_2(x_1) + \alpha_2 f_2(x_2) + \ldots + \alpha_n f_2(x_n) = 0.$$ Hence $f_1(x_1) = f_1(x_2) = \ldots = f_1(x_n) = f_2(x_1) = f_2(x_2) = \ldots = f_2(x_n) = 0$.

Let $y \in \mathbb{L}$. Then there exist $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and $y_1, y_2, \ldots, y_n \in X_0$ such that $y = \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n + \lambda x$. Therefore

$$f_1(y) = \lambda_1 \lambda_1$$ and $f_2(y) = \lambda_2 \lambda_2$. Since $\lambda_2 \neq 0$ then

$$f_1(y) = \lambda_1 \lambda_2$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2} \lambda_2$$

$$= \frac{\lambda_1}{\lambda_2} f_2(y).$$

Let $\mu = \frac{\lambda_1}{\lambda_2}$. Then $f_1 = \mu f_2$ and $\lambda_1 = \mu \lambda_2$.

**Definition 1.10** The core of a subset $E$ of $L$ is $\{x :$ for each $y \in \mathbb{L}$ there is an $\varepsilon > 0$ such that $x + ty \in E$ if $|t| < \varepsilon\}$.

**Definition 1.11** If $x, y \in \mathbb{L}$ then the line segment between them is $\{tx + (1-t)y : 0 \leq t \leq 1\}$. 
**Definition 1.12** A nonempty set $K$ in $L$ is convex if for each pair of points in $K$, the segment between them is in $K$.

**Theorem 1.6** The intersection of a family of convex sets is either empty or convex.

Proof: Let $A = \bigcap_{s \in S} E_s$ be the nonempty intersection of a family of convex sets. Suppose $x, y \in A$. Then $x, y \in E_s$ for each $s \in S$. Therefore, since each $E_s$ is convex, the line segment between $x$ and $y$ is contained in each $E_s$. Hence the line segment between $x$ and $y$ is contained in $\bigcap_{s \in S} E_s$. Consequently $A$ is convex.

**Corollary:** Each nonempty subset $E$ of a linear space $L$ is contained in a smallest convex set $k(E)$, called the convex hull of $E$. Clearly $k(E)$ is the intersection of all the convex sets in $L$ which contain $E$.

**Theorem 1.7** If $S$ and $T$ are convex sets in a linear space $L$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha S + \beta T$ is convex.

Proof: Let $x, y \in \alpha S + \beta T$. Then there exist $x, y \in S$ and $x_1, y_1 \in T$ such that $x = \alpha x_1 + \beta x_2$ and $y = \alpha y_1 + \beta y_2$.

Let $\mu \in \mathbb{R}$ such that $0 \leq \mu \leq 1$. Then $\mu x_1 + (1-\mu)y_1 \in S$ and $\mu x_2 + (1-\mu)y_2 \in T$. Therefore

$$\mu x + (1-\mu)y = \mu (x_1 + \beta x_2) + (1-\mu) (\alpha y_1 + \beta y_2)$$

$$= \alpha \mu x_1 + \alpha (1-\mu) y_1 + \beta \mu x_2 + \beta (1-\mu) y_2$$

$$= \alpha [\mu x_1 + (1-\mu)y_1] + \beta [\mu x_2 + (1-\mu)y_2]$$

so $\mu x + (1-\mu)y \in \alpha S + \beta T$. Hence $\alpha S + \beta T$ is convex.
Theorem 1.8 If $E$ is a nonempty subset of $L$ then

$$k(E) = \left\{ \sum_{i=1}^{n} t_i \cdot x_i : x_i \in E, \; t_i \geq 0, \; \sum_{i=1}^{n} t_i = 1, \; \text{and} \; n = 1, 2, \ldots \right\}.$$  

Proof: Let

$$A = \left\{ \sum_{i=1}^{n} t_i \cdot x_i : x_i \in E, \; t_i \geq 0, \; \sum_{i=1}^{n} t_i = 1, \; \text{and} \; n = 1, 2, \ldots \right\}.$$  

Suppose $x, y \in A$, so $x = \sum_{i=1}^{n} t_i \cdot x_i$ and $y = \sum_{i=1}^{n} s_i \cdot y_i$, where $x_i, y_i \in E$ and $t_i, s_i \geq 0, \; \sum_{i=1}^{n} t_i = 1, \; \sum_{i=1}^{n} s_i = 1$. Let

$$\mu \in \mathbb{R} \; \text{such that} \; 0 \leq \mu \leq 1.$$  

$$\mu x + (1-\mu)y = \mu \sum_{i=1}^{n} t_i \cdot x_i + (1-\mu) \sum_{i=1}^{n} s_i \cdot y_i.$$  

Therefore $\mu x + (1-\mu)y \in A$. Hence $A$ is convex, and clearly $E \subset A$ so $k(E) \subset A$.

Let $X$ be a convex set such that $E \subset X$. Let $x \in A$, then

$$x = \sum_{i=1}^{n} t_i \cdot x_i \; \text{where} \; x_i \in E \; \text{and} \; \sum_{i=1}^{n} t_i = 1, \; t_i \geq 0.$$  

Suppose $n = 1$, then $x = x_1$, so $x \in X$. Therefore $A \subset X$. Suppose that for $n = k, \; \sum_{i=1}^{k} t_i \cdot x_i \in X$ where $t_i \geq 0, \; \sum_{i=1}^{k} t_i = 1, \; \text{and} \; x_i \in E$. Consider

$$x = \sum_{i=1}^{k+1} t_i \cdot x_i + \sum_{i=k}^{k+1} t_i \cdot x_i,$$

where $t_i \geq 0$, $\sum_{i=k}^{k+1} t_i = 1, \; \text{and} \; x_i \in E$. Consider

$$t_{k+1} = 1 - \sum_{i=k}^{k+1} t_i \; \text{and} \; \sum_{i=1}^{k} t_i \cdot x_i \in X \; \text{since} \; \sum_{i=1}^{k} \frac{t_i}{\sum_{i=1}^{k} t_i} = 1.$$
Therefore, since $X$ is convex

$$t, x, +, \ldots + t, x + t, x = \sum_{i=k}^{n} \frac{t, x, +, \ldots + t, x, \in X}{\sum_{i=k}^{n}} + (1 - \sum_{i=k}^{n}) x, \in X.$$ 

This completes the induction process so if $n$ is a positive integer and $x = t, x, +, \ldots + t, x, n$ where $t, \succ 0$, $\sum_{i=n}^{n} t, = 1$, and $x, \in E$, then $x, \in X$. Therefore $A \subset X$. Hence $A$ is contained in the intersection of all the convex sets containing $E$, i.e. $A \subset K(E)$.

Therefore $K(E) = A$.

The following theorem, called Caratheodory's theorem, is true for $L = E$*. A proof of this theorem can be found in Eggleston (2, pp. 34-35).

Caratheodory's theorem "If $y$ is a point of $K(X)$ there is a set of $s$ points $x, \ldots, x$, all belonging to $X$ with $s \leq n+1$ such that $y$ is a point of the simplex whose vertices are $x, x, \ldots, x$, [i.e. there exist $\lambda, \ldots, \lambda, \in R$ $\lambda, \succ 0$, where $\lambda, + \lambda, + \lambda, + \ldots + \lambda, = 1$ such that $y = \lambda, x, +, \ldots + \lambda, x,$]."

Definition 1.13 A subset $A$ of a linear space $L$ is symmetric if $a \in A$ implies $-a \in A$.

Definition 1.14 A set $A$ is circled if and only if $aA \subset A$ whenever $|a| \leq 1$.

Theorem 1.9 If $A$ is circled then $A$ is symmetric.

Proof: Suppose $x \in A$. Then since $A$ is circled $(-1)A \subset A$ so $-x \in A$. Hence $A$ is symmetric.

Theorem 1.10 If $A$ is convex and circled then $A$ is circled and symmetric.
Proof: The proof follows directly from Theorem 1.9. The converse is not true, however, for consider the following circled and symmetric set in $E^2$ which is not convex.

$$A = \{(x,y): -1 \leq x \leq 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } -1 \leq y \leq 1\}$$

Fig. 1—Example of a circled symmetric set which is not convex

**Theorem 1.11** The smallest linear subspace containing a convex, circled set, $A$, is $\cup\{aA: a \in \mathbb{R}\}$.

Proof: Let $B = \cup\{aA: a \in \mathbb{R}\}$ and $L_o$ be the smallest linear subspace containing $A$. $L_o$ is the intersection of all the linear subspaces containing $A$. Suppose $x_o \in B$, then there exist $a_o \in \mathbb{R}$ and $x_o' \in A$ such that $x_o = a_o x_o'$. Since $x_o' \in A \subseteq L_o$, then $x_o = a_o x_o' \in L_o$. Therefore $B \subseteq L_o$.

