ABSTRACT VECTOR SPACES AND CERTAIN RELATED SYSTEMS

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ABSTRACT VECTOR SPACES AND CERTAIN RELATED SYSTEMS

THESIS

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CHAPTER I

INTRODUCTION

The purpose of this paper is to make a detailed study of vector spaces and a certain vector-like system. In Chapter II a vector-like system is studied, this system arising if one defines a vector to be a directed stroke from a point to a point (in Euclidean 3-space). It is shown that this system does not possess the properties which are desirable for a system to have if it is to be used in a study of vector analysis. Further, it is shown that this is not the actual system that is usually considered in vector analysis. A relationship is defined between the elements of this system and is used to partition the system. After the partitioning, another system is defined in Chapter III and it is shown that it has the properties which would be desirable for a system to have in a study of vectors. In Chapter IV the system defined in Chapter III is shown to be an ordinary vector space. A systematic study of a finite dimensional vector space is made in Chapter IV.

The properties of the real number system, complex number system, and notions of a field are assumed. A collection of objects will be referred to as a set and will be denoted by a capital letter. The set of real numbers will be denoted by R.

Definition 1.1 The symbol ϵ means "is an element of." Thus $t \epsilon T$ means that t is an element of the set T.

Definition 1.2 The set X is said to have $\mathcal{R}$ as a relation if for each pair (a,b) of elements of X, the phrase "a is in the relation $\mathcal{R}$ to b" is meaningful, being true or false depending upon the choice of a and b. The symbol $x \mathcal{R} y$ means that x is in the relation $\mathcal{R}$ to y.

Definition 1.3 The statement that Y is a subset of set X means that every element of Y is an element of X.

Definition 1.4 The statement that X is a partition of Y means that X is a collection of subsets of Y such that Y is the union of the sets of X and no two sets of X have an element in common.

Definition 1.5 The statement that a relation $\mathcal{R}$ is reflexive on the set X means that $x \mathcal{R} x$ for each $x \epsilon X$.

Definition 1.6 The statement that a relation $\mathcal{R}$ is symmetric on the set X means that if $x \epsilon X$, $y \epsilon X$, and $x \mathcal{R} y$, then also $y \mathcal{R} x$.

Definition 1.7 The statement that a relation $\mathcal{R}$ is transitive on set X means that if x, y, $z \epsilon X$ and $x \mathcal{R} y$ and $y \mathcal{R} z$, then also $x \mathcal{R} z$.

Definition 1.8 The statement that a relation $\mathcal{R}$ is an equivalence relation on X means that $\mathcal{R}$ is reflexive, symmetric and transitive.

Definition 1.9 If each of S and T is a set, then a mapping of S into T is a correspondence α between S and a subset of T that associates with each element of S a unique element of T. A mapping α of S into T is a mapping of S onto T if for each $b \in T$, there exists an $a \in S$ such that $a\alpha = b$. Thus if α is a mapping of S onto T, each element of T corresponds to some element of S.

Definition 1.10 The mapping α of S onto T is a 1-1 mapping of S onto T if for each $a, b \in S$, $a \neq b$ implies $a\alpha \neq b\alpha$.

Definition 1.11 The statement that * is an operation from $A \times B$ to C means that * is a mapping of $A \times B$ into C.

CHAPTER II

STUDY OF A CERTAIN VECTOR-LIKE SYSTEM

A system $\mathcal{S}\{S;\ (+),\ (\cdot)\}$ will now be defined. Let S denote the collection of all ordered pairs of points in Euclidean 3-space. Define (+) to be an operation from $S \times S$ to S and $(\cdot)$ to be an operation from $R \times S$ to S as follows: The statement that (A,B) (+) (C,D) = (E,F) means that E = A and $f_1 = b_1 + d_1 - c_1$, $f_2 = b_2 + d_2 - c_2$ and $f_3 = b_3 + d_3 - c_3$, and the statement that $k\ (\cdot)\ (A,B) = (D,C)$ means that $k \epsilon R$, D = A, and $c_1 = a_1 + k(b_1 - a_1)$, $c_2 = a_2 + k(b_2 - a_2)$ and $c_3 = a_3 + k(b_3 - a_3)$, where $A = (a_1,\ a_2,\ a_3)$, $B = (b_1,\ b_2,\ b_3)$, $C = (c_1,\ c_2,\ c_3)$, $D = (d_1,\ d_2,\ d_3)$, $E = (e_1,\ e_2,\ e_3)$, and $F = (f_1,\ f_2,\ f_3)$. The following nine theorems establish properties of the system $\mathcal{S}$ which are similar to those of an abstract vector space.

<u>Theorem 2.1</u>. If (A,B), (C,D), and (E,F) ϵ S, then (A,B) (+) [(C,D) (+) (E,F)] = [(A,B) (+) (C,D)] (+) (E,F).

<u>Proof</u>. By the definition of (+) it follows that (A,B) (+) (C,D) = (A,H) where $h_1 = b_1 + d_1 - c_1$, $h_2 = b_2 + d_2 - c_2$ and $h_3 = b_3 + d_3 - c_3$. Similarly it follows that (A,H) (+) (E,F) = (A,G) where $g_1 = b_1 + d_1 - c_1 + f_1 - e_1$, $g_2 = b_2 + d_2 - c_2 + f_2 - e_2$ and $g_3 = b_3 + d_3 - c_3 + f_3 - e_3$. Also by the definition of (+) it follows that (C,D) (+) (E,F) = (C,X) where $x_1 = d_1 + f_1 - e_1$, $x_2 = d_2 + f_2 - e_2$ and

$x_3 = d_3 + f_3 - e_3$, and (A,B) (+) (C,X) = (A,Y) where $y_1 = b_1 + d_1 + f_1 - e_1 - c_1$, $y_2 = b_2 + d_2 + f_2 - e_2 - c_2$, and $y_3 = b_3 + d_3 + f_3 - e_3 - c_3$. Hence it is clear that G = Y and the proof of the theorem is complete. This theorem implies that the operation (+) is associative.

<u>Theorem 2.2</u>. If (A,B) and (C,D) $\in$ S, k $\in$ R, then k (·) [(A,B) (+) (C,D)] = k (·) (A,B) (+) k (·) (C,D).

<u>Proof</u>. By the definitions of (+) and (·) it follows that k (·) (A,B) = (A,X), where $x_1 = a_1 + k(b_1 - a_1)$, $x_2 = a_2 + k(b_2 - a_2)$ and $x_3 = a_3 + k(b_3 - a_3)$; k (·) (C,D) = (C,Y), where $y_1 = c_1 + k(d_1 - c_1)$, $y_2 = c_2 + k(d_2 - c_2)$, and $y_3 = c_3 + k(d_3 - c_3)$. Also, (A,X) (+) (C,Y) = (A,Z), where $z_1 = a_1 + kb_1 - ka_1 + kd_1 - kc_1$, $z_2 = a_2 + kb_2 - ka_2 + kd_2 - kc_2$ and $z_3 = a_3 + kb_3 - ka_3 + kd_3 - kc_3$. It follows in a similar manner by the definition of (+) that (A,B) (+) (C,D) = (A,T), where $t_1 = b_1 + d_1 - c_1$, $t_2 = b_2 + d_2 - c_2$, and $t_3 = b_3 + d_3 - c_3$. Also, by the definition of (·) it follows that k (·) (A,T) = (A,P) where $p_1 = a_1 + kb_1 + kd_1 - kc_1 - ka_1$, $p_2 = a_2 + kb_2 + kd_2 - kc_2 - ka_2$, and $p_3 = a_3 + kb_3 + kd_3 - kc_3 - ka_3$. Hence P = Z and the proof is complete. It is established by this theorem that the operation (·) is distributive.

<u>Theorem 2.3</u>. If (A,B) $\in$ S and k $\in$ R, c $\in$ R, then (k + c) (·) (A,B) = k (·) (A,B) (+) c (·) (A,B).

