THE COMPARABILITY OF CARDINALS

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The purpose of this composition is to develop a rigorous, axiomatic proof of the comparability of the cardinals of infinite sets; that is, if each of $\alpha$ and $\beta$ denotes the cardinal of an infinite set, then exactly one of the following statements is true: $\alpha \leq \beta$, $\beta < \alpha$, or $\alpha = \beta$. Since the comparability of the cardinals of finite sets can be established so easily, they will not be considered in this paper. Generally, proofs of the comparability of cardinals are based on the Axiom of Choice, and the Theory of Ordinal Numbers; however, in this paper there will be no direct reference to either of these mathematical concepts.

Preparatory to the pursuance of the objective of this paper several elementary definitions and theorems concerning functions will be presented in order to provide a foundation for further development.

Definition 1. A function $f_1$ is a set of ordered pairs $(a,b)$ such that if each of the ordered pairs $(a,b)$ and $(c,d)$ is an element of $f_1$ and $a=c$, then $b=d$.

Definition 2. A reversible function $f_2$ is a function such that $f_2^{-1}$ is a function, where the ordered pair $(b,a)$ is an element of $f_2^{-1}$ if and only if $(a,b)$ is an element of $f_2$.

Definition 3. If a function $f$ is the set of ordered pairs $(a,b)$, then $c$ is an element of the domain of $f$ if and only if for some ordered pair $(a,b)$, $c=a$. And $d$ is an
element of the range of f if and only if for some ordered pair \((a, b)\), \(d = b\).

**Definition 4.** If each of \(f\) and \(g\) denotes a function, and the domain of \(f\) intersects the range of \(g\), then the composition of \(f\) and \(g\), \(fg\), is the set of all ordered pairs \((a, b)\) such that for some \(c\), \((a, c)\) is an element of \(f\), and \((c, b)\) is an element of \(g\).

**Theorem 1.** If each of \(f\) and \(g\) denotes a reversible function, then \(fg\) is a reversible function.

1.1 If the set of ordered pairs denoted by \(fg\) is not a function, then for some \(a\) the ordered pairs \((a, b)\) and \((a, c)\) are elements of \(fg\), where \(b \neq c\).

1.2 By definition of \(fg\) there exist two elements \(d\), and \(e\), such that \((a, d)\) and \((a, e)\) are elements of \(f\) and \((d, b)\) and \((e, c)\) are elements of \(g\).

1.3 Since \(f\) denotes a function, only one of the two ordered pairs \((a, d)\) and \((a, e)\) is an element of \(f\).

1.4 Thus, the assumption that the set of ordered pairs denoted by \(fg\) is not a function leads to a contradiction of the definition of \(fg\). Hence, \(fg\) is a function.

1.5 Clearly, by a similar procedure, it can readily be established that \((fg)^{-1}\) is a function.

**Theorem 2.** If each of \(f\), \(g\), \(h\) denotes a function, then \(f(gh) = f(gh)\).
2.1 If \((a,d)\) is an element of \(f(gh)\), then for some \(b\), \((b,d)\) is an element of \(gh\), and \((a,b)\) is an element of \(f\).

2.2 Since \((b,d)\) is an element of \(gh\) for some \(c\), \((b,c)\) is an element of \(g\), and \((c,d)\) is an element of \(h\).

2.3 Hence, \((a,b)\), \((b,c)\), \((c,d)\) are elements of \(f\), \(g\), \(h\) respectively.

2.4 Due to statement 2.3, \((a,c)\) is an element of \(fg\), and \((a,d)\) is an element of \((fg)h\).

2.5 Thus, if \((a,d)\) is an element of \(f(gh)\), then \((a,d)\) is an element of \((fg)h\).

2.6 Since this procedure can be followed to establish that if \((a,d)\) is an element of \((fg)h\), then \((a,d)\) is an element of \(f(gh)\); \(f(gh)=(fg)h\).

Remark: Since if \((a,b)\) is an element of a reversible function \(f\), \((b,a)\) is an element of \(f^{-1}\), thus if \(f(a)=b\), \(f^{-1}(b)=a\).

Now that the preliminary definitions and theorems have been presented, it is possible to construct the foundation of the argument, which consists of a discussion of a relation on a class of sets, equivalence.

Definition 5. If each of \(A\) and \(B\) denotes a set, and there exists a reversible function whose domain is \(A\) and whose
range is $B$, then the set denoted by $A$ will be said to be equivalent to the set denoted by $B$; or symbolically, $A \sim B$.

