ON ENVELOPES AND EXTRANEOUS LOCI OF
DIFFERENTIAL EQUATIONS OF ORDER
ONE AND HIGHER DEGREE

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CHAPTER I

INTRODUCTION

It is found in various books that the differential equations of order one may be implicitly written as
\[ F(x, y, p) = 0, \]
where \( p = \frac{dy}{dx} \), and have a solution of
\[ f(x, y, c) = 0. \]
The solution, referred to as the general solution, describes a family of curves that are called solution curves; whereby the coordinates and the direction at each point on a solution curve will satisfy the differential equation. Since these curves are obtained by assigning arbitrary values to \( c \), the question is raised as to whether or not there could exist a curve whose points would satisfy the differential equation, and yet not be one of the curves obtained in the general solution. Upon investigation, such a curve is found and is called the envelope of the differential equation. In addition to the envelope, other loci of points having unique properties are found associated with the differential equation and its solution curves. However, these loci are not necessarily solutions of the differential equation and are referred to as extraneous loci.

The purpose of this paper is to examine the properties of the envelope and the extraneous loci associated
with the solution curves of ordinary differential equations of the first order and degree greater than one.

The differential equation to be considered is of the form $F(x,y,p) = 0$ where $p = dy/dx$ and the function $F(x,y,p)$ is a polynomial in $p$. The general solution of $F(x,y,p) = 0$ is a family of curves expressed by the relation $f(x,y,c) = 0$ where $c$ is the parameter that determines a unique solution curve for each arbitrary value assigned it. The existence of the solution of $F(x,y,p) = 0$ is assumed in this paper.

In addition to assuming the basic relations in differential calculus, certain theorems will be stated without proof since they are needed as basic tools in the development of this paper. Also, for simplicity of writing, the notation $F_x$ will be used to denote $\partial F/\partial x$.

**Theorem 1.1.** Let $F(x,y,z)$ be a continuous function of the independent variables $x$, $y$, and $z$ with partial derivatives $F_x$, $F_y$, and $F_z$. For the system of values $(x_0,y_0,z_0)$, let $F(x_0,y_0,z_0) = 0$ and $F_p(x_0,y_0,z_0) \neq 0$. Then mark off an interval $z_1 \leq z \leq z_2$ about $z_0$ and a region $R$ containing $(x_0,y_0)$ such that for every $(x,y)$ in $R$ the equation $F(x,y,z) = 0$ is satisfied by just one value of $z$ in the interval $z_1 \leq z \leq z_2$. This value of $z$, which is denoted by $z = f(x,y)$, is a continuous function of $x,y$ and possesses continuous partial derivatives $f_x$, $f_y$, and $z_0 = f(x_0,y_0)$. 
The derivatives of $f$ are given by the equations

$$F_x + F_z f_x = 0 \text{ and } F_y + F_z f_y = 0.$$  

**Theorem 1.2.** If in a function $F(x,y,z)$ the variables are not independent of one another, but are subject to the condition $F = 0$, then the linear parts of the increments $(F_x dx, F_y dy, F_z dz)$ of the variables are likewise not independent of one another, but are connected by the condition $dF = 0$, that is, by the linear equation

$$F_x dx + F_y dy + F_z dz = 0.$$  

**Theorem 1.3.** Let $\alpha, \beta, \phi$ be functions of the variables $x, y, z$. These three functions will be functionally dependent, that is, there will exist a relation $F(\alpha, \beta, \phi) = 0$ which does not involve explicitly any of the variables $x, y, z$, if and only if the Jacobian vanishes identically. That is,

$$J = \frac{\partial(\alpha, \beta, \phi)}{\partial(x,y,z)} = 0.$$  

**Theorem 1.4.** Let there be given two functional relations, $F(x,y,z) = 0$ and $G(x,y,z) = 0$. Let these equations be satisfied for the set of values $x = x_0, y = y_0$, and $z = z_0$, and let the functions $F$ and $G$ and their first partial derivatives be continuous in the neighborhood of

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2. Ibid., p. 118.
this set of values. Suppose further that the Jacobian
\[ J = \frac{\partial(F,G)}{\partial(x,y)} \neq 0 \]
for this set of values. Then there exists one and only one system of continuous functions \( x = \phi(z) \) and \( y = \tau(z) \) satisfying the equations \( F = 0 \) and \( G = 0 \) identically and such that \( x_0 = \phi(z_0) \) and \( y_0 = \tau(z_0) \).4

Theorem 1.5. Let \( f(x,y,z) \) and all its partial derivatives up to and including those of order \( n \) be differentiable near \( (a,b,c) \), then
\[ f(a+h,b+k,c+l) = f(a,b,c) + \left( \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b} + \frac{\partial f}{\partial c} \right) + \frac{1}{2!} \left( \frac{\partial^2 f}{\partial a^2} + \frac{\partial^2 f}{\partial b^2} + \frac{\partial^2 f}{\partial c^2} \right) + \frac{1}{3!} \left( \frac{\partial^3 f}{\partial a^3} + \frac{\partial^3 f}{\partial b^3} + \frac{\partial^3 f}{\partial c^3} \right) + \ldots \]

where in the last term the values \( x = a + \epsilon_1 \), \( y = b + \epsilon_2 \), and \( z = c + \epsilon_3 \), for \( 0 < \epsilon < 1 \), are substituted after differentation.5

Theorem 1.5, known as Taylor's Theorem, is frequently written in a different form. This second form will be given since it is also used in this paper.

Corollary 1.1. Let \( f(x,y,z) \) and all of its first and second partial derivatives be differentiable near \( (a,b,c) \), then

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4 Ibid., p. 51.

\[
f(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) \\
+ f_z(a,b,c)(z-c) + \frac{1}{2}\left[f_{xx}(x_1,y_1,z_1)(x-a)^2 + f_{yy}(x_1,y_1,z_1)(y-b)^2 \\
+ f_{zz}(x_1,y_1,z_1)(z-c)^2 + 2f_{xy}(x_1,y_1,z_1)(x-a)(y-b) \\
+ 2f_{xz}(x_1,y_1,z_1)(x-a)(z-c) + 2f_{yz}(x_1,y_1,z_1)(y-b)(z-c)\right],
\]

where \((x_1,y_1,z_1)\) is some suitably chosen point between \((x,y,z)\) and \((a,b,c)\).
CHAPTER II

THE c-DISCRIMINANT LOCI

While this paper is primarily concerned with the family of curves associated with the general solution of a differential equation, the results of this chapter could be applied to any family of curves, \( f(x,y,c) \), whether the curves are solutions of a differential equation or not.

The study of the envelope and extraneous loci of \( f(x,y,c) \) will utilize the c-discriminant and the idea of double points on a curve. It will be shown that the c-discriminant will reveal much about the various loci related to \( f(x,y,c) \). In fact it will be shown, as first proposed by Cayley,\(^1\) that the c-discriminant will, in general, contain the envelope-locus as a factor once and only once, the node-locus as a factor twice and only twice, and the cusp-locus as a factor thrice and only thrice.

Before becoming involved with any discussion of the various loci of \( f(x,y,c) \), it will be necessary to define several terms.

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Definition 2.1. The c-discriminant is the relation obtained by eliminating the c term between the two relations: \(f(x,y,c) = 0\) and \(f_c(x,y,c) = 0\).

Definition 2.2. An envelope is a curve which touches every member of the family, and which, at each point is touched by some member of the family. Two curves are said to touch if they have the same direction at a common point.

Definition 2.3. A double point is a point on a curve that satisfies the conditions: \(f(x,y,c) = 0\), \(f_x(x,y,c) = 0\), and \(f_y(x,y,c) = 0\). The function \(f(x,y,c)\) is represented by the curve.

Definition 2.4. A nodal point is a point on a curve that is a double point and has two directions of the curve given at the point. A continuous curve made up of such points is called the node-locus.

Definition 2.5. A cuspidal point is a point on the curve that is a double point and has only one direction of the curve given at the point. A continuous curve made up of such points is called the cusp-locus.

Definition 2.6. The tac-locus is a locus of points such that two separate solution curves are tangent at each point on the locus.

Envelopes

After due deliberation, it would certainly appear that not every function \(f(x,y,c)\) would have an envelope for its family of curves. Therefore some conditions for
an envelope will be given. First, a necessary condition for the existence of an envelope will be stated.

**Theorem 2.1.** If \( f(x, y, c) = 0 \), where \( f(x, y, c) \) is twice differentiable, has an envelope \( E(x, y) \), then the points of the envelope must satisfy \( f(x, y, c) = 0 \) and \( f_c(x, y, c) = 0 \).

**Proof.** Let \( f(x, y, c) = 0 \) be the equation of the family of curves and \( E(x, y) = 0 \) be a curve tangent to each curve of \( f(x, y, c) \) at some point \( P_c \). Also each point of \( E(x, y) \) is a point \( P_c \) at which some curve of \( f(x, y, c) \) touches \( E(x, y) \). By Definition 2.2, \( E(x, y) \) is an envelope of \( f(x, y, c) \).