Let $x \in B$ and $x \in \mathbb{R}$. There exist elements, $x' \in A$ and $a \in \mathbb{R}$, such that $x = ax'$. Since $\lambda a \in \mathbb{R}$,
\[ \lambda x = \lambda (ax') = (\lambda a)x' \in B. \]

Therefore scalar multiples of \( B \) are elements of \( B \). It remains to be shown that \( B \) is closed with respect to addition. Let \( x, y \in B \). There exist \( a, b \in \mathbb{R} \) and \( x, y \in A \) such that \( x = ax \) and \( y = by \).

**Case I:** \( a = b = 0 \). Then \( x + y = 0 + 0 = 0 \in B \).

**Case II:** \( a + b = 0 \), \( a \neq 0 \). Suppose without loss of generality, that \( a > 0 \). Since \( A \) is circled, \( A \) is symmetric. Therefore \(-y, \in A \). Let \( \alpha = \frac{a}{a+b} \). \( \alpha x, + (1-\alpha)(-y) \in A \). Hence

\[
\begin{align*}
\lambda x + y &= ax, + by, \\
&= ax, -ay, \\
&= (a+a)[\alpha x, + (1-\alpha)(-y)] \in B.
\end{align*}
\]

**Case III:** \( a + b \neq 0 \). Suppose, without loss of generality, that \( a + b > 0 \).

If \( a > 0, b \geq 0 \) then, letting \( \beta = \frac{a}{a+b} \), \( \beta x, + (1-\beta)y \in A \). Therefore

\[
\begin{align*}
\lambda x + y &= ax, + by, \\
&= (a+b)[\beta x, + (1-\beta)y] \in B.
\end{align*}
\]

If \( a > 0, b < 0 \) then, letting \( \lambda = \frac{a}{a-b} \), \( \lambda x, + (1-\lambda)(-y) \in A \). Therefore

\[
\begin{align*}
\lambda x + y &= ax, + by, \\
&= (a-b)[\lambda x, + (1-\lambda)(-y)] \in B.
\end{align*}
\]

Hence, since \( \lambda x \in B \) and \( x + y \in B \) for every \( \lambda \in \mathbb{R} \) and \( x, y \in B \),
then $B$ is a linear subspace of $L$. Clearly $B$ contains $A$. Therefore $L \subseteq B$. Hence $L = B$.

**Theorem 1.12** A is convex and circled if and only if $A$ is convex and symmetric.

**Proof:** (only if) If $A$ is convex and circled, then $A$ is convex and symmetric as follows from Theorem 1.9.

(if) Suppose $A$ is convex and symmetric. Let $a \in \mathbb{R}$ such that $|a| \leq 1$ and let $x \in A$. Then $-x \in A$ since $A$ is symmetric.

Case I: $0 \leq a \leq 1$. Then $0 \leq a + 1 \leq 1$.

$$\frac{ax}{2} = \frac{ax + ax}{2}$$

$$\frac{ax}{2} + \frac{ax}{2} = \frac{ax + ax + x + x - x}{2}$$

$$= \frac{ax + x - x + ax + x}{2}$$

$$= \frac{a + 1}{2} x + (1 - a + 1)(-x) \in A.$$ 

Case II: $-1 < a < 0$. Then $0 < -a \leq 1$ so $-ax \in A$ and since $A$ is symmetric, then $ax = -(-ax) \in A$.

Therefore $A$ is circled and convex.

**Theorem 1.13** $A$ is circled and convex if and only if $ax + by \in A$ where $x, y \in A$ and $|a| + |b| \leq 1$.

**Proof:** (only if) Suppose $A$ is circled and convex. Let $a, b \in \mathbb{R}$ such that $|a| + |b| \leq 1$ and $x, y \in A$.

Case I: $a + b \neq 0$. $|a + b| \leq |a| + |b| \leq 1$, so

$$ax + bx = (a + b)x \in (a + b)A \subseteq A,$$ and

$$ay + by = (a + b)y \in (a + b)A \subseteq A.$$ Therefore
\[ ax + by = \frac{a + bx + a + bby}{a + b} = \frac{a^2 x + abx + aby + b^2 y}{a + b} = \frac{a^2 x + abx + a^2 y + aby + ab + b^2 y - a - y - aby}{a + b} = \frac{a^2 x + abx + a^2 y + aby + ab + b^2 y - a - y - aby}{a + b} = \frac{a^2 x + abx + ay + by - a - y - aby}{a + b} = \frac{a (ax + bx) + (1 - \frac{a}{a+b})(ay + by)}{a + b} \epsilon A. \]

Case II: \( a + b = 0 \). Then \( b = -a \) and \( \frac{|a|}{2} \leq 1 \) so \( |2a| \leq 1 \).

Hence \( 2ax \epsilon A \) and \(-2ay \epsilon A \).

\[ ax + by = ax - ay = \frac{1}{2}(2ax) + (1 - \frac{1}{2})(-2ay) \epsilon A. \]

Therefore, in either case, \( ax + by \epsilon A \).

(if) Suppose \( ax + by \epsilon A \) where \( x, y \epsilon A \) and \( |a| + |b| \leq 1 \).

Clearly \( A \) is convex for \( \alpha \epsilon R \) such that \( 0 \leq \alpha \leq 1 \). Then

\[ |a| + |1-\alpha| = 1, \]

so \( \alpha x + (1-\alpha)y \epsilon A \) for all \( x, y \epsilon A \). Let \( a \epsilon R \) such that \( |a| \leq 1 \) and let \( x \epsilon A \). \( 0 = \frac{1}{2}x + (-1)\frac{1}{2}x \epsilon A \). Let \( b = 1 - |a| \) then \( |a| + |b| = 1 \).

Hence \( ax = ax + b \cdot 0 \epsilon A \). Therefore \( A \) is convex and circled.

**Definition 1.14** The smallest circled set containing \( A \) is the circled extension of \( A \), denoted by \( \langle A \rangle \).

Clearly the circled extension of \( A \) is the intersection of all the circled sets containing \( A \).
Theorem 1.14 \( \mathcal{A}^* = \bigcup \{aA : |a| \leq 1\} \).

Proof: Let \( x \in \{aA : |a| \leq 1\} \). Then there exist \( x_i \in A \) and \( a_i \in \mathbb{R} \) such that \( |a_i| \leq 1 \), and \( x = a_i x_i \). Let \( B \) be a circled set containing \( A \). Since \( B \) is circled and \( x_i \in B \), then \( a_i x_i \in B \). Hence \( x \) is an element of the intersection of all the circled sets containing \( A \), i.e. \( x \in \mathcal{A}^* \). Therefore \( \bigcup \{aA : |a| \leq 1\} \subseteq \mathcal{A}^* \).

Theorem 1.15 The convex hull of a circled set is circled.

Proof: Let \( A \) be a circled set.

\[ k(A) = \left\{ \sum_{i \in \mathbb{N}} t_i x_i : t_i \geq 0, x_i \in A, \sum_{i \in \mathbb{N}} t_i = 1 \right\} \]

Let \( x \in k(A) \), \( x = t_1 x_1 + \ldots + t_n x_n \) where \( t_i \geq 0 \), \( x_i \in A \), \( \sum_{i \in \mathbb{N}} t_i = 1 \).

Let \( a \in \mathbb{R} \) such that \( |a| \leq 1 \). Since \( A \) is circled then \( a x_i \in A \), \( i = 1, \ldots, n \). Hence

\[ a x = a(t_1 x_1 + \ldots + t_n x_n) \]
\[ = at_1 x_1 + \ldots + at_n x_n \]
\[ = t_1 ax_1 + \ldots + t_n ax_n \in k(A). \]

Therefore \( k(A) \) is circled.

Theorem 1.16 The smallest convex circled set containing \( A \) is the convex hull of \( \mathcal{A}^* \) and is also

\[ \left\{ \sum_{i \in \mathbb{N}} a_i x_i : x_i \in A \text{ and } \sum_{i \in \mathbb{N}} |a_i| \leq 1 \text{ for } n = 1, 2, \ldots \right\}. \]

Proof: Clearly the smallest convex circled set containing \( A \) is the intersection of all the convex and circled
sets which contain \( A \); denote this set by \( B \). Since \( \mathcal{C}(A) \) is circled and convex and contains \( A \), \( B \subset \mathcal{C}(A) \). Let \( K \) be a circled convex set containing \( A \) so \( A \subset K \). Therefore 
\[
\mathcal{C}(A) \subset K. \quad \text{Hence } \mathcal{C}(A) \subset B. \quad \text{Therefore } B = \mathcal{C}(A).
\]

\[
\mathcal{C}(A) = \left\{ \sum_{i=n} t_i x_i : t_i \geq 0, x_i \in A, \sum_{i=n} t_i = 1, \text{ and } n = 1, 2, \ldots \right\}.
\]

Let \( C = \left\{ \sum_{i=n} a_i x_i : x_i \in A, \sum_{i=n} a_i \leq 1, \text{ and } n = 1, 2, \ldots \right\} \). Let 
\[
x = \sum_{i=n} t_i a_i x_i \text{ where } \sum_{i=n} t_i a_i \leq 1.
\]

Hence \( x \in C \). Therefore \( \mathcal{C}(A) \subset C \).