<u>Proof</u>. By the definition of (·) it follows that k (·) (A,B) = (A,X) where $x_1 = a_1 + k(b_1 - a_1)$, $x_2 =$

$a_2 + k(b_2 - a_2)$ and $x_3 = a_3 + k(b_3 - a_3)$. Also, by the same definition c (·) (A,B) = (A,Y), where $y_1 = a_1 + c(b_1 - a_1)$, $y_2 = a_2 + c(b_2 - a_2)$ and $y_3 = a_3 + c(b_3 - a_3)$. It follows from the definition of (+) that (A,X) (+) (A,Y) = (A,Z) where $z_1 = x_1 + y_1 - a_1$, $z_2 = x_2 + y_2 - a_2$ and $z_3 = x_3 + y_3 - a_3$. Also, since $k,c \in R$ then $(k + c)$ (·) (A,B) = (A,T), where $t_1 = a_1 + (k + c)(b_1 - a_1)$, $t_2 = a_2 + (k + c)(b_2 - a_2)$ and $t_3 = a_3 + (k + c)(b_3 - a_3)$. But $z_1 = a_1 + kb_1 - ka_1 + a_1 + cb_1 - a_1 - ca_1 = a_1 + (k + c)(b_1 - a_1)$, $z_2 = a_2 + kb_2 - ka_2 + a_2 + cb_2 - a_2 - ca_2 = a_2 + (k + c)(b_2 - a_2)$ and $z_3 = a_3 + kb_3 - ka_3 + a_3 + cb_3 - a_3 - ca_3 = a_3 + (k + c)(b_3 - a_3)$. It is clear that $z_1 = t_1$, $z_2 = t_2$, and $z_3 = t_3$. Hence Z = T and the proof of the theorem is complete. This theorem establishes a certain distributive property for the operation (·).

Theorem 2.4. If (A,B) $\in$ S and $k \in R$, $c \in R$, then $[c \cdot k]$ (·) (A,B) = c (·) [k (·) (A,B)].

Proof. It follows from the definition of (·) that $c \cdot k$ (·) (A,B) = (A,X) where $x_1 = a_1 + c \cdot k(b_1 - a_1)$, $x_2 = a_2 + c \cdot k(b_2 - a_2)$ and $x_3 = a_3 + c \cdot k(b_3 - a_3)$. Also, k (·) (A,B) = (A,T), where $t_1 = a_1 + k(b_1 - a_1)$, $t_2 = a_2 + k(b_2 - a_2)$ and $t_3 = a_3 + k(b_3 - a_3)$; and c (·) (A,T) = (A,Y), where $y_1 = a_1 + c(t_1 - a_1)$, $y_2 = a_2 + c(t_2 - a_2)$ and $y_3 = a_3 + c(t_3 - a_3)$. Also, $x_1 = a_1 + d(b_1 - a_1) = a_1 + c \cdot k(b_1 - a_1)$ and in a similar fashion $x_2 = c \cdot k(b_2 - a_2)$ and $x_3 = c \cdot k(b_3 - a_3)$. It follows then that $y_1 = a_1 +$

$c_1[a_1 + k(b_1 - a_1) - a_1] = a_1 + c'k(b_1 - a_1)$ and a similar expression is true for y_2 and y_3. Hence X = Y and the proof is complete.

<u>Theorem 2.5</u>. If $k \, (\cdot) \, (A,B) = (A,A)$ then either $k = 0$, or $B = A$.

<u>Proof</u>. Suppose $k \neq 0$ and $B \neq A$. Then by the definition of $(\cdot)$ it follows that $k \, (\cdot) \, (A,B) = (A,Y)$, where $y_1 = a_1 + k(b_1 - a_1)$, $y_2 = a_2 + k(b_2 - a_2)$ and $y_3 = a_3 + k(b_3 - a_3)$. But by the hypothesis $Y = A$, and so $y_1 = a_1$, $y_2 = a_2$ and $y_3 = a_3$. This means that $k(b_1 - a_1) = 0$, $k(b_2 - a_2) = 0$ and $k(b_3 - a_3) = 0$. Since $k \neq 0$, then $b_1 = a_1$, $b_2 = a_2$ and $b_3 = a_3$, which is contrary to assumption. Hence the theorem is valid. The proof of the converse follows in the next theorem.

<u>Theorem 2.6</u>. If $k = 0$ or $B = A$ then $k \, (\cdot) \, (A,B) = (A,A)$.

<u>Proof</u>.

Case I. Suppose $k = 0$. It follows from the definition of $(\cdot)$ that $k \, (\cdot) \, (A,B) = (A,X)$, where $x_1 = a_1 + 0(b_1 - a_1) = a_1$, $x_2 = a_2 + 0(b_2 - a_2) = a_2$ and $x_3 = a_3 + 0(b_3 - a_3) = a_3$. Hence X = A and the theorem is valid for this case.

Case II. Suppose $k \neq 0$ and $B = A$. By the definition of $(\cdot)$ then $k \, (\cdot) \, (A,A) = (A,X)$, where $x_1 = a_1 + k(a_1 - a_1) = a_1$, $x_2 = a_2 + k(a_2 - a_2) = a_2$ and $x_3 = k(a_3 - a_3) = a_3$. Hence X = A and the proof of the theorem is complete.

<u>Theorem 2.7</u>. If $k \, (\cdot) \, (A,B) = (A,B)$ and $B \neq A$ then $k = 1$.

<u>Proof</u>. Suppose $k \, (\cdot) \, (A,B) = (A,X)$. By the definition

of $(\cdot)$ it follows that $x_1 = a_1 + k(b_1 - a_1)$, $x_2 = a_2 + k(b_2 - a_2)$ and $x_3 = a_3 + k(b_3 - a_3)$. But by the hypothesis $X = B$, and so $b_1 = a_1 + k(b_1 - a_1)$, $b_2 = a_2 + k(b_2 - a_2)$ and $b_3 = a_3 + k(b_3 - a_3)$. Also, by the hypothesis $B \neq A$, thus then either $b_1 \neq a_1$, or $b_2 \neq a_2$, or $b_3 \neq a_3$. Suppose $b_1 \neq a_1$. Then it follows that $b_1 - a_1 = k(b_1 - a_1)$ and $k = 1$. The other cases are similar. Hence the theorem is true.

Theorem 2.8. If $k = 1$ then $k\ (\cdot)\ (A,B) = (A,B)$.

Proof. Suppose $k = 1$. Then by the definition of $(\cdot)$ it follows immediately that $k\ (\cdot)\ (A,B) = (A,X)$, where $x_1 = a_1 + (b_1 - a_1)$, $x_2 = a_2 + (b_2 - a_2)$ and $x_3 = a_3 + (b_3 - a_3)$. It is clear that $X = B$ and the proof is complete. Note that this property of the System $\mathscr{S}$ is a postulated property of a vector space.

Theorem 2.9. If $(A,X)\ (+)\ (A,Y) = (A,Y)$ then $X = A$.

Proof. Since by the hypothesis $(A,X)\ (+)\ (A,Y) = (A,Y)$, it follows by the definition of $(+)$ that $y_1 = x_1 + (y_1 - a_1)$, $y_2 = x_2 + (y_2 - a_2)$ and $y_3 = x_3 + (y_3 - a_3)$. Hence it is clear that $X = A$ and completes the proof of the theorem.

In the following theorem it is established that the system $\mathscr{S}$ does not have a certain very important property which every vector space possesses.

Theorem 2.10. There exist elements (A,B) and $(C,D) \in S$, such that $(A,B)\ (+)\ (C,D) \neq (C,D)\ (+)\ (A,B)$.

Proof. Consider the points $A(1,2,3)$, $B(2,3,4)$, $C(3,4,5)$ and $D(4,5,6)$ of E^3. By the definition of $(+)$, $(A,B)\ (+)$

$(C,D) = (A,X)$, where $x_1 = 3$, $x_2 = 4$ and $x_3 = 5$. Also by the same definition $(C,D) \oplus (A,B) = (C,Y)$, where $y_1 = 5$, $y_2 = 6$ and $y_3 = 7$. Hence $(A,X) \neq (C,Y)$ and the proof is complete. This theorem establishes that the operation $\oplus$ is not commutative. A relation is defined on the elements of the set S in the following definition which is used in connection with the study of another system in the following chapter.

<u>Definition 2.1</u> The statement that $(A,B) \sim (C,D)$ means that $b_1 - a_1 = d_1 - c_1$, $b_2 - a_2 = d_2 - c_2$ and $b_3 - a_3 = d_3 - c_3$.

<u>Theorem 2.11</u>. The relation defined in Definition 2.1 and denoted by $\sim$ is an equivalence relation on the set S.