**Definition 6.1**  
A relation is a set of ordered pairs $(a,b)$.

**6.2** If a relation $R$ is symmetric then if $(a,b)$ is an element of $R$, $(b,a)$ is an element of $R$.

**6.3** If a relation $R$ is transitive then if each of $(a,b)$ and $(b,c)$ is an element of $R$, $(a,c)$ is an element of $R$.

**6.4** If a relation $R$ is reflexive then if $(a,b)$ is an element of $R$, each of $(a,a)$ and $(b,b)$ is an element of $R$.

**Definition 7.**  
An equivalence relation is a relation that is symmetric, reflexive and transitive.

**Remark:**  
By virtue of the preceding definitions of the properties of reflexivity, symmetry and transitivity, if a relation $R$ is symmetric and transitive, then if $(a,b)$ is an element of $R$, $(b,a)$ is an element of $R$, consequently each of $(a,a)$ and $(b,b)$ is an element of $R$, and $R$ is also reflexive. Hence, if $R$ is symmetric and transitive, $R$ is an equivalence relation.

**Theorem 3.**  
The relation equivalence "$\sim\$" is an equivalence relation.
3.1 If each of $A$ and $B$ denotes a set and $A \sim B$, then there exists a reversible function $f$ whose domain is $A$, and whose range is $B$. Since $f$ is a reversible function, $f^{-1}$ is a reversible function whose domain is $B$ and whose range is $A$, thus $B \sim A$; "$\sim$" is symmetric.

3.2 If each of $A$, $B$, and $C$, denotes a set and $A \sim B$, and $B \sim C$; then there exists a reversible function $f$ whose domain is $A$ and whose range is $B$; and a reversible function $g$ whose domain is $B$ and whose range is $C$. If $a$ is an element of $A$, there exists an element of $B$, $b$, such that $(a,b)$ is an element of $f$. Since $b$ is an element of $B$, there exists an element of $C$, $c$ such that $(b,c)$ is an element of $g$. Hence, the domain of $fg$ is $A$. By a similar argument it can be established that the range of $fg$ is $C$. Thus, $A \sim C$; and "$\sim$" is transitive.

3.3 Since $\sim$ is symmetric and transitive, $\sim$ is an equivalence relation.

The next consideration, commonly referred to as Bernstein's Theorem is a theorem of utmost importance in Set Theory.

Theorem 8. If each of $A$ and $B$ denotes a set and $A \sim B_1$, and $B \sim A_1$, then $A \sim B$, where $B_1$ denotes a proper subset of $B$; and $A_1$ denotes a proper subset of $A$. 
Before proceeding with a proof of the theorem, two terms which will facilitate the development of the proof will be defined.

1. If each of \( A \) and \( B \) denotes a set, and if \( f \) denotes a function, then "\( f(A)=B \)" means that if \( a \) is an element of \( A \), there exists only one element of \( B \), \( b \), such that \( f(a)=b \); and if \( b \) is an element of \( B \), then there exists only one element of \( A \), \( a \), such that \( f(a)=b \).

2. If each of \( A \) and \( B \) denotes a set, then "\( A \supset B \)" means that \( B \) denotes a proper subset of the set denoted by \( A \).

8.1 Since \( A \sim B_1 \), and \( B \sim A_1 \); there exists a reversible function \( f \), and a reversible function \( g \), such that:
\[
f(B)=A_1, \quad g(A)=B_1
\]

8.2 Since \( A \supset A_1 \), and \( B \supset B_1 \), \( fg(A)=A_2 \), \( A_1 \supset A_2 \),
\[
fg(A_1)=A_3, \quad A_2 \supset A_3, \quad fg(A_2)=A_4, \quad A_3 \supset A_4.
\]
Thus, a sequence of subsets of \( A \) can be constructed such that: \( A=A_0 \supset A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots \), where \( n \) is a natural number. And since the composition of two reversible functions is a reversible function:

\[
A_0 \sim A_2 \sim A_4 \sim \cdots A_{2n} \sim \cdots
\]

\[
A_1 \sim A_3 \sim A_5 \sim \cdots A_{2n+1} \sim \cdots
\]

8.3 Since \( fg(A_{2n})=A_{2n+2} \), if \( A_{2n} \) is an element of \( A_{2n} \) there exists an element of \( A_{2n+2} \), \( a_{2n+2} \), such
that \( f_g(a_{2n}) = a_{2n+2} \) and for the same reason,
\( f_g(a_{2n+1}) = a_{2n+3} \). Hence if \( a_{2n} \) is an element of
\( A_{2n} \) and is not an element of \( A_{2n+1} \), \( f_g(a_{2n}) \) is not
an element of \( A_{2n+3} \). Thus \( f_g(A_{2n} - A_{2n+1}) = A_{2n+2} - A_{2n+3} \).