Let \( x \in C \), \( x = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \) where \( \sum_{i=n} a_i \leq 1 \) and 
\[
x_i \in A. \quad \text{Let}
\]

\[
k_i = \begin{cases} 
2 & \text{if } a_i \geq 0 \\
1 & \text{if } a_i < 0 
\end{cases}
\]

then 
\[
x = \frac{a_1}{\sum_{i=n} a_i} (-1)^{i-n} x_i + \frac{a_2}{\sum_{i=n} a_i} (-1)^{i-n} x_i + \ldots + \frac{a_n}{\sum_{i=n} a_i} (-1)^{i-n} x_n,
\]

if \( \sum_{i=n} a_i \neq 0 \). Let 
\[
t_j = \frac{a_j}{\sum_{i=n} a_i} \text{, } x_j = (-1)^{i-n} (\sum_{i=n} a_i) x_j \in A.
\]

Then \( x = \sum_{i=n} t_i x_i + \ldots + t_n x_n \) where \( \sum_{i=n} t_i = 1, t_i \geq 0 \) and \( x_i \in A \).

Hence \( x \in \mathcal{C}(A) \).
If $\sum a_i = 0$, then $a_1 = a_2 = \ldots = a_n = 0$. Therefore $a = 0$, and $0 \in \mathcal{A} \subseteq \mathcal{A}' \subseteq k\langle \mathcal{A}' \rangle$.

Therefore $C \subseteq k\langle \mathcal{A}' \rangle$. Hence $C = k\langle \mathcal{A}' \rangle$. 
CHAPTER BIBLIOGRAPHY


CHAPTER II

CONVEX SETS IN LINEAR

TOPOLOGICAL SPACES

In this chapter criteria will be given for a Hausdorff topology, in which the linear operations are continuous, for a linear space L. General topological properties of convex sets and linear spaces will then be examined, leading to theorems on supporting and separating hyperplanes.

**Definition 2.1** If a linear space L has a Hausdorff topology in which the linear operations are continuous (as functions of two variables), then L is called a linear topological space (LTS). If in addition every neighborhood of each point contains a convex open set, then L is called a locally convex linear topological space (LCS).

L is an LTS if and only if it has a neighborhood basis $\mathcal{U}$ at 0 such that (a) 0 is the only point common to all U in $\mathcal{U}$; (b) if $U,V\in\mathcal{U}$, then there is a $W\in\mathcal{U}$ such that $W\cap U\cap V$; (c) if $U\in\mathcal{U}$ and $|r|<1$, then $rU\subseteq U$; (symmetric star-shaped); (d) if $U\in\mathcal{U}$ there exists $V\in\mathcal{U}$ such that $V\cap U$; (e) 0 is a core point of each U in $\mathcal{U}$. L is also locally convex, if and only if $\mathcal{U}$ can also be chosen so that (f) every U in $\mathcal{U}$ is convex. (1, pp. 11-12).

**Remark:** If L is an LTS and $\mathcal{U}$ is a neighborhood basis of 0, then $\mathcal{U}_x = \{U+x:U\in\mathcal{U}\}$ is a neighborhood basis at x.

$E^*\mathcal{U}$ is an LTS which is not locally convex, under the topology determined by the following neighborhood basis of 0: $\mathcal{U}$ is the set of all
$U_r = \{(x, y) : (x, y) \text{ is in the interior of the circle of radius } r \text{ with center at } 0, \text{ or } x = 0 \text{ and } -2r < y < 2r, \text{ or } y = 0 \text{ and } -2r < x < 2r\}$, with $r > 0$.

If $x \in L$, then a neighborhood basis at $x$ is defined as in the above "Remark".

![Fig. 2—An LTS which is not an LCS](image)

**Theorem 2.1** If $X$ is an open set in an LTS $L$, and $r \neq 0$, then $rX$ is open.

**Proof:** Consider the function $f$ of scalar multiplication (which is continuous). Let $rx, e rX$. Now $f((\frac{1}{r}, rx)) = x, e X$. $X$ is a neighborhood of $x$, therefore there is an $e > 0$ and a neighborhood $V$ of $rx$, such that if $\frac{e}{r} \epsilon R$ and $\frac{|x - \frac{1}{r}|}{e} < e$, and $x \epsilon V$, then $f((\frac{e}{r}, x)) \epsilon X$. Since $\frac{1}{r}$ is clearly an element of any $\epsilon -$ neighborhood of itself, then if $x \epsilon V$, $f((\frac{1}{r}, x)) \epsilon X$. Hence $x \epsilon rX$. Therefore $V \epsilon rX$ so $rx$ is an interior point of $rX$. Hence $rX$ is open.
**Corollary:** If $X$ is a closed set in an LTS $L$, and $r \neq 0$, then $rX$ is closed.

**Proof:** $L \setminus X$ is open since $X$ is closed. $r(L \setminus X)$ is then an open set, as follows from the above theorem.

Hence $L \setminus r(L \setminus X)$ is a closed set.

**Theorem 2.2** Let $L$ be an LTS and $X, Y$ be subsets of $L$. If $X$ or $Y$ is open, then $X + Y$ is open.

**Proof:** Suppose without loss of generality that $X$ is open. Note that $X + Y = Y + X$.

Consider the continuous function $g : L \times L \to L$ of addition. Let $z_o \in X + Y$. There exist $x_o \in X$ and $y_o \in Y$ such that $z_o = x_o + y_o$. Since $g$ is continuous, and $g((z_o, -y_o)) = x_o \in X$, there exist neighborhoods $U$ of $x_o + y_o$ and $V$ of $-y_o$ such that if $x$ and $y$ are elements of $U$ and $V$ respectively, then $g((x, y)) \in X$.

Clearly $-y_o \in V$ so if $x \in U$ then $g((x, -y_o)) \in X$. But

$$x = g((x, -y_o), y_o), \text{ i.e. } (x+y_o)+y_o = x.$$ Therefore $x \in X + Y$. Hence $U \subseteq X + Y$ so $x_o + y_o = z_o$ is an interior point of $X + Y$ and $X + Y$ is open.

The converse is not true, for consider the following: In $\mathbb{F}$ let $X$ be the half open half closed interval $[0,1]$ and $Y$ be $(2,3]$, then $X + Y = (2,4)$ is open even though neither $X$ nor $Y$ is open.

**Theorem 2.3** If $X$ is open, then $k(X)$ is open.

**Proof:** $k(X) = \{\{t; x_i : x_i \in X, t_i \geq 0, \sum t_i = 1, \text{ and } n = 1, 2, \ldots, \text{ and } i \in n\}$. Let $x \in k(X)$. Then there exist $x_1, x_2, \ldots, x_n \in X$ and
Suppose \( t, \ldots, t_n \in \mathbb{R} \) such that \( t_i > 0, \sum_{i=1}^{n} t_i = 1 \), and \( x = \sum_{i=1}^{n} t_i x_i \). Suppose without loss of generality that each \( t_i \neq 0 \). Clearly 

\[ x \in (t, X + t_2 X + \ldots + t_n X) \cap k(X), \]

and by finite mathematical induction on Theorem 2.2, \( t, X + t_2 X + \ldots + t_n X \) is an open set. Hence \( x \) is an interior point of \( k(X) \) so \( k(X) \) is open.

If \( X \) is closed, \( k(X) \) is not necessarily closed, for consider the following example (2, p. 23):

Let \( X = \{(1,1), (-1,1), \ldots, (n,1), (-n,1), \ldots\} \) in \( \mathbb{R}^2 \).

\( k(X) \) contains all the points of the form \((0,y), 0 < y < 1\), but not the point \((0,0)\), so \( k(X) \) is not closed.

---

Fig. 3—Example of a closed set whose convex hull is not closed.

The converse of Theorem 2.3 is not true, for in \( E^1 \) let \( X = (0,1) \cup \{2\} \cup (3,4) \), which is not an open set, but \( k(X) \) is the open set \((0,4)\).
Theorem 2.4 The interior of a convex set is convex or empty.

Proof: Let \( X \) be a convex set. Denote the interior of \( X \) by \( X^o \). Suppose \( X^o \neq \emptyset \). \( X^o \) is an open subset of \( X \). By Theorem 2.3 \( k(X^o) \) is open, and \( k(X^o) \subseteq k(X) = X \). Hence every point of \( k(X^o) \) is an interior point of \( X \). Therefore \( k(X^o) \subseteq X^o \). But since \( X^o \subseteq k(X^o) \), then \( X^o = k(X^o) \) so \( X^o \) is convex.