<u>Proof</u>. Suppose any element $(A,B) \in S$ is chosen. It follows immediately that $(A,B) \sim (A,B)$ since $b_1 - a_1 = b_1 - a_1$, $b_2 - a_2 = b_2 - a_2$ and $b_3 - a_3 = b_3 - a_3$. Therefore the relation $\sim$ is reflexive. Next, consider any two elements (A,B) and (C,D) of the set S such that $(A,B) \sim (C,D)$. From the definition of $\sim$ it follows that $b_1 - a_1 = d_1 - c_1$, $b_2 - a_2 = d_2 - c_2$ and $b_3 - a_3 = d_3 - c_3$. Since $a,b,c,d \in R$ it is clear that $d_1 - c_1 = b_1 - a_1$, $d_2 - c_2 = b_2 - a_2$ and $d_3 - c_3 = b_3 - a_3$. Hence $(C,D) \sim (A,B)$. Therefore the relation $\sim$ is symmetric on the set S. Also consider any three elements (A,B), (C,D) and (E,F) of the set S, such that $(A,B) \sim (C,D)$ and $(C,D) \sim (E,F)$. By the definition of $\sim$ it follows that $b_1 - a_1 = d_1 - c_1$,

$b_2 - a_2 = d_2 - c_2$ and $b_3 - a_3 = d_3 - c_3$. Similarly, it is true that $d_1 - c_1 = f_1 - e_1$, $d_2 - c_2 = f_2 - e_2$ and $d_3 - c_3 = f_3 - e_3$. Since $a,b,c,d,e,f \in R$ it is clear that $b_1 - a_1 = f_1 - e_1$, $b_2 - a_2 = f_2 - e_2$ and $b_3 - a_3 = f_3 - e_3$. Hence the relation $\sim$ satisfies Definition 1.8 and is an equivalence relation on the set S.

Note the existence of a unique element $(N,N) \in S$ where $n_1 = 0$, $n_2 = 0$ and $n_3 = 0$. Suppose (A,B) is any element of set S. It follows by the definition of (+) that (A,B) (+) (N,N) = (A,B). Therefore the element (N,N) serves as a right additive identity element for every element $(A,B) \in S$. Notice also that there exists a subset $S_1 \in S$ such that $(A,B) \in S_1$ if and only if A = B, and that the subset S_1 is an infinite set. Since it has been established by Theorem 2.10 that the (+) operation is not commutative on the set S, it cannot be assumed that every right additive identity is also a left additive identity. Suppose there exists an element $(X,Y) \in S$ such that for any element $(A,B) \in S$, it is true that (X,Y) (+) (A,B) = (A,B). It follows immediately from Theorem 2.9 that this could occur if and only if X = Y = A. Hence X and Y depend upon A and no unique additive identity exists in the system $\mathcal{S}$.

CHAPTER III

A SYSTEM DERIVED FROM A VECTOR-LIKE SYSTEM

A system $\mathcal{S}'\ \{S'; [+], [\cdot]\}$ will now be defined in terms of the notions developed in Chapter II. Let S' denote the set of elements such that $\alpha \in S'$ if and only if α is a subset of S which belongs to the partition of S induced by the equivalence relation $\sim$. Define $[+]$ to be an operation from $S' \times S'$ to S' and $[\cdot]$ to be an operation from $R \times S'$ to S' as follows. The statement that $\alpha\ [+]\ \beta = \gamma$ means if $(A,B) \in \alpha$ and $(C,D) \in \beta$ then $(A,B)\ (+)\ (C,D) \in \gamma$. The statement that $k\ [\cdot]\ \alpha = \beta$ means that if $k \in R$ and $(A,B) \in \alpha$ then $k\ (\cdot)\ (A,B) \in \beta$. The first two of the following theorems show that $[+]$ and $[\cdot]$ are well defined.

<u>Theorem 3.1</u>. If $(A,B) \in \alpha$, $(C,D) \in \gamma$, $(E,F) \in \alpha$ and $(G,H) \in \gamma$ then $(A,B)\ (+)\ (C,D) \sim (E,F)\ (+)\ (G,H)$.

<u>Proof</u>. By the hypothesis $(A,B) \sim (E,F)$ and $(C,D) \sim (G,H)$. From the definition of $(+)$ it follows that $(A,B)\ (+)\ (C,D) = (A,X)$ and $(E,F)\ (+)\ (G,H) = (E,Y)$ where $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$, $x_3 = b_3 + d_3 - c_3$, $y_1 = f_1 + h_1 - g_1$, $y_2 = f_2 + h_2 - g_2$ and $y_3 = f_3 + h_3 - g_3$. Since $x_1 - a_1 = b_1 + d_1 - c_1 - a_1$ and $y_1 - e_1 = f_1 + h_1 - g_1 - e_1 = b_1 + d_1 - c_1 - a_1$, it follows that $x_1 - a_1 = y_1 - e_1$, and in a similar manner $x_2 - a_2 = y_2 - e_2$, and $x_3 - a_3 = y_3 - e_3$. Hence $(A,X) \sim (E,Y)$ and the proof is complete.

<u>Theorem 3.2</u>. If $(A,B) \in \alpha$, $(C,D) \in \alpha$ and $k \in R$ then $k\ (\cdot)\ (A,B) \sim k\ (\cdot)\ (C,D)$.

<u>Proof</u>. By the hypothesis $(A,B) \sim (C,D)$ and from the definition of $(\cdot)$, $k\ (\cdot)\ (A,B) = (A,X)$ and $k\ (\cdot)\ (C,D) = (C,Y)$, where $x_1 = a_1 + k(b_1 - a_1)$, $x_2 = a_2 + k(b_2 - a_2)$, $x_3 = a_3 + k(b_3 - a_3)$, $y_1 = c_1 + k(d_1 - c_1)$, $y_2 = c_2 + k(d_2 - c_2)$ and $y_3 = c_3 + k(d_3 - c_3)$. Since $x_1 = a_1 + k(b_1 - a_1)$ and $y_1 - c_1 = k(d_1 - c_1) = k(b_1 - a_1)$, then $y_1 - c_1 = x_1 - a_1$. It follows in a like fashion that $y_2 - c_2 = x_2 - a_2$ and $y_3 - c_3 = x_3 - a_3$. Hence $(A,X) \sim (C,Y)$ and the theorem is valid.

<u>Theorem 3.3</u>. If $\alpha \in S'$ and $\beta \in S'$ then $\alpha\ [+]\ \beta = \beta\ [+]\ \alpha$.

<u>Proof</u>. Let $\alpha\ [+]\ \beta = \gamma$ and $\beta\ [+]\ \alpha = \nu$. Then if $(A,B) \in \alpha$ and $(C,D) \in \beta$, it follows from the definition of $[+]$ that $(A,B)\ (+)\ (C,D) \in \beta$ and $(C,D)\ (+)\ (A,B) \in \nu$. Since $(A,B)\ (+)\ (C,D) = (A,X)$, where $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$, and $x_3 = b_3 + d_3 - c_3$, and $(C,D)\ (+)\ (A,B) = (C,Y)$, where $y_1 = d_1 + b_1 - a_1$, $y_2 = d_2 + b_2 - a_2$ and $y_3 = d_3 + b_3 - a_3$, it follows that $x_1 - a_1 = b_1 + d_1 - c_1 - a_1$, $x_2 - a_2 = b_2 + d_2 - c_2 - a_2$ and $x_3 - a_3 = b_3 + d_3 - c_3 - a_3$. Similarly, it follows that $y_1 - c_1 = d_1 + b_1 - a_1 - c_1$, $y_2 - c_2 = d_2 + b_2 - a_2 - c_2$ and $y_3 - c_3 = d_3 + b_3 - a_3 - c_3$. Hence, $x_1 - a_1 = y_1 - c_1$, $x_2 - a_2 = y_2 - c_2$ and $x_3 - a_3 = y_3 - c_3$. Therefore $(A,X) \sim (C,Y)$ and the proof of the theorem is complete. This theorem establishes that the operation $[+]$ is commutative.

Theorem 3.4. If $\alpha, \beta, \gamma \in S'$, then $[\alpha\ [+]\ \beta]\ [+]\ \gamma = \alpha\ [+]\ [\beta\ [+]\ \gamma]$.

Proof. Let $\alpha\ [+]\ \beta = \theta$ and $\beta\ [+]\ \gamma = \emptyset$. Suppose $(A,B) \in \alpha$, $(C,D) \in \beta$ and $(E,F) \in \gamma$. Then by definition it follows that $(A,B)\ (+)\ (C,D) \in \theta$ and $(C,D)\ (+)\ (E,F) \in \emptyset$. By definition $(A,B)\ (+)\ (C,D) = (A,X)$ where $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$ and $x_3 = b_3 + d_3 - c_3$. Similarly, $(C,D)\ (+)\ (E,F) = (C,Y)$ where $y_1 = d_1 + f_1 - e_1$, $y_2 = d_2 + f_2 - e_2$ and $y_3 = d_3 + f_3 - e_3$. Let $\theta\ [+]\ \gamma = \omega$ and $\alpha\ [+]\ \emptyset = \nu$. Then $(A,X)\ (+)\ (E,F) \in \omega$ and $(A,B)\ (+)\ (C,Y) \in \nu$. By the definition of $(+)$, $(A,X)\ (+)\ (E,F) = (A,T)$ where $t_1 = x_1 + f_1 - e_1$, $t_2 = x_2 + f_2 - e_2$ and $t_3 = x_3 + f_3 - e_3$. Likewise, $(A,B)\ (+)\ (C,Y) = (A,V)$ where $v_1 = b_1 + y_1 - c_1$, $v_2 = b_2 + y_2 - c_2$ and $v_3 = b_3 + y_3 - c_3$. It follows then that $t_1 - a_1 = v_1 - a_1$, $t_2 - a_2 = v_2 - a_2$ and $t_3 - a_3 = v_3 - a_3$. Hence the proof of the theorem is complete. This theorem establishes that the operation $[+]$ is associative. In the following theorem it is shown that the operation $[\cdot]$ is distributive with respect to the operation $[+]$.