Take \( c \) as a parameter and write the equation \( E(x, y) = 0 \) in parametric form. Thus,

\[
x = x(c) \quad \text{and} \quad y = y(c).
\]

From the above relations, the slope of the envelope at any point is

\[
\frac{dy}{dx} = \frac{dy}{dc} \cdot \frac{dc}{dx} \quad (1)
\]

Next, taking the total derivative of \( f(x, y, c) \) with respect to \( c \),

\[
\frac{f_x}{dc} \cdot dx + \frac{f_y}{dc} \cdot dy + f_c = 0. \quad (2)
\]

Taking the total derivative of \( f(x, y, c) \) with respect to \( x \),

\[
\frac{f_x}{dx} \cdot dx + \frac{f_y}{dy} \cdot dy + \frac{f_c}{dx} = 0. \quad (3)
\]

Let \( f(x, y, c_1) \) be one of the curves of the family \( f(x, y, c) \). Therefore \( c_1 \) is a constant and equation (3) becomes
(4) \[ f_x + f_{y} \frac{dy}{dx} = 0 \]

which holds for the single curve \( f(x,y,c_1) = 0 \). This gives the slope at any point on \( f(x,y,c_1) = 0 \); and at any point \( P_c \) on this curve, it should give the slope of \( E(x,y) = 0 \) by the hypothesis.

It has been shown that the envelope should satisfy both equation (1) and equation (4). Eliminating \( \frac{dy}{dx} \) from these two relations, the following relation is obtained:

(5) \[ f_x \frac{dx}{dc} + f_y \frac{dy}{dc} = 0. \]

Now it has been shown that both equations, (2) and (5), are satisfied by the point \( P_c \) on the envelope. Comparing these two relations, it must be concluded that \( f_c = 0 \) when evaluated at the point \( P_c \). Thus the theorem is proved.

The last theorem indicates that an envelope can be found by obtaining the c-discriminant by Definition 2.1, that is, if an envelope exists. If the existence of an envelope is overlooked, it might be expected that the c-discriminant will always yield an envelope. However, this is not always true. For example, the family of curves represented by the function \( f(x,y,c) = y^3 - (x - c)^2 = 0 \) has for its c-discriminant the relation \( y^3 = 0 \). Since this is a straight line with a slope of zero, \( f(x,y,c) \) would at some point have to have a slope of zero.
However, upon examining $y^3 - (x - c)^2 = 0$, it is found that a slope of zero does not exist at any point on the family.

Therefore it is necessary that a sufficiency condition be stated for the existence of an envelope.

**Theorem 2.2.** If $f(x,y,c) = 0$, $f_c(x,y,c) = 0$, $f_{cc}(x,y,c) \neq 0$, and $\begin{vmatrix} f_x & f_y \\ f_{cx} & f_{cy} \end{vmatrix} \neq 0$, where $f(x,y,c)$ is twice differentiable, then there exists an envelope of the function $f(x,y,c)$.

**Proof.** Since the Jacobian of $f(x,y,c) = 0$ and $f_c(x,y,c) = 0$ with respect to $x$ and $y$ is not zero, then $x$ and $y$ are functions of $c$ where $c$ is considered to be independent. By Theorem 1.4, both $f(x,y,c) = 0$ and $f_c(x,y,c) = 0$ can be written in parametric form, that is,

$$x = x(c) \quad \text{and} \quad y = y(c).$$

Since $f(x,y,c)$ is twice differentiable, $f_c(x,y,c)$ is differentiable and $x = x(c)$ and $y = y(c)$ have first derivatives.

Taking the total derivative of $f(x,y,c)$ with respect to $c$,

$$f \frac{dx}{dc} + f_y \frac{dy}{dc} + f_c = 0. \quad (7)$$

Next, taking the total derivative of $f_c(x,y,c)$ with respect to $c$,

$$f_{cx} \frac{dx}{dc} + f_{cy} \frac{dy}{dc} + f_{cc} = 0. \quad (8)$$
Since $f_{cc}(x,y,c) \neq 0$, $dx/dc$ and $dy/dc$ cannot both be equal to zero. Hence the curve $x = x(c)$ and $y = y(c)$ has a slope equal to the ratio of $dx/dc$ and $dy/dc$, that is,

$$\frac{dy}{dx} = \frac{dy/dc}{dx/dc}$$

Since $f_c(x,y,c) = 0$, equation (9) becomes

$$f_xdx + f_ydy + f_cdc = 0.$$  \hspace{1cm} (10)

Taking the total differential of $f(x,y,c) = 0$, (11)

$$f_xdx + f_ydy + f_cdc = 0.$$  

Holding $c$ constant, $dc = 0$. Thus for any one curve of the family,

$$f_xdx + f_ydy = 0.$$  \hspace{1cm} (12)

Since the Jacobian is unequal to zero, $f_x(x,y,c)$ and $f_y(x,y,c)$ cannot both be equal to zero, and the above equation determines the slope of the curve of the family with $c$ held constant.

Since $f_x(x,y,c)$ and $f_y(x,y,c)$ cannot both be zero and $dx/dc$ and $dy/dc$ cannot both be zero, it follows from equation (10) that if $dx/dc = 0$ and $dy/dc \neq 0$, then $f_x(x,y,c) \neq 0$ and $f_y(x,y,c) = 0$. Also if $dx/dc \neq 0$ and $dy/dc = 0$, then $f_x(x,y,c) = 0$ and $f_y(x,y,c) \neq 0$.

Comparing equations (10) and (12) after solving both relations for $-f_x/f_y$,

$$\frac{dy}{dx} = \frac{dy/dc}{dx/dc}$$

which holds for any point on $f(x,y,c) = 0$. 
Therefore, comparing equation (9) and equation (13), the curve \( f(x,y,c) = 0 \) and the curve \( x = x(c) \) and \( y = y(c) \) have the same direction at the same point. Hence the curve \( x = x(c) \) and \( y = y(c) \) is an envelope of \( f(x,y,c) \).

It should be noted that the condition \( f_{cc}(x,y,c) \neq 0 \) found in the above theorem is stated only to insure that \( dx/dc \) and \( dy/dc \) are not both zero. Thus there may be an envelope of \( f(x,y,c) \) when \( f_{cc}(x,y,c) = 0 \) and \( dx/dc \) and/or \( dy/dc \) is unequal to zero. Consider, for example, the function \( f(x,y,c) = (x-c)^4 + y^2 - 1 = 0 \). In this case \( f_{cc}(x,y,c) = 0 \) when \( x = c \) and the c-discriminant is found to be \( y^2 = 1 \). The function \( f(x,y,c) \) describes a family of circles with centers on the x-axis and having a radius of one. Thus it can be seen that \( y = 1 \) and \( y = -1 \) will be envelopes of the family of curves.

Conditions for Some Loci

Before investigating the involvement of the node and cusp loci with the c-discriminant, some conditions for the existence of these loci must be established.

Node-Locus

A necessary condition for the existence of a node-locus will first be given.

Theorem 2.3. If \( f(x,y,c) = 0 \) has a node-locus, then any point \( (\xi, \eta, \gamma) \) on the node-locus will satisfy \( f(x,y,c) = 0, f_x(x,y,c) = 0, f_y(x,y,c) = 0, \) and \( f_c(x,y,c) = 0 \).
Proof. From Definition 2.4, every point on the node-locus is a double point, that is, at each point on the node-locus, \( f(x,y,c) = 0 \), \( f_x(x,y,c) = 0 \), and \( f_y(x,y,c) = 0 \).

Taking the total derivative of \( f(x,y,c) = 0 \) with respect to \( c \),
\[
f_x \frac{dx}{dc} + f_y \frac{dy}{dc} + f_c = 0.
\]
Since \( f_x(x,y,c) = 0 \) and \( f_y(x,y,c) = 0 \), it follows that \( f_c(x,y,c) = 0 \).

Although it will not be of any immediate benefit in furthering the investigation of the node-locus, some interest should be taken in determining the direction of the node-locus at any point. Thus the slope may be determined by the conditions of the following theorem:

**Theorem 2.4.** If \( f(x,y,c) = 0 \) has a node-locus, then the direction of the locus at the point \((\xi, \eta, \gamma)\) on the locus is given by either
\[
\begin{align*}
\frac{f_{\eta\gamma}f_{\xi\xi} - f_{\eta\xi}f_{\xi\gamma}}{f_{\eta\xi}f_{\eta\eta} - f_{\xi\xi}f_{\xi\eta}}, & \quad \frac{f_{\xi\eta}f_{\xi\xi} - f_{\xi\gamma}f_{\xi\eta}}{f_{\eta\xi}f_{\eta\eta} - f_{\xi\xi}f_{\xi\eta}}, \\
\frac{f_{\xi\eta}f_{\eta\eta} - f_{\xi\gamma}f_{\eta\gamma}}{f_{\eta\xi}f_{\eta\eta} - f_{\xi\xi}f_{\xi\eta}},
\end{align*}
\]
or
\[
\begin{align*}
\frac{f_{\eta\gamma}f_{\xi\xi} - f_{\eta\xi}f_{\xi\gamma}}{f_{\eta\xi}f_{\eta\eta} - f_{\xi\xi}f_{\xi\eta}}, & \quad \frac{f_{\xi\eta}f_{\xi\xi} - f_{\xi\gamma}f_{\xi\eta}}{f_{\eta\xi}f_{\eta\eta} - f_{\xi\xi}f_{\xi\eta}}, \\
\frac{f_{\xi\eta}f_{\eta\eta} - f_{\xi\gamma}f_{\eta\gamma}}{f_{\eta\xi}f_{\eta\eta} - f_{\xi\xi}f_{\xi\eta}},
\end{align*}
\]

**Proof.** Let \((\xi, \eta, \gamma)\) be one point on the node-locus and \((\xi + \delta \xi, \eta + \delta \eta, \gamma + \delta \gamma)\) be another point on the node-locus.