Obviously, \( A_{2n+1} = A_{2n} \sim A_{2n+2} \).

If there exists a set \( K \) such that for all \( n \), \( A_n \supseteq K \),
then let \( T \) denote the identity function such that
\( T(K) = K \).

Furthermore, let \( f_{2n} \) denote the reversible function
whose domain is \( A_{2n} \sim A_{2n+1} \) and whose range is
\( A_{2n+2} \sim A_{2n+3} \); and let \( f_{2n+1} \) denote the reversible
function whose domain is \( A_{2n+1} \sim A_{2n+2} \) and whose range
is \( A_{2n+1} \sim A_{2n+2} \). Now, let \( F_1 = \bigcup f_{2n} \) and \( F_2 = \bigcup f_{2n+1} \).

Thus, if \( F = F_1 \cup F_2 \cup T \), \( F \) is a reversible function
whose domain is \( A \) and whose range is \( A_1 \), hence \( A \sim A_1 \).

And since \( \sim \) is an equivalence relation, \( A \sim B \).

Now that two methods of proving the equivalence of two sets
have been established, an application of the relation \( \sim \) will be
developed. Since \( \sim \) is an equivalence relation, it separates a class
of sets into equivalence classes, hence the following definition.

Definition 8. If \( A \) is a set, the cardinal of \( A \), \( \alpha \), denotes the
equivalence class of which the set \( A \) is an element.

Next, will be presented a collection of axioms, definitions,
and theorems which will ultimately establish the comparability of
cardinals.
Definition 9. If each of $A$ and $B$ is a set, and if $\alpha$ denotes the cardinal of $A$ and $\beta$ denotes the cardinal of $B$, then the statement that $\alpha < \beta$ means that there exists no reversible function $f$ such that $f(A) \supseteq B$.

Theorem 9. If $A$, $B$, and $C$ denote sets, and if $\alpha$, $\beta$, and $\gamma$ denote the cardinals of $A$, $B$, and $C$ respectively, then if $\alpha + \beta$ and $\beta + \gamma$, then $\alpha < \gamma$.

9.1 If $\alpha < \beta$, then there exists a reversible function $f$, such that $f(A) \supseteq B$.

9.2 If $\beta + \gamma$, then there exists a reversible function $g$, such that $g(B) \supseteq C$.

9.3 For some subset of $B$, $B'$, $g(B') = C$, and for some subset of $A$, $A'$, $f(A') = B'$.

9.4 Consequently $gf(A') = C$; and $\alpha + \gamma$.

Remark: Bernstein's Theorem can now be stated as: If $\alpha + \beta$ and $\beta + \alpha$, then $\alpha = \beta$, where $\alpha$ denotes the cardinal of the set $A$ in the original statement of Bernstein's Theorem, and $\beta$ denotes the cardinal of the set $B$.

Definition 10. If $A$ and $B$ denote two disjoint sets, then "$\alpha + \beta = \kappa$" means that $\kappa$ denotes the cardinal of $U$, where $U = A \cup B$.

Theorem 11. If $\alpha + \beta$, then $\alpha + \gamma = \beta + \gamma$.

11.1 Since $\alpha + \beta$, there exists a reversible function $f$, such that $f(A) = B$. 
Let $I$ denote the identity function such that $I(C) = C$.

Thus, $f \cup I$ is a reversible function $F$, such that $F(AUC) = BUC$; hence, $\alpha + \gamma = \beta + \gamma$ provided of course, that $A$ and $C$ are disjoint, and $B$ and $C$ are disjoint.

Definition 11. If $A$ denotes an infinite set then there exists a proper subset of $A$, denoted by $A^1$, such that $A \sim A^1$.

Theorem 12. If $A \supset B$, then $\alpha$

Obviously, the identity function $I$ is a function such that $I(A) = B$. Therefore $\alpha$

Definition 12. If $A$ denotes a set, and $\alpha$ denotes the cardinal of $A$, then $\alpha!$ denotes the cardinal of the set of all reversible functions whose domain and range is $A$.

Theorem 13. If $\alpha!$, then $\alpha! < (\alpha-1)! + (\alpha-1)!$.