Theorem 2.5 If a subset \( U \) of an LTS is convex, then \( U + U = 2U \).

Proof: Let \( U \) be a convex set in an LTS, and \( z \) be an element of \( U + U \). There exist \( x, y \in U \) such that \( z = x + y \). Since \( U \) is convex, \( \frac{1}{2} x + \frac{1}{2} y \in U \). Therefore

\[
\frac{1}{2} x + \frac{1}{2} y = 2(\frac{1}{2} x + \frac{1}{2} y) \in 2U.
\]

Hence \( U + U \subseteq 2U \).

Let \( z' \in 2U \). There exists an element \( x' \in U \) such that

\[
z' = 2x' = (1+1)x' = x' + x' \in U + U.
\]

Hence \( 2U \subseteq U + U \) so \( U + U = 2U \).

It can be verified that if \( U \) is open, then the converse is also true.

Theorem 2.6 If \( X \) is convex, then \( \overline{X} \) is convex.
Proof: Let \( x, y \in \mathbb{X} \). There exist nets \((x_n, n \in D, \lambda),\) and \((y_m, m \in E, \eta)\) in \( X \) which converge to \( x \) and \( y \) respectively, where \( D \) and \( E \) are directed sets. Let \( \prec \mathbb{R}, \preccurlyeq \prec \mathbb{1} \). Consider the net \((x_n + (1-\alpha)y_m, (n,m) \in D \times E, \gamma)\) where \((a,b) \prec (c,d)\) if and only if \( a \preccurlyeq c \) and \( b \preccurlyeq d \). Let \( U \) be a neighborhood of \( x + (1-\alpha)y \). Since addition is continuous, there exist neighborhoods \( U_1 \) and \( U_2 \) of \( x \) and \( (1-\alpha)y \) respectively, such that if \( x' \in U_1 \) and \( y' \in U_2 \), then \( x' + y' \in U \). Since scalar multiplication is continuous, there exist neighborhoods \( V_1 \) and \( V_2 \), of \( x \) and \( y \) respectively, such that if \( x'' \in V_1 \) and \( y'' \in V_2 \), then \( x'' \in U_1 \) and \( (1-\alpha)y'' \in U_2 \). There exist elements, \( N \in D \) and \( M \in E \), such that if \((n,m) \prec (N,M)\), then \( x_n + (1-\alpha)y_m \in U \). Hence \((x_n + (1-\alpha)y_m, (n,m) \in D \times E, \gamma)\) converges to \( x + (1-\alpha)y \). Therefore \( x + (1-\alpha)y \) is either an element of \( X \) or a limit point of \( X \). Consequently \( x + (1-\alpha)y \in \overline{X} \), so \( X \) is convex.

Definition 2.2 A hyperplane \( H \) is said to cut the set \( X \) if there are \( \prec \) points of \( X \) in each of the two open half-spaces into which \( H \) separates \( L \), the two open half-spaces being \( \{x \in f(x) > \lambda\} \) and \( \{x \in f(x) < \lambda\} \); the two closed half-spaces being \( \{x \in f(x) \geq \lambda\} \) and \( \{x \in f(x) \leq \lambda\} \), where \( \mathbb{H} = \{x \in f(x) = \lambda\} \).

Theorem 2.7 Let \( X \) be a convex set in an LTS such that \( X^\circ \neq \emptyset \) and \( H \) be a hyperplane such that \( X \notin H \), then \( H \) cuts \( X \) if and only if \( X^\circ \cap H \neq \emptyset \).

Proof: (only if) Suppose \( X \) is a convex set in an LTS such that \( X^\circ \neq \emptyset \), \( H \) is a hyperplane such that \( X \notin H \), and \( H \)
cuts X. Suppose there exists an element $b \in X^\circ$ such that $b \notin H$ (if this is not true, then $X^\circ \subset H$ so $X^\circ \cap H = X^\circ \neq \emptyset$) and, without loss of generality, that $f(b) > \lambda$. There is an $a \in X$ such that $f(a) < \lambda$. There exist positive numbers $\epsilon$, and $\epsilon_1$ such that $f(b) = \lambda + \epsilon$, and $f(a) = \lambda - \epsilon_1$. Then for $\alpha = \frac{\epsilon_1}{\epsilon + \epsilon_1}$,

$$f(\alpha a + (1-\alpha)b) = \alpha \epsilon + (1-\alpha)\lambda.$$ Since $1-\alpha \neq 0$, then $(1-\alpha)X^\circ$ is an open set by Theorem 2.1. Therefore $\{\alpha a\} + (1-\alpha)X^\circ$ is open, and

$$\alpha a + (1-\alpha)b \in \{\alpha a\} + (1-\alpha)X^\circ \subset X$$ by the convexity of $X$. Hence $\alpha a + (1-\alpha)b$ is an interior point of $X$, and since $f(\alpha a + (1-\alpha)b) = \lambda$, then $\alpha a + (1-\alpha)b \in X^\circ \cap H$.

(if) Suppose $X$ is a convex set, $X^\circ \neq \emptyset$, $H$ is a hyperplane such that $X \notin H$ and $X^\circ \cap H \neq \emptyset$. Let $a \in X^\circ \cap H$. There exist a neighborhood $U$ of $a$ such that $U \subset X$, and an element $b \in X$ such that $b \notin H$. Since addition and scalar multiplication are continuous as functions of two variables, and $\lambda a + (1-\lambda)b = a \in X^\circ$ which is a subset of $X$, there exists a positive number $\delta$, such that if $1-\delta < \delta < 1+\delta$, then $\delta a + (1-\delta)b \in U \subset X$. Therefore

$$x = (1+\delta)a + (1-(1+\delta))b = (1+\delta)a + (1-\delta)b \in U \subset X.$$ Case I: Suppose $f(b) < \lambda$. There exists an $\varepsilon > 0$ such that $f(b) = \lambda - \varepsilon$. Hence

$$f(x) = \frac{(1+\delta)f(a) - \delta f(b)}{2}$$

$$= \frac{\lambda + \delta \lambda - \delta \lambda + \delta \varepsilon}{2}$$

$$= \lambda + \frac{\delta \varepsilon}{2} > \lambda.$$
Therefore \( x, b \in X \) with \( f(x) > \lambda \), and \( f(b) < \lambda \). Hence \( H \) cuts \( X \).

Case II: Suppose \( f(b) > \lambda \). There exists an \( \epsilon > 0 \) such that \( f(b) = \lambda + \epsilon \).

\[
f(x) = \frac{(1+\delta) \cdot f(a) - \delta f(b)}{2}
\]

\[
= \frac{\lambda - \epsilon + \lambda - \delta \epsilon}{2}
\]

\[
= \lambda - \frac{\delta \epsilon}{2} < \lambda.
\]

Hence, since \( x, b \in X \) and \( f(x) < \lambda, f(b) > \lambda \), then \( H \) cuts \( X \).

**Definition 2.3** Let \( K \) be a set with 0 in its core; then the Minkowski functional, \( p(x) \), is defined for each \( x \in L \) by

\[
p(x) = \operatorname{glb}\left\{ r : x \in K \text{ and } r > 0 \right\}.
\]

**Definition 2.4** A set \( M \subseteq L \) is called a linear variety if \( \bar{M} = x_0 + L_0 \), where \( x_0 \) is a fixed element, and \( L_0 \) is a subspace of \( L \).

**Definition 2.5** If a set \( S \) is contained in one of the four half-spaces determined by a hyperplane \( H \), then \( S \) is said to lie on one side of \( H \). If \( S \) lies on one side of \( H \) and \( S \cap H = \emptyset \), then \( S \) is said to lie strictly on one side of \( H \).

**Definition 2.6** A functional (function with range in \( \mathbb{R} \)), \( p \), defined on a linear space \( L \) is subadditive if

\[ p(x+y) \leq p(x) + p(y) \]

for all \( x, y \in L \); is positive homogeneous if

\[ p(rx) = rp(x) \]

for each \( r > 0 \) and each \( x \in L \); and is sublinear if it has both the above properties. A sublinear functional \( p \) is a pre-norm (or seminorm) if

\[ p(\lambda x) = |\lambda| p(x) \]

for each \( \lambda \in \mathbb{R} \).
and \( x \in L \). A pre-norm is a norm whenever \( p(x) = 0 \) if and only if \( x = 0 \). If \( p \) is a norm, then \( p(x) \) is denoted by \( \| x \| \).