Theorem 3.5. If $\alpha \in S'$, $\beta \in S'$, and $k \in R$, then $k\ [\cdot]\ [\alpha\ [+]\ \beta] = (k\ [\cdot]\ \alpha)\ [+]\ (k\ [\cdot]\ \beta)$.

Proof. Let $\alpha\ [+]\ \beta = \gamma$. Suppose $(A,B) \in \alpha$ and $(C,D) \in \beta$. Then $(A,B)\ (+)\ (C,D) \in \gamma$. By definition of $(+)$, $(A,B)\ (+)\ (C,D) = (A,X)$ where $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$ and $x_3 = b_3 + d_3 - c_3$. It follows from the definition of $[\cdot]$ that $k\ [\cdot]\ \gamma = \rho$ such that if $(E,F) \in \gamma$ then $k\ (\cdot)\ (E,F)$

$\epsilon \rho$. Since $(A,X) \epsilon \gamma$, then $k \ (\cdot) \ (A,X) \epsilon \rho$ and $k \ (\cdot) \ (A,X) = (A,T)$ where $t_1 = a_1 + k(x_1 - a_1)$, $t_2 = a_2 + k(x_2 - a_2)$ and $t_3 = a_3 + k(x_3 - a_3)$. Let $k \ [\cdot] \ \alpha = \emptyset$ and $k \ [\cdot] \ \beta = \nu$, and suppose $(G,H) \epsilon \alpha$ and $(I,J) \epsilon \beta$. Then $k \ (\cdot) \ (G,H) \epsilon \emptyset$ and $k \ (\cdot) \ (I,J) \epsilon \nu$. Since $(A,B) \epsilon \alpha$ and $(C,D) \epsilon \beta$ then $k \ (\cdot) \ (A,B) \epsilon \emptyset$ and $k \ (\cdot) \ (C,D) \epsilon \nu$, where $k \ (\cdot) \ (A,B) = (A,Y)$ and $k \ (\cdot) \ (C,D) = (C,Z)$ such that $y_1 = a_1 + k(b_1 - a_1)$, $y_2 = a_2 + k(b_2 - a_2)$, $y_3 = a_3 + k(b_3 - a_3)$, $z_1 = c_1 + k(d_1 - c_1)$, $z_2 = c_2 + k(d_2 - c_2)$ and $z_3 = c_3 + k(d_3 - c_3)$. It follows from the definition of $[+]$ that $\emptyset \ [+] \ \nu = \omega$ such that if $(K,L) \epsilon \emptyset$ and $(M,N) \epsilon \nu$ then $(K,L) \ (+) \ (M,N) \epsilon \omega$. By the definition of $(+)$, $(A,Y) \ (+) \ (C,Z) = (A,S)$ where $s_1 = y_1 + z_1 - c_1$, $s_2 = y_2 + z_2 - c_2$ and $s_3 = y_3 + z_3 - c_3$. Since $t_1 - a_1 = kb_1 + kd_1 - kc_1 - ka_1$ and $s_1 - a_1 = kb_1 + kd_1 - kc_1 - ka_1$ it is clear that $t_1 - a_1 = s_1 - a_1$. In a similar manner it is true that $t_2 - a_2 = s_2 - a_2$ and $t_3 - a_3 = s_3 - a_3$. It follows that $(A,T) \sim (A,S)$ and therefore $\rho = \omega$. Hence the theorem is valid.

In the following theorem a distributive property of the operation $[\cdot]$ with respect to the operation $[+]$ is established.

<u>Theorem 3.6</u>. If $\alpha \epsilon S'$, $k \epsilon R$, and $c \epsilon R$, then $(k + c) \ [\cdot] \ \alpha = (k \ [\cdot] \ \alpha) \ [+] \ (c \ [\cdot] \ \alpha)$.

<u>Proof</u>. Let $k + c = d$ and $d \ [\cdot] \ \alpha = \beta$. Suppose $(A,B) \epsilon \alpha$. It follows then that $d \ (\cdot) \ (A,B) \epsilon \beta$ and $d \ (\cdot) \ (A,B) = (A,X)$ where $x_1 = a_1 + d(b_1 - a_1)$, $x_2 = a_2 + d(b_2 - a_2)$ and $x_3 = a_3 + d(b_3 - a_3)$. Let $k \ [\cdot] \ \alpha = \nu$ and $c \ [\cdot] \ \alpha = \omega$. Then

k [·] $(A,B) \in \nu$ and c [·] $(A,B) \in \omega$. Also k (·) $(A,B) = (A,Y)$ where $y_1 = a_1 + k(b_1 - a_1)$, $y_2 = a_2 + k(b_2 - a_2)$ and $y_3 = a_3 + k(b_3 - a_3)$, and c (·) $(A,B) = (A,Z)$ where $z_1 = a_1 + c(b_1 - a_1)$, $z_2 = a_2 + c(b_2 - a_2)$ and $z_3 = a_3 + c(b_3 - a_3)$. Let ν [+] $\omega = \gamma$. Then (A,Y) (+) $(A,Z) \in \gamma$, where (A,Y) (+) $(A,Z) = (A,T)$ such that $t_1 = y_1 + z_1 - a_1$, $t_2 = y_2 + z_2 - a_2$ and $t_3 = y_3 + z_3 - a_3$. Since $x_1 - a_1 = a_1 + d(b_1 - a_1) - a_1 = (k + c)(b_1 - a_1)$ and $t_1 - a_1 = y_1 + z_1 - a_1 - a_1 = (k + c)(b_1 - a_1)$, it is clear that $(A,X) \sim (A,T)$ and therefore $\beta = \gamma$. Hence the proof of the theorem is complete and a desirable property is established for the system $\mathcal{S}'$.

<u>Theorem 3.7</u>. If $\alpha \in S'$ and $c,k \in R$ then $(c \cdot k)$ [·] α = c [·] (k [·] α).

<u>Proof</u>. Let $c \cdot k = d$ and d [·] $\alpha = \gamma$, and suppose that $(A,B) \in \alpha$. Then d [·] $(A,B) \in \gamma$ where d (·) $(A,B) = (A,X)$ such that $x_1 = a_1 + d(b_1 - a_1)$, $x_2 = a_2 + d(b_2 - a_2)$ and $x_3 = a_3 + d(b_3 - a_3)$. Let k [·] $\alpha = \beta$, and suppose that $(C,D) \in \alpha$. Then k (·) $(C,D) \in \beta$. Since $(A,B) \in \alpha$ then k (·) $(A,B) \in \beta$, where k (·) $(A,B) = (A,Y)$ such that $y_1 = a_1 + k(b_1 - a_1)$, $y_2 = a_2 + k(b_2 - a_2)$ and $y_3 = a_3 + k(b_3 - a_3)$. Let c [·] $\beta = \nu$ and suppose that $(E,F) \in \beta$. Then c (·) $(E,F) \in \nu$. Since $(A,Y) \in \beta$, then c (·) $(A,Y) \in \nu$, where c (·) $(A,Y) = (A,T)$ such that $t_1 = a_1 + c(y_1 - a_1)$, $t_2 = a_2 + c(y_2 - a_2)$ and $t_3 = a_3 + c(y_3 - a_3)$. It follows that $x_1 - a_1 = ck(b_1 - a_1)$ and $t_1 - a_1 = ck(b_1 - a_1)$. Therefore it is clear that $x_1 - a_1 = t_1 - a_1$ and similarly it is true that $x_2 - a_2 = t_2 - a_2$,

and $x_3 - a_3 = t_3 - a_3$. It follows then that $(A,T) \sim (A,X)$ and $\gamma = \nu$. Hence the proof of the theorem is complete. In the following theorem a property regarding the multiplicative identity element in R and any element of the set S' is established.