From Theorem 2.3, any point on the node-locus will satisfy
\[
\begin{align*}
(14) & \quad f(x,y,c) = 0, \\
(15) & \quad f_x(x,y,c) = 0, \\
(16) & \quad f_y(x,y,c) = 0, \\
(17) & \quad f_c(x,y,c) = 0.
\end{align*}
\]
Taking the total differentials of equations (15), (16), and (17), the following relations are obtained:

\[(18) \quad f_{xx} \Delta x + f_{xy} \Delta y + f_{xc} \Delta c = 0,\]
\[(19) \quad f_{xy} \Delta x + f_{yy} \Delta y + f_{yc} \Delta c = 0,\]
\[(20) \quad f_{xc} \Delta x + f_{yc} \Delta y + f_{cc} \Delta c = 0.\]

At \((\xi, \eta, \gamma)\), equations (18), (19), and (20) become

\[(21) \quad f_{\xi \xi} \partial \xi + f_{\xi \eta} \partial \eta + f_{\xi \gamma} \partial \gamma = 0,\]
\[(22) \quad f_{\eta \eta} \partial \xi + f_{\eta \eta} \partial \eta + f_{\eta \gamma} \partial \gamma = 0,\]
\[(23) \quad f_{\gamma \gamma} \partial \xi + f_{\gamma \gamma} \partial \eta + f_{\gamma \gamma} \partial \gamma = 0.\]

Eliminating \(\partial \gamma\) from any two of equations (21), (22), and (23), \(\partial \eta / \partial \xi\), which is the direction of the node-locus at \((\xi, \eta, \gamma)\), is determined.

Thus, from (21) and (22),

\[\frac{\partial \eta}{\partial \xi} = \frac{f_{\xi \gamma} ^{2} f_{\xi \eta} - f_{\eta \gamma} f_{\xi \xi}}{f_{\eta \eta} ^{2} f_{\xi \eta} - f_{\xi \gamma} f_{\eta \eta}}.\]

Equations (21) and (23) result in

\[\frac{\partial \eta}{\partial \xi} = \frac{f_{\xi \gamma} ^{2} f_{\xi \eta} - f_{\gamma \eta} f_{\xi \xi}}{f_{\gamma \gamma} ^{2} f_{\xi \eta} - f_{\xi \gamma} f_{\gamma \gamma}}.\]

And from equations (22) and (23),

\[\frac{\partial \eta}{\partial \xi} = \frac{f_{\xi \gamma} ^{2} f_{\eta \eta} - f_{\xi \eta} f_{\gamma \gamma}}{f_{\eta \eta} ^{2} f_{\eta \gamma} - f_{\eta \gamma} f_{\gamma \gamma}}.\]

Since the coordinates of the node depend upon \(c\), by Theorem 1.3 \(\partial (f_{\xi}, f_{\eta}, f_{\gamma}) = 0\) must hold to insure consistency.
Cusp-Locus

In examining the cusp-locus, it will first be necessary to state a necessary condition for a function to have a cusp-locus.

Theorem 2.5. If \( f(x,y,c) \) has a cusp-locus, then any point \((\xi, \eta, \gamma)\) on the cusp-locus will satisfy \( f(x,y,c) = 0 \), \( f_x(x,y,c) = 0 \), \( f_y(x,y,c) = 0 \), \( f_c(x,y,c) = 0 \), and \( f_{xx}f_{yy} - f_{xy}^2 = 0 \).

Proof. By Definition 2.5, each point of the cusp-locus is a double point. Since \((\xi, \eta, \gamma)\) is a point on the cusp-locus and a double point, it follows from Theorem 2.3 that \( f_x(x,y,c) = 0 \), \( f_y(x,y,c) = 0 \), and \( f_c(x,y,c) = 0 \) at \((\xi, \eta, \gamma)\). It remains to be shown that \( f_{xx}f_{yy} - f_{xy}^2 = 0 \) at \((\xi, \eta, \gamma)\).

Expanding \( f(x,y,c) \) by Theorem 1.5 for values of \( x \), \( y \), and \( c \) near the cusp point \((\xi, \eta, \gamma)\),

\[
f(x,y,c) = f(\xi, \eta, \gamma) + f_x(\xi, \eta, \gamma)(x-\xi) + f_y(\xi, \eta, \gamma)(y-\eta) + f_c(\xi, \eta, \gamma)(c-\gamma) + 1/2 \left[ f_{xx}(x_1,y_1,c_1)(x-\xi)^2 + f_{yy}(x_1,y_1,c_1)(y-\eta)^2 + f_{cc}(x_1,y_1,c_1)(c-\gamma)^2 + 2f_{xy}(x_1,y_1,c_1)(x-\xi)(y-\eta) + 2f_{xc}(x_1,y_1,c_1)(x-\xi)(c-\gamma) + 2f_{yc}(x_1,y_1,c_1)(y-\eta)(c-\gamma) \right],
\]

where \((x_1,y_1,c_1)\) is some point on the cusp-locus between the points \((x,y,c)\) and \((\xi, \eta, \gamma)\). When \( c = \gamma \) and since \( f(x,y,c) = 0 \), \( f_x(x,y,c) = 0 \), \( f_y(x,y,c) = 0 \), and \( f_c(x,y,c) = 0 \) at the cusp point \((\xi, \eta, \gamma)\), the above expansion becomes
\[ f(x,y,c) = \frac{1}{2} \left[ f_{xx}(x_1,y_1,c_1)(x-\xi)^2 + 2f_{xy}(x_1,y_1,c_1)(x-\xi)(y-\eta) \\
+ f_{yy}(x_1,y_1,c_1)(y-\eta)^2 \right]. \]

Now let \( x-\xi = \Delta x \) and \( y-\eta = \Delta y \). Upon substituting these values in the above relation,

\[ f_{xx}(x_1,y_1,c_1)\Delta x^2 + 2f_{xy}(x_1,y_1,c_1)\Delta x \Delta y + f_{yy}(x_1,y_1,c_1)\Delta y^2 = 0. \]

Dividing by \( \Delta x^2 \) and then solving for \( \Delta y/\Delta x \),

\[ \frac{\Delta y}{\Delta x} = \frac{-f_{xy} + \sqrt{f_{xy}^2 - f_{xx}f_{yy}}}{f_{yy}}. \]

The slope of \( f(x,y,c) \) at the cusp point is given by \( \Delta y/\Delta x \).

Since there is only one direction at the cusp, the term within the radical must be zero, that is,

\[ f_{xy}^2 - f_{xx}f_{yy} = 0. \]

**Corollary 2.1.** If \( f(x,y,c) \) has a cusp-locus, then the direction of the function at the cusp point is given by \( \Delta y/\Delta x = -f_{xy}/f_{yy} \).

**Proof.** It follows from Theorem 2.5 that \( \Delta y/\Delta x = -f_{xy}/f_{yy} \), since \( f_{xy}^2 - f_{xx}f_{yy} = 0 \).

Another property of the cusp-locus to be determined is the direction of the locus at the point \((\xi,\eta,\gamma)\).

**Remark 2.1.** To determine the direction of the cusp-locus at the point \((\xi,\eta,\gamma)\).

From the necessary conditions of a cusp-locus,

\[ f_{\xi\xi,\eta\eta} - f_{\xi\eta}^2 = 0 \]

at the point \((\xi,\eta,\gamma)\) on the cusp-locus. Let \( G \) be a
function of \((x,y,c)\) such that at the point \((\xi,\eta,\gamma)\)

\[
G(\xi,\eta,\gamma) = f_{\xi\xi} \eta - f_{\xi\gamma}^2 = 0.
\]

Taking the total differential of \(G\),

\[
(25) \quad G_{\xi} \partial \xi + G_{\eta} \partial \eta + G_{\gamma} \partial \gamma = 0.
\]

Since the cusp point is a double point, equations (21), (22), and (23) from Theorem 2.4 hold at the point \((\xi,\eta,\gamma)\).

(21) \hspace{1cm} f_{\xi\xi} \partial \xi + f_{\xi\eta} \partial \eta + f_{\xi\gamma} \partial \gamma = 0

(22) \hspace{1cm} f_{\eta\xi} \partial \xi + f_{\eta\eta} \partial \eta + f_{\eta\gamma} \partial \gamma = 0

(23) \hspace{1cm} f_{\gamma\xi} \partial \xi + f_{\gamma\eta} \partial \eta + f_{\gamma\gamma} \partial \gamma = 0

Eliminating \(\partial \gamma\) from equation (25) and any one of the equations (21), (22), and (23), the direction of the cusp-locus at \((\xi,\eta,\gamma)\), \(\partial \eta / \partial \xi\), is obtained.

Some additional properties of the cusp-locus must be stated before beginning a study of the cusp-locus as related to the c-discriminant.

**Theorem 2.6.** If \(f(x,y,c)\) has a cusp-locus, then the following relations hold at the point \((\xi,\eta,\gamma)\) on the cusp-locus:

\[
\frac{f_{\xi\xi} + f_{\xi\gamma}}{f_{\xi\eta}} = \frac{f_{\xi\eta} + f_{\eta\eta}}{f_{\eta\eta}}, \quad \frac{f_{\xi\eta} + f_{\eta\gamma}}{f_{\eta\eta}} = \frac{f_{\xi\gamma} + f_{\gamma\gamma}}{f_{\eta\eta}},
\]

and \(\frac{f_{\xi\xi} + f_{\xi\gamma}}{f_{\xi\eta}} = \frac{f_{\xi\gamma} + f_{\gamma\gamma}}{f_{\eta\gamma}}\).