**Definition 2.7** A linear space \( L \) on which a norm is defined is called a normed linear space.

A normed linear space is an LTS with a neighborhood basis, of open sets, of a point \( x \in L \) the family of all sets of the form \( \{ y : \| x - y \| < \varepsilon \} \), for \( \varepsilon > 0 \).

\[ L = \{ f : [0,1] \to R \mid f \text{ is continuous on } [0,1] \} \]

with \( [f_1 + f_2](x) = f_1(x) + f_2(x) \) and \( [\lambda f_1](x) = \lambda f_1(x) \), is a linear space over the field of real numbers, \( R \). \( L \) is a normed linear space with \( \| f \| = \operatorname{lub} \{ |f(x)| : x \in [0,1] \} \). Therefore \( L \) is an LTS with neighborhood basis defined as above.

Several theorems will be accepted for use in later proofs, for proofs of these theorems, see Taylor (3, pp. 135, 139, 141-142, 144).

**Theorem 3.42-C.** Let \( X \) be a topological linear space, and let \( \rho \) be the Minkowski functional of a set \( K \) in \( X \), where \( K \) is convex, absorbing, and contains \( 0 \) \( [0 \) is a core point of \( K \)]. Let \( K_1 = \{ x : \rho(x) < 1 \} \).

\[ K_1 = \{ x : \rho(x) < 1 \} \]

Then

a. \( \operatorname{int}(K) \subseteq K_1 \subseteq K \subseteq \overline{K} \).

b. \( K = K_1 \) if \( K \) is open.

c. \( K = K_1 \) if \( K \) is closed.

d. If \( \rho \) is continuous, \( K_1 = \operatorname{int}(K) \) and \( K_1 = \overline{K} \).

e. \( \rho \) is continuous if and only if \( 0 \in \operatorname{int}(K) \).

**Theorem 3.5-E.** Let \( X \) be a topological linear space and \( M \) a hyperplane containing \( 0 \). Then either \( M \) is a closed set, or it is everywhere dense in \( X \). Suppose \( M = \{ x : x'(x) = 0 \} \), where \( x' \) is a fixed element of \( X' [X'] \). Then \( M \) is closed if and only if \( x' \) is continuous.

**Theorem 3.6-C.** Let \( X \) be a topological linear space and \( M \) a hyperplane. Suppose that \( S \) is a set having at least one interior point and lying on one side of \( M \). Then \( M \) is closed; the interior of \( S \) lies strictly
on one side of $M$, and the closure $\overline{S}$ also lies on one side of $M$.

**Theorem 3.6-D.** Let $K$ be a convex absorbing set which contains $0$, and let $p$ be its Minkowski functional. Let $L$ be a linear variety which does not intersect $K$. Then there exists a hyperplane $M$ with equation $x'(x) = 1$ such that $M$ contains $L$ and $x'(x) < p(x)$ for every $x$. The set $K$ lies in the half space $\{x : x'(x) < 1\}$. If $X$ is a topological linear space and $K$ has interior points, $x'$ is continuous and $\overline{M}$ is closed. In this case the closure of $K$ lies in the half space $\{x : x'(x) < 1\}$, and the interior of $K$ lies in the open half space $\{x : x'(x) < 1\}$. In particular, if $K$ is open, $K$ does not intersect the closed hyperplane $\overline{M}$.

**Theorem 3.6-E.** Let $K$ be a nonempty open convex set in a topological linear space $X$, and let $L$ be a linear variety which does not intersect $K$. Then, there exists a closed hyperplane $M$ which contains $K$ and is such that $K$ lies strictly on one side of $M$.

**Theorem 2.8** If $K_1$ and $K_2$ are nonempty, nonintersecting convex sets and if $K_1$ is open, then there exists a closed hyperplane $H$ such that $K_1$ is in one of the two closed half-spaces determined by $H$ and $K_2$ is in the other. If $K_2$ is also open, then $H$ can be chosen so that $K_1$ and $K_2$ are strictly on opposite sides of $H$.

**Proof:** Let $K = K_1 + (-1)K_2$. $K$ is convex by Theorem 1.7, nonempty, open by Theorem 2.2, and does not contain $0$. Therefore (Theorem 3.6-E) there exists a closed hyperplane $M$ through $0$ such that $K$ lies strictly on one side of $M$.

$M = \{x : f(x) = 0\}$, where $f$ is a continuous (Theorem 3.5-E) element of $L'$. Let $x_1 \in K_1$ and $x_2 \in K_2$. Then $x = x_1 - x_2 \in K$. Assume, without loss of generality, that $f(x_1) = f(x_1) - f(x_2) < 0$. Hence $f(x_1) < f(x_2)$. Therefore $\text{lub}\{f(x) : x \in K_1\} < \text{glb}\{f(x) : x \in K_2\}$. 


There exists a \( c \in \mathbb{R} \) such that \( \text{lub}\{f(x): x \in K\} \leq c \leq \text{glb}\{f(x): x \in K\} \).

Let \( H = \{x: f(x) = c\} \). \( H \) is closed and \( K_c = \{x: f(x) \leq c\} \), \( K_c \subseteq \{x: f(x) \geq c\} \).

Since \( K_a \) is open, it has an interior point and therefore (Theorem 3.6-C) lies strictly on one side of \( H \). If \( K_a \) is open, then in like manner \( K_c \) lies strictly on the other side of \( H \).

**Definition 2.8** If \( S \subseteq \mathbb{L} \), a support of \( S \) is a hyperplane \( H \) such that \( S \) lies on one side of \( H \) and \( S \cap H \neq \emptyset \). If \( x_0 \in S \cap H \), it is said that \( S \) is supported by \( H \) at \( x_0 \).

**Definition 2.9** A closed convex set with a nonvacuous interior is called a convex body.

**Lemma 2.1** Let \( K \) be a convex set in an LCS \( L \). Suppose \( x_0 \in K^o \) and \( y_0 \in \overline{K} \). Then every point \( y \) expressible in the form \( y = \alpha x_0 + (1-\alpha)y_0 \), \( 0 < \alpha < 1 \), is an interior point of \( S \).

**Proof:** Let \( y = \alpha x_0 + (1-\alpha)y_0 \). Then \( x_0 = \frac{\alpha y + (\alpha-1)y_0}{\alpha} \).

There exists a neighborhood \( U \) of \( x_0 \) such that \( U \subseteq K \). There exist neighborhoods \( V_1 \) and \( V_2 \) of \( y \) and \( y_0 \) respectively, such that if \( x \in V_1 \) and \( x' \in V_2 \), then \( \frac{\alpha x + (\alpha-1)x'}{\alpha} \in U \subseteq K \). Since \( y_0 \in \overline{K} \), there exists a \( y' \in V_2 \cap K \). Therefore \( z = \frac{\alpha x + (\alpha-1)y'}{\alpha} \in U \subseteq K \).

\( x = \alpha z + (1-\alpha)y' \) is an element of \( K \), since \( K \) is convex. Hence \( V_1 \subseteq K \), so \( y \) is an interior point of \( K \).

**Corollary:** If \( K \neq \emptyset \), then \( \overline{K}^o = \overline{K} \).
Theorem 2.9  If K is a convex body in an LTS, then a support of K is closed and K is supported at every boundary point.

Proof: Suppose H is a support of K. Then H is closed as follows from Theorem 3.6-C. Let $x_0$ be a boundary point of K, i.e. $x_0 \in \partial K \setminus K$. There exists an $x_1 \in K$ and a convex open set $V_0$ containing 0 such that $K^o = x_1 + V_0$ (Theorems 1.7 and 2.2). $K = \overline{K^o} = x_1 + \overrightarrow{V_0}$. There exists an $x_0' \in \overline{V_0} \cap \overline{L \setminus V_0}$ such that $x_0 = x_1 + x_0'$. Let $p(x)$ be the Minkowski functional of $V_0$. There exists a hyperplane $M$ with equation $f(x) = 1$ such that $x_0 \in M$ and $f(x) < p(x)$ for all $x \in L$. $V_0 \subseteq \{x : f(x) \leq 1\}$ and $\overline{V_0} \subseteq \{x : f(x) \leq 1\}$ as follows from Theorem 3.6-D, with $x_0'$ being the linear variety. Let $H = \{x : f(x) = f(x_1) + 1\} = x_1 + M$. Since $f(x_0) = f(x_1) + f(x_0') = f(x_1) + 1$, $x_0 \in H$. If $x \in K$ then there exists an $x' \in \overline{V_0}$ such that $x = x_1 + x'$. $f(x) = f(x_1) + f(x') \leq f(x_1) + 1$. Hence H is a support hyperplane at $x_0$.

Theorem 2.10  Let K be a convex body in an LTS L, K \not\subseteq L. Consider the closed half-spaces containing K and determined by the supports of K. The intersection of all these half-spaces is K.

Proof: Let A be the intersection of all the closed half-spaces containing K and determined by the supports of K. Clearly $K \subseteq A$. Let $x_1 \in A$ and suppose by way of contradiction that $x_1 \notin K$. There exist an $x_0 \in K$ and a convex open set
$V_0$ containing 0 such that $K^o = x_0 + V_0$. Let $p(x)$ be the Minkowski functional of $V_0$. There exists a $z \in V_0$ such that $x_i = x_0 + z$ ($K = x_0 + V_0$). Since $\overline{V_0} = \{x : p(x) \leq 1\}$, $p(z) > 1$.