Theorem 3.8. If $\alpha \in S'$, then $1\ [\cdot]\ \alpha = \alpha$.

Proof. By the definition of $[\cdot]$ it follows that $1\ [\cdot]\ \alpha = \beta$ such that if $(A,B) \in \alpha$ then $1\ (\cdot)\ (A,B) \in \beta$. Since by a theorem in Chapter II, $1\ (\cdot)\ (A,B) = (A,B)$ it is clear that the element (A,B) is common to α and β. Hence $\alpha = \beta$ and the proof is complete.

Let η denote the element of S' such that $(A,B) \in \eta$ if and only if $B = A$. The next four theorems establish some properties regarding the element η of the system $\mathcal{S}'$.

Theorem 3.9. If $\alpha \in S'$, then $0\ [\cdot]\ \alpha = \eta$.

Proof. It follows from the definition of $[\cdot]$ that $0\ [\cdot]\ \alpha = \beta$ such that if $(A,B) \in \alpha$ then $0\ (\cdot)\ (A,B) \in \beta$. Since $0\ (\cdot)\ (A,B) = (A,A)$ by a previous theorem and $(A,A) \in \eta$, it is clear that $\beta = \eta$ and the proof of the theorem is complete.

Theorem 3.10. If $k\ [\cdot]\ \alpha = \eta$, where $k \in R$ and $\alpha \in S'$ then either $k = 0$ or $\alpha = \eta$.

Proof. Suppose $k \neq 0$ and $\alpha \neq \eta$. From the definition of $[\cdot]$ it follows that if $(A,B) \in \alpha$ then $k\ [\cdot]\ (A,B) \in \eta$. Also $k\ (\cdot)\ (A,B) = (A,X)$ such that $x_1 = a_1 + k(b_1 - a_1)$, $x_2 = a_2 + k(b_2 - a_2)$ and $x_3 = a_3 + k(b_3 - a_3)$. By hypothesis $(A,X) \in \eta$, and so $x_1 - a_1 = 0$, $x_2 - a_2 = 0$ and $x_3 - a_3 = 0$.

This means that $k(b_1 - a_1) = 0$, $k(b_2 - a_2) = 0$ and $k(b_3 - a_3) = 0$. Since $k \neq 0$, then $b_1 = a_1$, $b_2 = a_2$ and $b_3 = a_3$, which is contrary to assumption and the theorem holds.

Theorem 3.11. If $k = 0$ or $\alpha = \eta$ where $\alpha, \eta \in S'$ then $k\ [\cdot]\ \alpha = \eta$.

Proof.

Case I. Suppose $k = 0$. Let $k\ [\cdot]\ \alpha = \beta$, and suppose $(A,B) \in \alpha$. Then $k\ (\cdot)\ (A,B) \in \beta$. Since $k\ (\cdot)\ (A,B) = (A,X)$ such that $x_1 = a_1 + k(b_1 - a_1)$, $x_2 = a_2 + k(b_2 - a_2)$ and $x_3 = a_3 + k(b_3 - a_3)$ and by assumption $k = 0$, then $X = A$ and so $\beta = \eta$ and the proof of Case I is complete.

Case II. Suppose $k \neq 0$ but $\alpha = \eta$. Let $k\ [\cdot]\ \alpha = \nu$. It follows from the definition of $[\cdot]$ that if $(A,B) \in \alpha$ then $k\ (\cdot)\ (A,B) \in \nu$. By assumption $\alpha = \eta$ so $k\ (\cdot)\ (A,A) \in \nu$, where $(A,A) \in \alpha$. Since by a previous theorem $k\ (\cdot)\ (A,A) = (A,A)$ it follows that $\nu = \eta$ and the theorem is valid.

Theorem 3.12. If $(A,B)\ (+)\ (C,D) \in \eta$ then $(B,A) \sim (C,D)$.

Proof. By the hypothesis $(A,B)\ (+)\ (C,D) \in \eta$ and by the definition of $(+)$ it follows that $(A,B)\ (+)\ (C,D) = (A,X)$ such that $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$ and $x_3 = b_3 + d_3 - c_3$. Since $a_1 = x_1$, $a_2 = x_2$, and $a_3 = x_3$, it follows that $a_1 - b_1 = d_1 - c_1$, $a_2 - b_2 = d_2 - c_2$ and $a_3 - b_3 = d_3 - c_3$, which implies that $(B,A) \sim (C,D)$ and completes the proof of the theorem.

Consider the possibility of the existence of an element $\alpha \in S'$ such that for any element $\beta \in S'$ it is true that $\beta\ [+]\ \alpha = \beta$. From the definitions of $[+]$ and $(+)$ it follows that if $(A,B) \in \beta$ and $(C,D) \in \alpha$ then $(A,B) \oplus (C,D) \in \nu$, where ν is to be equal to β. Thus $(A,B) \sim (A,B) \oplus (C,D)$. But by the definition of $\oplus$ it follows that $(A,B) \oplus (C,D) =$ (A,X) such that $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$ and $x_3 = b_3 + d_3 - c_3$. It is clear that if $X = B$ then $(C,D) \in \eta$. Therefore for any element $\beta \in S'$, then β [+] $\eta = \beta$.

Consider η [+] β for any $\beta \in S'$. From the definition of [+] it follows that η [+] $\beta = \nu$ such that if $(A,B) \in \eta$ and $(C,D) \in \beta$ then $(A,B) \oplus (C,D) \in \nu$. Since $(A,B) \oplus (C,D) =$ (A,X) such that $x_1 = b_1 + d_1 - c_1$, $x_2 = b_2 + d_2 - c_2$ and $x_3 = b_3 + d_3 - c_3$ and $B = A$, then $x_1 - a_1 = d_1 - c_1$, $x_2 - a_2 = d_2 - c_2$ and $x_3 - a_3 = d_3 - c_3$. Hence, it follows that $(A,X) \sim (C,D)$ and $\nu = \beta$. Therefore η is a unique left or right additive identity element for the system $\mathcal{S}'$.

CHAPTER IV

ABSTRACT VECTOR SPACE

In this chapter the notion of an abstract vector space is defined and studied, and the system of Chapter III is shown to be a vector space. Three functions related to vector spaces are studied, namely, inner products, norms and linear transformations.

<u>Definition 4.1</u> The statement that $\mathcal{V}\{V;\oplus,\odot\}$ is a vector space over the field $\mathcal{F}\{F;+,\cdot\}$ means that $\oplus$ is an operation from $V \times V$ to V, and $\odot$ is an operation from $F \times V$ to V such that:

(i) if $x \in V$ and $y \in V$, then $x \oplus y = y \oplus x$,

(ii) if $x \in V$, $y \in V$, and $z \in V$, then $x \oplus (y \oplus z) =$ $(x \oplus y) \oplus z$,

(iii) there exists $N \in V$ such that if $x \in V$, then $N \oplus x = x$,

(iv) if $x \in V$, then there exists $-x \in V$ such that $x \oplus - x = N$,

(v) if $x \in V$, $y \in V$, and $a \in F$, then $a \odot (x \oplus y)$ $a \odot x \oplus a \odot y$,

(vi) if $a \in F$, $b \in F$, and $x \in V$, then $(a \cdot b) \odot x$ $a \odot (b \odot x)$,

(vii) if $a \in F$, $b \in F$, and $x \in V$, then $(a + b) \odot x$ $a \odot x \oplus b \odot x$,

(viii) if 1 is the multiplicative identity element in F and $x \in V$, then $1 \odot x = x$.

Theorem 4.1. Suppose $\mathcal{V}\{V; \oplus, \odot\}$ is an abstract vector space over $\mathcal{F}\{F; +, \cdot\}$. Then

(A) if $N' \in V$ and there exists $x \in V$ such that $N' \oplus x = x$, then $N' = N$,

(B) if $x \in V$, there exists a unique element $x' \in V$ such that $x \oplus x' = N$, and

(C) if $x \oplus y = N$, then $y = -x$.

Proof. Let x be an element of V such that $N' \oplus x = x$. By (iii) $N \oplus x = x$, and so $N' \oplus x = N \oplus x$. By (iv) there exists $-x \in V$ such that $x \oplus -x = N$. It follows that $(N' \oplus x) \oplus -x = (N \oplus x) \oplus -x$. But by (ii) $N' \oplus (x \oplus -x) = N \oplus (x \oplus -x)$, and by (iv) $N' \oplus N = N \oplus N$. Hence, by (iii), $N' = N$ and part (A) of the theorem holds.