**Proof.** It was shown in Remark 2.1 that equations (21), (22), (23), and (24) hold at the point \((\xi,\eta,\gamma)\) on the cusp-locus. Thus,
Eliminating $\partial \eta$ from equations (21) and (22),

$$(f_{\xi,\xi} - f_{\xi,\eta}^2)\partial \xi + (f_{\xi,\eta}^2 f_{\eta,\eta} + f_{\xi,\eta} f_{\eta,\eta})\partial \gamma = 0$$

is obtained. Comparing equations (24) and (26),

$$f_{\xi,\xi}^\eta f_{\eta,\eta} - f_{\xi,\eta} f_{\eta,\eta} = 0.$$

The next relation is obtained by combining equations (24) and (27) which proves part of the conclusion.

$$(f_{\xi,\eta} + f_{\xi,\eta}) f_{\eta,\eta} + f_{\xi,\eta} + f_{\eta,\eta} = 0.$$

Next, eliminating $\partial \eta$ from equations (23) and (22),

$$(f_{\xi,\eta} f_{\eta,\eta} - f_{\xi,\eta} f_{\eta,\eta})\partial \xi + (f_{\eta,\eta}^2 f_{\eta,\eta} + f_{\eta,\eta} f_{\eta,\eta})\partial \gamma = 0.$$

Comparing equations (27) and (29),

$$f_{\eta,\eta}^2 - f_{\eta,\eta} f_{\eta,\eta} = 0$$

is obtained. Then upon combining equations (27) and (30),

$$f_{\xi,\eta} + f_{\xi,\eta} = f_{\xi,\eta} + f_{\xi,\eta}.$$

A third relation is obtained by comparing equations (28) and (31), that is,

$$f_{\xi,\eta} + f_{\xi,\eta} = f_{\xi,\eta} + f_{\xi,\eta}.$$
From equations (28), (31), and (32), the following proportions are obtained which are in a convenient form for obtaining various relations involving second derivatives of \( f(x,y,c) \):

\[
\begin{align*}
  f_{\xi\xi} &= k_1 f_{\xi\eta} \quad f_{\xi\eta} = k_2 f_{\xi\eta} \quad f_{\xi\eta} = k_3 f_{\xi\eta} \\
  f_{\eta\eta} &= k_1 f_{\eta\eta} \quad f_{\eta\eta} = k_2 f_{\eta\eta} \quad f_{\eta\eta} = k_3 f_{\eta\eta} \\
  f_{\eta\eta} &= k_1 f_{\eta\eta} \quad f_{\eta\eta} = k_2 f_{\eta\eta} \quad f_{\eta\eta} = k_3 f_{\eta\eta}
\end{align*}
\]

Factors of the c-Discriminant

As stated in Definition 2.1, the c-discriminant is the relation resulting from the elimination of the c term between \( f(x,y,c) = 0 \) and \( f_c(x,y,c) = 0 \). To investigate the envelope, node-locus, and cusp-locus as factors of the c-discriminant, it will be necessary at this time to establish a general form of the c-discriminant. Here the function \( f(x,y,c) \) is a polynomial in c.

**Definition 2.7.** The general form of the c-discriminant is \( \Xi = A_m^{-1} f(x,y,r_1) f(x,y,r_2) \ldots f(x,y,r_{m-1}) \), where \( r_1, r_2, r_3, \ldots, r_{m-1} \) are the roots of \( f_c(x,y,c) = 0 \) and \( A \) is a function of \( x \) and \( y \).

**Envelope**

The general method for obtaining the envelope is to find the c-discriminant. However, it has been shown that the c-discriminant does not always contain an envelope. It would be of interest then to find out how the envelope
is related to the c-discriminant so that upon inspection the discriminant would indicate the presence or absence of an envelope. Therefore, it will be shown that the c-discriminant will, in general, contain the envelope as a factor once and only once.

**Theorem 2.7.** If \((\xi, \eta)\) are the coordinates of any point on the envelope of the curves \(f(x,y,c) = 0\), and if \((x,y)\) is set equal to \((\xi, \eta)\) in the c-discriminant of \(f(x,y,c)\), then this discriminant will vanish; and consequently this discriminant will in general contain the envelope-locus once, and only once, as a factor.

**Proof.** From Definition 2.7, the c-discriminant is

\[
E = A_m^{m-1} f(x,y,c_1) f(x,y,c_2) \ldots f(x,y,c_{m-1}).
\]

Letting

\[
Q = A_m^{m-1} f(x,y,c_2) f(x,y,c_3) \ldots f(x,y,c_{m-1}),
\]

the discriminant is

\[
E = Qf(x,y,c_1).
\]

Since \((\xi, \eta)\) is a point on the envelope, it is known that the two equations \(f(\xi, \eta, c) = 0\) and \(f_c(\xi, \eta, c) = 0\) are satisfied by a common value of \(c\), say \(c_1\). Thus when \((x,y)\) is equal to \((\xi, \eta)\), the parameter is \(c_1\) and \(f(\xi, \eta, c_1) = 0\) and \(f_c(\xi, \eta, c_1) = 0\). Therefore, when \((x,y)\) is equal to \((\xi, \eta)\), \(E = Qf(\xi, \eta, c_1)\). Since \(f(\xi, \eta, c_1) = 0\), \(E = 0\). Thus the discriminant vanishes when \((x,y)\) is set equal to any point \((\xi, \eta)\) on the envelope. Hence the discriminant must contain the envelope-locus as a factor at least once.

Taking the partial derivative of the discriminant with respect to \(x\),
\[
\frac{\delta \Xi}{\delta x} = Q \frac{\partial f(x, y, c)}{\partial x} + f(x, y, c_1) \frac{\delta \phi}{\delta x}.
\]

Since \( f(x, y, c) = 0 \) when \( x = \zeta \) and \( y = \eta \), the above becomes
\[
\frac{\delta \Xi}{\delta x} = Q \frac{\partial f(x, y, c)}{\partial x}.
\]

Similarly, it can be shown that
\[
\frac{\delta \Xi}{\delta y} = Q \frac{\partial f(x, y, c)}{\partial y}.
\]

If \( \delta \Xi/\delta x \) and/or \( \delta \Xi/\delta y \) does not vanish when \( x = \zeta \) and \( y = \eta \), then the discriminant contains the envelope-locus once and only once since all other factors of the discriminant are the factors of \( Q \).

Let \( \varphi = 0 \) be the envelope-locus and \( \Xi = \varphi^n R \), where \( n \) is the number of times the envelope-locus is a factor of the discriminant and \( R \) represents all of the factors of the discriminant that do not represent an envelope-locus.

Taking the partial derivative of the discriminant with respect to \( x \),
\[
\frac{\delta \Xi}{\delta x} = n \varphi^{n-1} \frac{\delta \varphi R}{\delta x} + \varphi^n \frac{\delta R}{\delta x}.
\]

If \( n = 1 \), the above relation becomes
\[
\frac{\delta \Xi}{\delta x} = \frac{\delta \varphi R}{\delta x} + \frac{\delta R}{\delta x}.
\]

Since \( \varphi = 0 \),
\[
\frac{\delta \Xi}{\delta x} = \frac{\delta \varphi R}{\delta x}.
\]

Since for this case the discriminant contains the envelope-locus only once, \( \delta \varphi/\delta x \neq 0 \) and therefore \( \delta \Xi/\delta x \neq 0 \).
Similarly, it can be shown that $\frac{\delta \Sigma}{\delta y} \neq 0$ if the envelope-locus is a factor only once.

If $n$ is greater than one, then

$$\frac{\delta \Sigma}{\delta x} = n\varphi^{n-1}\frac{\delta \varphi}{\delta x} + \varphi^n \frac{\delta R}{\delta x}.$$  

Factoring out $\varphi$,

$$\frac{\delta \Sigma}{\delta x} = \varphi(n\varphi^{n-2}\frac{\delta \varphi}{\delta x} + \varphi^{n-1}\frac{\delta R}{\delta x}).$$  

And since $\varphi = 0$,

$$\frac{\delta \Sigma}{\delta x} = 0.$$

Since it is possible that either $f_x(x, y, c_1) = 0$ or $f_y(x, y, c_1) = 0$, $\delta \Sigma/\delta x$ or $\delta \Sigma/\delta y$ could be equal to zero while the envelope-locus is a factor of the c-discriminant only once. Thus the conclusion must be that the c-discriminant will, in general, contain the envelope-locus as a factor once and only once.

**Node-Locus**

The first extraneous locus to be considered as a factor of the c-discriminant is the node-locus. The determination of the relationship between the node-locus and the discriminant will help in detecting the presence of a node-locus in the c-discriminant. It will be shown that the c-discriminant will, in general, contain the node-locus as a factor twice and only twice.

**Theorem 2.8.** If $(\xi, \eta)$ is any point on the node-locus of the family of curves $f(x, y, c) = 0$, then the discriminant
\( E = 0, \delta E/\delta x = 0, \delta E/\delta y = 0, \) when \( x = \xi \) and \( y = \eta \), and \( E \) will in general contain the node-locus twice, and twice only, as a factor.