There exists an $\varepsilon > 0$ such that $p(z) = 1 + \varepsilon$. Since $p(\frac{1}{1+\varepsilon} z) = 1$, then $\frac{1}{1+\varepsilon} z \in \overline{V_0} \cap \overline{L \setminus V_0}$. Hence there is a support hyperplane $M$ through $\frac{1}{1+\varepsilon} z$ such that $f(\frac{1}{1+\varepsilon} z) = 1$ and if $x \in \overline{V_0}$ then $f(x) \leq 1$.

\[
\frac{f(\frac{1}{1+\varepsilon} z)}{1+\varepsilon} = \frac{1}{1+\varepsilon} f(z) = 1,
\]

Hence $f(z) = 1 + \varepsilon$.

Let $H = \{x : f(x) = f(x_0) + 1\}$. If $x \in K$ then there exists an $x' \in \overline{V_0}$ such that $x = x_0 + x'$.

\[
f(x) = f(x_0) + f(x') \leq f(x_0) + 1.
\]

\[
f(x_0 + \frac{1}{1+\varepsilon} z) = f(x_0) + 1.
\]

Hence $H$ is a support of $K$ at $x_0 + \frac{1}{1+\varepsilon} z$.

\[
f(x_i) = f(x_0 + z) = f(x_0) + 1 + \varepsilon > f(x_0) + 1.
\]

Therefore $x_i \notin H$. This contradicts the assumption that $x_i \in A$.

Hence $x_i \in K$. Therefore $A \subset K$ so $A = K$. 

CHAPTER BIBLIOGRAPHY


CHAPTER III

COMPACT CONVEX SETS

In this chapter topological properties, leading to the Krein-Mil'man theorem, will be examined.

**Definition 3.1** A family \( a \) of open sets is an open cover of a set \( X \), if every element of \( X \) is in some member of \( a \), i.e. \( X \subseteq \bigcup \{ A \colon A \in a \} \). A subfamily of \( a \) which also covers \( X \) is called a subcover of \( X \).

**Definition 3.2** A subset \( X \) of a topological space is compact if and only if every open cover of \( X \) has a finite subcover. For proof of the following see Kelly (2, p. 136).

A topological space [and consequently any subset of a topological space under the relative topology] is compact if and only if each family of closed sets which has the finite intersection property [a family of sets in which the intersection of any finite subfamily is not empty] has a non-void intersection.

\( X \) is compact if and only if each net in \( X \) has a subnet which converges to some point of \( X \).

**Theorem 3.1** If \( X \) is closed and \( Y \) is compact in an LTS, then \( X+Y \) is closed.

Proof: Let \( z \) be a limit point of \( X+Y \). There exists a net \( (a_n+b_n, n \in D, \varepsilon) \) of elements of \( X+Y \), where \( a_n \in X \) and \( b_n \in Y \), which converges to \( z \). Since \( Y \) is compact, there exists a subnet \( (b_m, m \in E, \varepsilon) \) of \( (b_n, n \in D, \varepsilon) \) which converges to a point \( y \in Y \). Let \( (x_m, m \in E, \varepsilon) \) be the corresponding subnet of
Let $U$ be a neighborhood of $z-y$. There exists a neighborhood $V$ of $0$ such that $U = z - y + V$, and a neighborhood $V_x$ of $0$ such that $V_x + V_x \subset V$. Since $(-1)V_x \subset V$, and $(-1)V_x$ is an open set containing $0$, $y + (-1)V_x$ is a neighborhood of $y$. Therefore there exists an $N, \epsilon \in \mathbb{E}$ such that if $n \in \mathbb{N}$, then $y_n \in y + (-1)V_x$. Since $z + V_x$ is a neighborhood of $z$, there exists an $N_z, \epsilon \in \mathbb{E}$ such that if $n \in \mathbb{N}$, then $x_n + y_n \in z + V_x$. Therefore there exists an $N, \epsilon \in \mathbb{E}$ such that if $n \in \mathbb{N}$, then $y_n = y - v_n$, and $x_n + y_n = z + v_n$. Therefore

$$x_n = z - y_n + v_n \quad = z - y - (-v_n) + v_n \quad = z - y + (v_n + v_n) \in (z - y + V_x + V_x) \subseteq z - y + V = U.$$ 

Hence $(x_n, m \in \mathbb{E}, n) \in X + Y$ converges to $z - y$. Therefore $z - y$ is a limit point of $X$, and since $X$ is closed, $z - y \in X$. Since $(z - y) + y = z$, then $z \in X + Y$. Consequently $X + Y$ is closed.

Compactness is a necessary condition, for consider the following example in $\mathbb{E}^2$.

Let $X = \left\{ \left(-1, \frac{1}{n}\right), \left(-\frac{1}{2}, \frac{1}{n}\right), \ldots, \left(-\frac{1}{n}, \frac{1}{n}\right), \ldots \right\}$

$$Y = \left\{ \left(1, \frac{1}{n}\right), \left(2, \frac{1}{n}\right), \ldots, \left(n, \frac{1}{n}\right), \ldots \right\}.$$ 

Both $X$ and $Y$ are closed but neither $X$ nor $Y$ is compact. \{(0,2), (0,1), (0,2), \ldots, (0, 2), \ldots \} \in X + Y$. $(0,0)$ is a limit
point of this set, and therefore of \( X+Y \) and \((0,0) \notin X+Y \).

Therefore \( X+Y \) is not closed.

**Lemma 3.1** Let \( A_1, A_2, \ldots, A_n \) be convex sets in an LTS.

Let \( A = A_1 \cup A_2 \cup \ldots \cup A_n \). Then \( k(A) = \left\{ \sum t_i x_i : x_i \in A_i, \sum t_i = 1 \right\} \).

**Proof:** Let \( X = \left\{ \sum t_i x_i : x_i \in A_i, \sum t_i = 1 \right\} \). It follows from Theorem 1.8 that \( X \cap k(A) \). Suppose \( x \in k(A) \), then \( x = t, x_1 + t_2 x_2 + \ldots + t_n x_n \) where \( t_i > 0, \sum t_i = 1 \) and \( x_i \in A_i \). Let \( A_j \) be the subset (possibly empty) of the set of \( x_i \) in the expansion of \( x \) such that \( A_j \cap A_j \) for \( j = 1, 2, \ldots, n \).

If \( A_j \neq \emptyset \), say \( A_j = \{ x_j, x_{j1}, \ldots, x_{jn} \} \), define \( t_j = \sum_{i \in k} t_{ij}t_i \) where \( t_{ij} \) is the coefficient of \( x_{ij} \) in the expansion of \( x \). Define \( x_j = \sum_{i \in k} \left( \frac{t_{ij}x_i}{t_j} \right) \). Now \( t_j > 0, \sum_{i \in k} \frac{t_{ij}x_i}{t_j} = 1 \), and \( x_j \in A_j \), so \( x_j \in A_j \) since \( k(A_j) = A_j \).

If \( A_j = \emptyset \), define \( t_j = 0 \) and let \( x_j' \) be an element of \( A_j \).

\[ t_j x_j = t_j, x_{j1} + t_j, x_{j1} + \ldots + t_j, x_{jn}. \]

Clearly \( x = \sum_{j \in n} t_j x_j' \) where \( t_j > 0, \sum_{j \in n} t_j = 1 \) and \( x_j \in A_j \). Hence \( x \in X \) so \( k(A) \cap X \). Therefore \( k(A) = X \).

**Theorem 3.2** Let \( A_1, A_2, \ldots, A_n \) be compact convex sets in an LTS. Let \( A = A_1 \cup A_2 \cup \ldots \cup A_n \). Then \( k(A) \) is compact.
Proof: Let \((z_m, m \in D, \gamma)\) be a net of elements of \(k(A)\).