For proof of part (B), note that by (iv) there exists $-x \in V$ such that $x \oplus -x = N$. Let $-x'$ be an element of V such that $x \oplus -x' = N$, and so $x \oplus -x = x \oplus -x'$. It follows then that $-x \oplus (x \oplus -x) = -x \oplus (x \oplus -x')$ and by (ii) it follows that $(-x \oplus x) \oplus -x = (-x \oplus x) \oplus -x'$. But by (iv) $N \oplus -x = N \oplus -x'$, and by (iii) $-x = -x'$. Hence, part (B) is valid.

For proof of part (C), consider the hypothesis which states that $x, y \in V$ and $x \oplus y = N$. By property (iv) $x \oplus -x = N$, and so, $x \oplus y = x \oplus -x$. It follows then that $-x \oplus (x \oplus y) = -x \oplus (x \oplus -x)$, and by (ii) then $(-x \oplus x)$

$\oplus$ y = (-x $\oplus$ x) $\oplus$ -x. It follows by (iv) that N $\oplus$ y = N $\oplus$ -x and by (iii) that y = -x. Hence, the proof of the theorem is complete.

It follows from the definition of a vector space that the mathematical system $\mathcal{S}'\{S'; [+], [\cdot]\}$, studied in Chapter III, is a vector space over the field $\mathcal{R}\{R; +, \cdot\}$. The element $\eta \in S'$ satisfies property (iii) of Definition 4.1 and has the properties of the element N in Theorem 4.1. Since the system $\mathcal{S}\{S; (+), (\cdot)\}$ studied in Chapter II does not have property (i) of Definition 4.1 and property (i) of Theorem 4.1, it does not constitute a vector space over a field. Note also that it is not the system ordinarily used in a study of vectors in E^3.

Definition 4.2 The statement that Q is an inner product defined on the vector space $\mathcal{V}\{V; \oplus, \diamond\}$ over the field $\mathcal{C}\{C; +, \cdot\}$ means that Q is a function from $V \times V$ to C such that:

(i) Q (x, x) $\geq$ 0 and = 0 if and only if x = N,

(ii) Q (x $\oplus$ y, z) = Q (x, z) + Q (y,z),

(iii) Q (c $\diamond$ x, y) = c$\cdot$Q (x, y), and

(iv) Q (x, y) = $\overline{Q\ (y,\ x)}$.

Theorem 4.2. If x, y, and $z \in V$, $c \in C$, where V is a collection of elements belonging to any vector space $\mathcal{V}\{V; \oplus, \diamond\}$ over $\mathcal{C}\{C; +, \cdot\}$, then Q (x, y $\oplus$ z) = Q (x, y) + Q (x, z) and Q (x, c $\diamond$ y) = $\bar{c}$ Q (x, y).

Proof. Denote the elements x,y and $z \in C$, which are n-tuples of complex numbers as follows:

x: $[\alpha_1, \alpha_2, \ldots, \alpha_n]$, y: $[\beta_1, \beta_2, \ldots, \beta_n]$, and z: $[\gamma_1, \gamma_2, \ldots, \gamma_n]$. It follows from the definition of an inner product function for a vector space that $Q(x,y) = \alpha_1\overline{\beta}_1 + \alpha_2\overline{\beta}_2 + \ldots + \alpha_n\overline{\beta}_n$, and $Q(x,z) = \alpha_1\overline{\gamma}_1 + \alpha_2\overline{\gamma}_2 + \ldots + \alpha_n\overline{\gamma}_n$. Since $y \oplus z = t$, such that $t \epsilon V$ and $\overline{t} = \overline{y} \oplus \overline{z}$, it follows that $Q(x, y \oplus z) = Q(x,t) = \alpha_1(\overline{\beta_1 + \gamma_1}) + \alpha_2(\overline{\beta_2 + \gamma_2}) + \ldots + \alpha_n(\overline{\beta_n + \gamma_n}) = \alpha_1\overline{\beta}_1 + \alpha_1\overline{\gamma}_1 + \alpha_2\overline{\beta}_2 + \alpha_2\overline{\gamma}_2 + \ldots + \alpha_n\overline{\beta}_n + \alpha_n\overline{\gamma}_n = Q(x,y) + Q(x,z)$. Similarly by the definition of Q and $\diamond$, it follows that $Q(x, c \diamond y)$

$= \alpha_1\overline{c\beta}_1 + \alpha_2\overline{c\beta}_2 + \ldots + \alpha_n\overline{c\beta}_n$

$= \overline{c}[\alpha_1\overline{\beta}_1 + \alpha_2\overline{\beta}_2 + \ldots + \alpha_n\overline{\beta}_n]$

$= \overline{c}Q(x,y)$. Hence the proof of the theorem is complete.

<u>Theorem 4.3</u>. If $x,y \epsilon V$, where V is the collection of elements belonging to any inner product vector space $\mathcal{V}\{V;\oplus,\diamond\}$ over $\mathcal{C}\{C;+,\cdot\}$, then

$$|Q(x,y)|^2 \leq Q(x,x)\ Q(y,y).$$

<u>Proof</u>. Suppose y is $N \epsilon V$, then it follows from the definition of an inner product that $Q(y,y) = 0$ and $Q(x,y) = 0$ for any $x \epsilon V$, and therefore for this special case $|Q(x,y)|^2 = Q(x,x)\ Q(y,y)$. By considering the more general case, where $x,y \epsilon V$ and $y \neq N$, $c \epsilon C$, examine the inequality

$0 \leq Q\ (x, c \diamond y), (x, c \diamond y)$

$\quad = Q(x,x) + Q(c \diamond y, x) + Q(x, c \diamond y) + Q(c \diamond y, c \diamond y)$

$$= Q(x,x) + c[Q(y,x)] + \bar{c}[Q(x,y)] + c\bar{c}[Q(y,y)] + \frac{Q(x,y)}{Q(y,y)} - \frac{Q(x,y)}{Q(y,y)}$$

$$= \left[\frac{\bar{c}\ \{Q(y,y)\}^{1/2}}{1} + \frac{Q(x,y)}{\{Q(y,y)\}^{1/2}}\right]\left[\frac{c\ \{Q(y,y)\}^{1/2}}{1} + \frac{Q(y,x)}{\{Q(y,y)\}^{1/2}}\right]$$

$$+ \frac{Q(x,x)\ Q(y,y) - |Q(x,y)|^2}{Q(y,y)}.$$ If $c = -\frac{Q(x,y)}{Q(y,y)}$, then it follows that $0 \leq \frac{Q(x,x)\ Q(y,y) - |Q(x,y)|^2}{Q(y,y)}$, which is positive or equivalent to zero if and only if $0 \leq Q(x,x)\ Q(y,y) - |Q(x,y)|^2$. Hence it follows that $|Q(x,y)|^2 \leq Q(x,x)\ Q(y,y)$ and the proof of the theorem is complete.

<u>Definition 4.3</u> The statement that the set of n vectors, $x_1, x_2, \ldots, x_n$, belonging to a vector space $\mathcal{V}\{V; \oplus, \diamond\}$ over $\mathcal{F}\{F;+,\cdot\}$ is a linearly independent set means that if $c_1 \diamond x_1 \oplus c_2 \diamond x_2 \oplus \ldots \oplus c_n \diamond x_n = N$, where $c_r \in F$, then $c_1 = c_2 = \ldots = c_n = 0$. If a set is not linearly independent, it is said to be linearly dependent.

<u>Definition 4.4</u> The statement that $\mathcal{V}\{V; \oplus, \diamond\}$ is an n-dimensional vector space over $\mathcal{F}\{F;+,\cdot\}$ means that Definition 4.1 is satisfied and there exists a set of n linearly independent vectors in the space $\mathcal{V}$, and there does not exist a set of (n + 1) linearly independent vectors in $\mathcal{V}$.

<u>Definition 4.5</u> The statement that a set of vectors $x_1, x_2, \ldots, x_n$ of a vector space $\mathcal{V}\{V; \oplus, \diamond\}$ over $\mathcal{F}\{F;+,\cdot\}$ is a basis for $\mathcal{V}$, or spans $\mathcal{V}$, means that:

(i) if $y \in V$ then y may be expressed as a linear combination of the set $x_1, x_2, \ldots, x_n$,

(ii) $x_1, x_2, \ldots, x_n$ is a linearly independent set.

Definition 4.6 The statement that $\mathcal{N}$ is a norm for a vector space $\mathcal{V}\{V; \oplus, \odot\}$ over $\mathcal{C}\{C;+,\cdot\}$ means that $\mathcal{N}$ is a function from V to R such that if $x, y \in V$ and $a \in C$, then:

(i) $\mathcal{N}(x) \geq 0$ and $\mathcal{N}(x) = 0$ if and only if $x = N$,

(ii) $\mathcal{N}(x \oplus y) \leq \mathcal{N}(x) + \mathcal{N}(y)$, and

(iii) $\mathcal{N}(a \odot y) = |a| \cdot \mathcal{N}(x)$.