Proof. If \((\xi, \eta)\) is a point on the node-locus, then \( x = \xi, y = \eta, \) and \( c = c_1(x, y), \) where \( c_1(\xi, \eta) = \gamma, \) will satisfy the equations \( f(x, y, c) = 0, f_x(x, y, c) = 0, f_y(x, y, c) = 0, \) and \( f_c(x, y, c) = 0. \)

It was shown in Theorem 2.7 that

\[ E = q f(x, y, c_1). \]

Taking the derivative of \( E \) with respect to \( x \) and then \( y, \)

\[ \frac{\delta E}{\delta x} = Q f_x(x, y, c_1) + f(x, y, c_1) \frac{\delta q}{\delta x}, \]
\[ \frac{\delta E}{\delta y} = Q f_y(x, y, c_1) + f(x, y, c_1) \frac{\delta q}{\delta y}. \]

Since \( f(x, y, c_1) = 0, f_x(x, y, c_1) = 0, \) and \( f_y(x, y, c_1) = 0, \) the preceding equations become

\[ \frac{\delta E}{\delta x} = 0, \]
\[ \frac{\delta E}{\delta y} = 0. \]

Since \( E = 0 \) at \((\xi, \eta), \) it follows that the discriminant \( E \) contains the node-locus as a factor at least once.

Now let \( \varphi \) be the node-locus factor, that is, \( \varphi = 0. \)

Then

\[ E = \varphi R, \]

where \( R \) represents the remaining factors of the discriminant. Taking the derivative of \( E \) with respect to \( x, \)

\[ \frac{\delta E}{\delta x} = \frac{\delta \varphi R}{\delta x} + \frac{\delta R \varphi}{\delta x}. \]
Since \( \phi = 0 \) and \( \delta z/\delta x = 0 \), the last equation becomes

\[
\frac{\delta \varphi R}{\delta x} = 0.
\]

Since \( \delta \varphi/\delta x \) will not in general vanish at \((\xi, \eta)\), \( R \) must vanish, that is, \( R \) must contain the node-locus \( \phi \) as a factor at least once.

Now it can be stated that \( E \) will contain the node-locus at least twice since \( E = \varphi R \).

If \( E \) contained the node-locus more than twice, then \( \delta^2 E/\delta x^2 \), \( \delta^2 E/\delta x \delta y \), and \( \delta^2 E/\delta y^2 \) would vanish at \((\xi, \eta)\).

Taking the derivative of \( \delta E/\delta x \) with respect to \( x \),

\[
\frac{\delta^2 E}{\delta x^2} = \frac{\varphi \delta^2 R}{\delta x^2} + 2 \frac{\delta R}{\delta x} \frac{\delta \varphi}{\delta x} + \frac{\delta^2 \varphi}{\delta x^2}.
\]

Since \( \varphi \) and \( R \) both vanish, the above becomes

\[
\frac{\delta^2 E}{\delta x^2} = 2 \frac{\delta R}{\delta x} \frac{\delta \varphi}{\delta x}.
\]

In general, \( \delta R/\delta x \) and \( \delta \varphi/\delta x \) will not vanish. Therefore, \( \delta^2 E/\delta x^2 \) will not vanish and it can be similarly shown that \( \delta^2 E/\delta y^2 \) and \( \delta^2 E/\delta x \delta y \) will not vanish at \((\xi, \eta)\).

Therefore, in general, the c-discriminant will contain the node-locus twice, and only twice, as a factor.

**Cusp-Locus**

The cusp-locus is also found to be a part of the c-discriminant. In order to distinguish between the factors of the discriminant that represent envelopes, node-loci, and cusp-loci, it will be shown that the
cusp-locus will occur in the c-discriminant, in general, as a factor three times and only three times.

**Theorem 2.9.** If \( f(x,y,c) \) has a cusp-locus and \((\xi,\eta,\gamma)\) is a point on the cusp-locus, then the c-discriminant \( \Xi \), \( \partial \Xi / \partial x, \partial \Xi / \partial y, \partial^2 \Xi / \partial x^2, \partial^2 \Xi / \partial y^2 \), and \( \partial^2 \Xi / \partial x \partial y \) will all vanish when \( x = \xi, y = \eta, \) and \( c_1 = \gamma \); and the c-discriminant will in general contain the cusp-locus three times, and only three times, as a factor.

**Proof.** From Theorems 2.5 and 2.6, the following relations hold at the point \((\xi,\eta,\gamma)\) on the cusp-locus:

(34) \( f = 0, \quad f_x = 0, \quad f_y = 0, \quad f_c = 0, \)

and

(35) \[
\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial y}.
\]

Since the conditions of (34) are the same for both the node-locus and the cusp-locus, it can be shown that \( \Xi = 0, \partial \Xi / \partial x = 0, \) and \( \partial \Xi / \partial y = 0 \) will hold at \((\xi,\eta,\gamma)\) for the cusp-locus just as they did for the node-locus in Theorem 2.8.

From Theorem 2.8, the second partial derivative of \( \Xi \) with respect to \( x \) is given as

\[
\frac{\partial^2 \Xi}{\partial x^2} = Q\left[ f_{xx}(x,y,c_1) + f_{xc_1}(x,y,c_1)\frac{\partial c_1}{\partial x} \right] + 2f_x(x,y,c_1)\frac{\partial Q}{\partial x} + f_{c_1}(x,y,c_1)\frac{\partial^2 Q}{\partial x^2} + f(x,y,c_1)\frac{\partial^2 Q}{\partial x^2}.
\]
From the conditions of (34), the last relation becomes

\[ \frac{\delta^2}{\delta x^2} = Q \left[ f_{xx}(x,y,c_1) - f_{x0}(x,y,c_1)^2 \right]. \]

By evaluating equation (36) at \((\xi, \eta, \gamma)\) and applying the conditions of (33), equation (36) becomes \(\delta^2 / \delta x^2 = 0\). Similarly, it can be shown that \(\delta^2 / \delta \xi^2 = 0\) and \(\delta^2 / \delta \eta \delta \xi = 0\). Thus the discriminant \(E\) contains the cusp-locus as a factor at least twice.

If \(\phi\) is the cusp-locus, then \(E = \phi^2 S\) where \(S\) represents all of the remaining factors of the discriminant. Taking the partial derivative of \(E\) with respect to \(x\),

\[ \frac{\delta E}{\delta x} = 2S \phi \frac{\delta \phi}{\delta x} + \phi^2 \frac{\delta S}{\delta x}. \]

Now taking the second partial derivative of \(E\) with respect to \(x\),

\[ \frac{\delta^2 E}{\delta x^2} = 2S \phi \frac{\delta^2 \phi}{\delta x^2} + 2S \left[ \frac{\delta \phi}{\delta x} \right]^2 + 4 \phi \frac{\delta S}{\delta x} \frac{\delta \phi}{\delta x} + \phi^2 \frac{\delta^2 S}{\delta x^2} \]

Since \(\phi = 0\), equation (37) becomes

\[ \frac{\delta^2 E}{\delta x^2} = 2S \left[ \frac{\delta \phi}{\delta x} \right]^2. \]

Since \(\delta \phi / \delta x\) does not in general vanish at \((\xi, \eta, \gamma)\) and \(\delta^2 E / \delta \eta^2 = 0\), \(S = 0\). Thus \(S\) must contain the cusp-locus as a factor at least once. Since \(E = \phi^2 S\), the discriminant must contain the cusp-locus at least three times.

Taking the third derivative of \(E\) with respect to \(x\),

\[ \frac{\delta^3 E}{\delta x^3} = 4S \phi \frac{\delta^2 \phi}{\delta x^2} + 2 \left[ \frac{\delta \phi}{\delta x} \right]^2 \frac{\delta S}{\delta x}. \]
Since $S = 0$, the last relation becomes
\[ \frac{\delta^3 \xi}{\delta x^3} = 2 \left( \frac{\delta \psi}{\delta x} \right)^2 \frac{\delta S}{\delta x}. \]

In general $\delta \psi/\delta x$ and $\delta S/\delta x$ will not vanish at $(\xi, \eta, \gamma)$. Therefore the discriminant will not in general contain the cusp-locus more than three times.

Thus it is concluded that, in general, the c-discriminant will contain the cusp-locus three times, and only three times, as a factor.
CHAPTER III

THE $p$-DISCRIMINANT LOCI

Although the family of curves discussed in the past chapter was not necessarily a solution of a differential equation, it certainly can represent the general solution of a differential equation. Thus, if $f(x,y,c) = 0$ is the solution of $F(x,y,p) = 0$, then it has already been shown that the $c$-discriminant will contain the envelope, the node-locus, and/or the cusp-locus if any of these loci exist. Therefore it was through the function of the solution, $f(x,y,c)$, that the various loci of $F(x,y,p)$ were determined. It would seem that some of these results could be obtained through an examination of the differential equation itself. There is a method which is very similar to the method used with the solution function. That is, find the $p$-discriminant by eliminating $p$ between $F(x,y,p) = 0$ and $F_p(x,y,p) = 0$. This method is used to find the envelope and it also is found to reveal the cusp-locus and the tac-locus, but not the node-locus.

It is noticed that finding the $p$-discriminant is much the same as finding the $c$-discriminant, with the only apparent difference being in $p$ and $c$. This would lead one to believe that the proofs for the conditions of an envelope
of \( f(x,y,c) \) in Chapter II would suffice for proving the conditions of an envelope of \( F(x,y,p) \) simply by inserting \( p \) in place of \( c \). However this is not the case since \( c \) has a constant value for one curve of the family and certainly \( p \) takes on several values for this same curve. Difficulties arise in attempting to state both necessary and sufficient conditions for an envelope of \( F(x,y,p) \). Therefore, in this paper only a necessary condition for the envelope of the differential equation \( F(x,y,p) \) will be given.