13.1 Since $\alpha > 1$, $A$ contains two elements, $a, a'$.

13.2 Let $A!$ denote the set of reversible functions whose domain and range is $A$.

13.3 Let $F$ denote the subset of $A!$ such that if $f$ is an element of $F$, $f(a) = a$.

13.4 Let $\overline{\text{1}}$ denote the subset of $A!$ such that if $f$ is an element of $\overline{\text{1}}$, $f(a) = a^1$. 
Since each element of $F$ is the union of the ordered pair $(a,a)$ and a reversible function whose domain and range is $A-a$, and since the cardinal of the set of all such functions is $(\alpha-1)^{\alpha-1}$, the cardinal of $F$ is $(\alpha-1)!$. And for the same reasons, the cardinal of $F^1$ is $(\alpha-1)!$.

Since $F$ and $F^1$ are disjoint sets, the cardinal of $F \cup F^1$ is $(\alpha-1)! + (\alpha-1)!$. And since $A! \supset F \cup F^1$, by Theorem 12, $\alpha! < (\alpha-1)! + (\alpha-1)!$.

Remark: Since if $A$ denotes an infinite set, there exists a proper subset of $A$, $B$, such that $B \sim A$, and since $B$ is a proper subset of $A$, there exists an element of $A$, $a$, such that $B$ is a subset of $A-a$.

Hence, if $(\alpha-1)$ denotes the cardinal of $A-a$, $(\alpha-1) + \rho + \alpha$ and $(\alpha-1) < \alpha$, and by Bernstein's Theorem, $(\alpha-1) = \alpha$.

Axiom 1: If each of $\alpha$ and $\beta$ denotes a cardinal, and if $\alpha < \beta$, then $\alpha! < \beta!$.

Theorem 14: $\alpha! < \beta!$ if and only if $\alpha < \beta$.

14.1 From axiom 1, if $\alpha < \beta$, then $\alpha! < \beta!$.

14.2 If $\alpha < \beta$, then there exists a reversible function $h$ such that if $b$ is an element of $B$, there exists only one element of $A$, $a$, such that $b = h(a)$. 
If each of $b$ and $b'$ denotes an element of $B$, and if $f$ denotes a reversible function whose range and domain is $B$, then if $f(b)=b'$, $fh(a)=h(a')$, and $h^{-1}fh(a)=a'$. And if $f(b)=b'$ and $f'(b)=b''$, where $b''\neq b'$, then $h(a)=b$, $h(a')=b'$, $h(a'')=b''$. Furthermore, since $h$ is a reversible function $a' \neq a''$. Consequently, $h^{-1}fh(a)=a'$ and $h^{-1}fh(a)=a''$, and $h^{-1}fh(a)\neq h^{-1}fh(a)$.

Hence, there exists a reversible function $F$ whose domain is the set of reversible functions whose range and domain is $A$, and whose range contains the set of reversible functions whose domain and range is $B$. Thus, $\alpha!$.

**Theorem 15:**
If $\alpha$ denotes an infinite cardinal, then $\alpha!=\alpha!+\alpha!$.

Since $\alpha!=(\alpha-1)!$ and $\alpha!+\alpha!=(\alpha-1)!+(\alpha-1)!$, and $\alpha!=(\alpha-1)!+(\alpha-1)!$, $\alpha!=\alpha!+\alpha!$. Obviously, $\alpha!+\alpha!=\alpha!$. Consequently, $\alpha!=\alpha!+\alpha!$.

**Definition 13:**
If $Q$ denotes a class of pair-wise disjoint sets $A_i$, and if the cardinal of $Q$ is $\alpha$, and if for each $i$, the cardinal of $A_i$ is $B$, then if $T$ denotes the union of all the elements of $Q$, the cardinal of $T$ is $\alpha$.

**Theorem 16:**
If $\alpha$ denotes the cardinal of an infinite set, then $(\alpha!)!=(\alpha!)!\cdot(\alpha!)!$.

**16.1**
Obviously, $(\alpha!)!\cdot(\alpha!)!$. $(\alpha!)!$.
16.2 Let $A!$ and $B!$ denote two disjoint sets each of whose cardinal is $\alpha!$.

16.3 Let $F$ denote the set of reversible functions each of whose domain and range is $A!$, and let $G$ denote the set of reversible functions each of whose domain and range is $B!$.

16.4 If $f'$ is an element of $F$, then for each $g$ in $G$, $f' \cup g$ would be a reversible function whose domain and range is $A! \cup B!$. And since the cardinal of $F$ is $(\alpha!)!$ and the cardinal of $G$ is $(\alpha!)!$, the cardinal of all such functions $f \cup g$ would be $(\alpha!)! \cdot (\alpha!)!$.