Consider a mathematical system $\mathcal{S}^n\{S^n; \dot{\oplus}, \dot{\odot}\}$, where S^n is the set of all ordered n-tuples of real numbers. Define $\dot{\oplus}$ to be an operation from $S^n \times S^n$ to S^n such that $K \dot{\oplus} T = X$, where $X \in S^n$ and $x_1 = k_1 + t_1$, $x_2 = k_2 + t_2$, $\ldots$, $x_n = k_n + t_n$. In addition, define $\dot{\odot}$ to be an operation from $R \times S^n$ to S^n such that if $c \in R$, then $c \dot{\odot} K = Y$, where $Y \in S^n$ and $y_1 = ck_1$, $y_2 = ck_2$, $\ldots$, $y_n = ck_n$.

Theorem 4.4. The system $\mathcal{S}^n\{S^n; \dot{\oplus}, \dot{\odot}\}$ is a vector space over $\mathcal{R}\{R;+,\cdot\}$.

Proof. It follows from the definition of $\dot{\oplus}$ that if $X \in S^n$, $Y \in S^n$ and $Z \in S^n$ then $X \dot{\oplus} Y = Y \dot{\oplus} X$, and $X \dot{\oplus} (Y \dot{\oplus} Z) = (X \dot{\oplus} Y) \dot{\oplus} Z$. Note the existence of a certain element $U \in S^n$ such that $u_1 = u_2 = \ldots = u_n = 0$. By the definition of $\dot{\oplus}$ it follows that if $X \in S^n$ then $U \dot{\oplus} X = X$ and there exists an $X' \in S^n$ such that $X \dot{\oplus} X' = U$. Here $x'_1 = -x_1$, $x'_2 = -x_2$, $\ldots$, $x'_n = -x_n$. In addition it follows that if $X \in S^n$,

$Y \in S^n$, $a \in R$, and $b \in R$, then $a \dot{\diamond} (X \dot{\diamond} Y) = a \dot{\diamond} X \dot{\oplus} a \dot{\diamond} Y$, $(a \cdot b) \dot{\diamond} X = a \dot{\diamond} (b \dot{\diamond} X)$, $(a + b) \dot{\diamond} X = a \dot{\diamond} X \dot{\oplus} b \dot{\diamond} X$ and $1 \dot{\diamond} X = X$. Hence Definition 4.1 is satisfied and $\mathcal{S}^n$ is a vector space over $\mathcal{R}\{R;+,\cdot\}$. Observe that the element $U \in S^n$ serves as the element N in Definition 4.1.

Define a function Q for the vector space $\mathcal{S}^n$ as follows: If $X, Y \in S^n$, then $Q(X,Y) = x_1y_1 + x_2y_2 + \dots + x_ny_n$. It follows from this definition of Q and the previous definitions of $\dot{\oplus}$ and $\dot{\diamond}$ that if X, Y, and $Z \in S^n$, and $c \in R$, then:

(i) $Q(X,X) \geq 0$ and $Q(X,X) = 0$ if and only if $X = U$,

(ii) $Q(X \dot{\oplus} Y, Z) = Q(X,Y) + Q(Y,Z)$,

(iii) $Q(c \dot{\diamond} X, Y) = cQ(X,Y)$, and

(iv) $Q(X,Y) = \overline{Q(Y,X)}$.

Hence, Q satisfies Definition 4.2 and is an inner product function for the vector space $\mathcal{S}^n$.

Define a function $\mathcal{N}$ for the vector space $\mathcal{S}^n$ as follows: If $X \in S^n$, then $\mathcal{N}(X) = [Q(X,X)]^{1/2}$. Note that the function $\mathcal{N}$ defined in this manner satisfies Definition 4.6 and is a norm for the vector space $\mathcal{S}^n$.

It is a consequence of later theorems that the vector space $\mathcal{S}^n$ is a n-dimensional vector space over the field .

Notice the existence of a set of elements

$$e_1, e_2, \dots, e_n \in S^n \text{ such that } e_i = \begin{bmatrix} e_{i1} \\ e_{i2} \\ \cdot \\ \cdot \\ \cdot \\ e_{in} \end{bmatrix},$$

where $e_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j, \end{cases}$, which are linearly independent according to Definition 4.3 and serve as a basis for $\mathcal{S}^n$.

The mathematical system $\mathcal{S}^n \{S^n, \dot{\oplus}, \dot{\Diamond}, Q, \mathcal{n}\}$, is the system usually referred to as Euclidean n-space and denoted by E^n.

<u>Theorem 4.5</u>. If the vectors $u_1, u_2, u_3, \ldots, u_n$ are linearly dependent in the vector space $\mathcal{V}\{V; \oplus, \Diamond\}$ over $\mathcal{F}\{F;+,\cdot\}$, then the vectors $u_1, u_2, u_3, \ldots, u_n, u_{n+1}, u_{n+2}, \ldots, u_m$ are linearly dependent.

<u>Proof</u>. By the hypothesis and the definition of linearly dependent vectors there exists a set $c_1, c_2, \ldots, c_n \epsilon F$, and not all zero, such that $c_1u_1 + c_2u_2 + \ldots + c_nu_n = N$. It follows that $c_{n+1}u_{n+1} + \ldots + c_mu_m = N$ by simply letting $c_{n+1} = \ldots = c_m = 0$. Hence the set of vectors $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_m$ is a linearly dependent set and the theorem is valid.

<u>Theorem 4.6</u>. If n vectors span a vector space containing k linearly independent vectors, then $n \geq k$.

<u>Proof</u>. By the hypothesis, there exists a subset A of set V composed of $a_1, a_2, \ldots, a_n$, which spans the vector space $\mathcal{V}\{V; \oplus, \Diamond\}$ over $\mathcal{F}\{F;+,\cdot\}$, and also V contains a set B of k linearly independent vectors $b_1, b_2, \ldots, b_k$. Since $b_1 \neq N$, it may be expressed as a linear combination of elements of set A. It follows from the definition of dependent vectors that the set $b_1, a_1, a_2, \ldots,$

a_n is linearly dependent and that some a_i of this set is a linear combination of the other vectors of this set. Replacing a_i by b_1 it follows that the set composed of $b_1, a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ still spans the vector space. Continue this procedure until

(a) all elements of set A are replaced by all of the elements of set B, or

(b) all elements of set B and some elements of set A are used to span the vector space, or

(c) all elements of set A are replaced by elements of set B and some elements of B remain.

If (a) occurs, then $n = k$, if (b) occurs, then $n > k$. If (c) occurs then the set $b_1 b_2 \ldots b_n, b_{n+1}$ is linearly dependent which is contrary to the hypothesis. Hence $n \geq k$ and the theorem is valid.

<u>Theorem 4.7</u>. Any $n + 1$ vectors in E^n constitute a linearly dependent set.

<u>Proof</u>. If $n + 1$ vectors in E^n were linearly independent, then by the previous theorem $n \geq n + 1$, which is absurd. Hence any $n + 1$ vectors of E^n must be linearly dependent and the theorem holds.

<u>Theorem 4.8</u>. All bases of a finite dimensional vector space contain the same number of vectors.

<u>Proof</u>. Let $u_1, u_2, \ldots, u_n$ and $e_1, e_2, \ldots, e_k$ form sets U and E respectively and be bases of a common vector space. Since the set U spans the vector space and

the set E is linearly independent, it follows by Theorem 4.6 that $n \geq k$. Similarly, the set E spans the same vector space and U is a set of linearly independent vectors, so that $k \geq n$. Hence these conditions establish that $n = k$, and the proof of the theorem is complete.

<u>Theorem 4.9</u>. If there exists a set of n linearly independent vectors $e_1, e_2, \ldots, e_n$ of a vector space which spans the space $\mathcal{V}\{V; \oplus, \odot\}$ over $\mathcal{R}\ \{R; +, \cdot\}$, then the set forms a basis for the vector space and if $x \in V$, then x has a unique representation as follows: $x = a_1 \odot e_1 \oplus a_2 \odot e_2 \oplus \ldots \oplus a_n \odot e_n$, where the a_i's $\in R$.