**Lemma 3.1.** If \( f(x,y,c) > 0 \) and by the implicit function theorem may be written as \( y = G(x,c) \) and if \( f_c(x,y,c) = 0 \), then \( G_c(x,c) = 0 \).

**Proof.** Taking the derivative of \( f(x,y,c) \) with respect to \( c \) while holding \( x \) constant, \( f_y \frac{dy}{dc} + f_c = 0 \). Upon rearranging this,

\[
\frac{dy}{dc} = \frac{-f_c}{f_y}.
\]

Taking the derivative of \( y = G(x,c) \) with respect to \( c \) while holding \( x \) constant,

\[
\frac{dy}{dc} = G_c(x,c).
\]

Eliminating \( \frac{dy}{dc} \) between equations (1) and (2),

\[
G_c(x,c) = \frac{-f_c}{f_y}.
\]

If an envelope of \( f(x,y,c) \) exists, then by Theorem 2.1 \( f_c(x,y,c) = 0 \). Thus it is seen from equation (3) that if an envelope exists, then \( G_c(x,c) = 0 \).
Theorem 3.1. If the differential equation \( F(x, y, p) \) has an envelope, then each point on the envelope must satisfy \( F(x, y, p) = 0 \) and \( F_p(x, y, p) = 0 \).

Proof. First consider the differential equation

(4) \[ F(x, y, p) = 0 \]

whose general solution consists of a family of curves

(5) \[ f(x, y, c) = 0. \]

By Theorem 1.1, equation (5) may be written in the form

(6) \[ y = G(x, c) \]

where \( G \) is a function of the independent variables \( x \) and \( c \).

Taking the derivative of \( y \) with respect to \( x \),

(7) \[ \frac{dy}{dx} = G_\times(x, c). \]

Since \( p = \frac{dy}{dx} \), equation (7) becomes

(8) \[ p = G_\times(x, c). \]

If equation (8) were solved for \( c \), the results would be \( c = \varphi(x, p) \). Substituting this in equation (6),

(9) \[ y = G[x, \varphi(x, p)] \] or \( y = \alpha(x, p) \) where \( x \) and \( p \) are independent variables.

Differentiating equation (6) with respect to \( p \),

(10) \[ F_x dx + F_y dy + F_p = 0. \]

Since \( x \) and \( p \) are independent variables, (10) becomes
(11) \[ \frac{F_y}{dp} + F_p = 0. \]

Eliminating \(dy/dp\) between (9) and (11),

\[ F_y \left[ G_c(x,c) \frac{dc}{dp} \right] + F_p = 0. \]

If an envelope exists, \(G_c(x,c) = 0\) by Lemma 3.1. Therefore, if an envelope exists,

\[ F_p = 0. \]

**Loci of Contacts of Parallel Tangents**

To investigate the envelope and extraneous loci associated with the \(p\)-discriminant, it will be necessary to entertain the idea of loci of contacts of parallel tangents. These loci are determined by the points on the curves of the family having equal values of \(p\). When considering \(p\) as a constant, \(F(x,y,p)\) behaves in the same manner as did \(f(x,y,c)\) in Chapter II. To avoid confusion, \(a\) can be used as the parameter when \(p\) is a constant. Thus \(F(x,y,p)\) can be represented by \(F(x,y,a)\) when \(p\) is the constant \(a\).

It will first be shown that the loci of contacts of parallel tangents will touch the envelope-locus. That is, there will be at least one point where the direction of the loci of contacts of parallel tangents is the same as the direction of the envelope-locus.

**Theorem 3.2.** If \(f(x,y,c)\) has an envelope, the point \((\xi, \eta)\) is on the envelope, and \((\xi, \eta)\) is a point on the contacts of parallel tangents, then the locus of contacts of
parallel tangents touches the envelope-locus. That is, both loci have the same direction at \((\zeta, \eta)\).

**Proof.** Let \( \alpha \) be the angle between the \( x \)-axis and the tangent to \( f(x, y, \gamma) = 0 \) at \((\zeta, \eta)\) where \((\zeta, \eta)\) is a point on the envelope and \( \gamma \) is a constant parameter.

If the parameter of \( f(x, y, \gamma) = 0 \) is a constant, say \( \gamma \), then it has been shown in Theorem 2.1 that the slope of the curve at any point \((x, y)\) is equal to \(-\frac{f_x}{f_y}\). Thus for the curve \( f(x, y, \gamma) = 0 \), \( \tan \alpha = -\frac{f_x}{f_y} \) when \( x = \zeta \) and \( y = \eta \).

Similarly, for the curve \( f(x, y, \gamma + \delta \gamma) = 0 \), \( \tan \alpha = -\frac{f_x}{f_y} \) when \( x = \zeta + \delta \zeta \) and \( y = \eta + \delta \eta \).

Since \( f(\zeta + \delta \zeta, \eta + \delta \eta, \gamma + \delta \gamma) = 0 \) and \( f(\zeta, \eta, \gamma) = 0 \), the total differential of \( f \) is equal to zero, that is,

\[
\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \gamma} = 0
\]

Since \((\zeta, \eta)\) is a point on the envelope, then by Theorem 2.1 \( \frac{\partial f}{\partial \gamma} = 0 \). Thus equation (14) becomes

\[
\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial \eta} = 0.
\]

Solving equation (15) for \( \frac{\partial \eta}{\partial \zeta} \), which gives the slope of the loci of contacts of parallel tangents,

\[
\frac{\partial \eta}{\partial \zeta} = -\frac{\partial f}{\partial \eta}.
\]

Therefore it is found that the direction of the locus of contacts of parallel tangents is equal to \(-\frac{f_\zeta}{f_\eta}\). It has previously been shown that this is also the direction of the envelope of \( f(x, y, \gamma) = 0 \) at \((\zeta, \eta)\). Thus the loci
of contacts of parallel tangents touch the envelope. Also all points on the locus of contacts of parallel tangents satisfy \( F(x, y, a) = 0 \) where \( a = \tan \alpha \).

Next it will be shown that the loci of contacts of parallel tangents touch the cusp-locus of \( f(x, y, c) \).

**Theorem 3.3.** If \( f(x, y, c) = 0 \) has a cusp-locus, then the loci of contacts of parallel tangents touch the cusp-locus.

**Proof.** If \((\xi, \eta, \gamma)\) is a point on the cusp-locus of the function \( f(x, y, c) = 0 \), then by Theorem 2.5, \( f_x = 0 \), \( f_y = 0 \), and \( f_c = 0 \) at \((\xi, \eta, \gamma)\). Also it has been shown in Corollary 2.1 that the slope of \( f(x, y, c) \) at the cusp point \((\xi, \eta, \gamma)\) is given by \(-f_{xx}/f_{xy}\). Thus the slope at \((\xi, \eta, \gamma)\) is \(-f_{\xi\xi}/f_{\xi\eta}\).

Let \((\xi + \delta\xi, \eta + \delta\eta, \gamma + \delta\gamma)\) be a point where the tangent is parallel to the tangent at \((\xi, \eta, \gamma)\). Since the slope at any point on \( f(x, y, \gamma + \delta\gamma) \) is given by \(-f_x/f_y\), the slope at this point is \(-f_x/f_y\) where \( x = \xi + \delta\xi \) and \( y = \eta + \delta\eta \).

Since the tangents are parallel, \(-f_{\xi\xi}/f_{\xi\eta} = -f_x/f_y\) where \( x = \xi + \delta\xi \) and \( y = \eta + \delta\eta \). Expanding \( f_x \) and \( f_y \) by Theorem 1.5, Taylor's Theorem, the following results are obtained:

\[
(16) \quad \frac{f_{\xi\xi}}{f_{\xi\eta}} = \frac{f_{\xi} + f_{\xi\xi} \delta\xi + f_{\xi\eta} \delta\eta + f_{\xi\gamma} \delta\gamma + 1/2 R}{f_{\xi} + f_{\xi\xi} \delta\xi + f_{\xi\eta} \delta\eta + f_{\xi\gamma} \delta\gamma + 1/2 S}
\]

where \( R = f_{\xi\xi\xi} \delta\xi^2 + f_{\xi\eta\eta} \delta\eta^2 + f_{\xi\gamma\gamma} \delta\gamma^2 + 2f_{\xi\xi\eta} \delta\xi \delta\eta + 2f_{\xi\xi\gamma} \delta\xi \delta\gamma + 2f_{\xi\eta\gamma} \delta\eta \delta\gamma \)
and \( S = f_{\xi \xi} \eta^2 + f_{\eta \eta} \eta^2 + f_{\eta \eta \eta} \eta^2 + 2f_{\xi \xi} \eta \xi \eta \eta + 2f_{\xi \eta} \eta \xi \eta \eta \eta . \)

From Remark 2.1, \( f_{\xi \xi} \eta \eta + f_{\eta \eta} \eta \xi \eta \eta + f_{\xi \eta} \eta \xi \eta \eta \eta = 0 \) and \( f_{\xi \eta} \eta \xi \eta \eta + f_{\eta \eta} \eta \xi \eta \eta = 0 . \) Therefore, equation (16) becomes \( R - 2f_{\xi} \eta = 0 . \)