For each \(j \in D\), \(z_j = t_j x_j + t_2 x_2 + \ldots + t_n x_n\) where \(t_j \geq 0\), \(\Sigma t_j = 1\), \(i \in n\) and \(x_j \in A_j\). Since \([0,1]\) is a compact set in \(R\), there exists a subnet \((t_{n_1}^{'}, n \in E, \gamma)\) of \((t_m, m \in D, \gamma)\) which converges to a point \(t \in [0,1]\). Let \((x_{n_1}^{'}, n \in E, \gamma), (t_{n_2}^{'}, n \in E, \gamma), (x_{n_2}^{'}, n \in E, \gamma), \ldots, (t_{n_k}^{'}, n \in E, \gamma), (x_{n_k}^{'}, n \in E, \gamma)\) be the corresponding subnets of \((x_m, m \in D, \gamma)), (t_m, m \in D, \gamma)), (x_m, m \in D, \gamma), \ldots, (t_m, m \in D, \gamma), (x_m, m \in D, \gamma), (x_m, m \in D, \gamma).\)

Since \(A_i\) is compact, so there exists a subnet \((x_{n_k}^{'}, k \in E, \gamma)\) which converges to a point \(x \in A_i\). Let \((t_{n_k}^{'}, k \in E, \gamma), (t_{n_{k+1}}^{'}, k \in E, \gamma), (x_{n_{k+1}}^{'}, k \in E, \gamma), \ldots, (t_{n'}^{'}, k \in E, \gamma), (x_{n'}^{'}, k \in E, \gamma)\) be the corresponding subnets of \((t_n, n \in E, \gamma), (t_{n_1}^{'}, n \in E, \gamma), (x_n, n \in E, \gamma), \ldots, (t_n, n \in E, \gamma), (x_n, n \in E, \gamma)\). Note that \((t_{n_k}^{'}, k \in E, \gamma)\) converges to \(t\), and \(\Sigma t_{n_k}^{'}, k \in n\) this process subnets \((t_{n_k}^{'}, s \in D, \gamma)\) and \((x_{n_k}^{'}, s \in D, \gamma)\) are found such that \((t_{n_k}^{'}, s \in D, \gamma)\) converges to \(t \in [0,1]\) and \((x_{n_k}^{'}, s \in D, \gamma)\) converges to \(x \in A_i\). Note that \(t \geq 0\), and \(\Sigma t \leq 1\). Therefore \(t, x, +t_2 x_2 + \ldots + t_n x_n k(A)\). The subnet \((t_{n_k}^{'}, x_{n_k}^{'}, s \in D, \gamma)\) of \((z_m, m \in D, \gamma)\) is readily seen to converge to \(t, x, +t_2 x_2 + \ldots + t_n x_n\). Therefore \(k(A)\) is compact.

**Definition 3.3** A metric space is a set \(X\), for which a function, \(d\), is defined, whose domain is \(X \times X\) and whose range is a subset of \(\mathbb{R}\), which satisfies the following conditions:

(a) \(d(x, y) = d(y, x)\) for all \(x, y \in X\),

(b) \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\),
(c) \(d(x, y) = 0\) if and only if \(x = y\),
(d) \(d(x, y) \geq 0\) for all \(x, y \in X\).

**Definition 3.4** A sequence \(\{x_n\}\) in a metric space \(X\) is Cauchy if, given \(\varepsilon > 0\), there exists a positive integer \(N\) such that if \(n\) and \(m\) are positive integers greater than \(N\), then \(d(x_n, x_m) < \varepsilon\). If every Cauchy sequence in \(X\) converges to some point of \(X\), then \(X\) is said to be a complete metric space. Otherwise \(X\) is said to be an incomplete metric space.

If \(X\) is an incomplete metric space then there exists a complete metric space \(Y\) such that \(X \subset Y\) and \(X\) is dense in \(Y\), i.e. \(\overline{X} = Y\). \(Y\) is called the completion of \(X\) and is denoted by \(\widehat{X}\). See Taylor (4, pp. 74-75).

**Definition 3.5** If \(\widehat{S}\) is compact, then \(S\) is called precompact.

For proof of the following, see Taylor (4, p. 78).

"A nonempty set \(S\) in a metric space \(X\) is precompact if and only if to each \(\varepsilon > 0\) corresponds some finite set \(x_1, \ldots, x_n\) of points of \(X\) such that \(S\) is contained in the union of the spheres of radius \(\varepsilon\) with centers at \(x_1, \ldots, x_n\). [The sphere (open sphere) with radius \(\varepsilon\) and center at \(x_0\) is \(\{y \in X : d(x_0, y) < \varepsilon\}\}.]

**Definition 3.6** If a normed linear space \(L\) is complete under the metric \(\|x - y\|\), then \(L\) is called a Banach space. A normed linear space is an LTS with a neighborhood basis of open sets of a point \(x \in L\) the family of sets \(\{y : \|x - y\| < \varepsilon\}\)."
Theorem 3.4: In a normed linear space \( L \), the convex hull of a precompact set \( S \) is precompact.

Proof: Suppose \( S \) is a precompact set in \( L \). Let \( \epsilon > 0 \).

There exist \( x_1, x_2, \ldots, x_n \in L \) such that \( S \) is contained in the union \( B \) of the open spheres of radii \( \frac{\epsilon}{2} \) with centers \( x_1, x_2, x_3, \ldots, x_n \). Let \( A = \{ x_1, x_2, \ldots, x_n \} \). Clearly \( k(S) \subseteq k(B) \).

Suppose \( z \in k(B) \), then \( z = t_1 z_1 + t_2 z_2 + \cdots + t_m z_m \) where \( t_i > 0 \), \( \Sigma t_i = 1 \) and \( z_i \in B \). For each \( z_i \) there is an \( x_i \in A \) such that \( \| x_i - z_i \| < \frac{\epsilon}{2} \).

Let \( x = \Sigma t_i x_i \in k(A) \)

\[
\| x - z \| = \| \Sigma t_i x_i - \Sigma t_i z_i \|
\]

\[
= \| \Sigma t_i (x_i - z_i) \|
\]

\[
\leq \Sigma t_i \| x_i - z_i \|
\]

\[
< \frac{\epsilon}{2} \Sigma t_i
\]

\[
= \frac{\epsilon}{2}
\]

Therefore \( k(B) \) is contained in the union of open spheres of radii \( \frac{\epsilon}{2} \) with centers in \( k(A) \). Since \( k(A) \) is compact, \( k(A) \) is precompact. Therefore there exists a finite set \( y_1, \ldots, y_k \) of elements of \( L \) such that \( k(A) \) is contained in the union of open spheres of radii \( \frac{\epsilon}{2} \) with centers at \( y_1, \ldots, y_k \). Clearly \( k(B) \) is contained in the union of all spheres of radii \( \frac{\epsilon}{2} \) with centers at \( y_1, \ldots, y_k \). Since \( k(S) \subseteq k(B) \), then \( k(S) \) is
contained in the union of all spheres of radii \( \epsilon \) with centers at \( y, \ldots, y_n \). Hence \( k(S) \) is precompact.

**Definition 3.7** The closed convex hull of a set \( X \), denoted by \( K(X) \), is the closure of \( k(X) \).

**Theorem 3.5** In a Banach space \( L \), the closed convex hull of a precompact set is compact.

**Proof:** Let \( X \) be a precompact set in a Banach space \( L \). As follows from the preceding theorem, \( k(X) \) is precompact. By definition, \( \overline{k(X)} \) is compact. Since \( L \) is complete,

\[
k(X) = \overline{k(X)} = K(X).
\]
Therefore \( K(X) \) is compact.

**Theorem 3.6** Let \( A \) and \( B \) be disjoint sets in an LTS \( L \). Assume \( A \) compact and \( B \) closed. Then there exists a neighborhood \( V \) of 0 such that \( A+V \) and \( B+V \) are disjoint.

**Proof:** Assume by way of contradiction that

\[
(A+V) \cap (B+V) \neq \emptyset
\]
no matter how \( V \) is chosen as a neighborhood of 0. If \( V \) is symmetric, then \( A \cap (B+V+V) \neq \emptyset \), for there exists \( a \in A, b \in B \) and \( v, v_1 \in V \) such that \( a+v = b+v_2 \). Hence

\[
a = b+v_1 -v_1 \in B+V+V.
\]

Let \( V, V_1, \ldots, V_n \) be symmetric neighborhoods of 0.

\[
V = \cap V_i \text{ is a symmetric neighborhood of } 0. \text{ Therefore if not } A \cap (B+V+V) \neq \emptyset.
\]
\[ A \cap (B+V+V) = A \cap \bigcap_{i=1}^{n} (B+V_i+V_i) \]
\[ = \bigcap_{i=1}^{n} [A \cap (B+V_i+V_i)] \neq \emptyset. \]

Hence \( \bigcap_{i=1}^{n} [A \cap (B+V_i+V_i)] \neq \emptyset \). Note that \( \bigcap_{i=1}^{n} [A \cap (B+V_i+V_i)] \subseteq A. \)

Therefore, since \( A \) is compact, there exists an \( x_0 \in \bigcap_{i=1}^{n} [A \cap (B+V_i+V_i)] \) where \( U \) is the neighborhood basis at \( 0. \)

Hence for any symmetric neighborhood \( V \) of \( 0, x_0+V \) must intersect \( A \cap (B+V+V). \)