16.6 Thus, since the set of reversible functions $f \cup g$ is a subset of the set of all reversible functions whose domain and range is $A! \cup B!$, $(\alpha!+\alpha!)! \not\leftrightarrow (\alpha!)! \cdot (\alpha!)!$, where $(\alpha!+\alpha!)!$ is the cardinal of the set of reversible functions each of whose domain and range is $A! \cup B!$.

16.7 Since $\alpha! \neq \alpha!+\alpha!$, by Theorem 14, $(\alpha!)! \not\leftrightarrow (\alpha!+\alpha!)!$. And, since $\not\leftrightarrow$ is transitive, $(\alpha!)! \not\leftrightarrow (\alpha!)! \cdot (\alpha!)!$.

16.8 Since $(\alpha!)! \not\leftrightarrow (\alpha!)! \cdot (\alpha!)!$, and $(\alpha!)! \cdot (\alpha!)! \cdot (\alpha!)!$ by Bernstein's Theorem, $(\alpha!)!=(\alpha!)! \cdot (\alpha!)!$.

Theorem 17: If $\alpha$ and $\beta$ denote infinite cardinals, then either $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$. Obviously, either $\alpha < \beta$, or $\alpha + \beta$, and $\beta < \alpha$, or $\beta + \alpha$. From
Bernstein's Theorem, if $\alpha \leq \beta$, and $\beta \neq \alpha$, then $\alpha = \beta$. Hence either $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$.

Although Theorem 17 establishes a "type" of comparability for infinite cardinals, it does not establish the anti-symmetric property which is commonly implied by "comparability"; that is, if $\alpha < \beta$, then $\beta \neq \alpha$. Usually this property is established by the use of Zermelo's Well-ordering Theorem, which is dependent on the Axiom of Choice. However, in order to obtain a dichotomy between cardinals and ordinal numbers, another axiom will be introduced. Perhaps it should be noted that all results obtained thus far are not dependent on Axiom 1, however, it was introduced early for the sake of compactness.

**Axiom 2:** If $\alpha$ and $\beta$ denote cardinals, and if $\beta < \alpha$, and $\alpha \cdot \alpha = \alpha$, then $\alpha \cdot \beta = \alpha$.

**Theorem 18.** If $\alpha$, $\beta$, and $\gamma$ denote cardinals, and $\alpha = \beta$, then $\alpha \cdot \gamma = \beta \cdot \gamma$.

**Lemma:** If each of $\alpha$ and $\beta$ denotes a cardinal, and if $\alpha \cdot \alpha = \alpha$, and $\beta < \alpha$, then $\alpha = \beta + \alpha$.

1. $\alpha \cdot \beta = \beta + (\alpha - 1) \cdot \beta$
2. Since $\alpha = (\alpha - 1)$, $\alpha \cdot \beta = \beta + \alpha \cdot \beta$
3. By Axiom 2, $\alpha \cdot \beta = \alpha$, and $\alpha = \beta + \alpha \cdot \beta$

**Theorem 19:** If $\alpha$ and $\beta$ denote infinite cardinals, then if $\alpha < \beta$, $\beta \neq \alpha$.

19.1 If $\alpha < \beta$, then from Axiom 1, it follows that $\alpha! < \beta!$ and $(\alpha!)! < (\beta!)!$. 

19.2 Let $\mu = (\alpha)! + (\beta)!$.

19.3 Since $((\beta)!)! = (\beta)! \cdot (\beta)!$, and $(\alpha)! < (\beta)!$, $(\beta)! \neq \mu$. By Theorem 12, $\mu \neq (\alpha)!$. Hence, since $\mathcal{A}$ is transitive, $(\beta)! \neq (\alpha)!$.

19.4 From Theorem 14, it follows that $\alpha! < \beta!$.

Furthermore, if $\beta! < \alpha!$, then $(\beta)! < (\alpha)!$.

Hence $\beta! \neq \alpha!$.

19.5 Similarly, $\alpha \leq \beta$, and $\beta \neq \alpha$.

19.6 Thus, if $\alpha < \beta$, then $\beta \neq \alpha$.

In conclusion, since Theorem 17 establishes that if $\alpha$ and $\beta$ denote infinite cardinals, either $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$; and since Theorem 19 establishes that if $\alpha < \beta$, then $\beta \neq \alpha$; infinite cardinals are comparable. That is, if $\alpha$ and $\beta$ denote infinite cardinals then exactly one of the following statements is true: $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$. 