By substituting the values of \( R \) and \( S \) found above in this relation, the following equation is obtained:

\[
\begin{align*}
&f_{\xi \xi \xi} \eta^2 + f_{\xi \eta \eta} \eta^2 + f_{\eta \eta \eta} \eta^2 + 2f_{\xi \xi} \eta \xi \eta \eta + 2f_{\xi \eta} \eta \xi \eta \eta \eta + 2f_{\xi \eta} \eta \xi \eta \eta \eta \eta + 2f_{\xi \eta} \eta \xi \eta \eta \eta \eta \eta \\
&+ 2f_{\eta \eta} \eta \xi \eta \eta \eta \eta \eta - 2f_{\xi \xi} \eta \xi \eta \eta \eta \eta - 2f_{\xi \eta} \eta \xi \eta \eta \eta \eta - 2f_{\xi \eta} \eta \xi \eta \eta \eta \eta \eta - 2f_{\xi \eta} \eta \xi \eta \eta \eta \eta \eta - 2f_{\xi \eta} \eta \xi \eta \eta \eta \eta \eta = 0.
\end{align*}
\]

Let \( A, B, C, D, E, \) and \( F \) take on the following values:

\[
\begin{align*}
A &= f_{\xi \xi} - f_{\xi} f_{\xi \xi} \\
B &= f_{\xi \eta} - f_{\xi} f_{\xi \eta} \\
C &= f_{\xi \eta \eta} - f_{\xi} f_{\xi \eta \eta} \\
D &= f_{\xi \eta \eta \eta} - f_{\xi} f_{\xi \eta \eta \eta} \\
E &= f_{\xi \xi \eta} - f_{\xi} f_{\xi \xi \eta} \\
F &= f_{\xi \eta \eta \eta} - f_{\xi} f_{\xi \eta \eta \eta} .
\end{align*}
\]

Thus the above equation becomes, after the substitution of \( A, B, C, D, E, \) and \( F, \)

\[
(17) \quad A \eta^2 + B \eta + C \eta^2 + 2D \eta \eta \eta + 2E \eta \xi \eta \eta \eta + 2F \eta \xi \eta \eta \eta \eta = 0.
\]
This relation is satisfied by any point on the locus of contacts of parallel tangents.

Since \((\xi, \eta)\) is a point on the curve \(f(x, y, \gamma)\) and 
\((\xi + \delta\xi, \eta + \delta\eta)\) is a point on \(f(x, y, \gamma + \delta\gamma)\), \(f(\xi, \eta, \gamma) = 0\) and 
\(f(\xi + \delta\xi, \eta + \delta\eta, \gamma + \delta\gamma) = 0\). Expanding \(f(\xi + \delta\xi, \eta + \delta\eta, \gamma + \delta\gamma) = 0\) 
by Theorem 1.5, the following relation is obtained:

\[
\begin{align*}
&f_{\xi\xi}(\xi, \eta, \gamma)\delta\xi^2 + f_{\eta\eta}(\xi, \eta, \gamma)\delta\eta^2 + f_{\gamma\gamma}(\xi, \eta, \gamma)\delta\gamma^2 \\
&+ 2f_{\xi\eta}(\xi, \eta, \gamma)\delta\xi\delta\eta + 2f_{\xi\gamma}(\xi, \eta, \gamma)\delta\xi\delta\gamma + 2f_{\eta\gamma}(\xi, \eta, \gamma)\delta\eta\delta\gamma = 0.
\end{align*}
\]

By Theorem 2.6, \(f_{\xi\xi}f_{\eta\eta} - f_{\xi\eta}^2 = 0\), \(f_{\xi\xi}f_{\gamma\gamma} - f_{\xi\gamma}^2 = 0\), 
and \(f_{\xi\xi}f_{\eta\gamma} - f_{\xi\eta}f_{\xi\gamma} = 0\) hold at the cusp point \((\xi, \eta, \gamma)\). 
Using these conditions and by multiplying by \(f_{\xi\xi}\), the above equation becomes

\[
\begin{align*}
&f_{\xi\xi}^2\delta\xi^2 + f_{\xi\eta}^2\delta\eta^2 + f_{\xi\gamma}^2\delta\gamma^2 + 2f_{\xi\xi}f_{\eta\gamma}\delta\xi\delta\eta \\
&+ 2f_{\xi\xi}f_{\xi\gamma}\delta\xi\delta\gamma + 2f_{\eta\gamma}f_{\xi\gamma} = 0.
\end{align*}
\]

By factoring, the above relation becomes

\[(18) \quad f_{\xi\xi}\delta\xi + f_{\xi\eta}\delta\eta + f_{\xi\gamma}\delta\gamma = 0.\]

Equation (18) is satisfied by any point on the locus of contacts of parallel tangents.

The direction of the locus of contacts of parallel tangents is obtained by eliminating \(\delta\gamma\) from (17) and (18). 
Thus from (18), \(\delta\gamma = -\frac{f_{\xi\eta}\delta\xi - f_{\xi\xi}\delta\eta}{f_{\xi\gamma}}\) and upon substituting 
this in (17), the following is obtained after some simplification:
(19) \[(Af\xi\eta^2 + Cf\xi\eta^2 - 2Ef\eta^2)\partial\xi^2\]
\[+ (2Cf\xi^2\eta - 2Df\eta^2\xi + 2Ef\xi\eta + 2f^2\xi^2)\partial\xi\partial\eta\]
\[+ (Bf\xi^2 + Cf\xi^2 - 2Df\eta^2\xi\eta)\partial\eta^2 = 0.\]

In equation (19) let \(L\) be the coefficient of \(\partial\xi\), \(M\) the coefficient of \(\partial\xi\partial\eta\), and \(N\) the coefficient of \(\partial\eta^2\). This results in the equation
\[L\partial\xi^2 + M\partial\xi\partial\eta + N\partial\eta^2 = 0.\]

Solving this relation for \(\partial\xi/\partial\eta\),
\[\frac{\partial\xi}{\partial\eta} = \frac{-M \pm \sqrt{M^2 - 4LN}}{2L}.\]

Since the locus has only one direction at \((\xi, \eta, \gamma)\),
\[M^2 - 4LN = 0.\] Therefore, \(\partial\xi/\partial\eta = -M/2L\) or
\[L\partial\xi + M\partial\eta = 0.\]

Inserting the values of \(L\) and \(M\), the above relation gives the direction of the locus of contacts of parallel tangents at the point \((\xi, \eta, \gamma)\).

(20) \[(Af\xi\eta^2 - 2Ef\xi\eta + Cf\xi^2)\partial\xi\]
\[+ (Cf\xi\eta - Df\xi\eta - Ef\xi\eta + Ef\eta^2)\partial\eta = 0.\]

The conditions at the cusp point \((\xi, \eta, \gamma)\),
(21) \[f_{\xi\eta} = f_{\xi\eta} = f_{\eta\eta} = f_{\eta\eta}\]
are given by Theorem 2.6. Thus the relations for the cusp at \((\xi + \partial\xi, \eta + \partial\eta)\), when \(c = \gamma + \partial\gamma\), is obtained by taking the total differential of (21). That is,
where $D$ represents $\frac{\partial \xi}{\partial \xi} + \frac{\partial \eta}{\partial \eta} + \frac{\partial \gamma}{\partial \gamma}$. The above relation becomes

$$\frac{f_{\xi \xi} f_{\xi \eta} - f_{\xi \xi} f_{\xi \eta}}{f_{\xi \eta}} = \frac{f_{\eta \eta} f_{\xi \eta} - f_{\xi \eta} f_{\eta \eta}}{f_{\eta \eta}}$$

Performing the operation as indicated by $D$,

$$\frac{f_{\xi \xi} \frac{\partial \xi}{\partial \xi} + f_{\xi \eta} \frac{\partial \eta}{\partial \eta} + f_{\xi \gamma} \frac{\partial \gamma}{\partial \gamma}}{f_{\xi \eta}} - \frac{f_{\xi \xi} f_{\xi \eta} \frac{\partial \xi}{\partial \xi} + f_{\xi \xi} f_{\xi \eta} \frac{\partial \xi}{\partial \eta} + f_{\xi \xi} f_{\xi \eta} \frac{\partial \xi}{\partial \gamma}}{f_{\xi \eta}}$$

$$= \frac{f_{\eta \eta} \frac{\partial \xi}{\partial \eta} + f_{\eta \eta} \frac{\partial \eta}{\partial \eta} + f_{\eta \eta} \frac{\partial \gamma}{\partial \gamma}}{f_{\eta \eta}} - \frac{f_{\eta \eta} f_{\eta \eta} \frac{\partial \xi}{\partial \eta} + f_{\eta \eta} f_{\eta \eta} \frac{\partial \eta}{\partial \eta} + f_{\eta \eta} f_{\eta \eta} \frac{\partial \eta}{\partial \gamma}}{f_{\eta \eta}}$$

A better form of this relation is obtained by factoring out the $\frac{\partial \xi}{\partial \xi}$, $\frac{\partial \eta}{\partial \eta}$, and $\frac{\partial \gamma}{\partial \gamma}$ terms. Doing this, the following is obtained:

$$\frac{(f_{\xi \xi} f_{\xi \eta} - f_{\xi \xi} f_{\xi \eta}) \frac{\partial \xi}{\partial \xi}}{f_{\xi \eta}^2} + \frac{(f_{\xi \xi} f_{\xi \eta} - f_{\xi \xi} f_{\xi \eta}) \frac{\partial \xi}{\partial \eta}}{f_{\xi \eta}^2} + \frac{(f_{\xi \xi} f_{\xi \eta} - f_{\xi \xi} f_{\xi \eta}) \frac{\partial \xi}{\partial \gamma}}{f_{\xi \eta}^2}$$

$$+ \frac{(f_{\eta \eta} f_{\eta \eta} - f_{\eta \eta} f_{\eta \eta}) \frac{\partial \eta}{\partial \eta}}{f_{\eta \eta}^2} + \frac{(f_{\eta \eta} f_{\eta \eta} - f_{\eta \eta} f_{\eta \eta}) \frac{\partial \eta}{\partial \gamma}}{f_{\eta \eta}^2}$$

$$+ \frac{(f_{\eta \eta} f_{\eta \eta} - f_{\eta \eta} f_{\eta \eta}) \frac{\partial \gamma}{\partial \gamma}}{f_{\eta \eta}^2}$$
Letting $A$, $B$, $C$, $D$, $E$, and $F$ have the same meaning as stated earlier, this last equation can be written in the simpler form of

$$\frac{\partial}{\partial \eta} \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \eta}.$$  

Equation (22) is therefore satisfied by the point $(\xi, \eta, \gamma)$ on the cusp-locus.