Let \( U \) be a neighborhood of \( x_0 \). There exists a symmetric neighborhood \( V_0 \) of \( 0 \) such that \( x_0+V_0 \subset U. \) There exists a symmetric neighborhood \( V_1 \) of \( 0 \) such that \( V_1+V_0 \subset V_0, \) and there exists a symmetric neighborhood \( V_2 \) of \( 0 \) such that \( V_2+V_0 \subset V_1 \subset V_0. \) Hence there exist \( v_1, v_2, v_3 \in V_2 \) and \( b \in B \) such that \( x_0+v_1 = b+v_1+v_2. \) Note that \( -(v_2+v_3) \in V_1. \) Hence \( b = x_0+v_1-(v_2+v_3). \) Therefore \( b \in x_0+V_1+V_0 \subset x_0+V_0 \subset U. \) Hence \( x_0 \) is a limit point of \( B, \) and therefore an element of \( B \) since \( B \) is closed. This contradicts the hypothesis that \( A \cap B = \emptyset. \) Therefore there exists a neighborhood \( V \) of \( 0 \) such that \( (A+V) \cap (B+V) = \emptyset. \)

Compactness is necessary, for consider the following example in \( \mathbb{R}^2 \) under the usual topology.

\[ A = \{(1,\frac{1}{2}), (2,\frac{1}{2}), \ldots, (n,\frac{1}{2n}), \ldots\} \]

\[ B = \{(1,\frac{1}{3}), (2,\frac{1}{3}), \ldots, (n,\frac{1}{n+1}), \ldots\} \]
Clearly there is no neighborhood $V$, of $0$ such that
\[(A+V)\cap(B+V) = \emptyset.\]

Fig. 4—An example showing necessity of compactness in
Theorem 3.6.

If $A$ or $B$ were finite, and therefore compact, in the
above example, say $A = \{(1,1),(2,1),\ldots,(n,1)\}$, then
$V = \{(x,y):\sqrt{x^2+y^2} < \frac{1}{2}\frac{1}{\sqrt{n(n+1)}}\}$ is a neighborhood of $0$ such that
$(A+V)\cap(B+V) = \emptyset$.

**Theorem 3.7** Let $L$ be an LCS. Let $K_1,K_2$ be nonempty,
nonintersecting convex sets in $L$ with $K_1$ closed and $K_2$ com-
 pact. Then there exists a closed hyperplane $H$ such that $K_1$
and $K_2$ lie strictly on opposite sides of $H$.

Proof: From the previous theorem, there exists an open
neighborhood $U$, of $0$ such that $(K_1+U)\cap(K_2+U) = \emptyset$. Since $L$
is locally convex, then there exists a convex neighborhood
$V$, of $0$ such that $V\subset U$. Then $K_1+V$ and $K_2+V$ are open, convex,
and \((K_1 + V) \cap (K_2 + V) = \emptyset\). As follows from Theorem 2.8, there exists a closed hyperplane \(H\) such that \(K_1 + V\) and \(K_2 + V\) lie strictly on opposite sides of \(H\). There exists an \(f \in L^\#\) and an \(\alpha \in \mathbb{R}\), such that \(H = \{x : f(x) = \alpha\}\). Suppose, without loss of generality, that \(K_1 + V \subset \{x : f(x) < \alpha\}\) and \(K_2 + V \subset \{x : f(x) > \alpha\}\).

Let \(x_1 \in K_1\), \(x_2 \in K_2\). Then, since \(0 \in V\),

\[
f(x_1) = f(x_1 + 0) < \alpha \quad < f(x_2 + 0) = f(x_2).\]

Therefore \(K_1\) and \(K_2\) lie strictly on opposite sides of \(H\).

**Theorem 3.8** If \(K\) is a closed convex set in an LTS \(L\), then \(K\) is the intersection of all the closed half-spaces which contain \(K\).

**Proof:** Let \(A\) be the intersection of all the closed half-spaces which contain \(K\). Clearly \(K \subseteq A\). Suppose, by way of contradiction, that there exists an \(x_0 \notin A\) such that \(x_0 \notin K\).

Now \(\{x_0\}\) is compact and convex, \(K\) is closed, and \(\{x_0\} \cap K = \emptyset\).

Hence from the preceding theorem, there exists a closed hyperplane \(H\) such that \(K\) and \(\{x_0\}\) lie strictly on opposite sides of \(H\). \(H = \{x : f(x) = \alpha\}\). Suppose, without loss of generality, that \(K \subset \{x : f(x) < \alpha\}\) and \(K \subset \{x : f(x) > \alpha\}\). Then \(f(x_0) > \alpha\).

Hence, since \(x_0 \notin \{x : f(x) < \alpha\}\), then \(x_0 \notin A\). This contradicts the assumption that \(x_0 \notin A\) and \(x_0 \notin K\). Therefore \(x_0 \notin K\) so \(A \cap K = \emptyset\).

Hence \(A = K\).
Theorem 3.2 Let $K$ be a nonempty compact convex set in an LCS $L$. Consider the closed half-spaces which contain $K$ and are determined by the supports of $K$. The intersection of these half-spaces is $K$.

Proof: Let $A$ be the intersection of the closed half-spaces which contain $K$ and are determined by the supports of $K$. Clearly $K \subseteq A$. Let $x_0 \in L$, $x_0 \notin K$. Since $K$ is compact and $\{x_0\}$ is closed, there exists a closed hyperplane $M = \{x : f(x) = \lambda\}$ such that $K$ and $\{x_0\}$ lie strictly on opposite sides of $M$, as follows from Theorem 3.7. As follows from Theorem 3.6-D, $f$ is continuous. Suppose, without loss of generality, that $K \subseteq \{x : f(x) < \lambda\}$. Let $D = \{x : f(x) < \lambda\}$, and $\beta = \text{lub } D$. Since $K$ is compact and $f$ is continuous, then $D$ is compact (4, p. 63). Therefore $D$ is closed, and since $\beta$ is obviously a limit point of $D$, $\beta \in D$. Hence there exists an $x' \in K$ such that $f(x') = \beta$. If $x \in K$ then $f(x) < \beta$. Since $f(x_0) > \lambda$, $f(x_0) > \beta$. Therefore $\mathcal{H} = \{x : f(x) = \beta\}$ is a supporting hyperplane of $K$ and $x_0 \in \{x : f(x) > \beta\}$. Hence $x_0 \notin A$. Therefore $A \subseteq K$, so $A = K$.

Definition 3.8 Let $X$ be a convex subset of a linear space $L$. A point $x \in X$ is a passing point of $X$ if $x$ belongs to an open segment which is contained in $X$. A point of $X$ which is not a passing point of $X$ is an extreme point of $X$.

Lemma 3.2 If $K$ is a compact convex set in an LCS and $H$ is a closed support hyperplane of $K$, then $H$ contains an extreme point of $K$. 
Proof: Let \( K \) be a compact convex set in an LCS and let \( H \) be a closed support hyperplane of \( K \). Since \( K \) is compact, \( K \) is closed, so \( K' = K \cap H \subseteq K \) is closed and therefore compact. Clearly \( K' \) is convex. Day (1, pp. 14-78) shows that every compact convex set in an LCS contains an extreme point. Hence \( H \) contains an extreme point \( z \) of \( K' \). It is readily seen that \( z \) is an extreme point of \( K \). Consequently \( H \) contains an extreme point of \( K \).

For different proofs of the following theorem, called the Krein-Mil'man theorem, see Day (1, pp. 78-79) or Krein and Mil'man (3).

**Theorem 3.10** Let \( K \) be a compact, convex set in an LCS and let \( E \) be the set of extreme points of \( K \). Then \( K = \text{conv}(E) \).

**Proof:** Since \( E \subseteq K \), then \( K(E) \) is a subset of the closed convex hull of \( K \). Therefore, since \( K \) is the closed convex hull of \( K \), \( K(E) \subseteq K \). Suppose, be way of contradiction, that there exists a point \( x \in K \) such that \( x \notin K(E) \). It follows from Theorem 3.7 that there exists a closed hyperplane \( M = \{ x : f(x) = \lambda \} \), such that \( \{ x_0 \} \) and \( K(E) \) lie strictly on opposite sides of \( M \). Suppose, without loss of generality, that \( f(x_0) < \lambda \). Since \( M \) is closed, \( f \) is continuous. Therefore, since \( K \) is compact, \( A = \{ \alpha : \text{for some } x \in K, f(x) = \alpha \} \) is compact, and consequently, is closed and bounded. Let \( c \) be the \( \text{glb} \) of \( A \). Clearly \( c \in A \). Hence there exists a \( z \in K \) such that \( f(z) = c \) and if \( x \in K, f(x) > c \). Therefore
$H = \{x: f(x) = c\}$ is a support hyperplane of $K$ at $z$. $H$ therefore contains an extreme point $y$ of $K$. This implies $f(y) = c < \lambda$ which is the desired contradiction, since if $x \in E, f(x) > \lambda$. Consequently if $x \in K$, then $x \in K(E)$ so $K \subset K(E)$. Therefore $K = K(E)$. 
CHAPTER BIBLIOGRAPHY


BIBLIOGRAPHY

Books


ARTICLES