From Theorem 2.6, the points on the cusp-locus satisfy

$$f_{\xi\zeta} \frac{\partial}{\partial \zeta} + f_{\xi\eta} \frac{\partial}{\partial \eta} + f_{\xi\gamma} \frac{\partial}{\partial \gamma} = 0.  

The direction of the cusp-locus may be determined by eliminating $\frac{\partial}{\partial \gamma}$ from (22) and (23). Using the first and third relations in (22) and with equation (23), the direction of the cusp-locus at $(\xi, \eta, \gamma)$ is given by

$$\frac{(Af_{\xi\gamma}^2 - 2Ef_{\xi\gamma}f_{\xi\eta} + Cf_{\xi\eta}^2)}{(Cf_{\xi\xi}f_{\xi\eta} - Df_{\xi\xi}f_{\xi\gamma} - Ef_{\xi\eta}f_{\xi\gamma} + Ef_{\xi\eta}f_{\xi\eta} + Ef_{\xi\eta}f_{\xi\gamma}^2)} = 0.  

Comparing equations (20) and (24), it is seen that the directions of the cusp-locus and the locus of contact points of parallel tangents are the same at their common point $(\xi, \eta, \gamma)$. Thus it is concluded that the loci of contacts of parallel tangents touch the cusp-locus.
In order to find some relation between the tac-locus and the \( p \)-discriminant, it will first be shown that the tac-locus is a part of the node-locus of the loci of contact points of parallel tangents of \( f(x,y,c) = 0 \).

**Theorem 3.4.** If \( f(x,y,c) = 0 \) has a tac-locus, then the tac-locus is, in general, a part of the node-locus of the loci of contacts of parallel tangents to the family of curves \( f(x,y,c) = 0 \).

**Proof.** Let \((\xi,\eta)\) be a point on the tac-locus such that it will satisfy both relations, \( f(\xi,\eta,\gamma_1) = 0 \) and \( f(\xi,\eta,\gamma_2) = 0 \). The slope at any point on the curve \( f(x,y,\gamma_1) = 0 \) is given by \( \frac{dy}{dx} = -\frac{f_x}{f_y} \). Thus at \((\xi,\eta)\) the slope is \( -\frac{f_x(\xi,\eta,\gamma_1)}{f_y(\xi,\eta,\gamma_1)} \). Similarly, the slope of \( f(x,y,\gamma_2) = 0 \) at \((\xi,\eta)\) is \( -\frac{f_x(\xi,\eta,\gamma_2)}{f_y(\xi,\eta,\gamma_2)} \). The slopes of both curves at \((\xi,\eta)\) are equal, thus

\[
\frac{f_x(\xi,\eta,\gamma_1)}{f_y(\xi,\eta,\gamma_1)} = \frac{f_x(\xi,\eta,\gamma_2)}{f_y(\xi,\eta,\gamma_2)}.
\]

Since \( -\frac{f_x}{f_y} \) is the slope at any point on a given curve of the family, let \( \tan \alpha \) be the direction at the points \((x,y,\gamma_1)\) and \((x,y,\gamma_2)\). Thus the slope of the curve at any point on the locus of contacts of parallel tangents is equal to \( \tan \alpha \) where \( \tan \alpha = -\frac{f_x}{f_y} \).

Since \( -\frac{f_x}{f_y} \) remains constant, take its total differential and obtain

\[
\frac{\partial}{\partial x} f_x \cdot dx + \frac{\partial}{\partial y} f_y \cdot dy + \frac{\partial}{\partial c} f_y \cdot dc = 0.
\]
Taking the total differential of \( f(x, y, c) = 0 \),

\[
\frac{\partial f}{\partial x} \frac{\partial x}{\partial c} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial c} + \frac{\partial f}{\partial c} \frac{\partial c}{\partial c} = 0.
\]

Eliminating \( \partial c \) from the last two equations,

\[
\frac{\partial y}{\partial x} = \frac{x \frac{\partial f}{\partial c} \left( \frac{f_x}{f_y} \right) - y \frac{\partial c}{\partial x} \left( \frac{f_x}{f_y} \right)}{c \frac{\partial f}{\partial y} \left( \frac{f_x}{f_y} \right) - y \frac{\partial c}{\partial c} \left( \frac{f_x}{f_y} \right)},
\]

which gives the direction of the locus of contacts of parallel tangents. Thus, at some point \((\xi, \eta, c)\),

\[
\frac{\partial \eta}{\partial \xi} = \frac{x \frac{\partial f}{\partial c} \left( \frac{f_\xi/f_\eta}{f_\eta} \right) - y \frac{\partial c}{\partial \xi} \left( \frac{f_\xi/f_\eta}{f_\eta} \right)}{c \frac{\partial f}{\partial \eta} \left( \frac{f_\xi/f_\eta}{f_\eta} \right) - y \frac{\partial c}{\partial \eta} \left( \frac{f_\xi/f_\eta}{f_\eta} \right)}.
\]

The direction of the locus of contacts of parallel tangents will, in general, be different when \( c \) is given different values. That is, when \( c = \gamma_1 \) a different slope is obtained than when \( c = \gamma_2 \).

Thus at a point \((\xi, \eta)\) on the tac-locus, the locus of contacts of parallel tangents has two directions. This indicates that the point on the tac-locus is also a nodal point on the locus of contacts of parallel tangents. Thus it is concluded that the tac-locus is at least a part of the node-locus of the contacts of parallel tangents.

Factors of the \( p \)-Discriminant

The last three theorems will enable one to now show the relations between the envelope, cusp-locus, tac-locus, and the \( p \)-discriminant. It will be shown, in general, that the \( p \)-discriminant will contain the envelope-locus as a
factor once, the cusp-locus as a factor once, and the tac-locus as a factor twice.

**Theorem 3.5.** If the differential equation \( F(x,y,p) = 0 \) has for its solution the family of curves \( f(x,y,c) = 0 \) and if an envelope, cusp-locus, and tac-locus exist for \( f(x,y,c) = 0 \), then the \( p \)-discriminant contains, in general, the envelope-locus as a factor once, the cusp-locus as a factor once, and the tac-locus twice as a factor.

**Proof.** Since \( F(x,y,p) = 0 \) is the relation representing the differential equation of the family of solution curves \( f(x,y,c) = 0 \), \( F(x,y,a) = 0 \) represents the loci of contacts of parallel tangents. It was stated in Theorem 3.2 that the loci of contacts of parallel tangents touch the envelope of the curves \( f(x,y,c) = 0 \). Therefore \( F(x,y,a) = 0 \) contains the envelope as a factor. Since \( a \) remains constant for a given locus of points of contacts of parallel tangents, \( F(x,y,a) = 0 \) may be treated in the same manner as \( f(x,y,c) = 0 \) was treated. That is, the \( a \)-discriminant would be found in the same manner and it is concluded that the \( a \)-discriminant will, in general, contain the envelope-locus once and only once. Now since \( a \) represents the slope at a point on the family, it is actually an arbitrary value of \( p \). Therefore, the \( a \)-discriminant is the same as the \( p \)-discriminant. Thus it is concluded that the \( p \)-discriminant will, in general, contain the envelope-locus once and only once.
Similarly, it has been shown that $F(x, y, a) = 0$ touches the cusp-locus. Therefore, the cusp-locus is a part of the envelope of the curves $F(x, y, a) = 0$. As above, the $a$-discriminant contains, in general, the envelope locus once and only once. Since the cusp-locus of $f(x, y, c) = 0$ is a part of the envelope of $F(x, y, a) = 0$, the $a$-discriminant will, in general, contain the cusp-locus once and only once. Thus it is concluded that the $p$-discriminant will, in general, contain the cusp-locus once and only once.

It has been shown that the tac-locus of the curves $f(x, y, c) = 0$ is, in general, a part of the node-locus of the contacts of parallel tangents, $F(x, y, a) = 0$. Since the $a$-discriminant has been shown to contain, in general, the node-locus as a factor twice and only twice, it is concluded that the $p$-discriminant will, in general, contain the tac-locus as a factor twice and only twice.
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