<u>Proof</u>. By the hypothesis if $x \in V$ then $x = a_1 \odot e_1 \oplus a_2 \odot e_2 \oplus \ldots \oplus a_n \odot e_n$, where $a_i \in R$. Suppose $x = b_1 \odot e_1 \oplus b_2 \odot e_2 \oplus \ldots \oplus b_n \odot e_n$ and each $b_i \in R$. It follows that $N = (a_1 - b_1) \odot e_1 \oplus (a_2 - b_2) \odot e_2 \oplus \ldots \oplus (a_n - b_n) \odot e_n$. Hence, $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$ and the proof of the theorem is complete.

<u>Theorem 4.10</u>. If an inner product vector space $\mathcal{V}\{V; \oplus, \odot\}$ is spanned by a collection of linearly independent vectors $x_1, x_2, \ldots, x_n$, then there exists a set of orthonormal vectors $b_1, b_2, \ldots, b_n$ that span the same space.

<u>Proof</u>. Construct a set $y_1, y_2, \ldots, y_n \in V$ by a method known as the Gram-Schmidt process. In order to perform this construction choose

$y_1 = x_1$,

$y_2 = x_2 - \frac{Q(x_2,y_1)}{Q(y_1,y_1)} y_1,$

$y_3 = x_3 - \frac{Q(x_3,y_1)}{Q(y_1,y_1)} y_1 - \frac{Q(x_3,y_2)}{Q(y_2,y_2)} y_2, \ldots ,$ in general

$y_k = x_k - \sum_{i=1}^{k-1} \frac{Q(x_k,y_i)}{Q(y_i,y_i)} y_i.$ The next part of the proof follows by induction. It is clear that properties (1) and (2) hold for $j = 1$. Assume the following properties hold for $j < k$:

(1) The vector space spanned by $y_1, y_2, \ldots , y_j$ is the same vector space as the vector space spanned by $x_1, x_2, \ldots , x_j$.

(2) The set $y_1, y_2, \ldots , y_j$ is orthogonal.

The next step is to show that (1) and (2) hold when $j = k$. Notice that each $y_i \neq N$. This follows because if $y_i = N$ then by (1) the set $x_1, x_2, \ldots , x_n$ would be dependent and could not be a basis for a vector space. It follows from the general definition of y_k and (2) that if $j < k$ then $Q(y_k,y_j) = Q(x_k,y_j) - Q(x_k,y_j) = 0$. Hence $y_1, y_2, \ldots , y_k$ are mutually orthogonal. By the definition of y_k it is implied that x_k is an element of the set $x_1, x_2, \ldots , x_k, \ldots , x_n$. Since $y_k \neq N$ and the assumption of (1) it follows that property (2) is also valid for $j = k$. This part of the proof establishes that if the construction can be performed k times, then there exists a mutually orthogonal set $y_1, y_2, \ldots, y_k, y_{k+1}$. It follows that an orthonormal basis $b_1, b_2, \ldots, b_n$ can be formed by taking

$b_i = \dfrac{y_i}{[Q(y_i, y_i)]^{1/2}}$, which completes the proof of the theorem.

Theorem 4.11. Any finite dimensional vector space is isomorphic with the vector space $\mathcal{V}\{V; \oplus, \diamond\}$ of ordered n-tuples over the same field.

Proof. Consider the basis set of orthonormal vectors $b_1, b_2, \ldots, b_n$ derived in the proof of Theorem 4.10. Since the set $b_1, b_2, \ldots, b_n$ constitutes a basis for an inner product vector space $\mathcal{V}\{V; \oplus, \diamond\}$ over $\mathcal{F}\{F; +, \cdot\}$, then any $x \in V$ has a unique representation $x = k_1 b_1 + k_2 b_2 + \ldots + k_n b_n$ where each $k_i \in F$. Consider the element $X \in E^n$, where X is the ordered n-tuple of real numbers $k_1, k_2, \ldots, k_n$. It follows that $x \leftrightarrow X$ is a one-to-one correspondence between the elements of V and the elements of E^n. Also, this correspondence is an isomorphism between E^n and , since if $x, y \in V$, then $x \oplus y \leftrightarrow X \diamond Y$, and if $c \in F$, then $c \diamond x = c \diamond X$. This completes the proof of the theorem.

Definition 4.8 The statement that T is a linear transformation from $\mathcal{V}\{V; \oplus, \diamond\}$ over $\mathcal{F}\{F; +, \cdot\}$ to $\mathcal{W}\{W; (+), (\cdot)\}$ over $\mathcal{F}\{F; +, \cdot\}$ means if $x, y \in V$, $k \in F$ then T satisfies the following properties:

(i) $T(x \oplus y) = Tx \, (+) \, Ty$

(ii) $T(k \diamond y) = k \, (\cdot) \, Ty$.

Theorem 4.12. If the set $x_1, x_2, \ldots, x_n$ forms a basis for a vector space $\mathcal{V}\{V; \oplus, \diamond\}$ over $\mathcal{F}\{F; +, \cdot\}$ and

$y_1, y_2, \ldots, y_n$ is any ordered set of elements of a vector space $\mathcal{W}\{W; (+), (\cdot)\}$ over $\mathcal{F}$, then there exists one and only one linear transformation T from V to W such that $T(x_i) = y_i$, for $i = 1, 2, \ldots, n$.

Proof. By the hypothesis if $z_1, z_2 \epsilon V$ and $a_i, b_i \epsilon F$, then $z_1 = a_1x_1 \oplus a_2x_2 \oplus \ldots \oplus a_nx_n$ and $z_2 = b_1x_1 \oplus b_2x_2 \oplus \ldots \oplus b_nx_n$. Define $T(z_1) = a_1y_1 (+) a_2y_2 (+) \ldots (+) a_ny_n$ and $T(z_2) = b_1y_1 (+) b_2y_2 (+) \ldots (+) b_ny_n$. It follows that

$$\begin{aligned}(z_1 \oplus z_2) &= (a_1x_1 \oplus \ldots \oplus a_nx_n) \oplus (b_1x_1 \oplus \ldots \oplus b_nx_n) \\ &= [(a_1x_1 \oplus b_1x_1) \oplus \ldots \oplus (a_nx_n \oplus b_nx_n)] \\ &= [(a_1 + b_1)x_1 \oplus \ldots \oplus (a_n + b_n)x_n].\end{aligned}$$

By the definition of $T(z_1)$ and $T(z_2)$ it follows that

$$\begin{aligned}T(z_1 \oplus z_2) &= T[(a_1 + b_1)x_1 \oplus \ldots \oplus (a_n + b_n)x_n] \\ &= (a_1 + b_1)y_1 (+) \ldots (+) (a_n + b_n)y_n \\ &= (a_1y_1 (+) \ldots (+) a_ny_n) (+) (b_1y_1 (+) \ldots (+) b_ny_n) \\ &= T(z_1) (+) T(z_2).\end{aligned}$$

If $k \epsilon F$ and $z_1 \epsilon V$ then it follows that $k \odot z_1 = (k \odot a_1x_1 \oplus k \odot a_2x_2 \oplus \ldots \oplus k \odot a_nx_n)$. By the definition of $T(z_1)$ it follows that

$$\begin{aligned}T(k \odot z_1) &= T(k \odot a_1x_1 \oplus k \odot a_2x_2 \oplus \ldots \oplus k \odot a_nx_n) \\ &= k (\cdot) a_1y_1 (+) k (\cdot) a_2y_2 (+) \ldots (+) k (\cdot) a_ny_n \\ &= k (\cdot) [(a_1y_1 (+) a_2y_2 (+) \ldots (+) a_ny_n)] \\ &= k (\cdot) T(z_1).\end{aligned}$$

Hence, by Definition 4.8 it follows that T is a linear transformation. To establish that T is unique, suppose there exists a linear transformation T' such that if $z_1 \epsilon V$, then $T'(z_1) = y$ and in general $T'(z_i) = y_i$, for $i = 1, 2,$

..., n. If $z_1 \epsilon V$ it follows that $T'(z_1) = T'(a_1x_1 \oplus \ldots \oplus a_nx_n)$. By the assumption that T' is a linear transformation it follows that if z_1 is any element of V, then

$$T'(z_1) = a_1[T'(x_1)] \oplus \ldots \oplus a_n[T'(x_n)]$$
$$= a_1y_1 \ (+) \ \ldots \ (+) \ a_ny_n$$
$$= T(z_1).$$

Hence T' = T and the proof of the theorem is complete.

The following theorem characterizes linear transformations from E^n to E^n.

Theorem 4.13.

(1) If T is a linear transformation from E^n to E^n then there exists an n by n matrix A such that

if $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \epsilon E^n$ then T(X) is the product of the matrix A and the matrix X.

(2) If A is an n by n matrix of real numbers then there exists a linear transformation T from E^n to E^n such that

if $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \epsilon E^n$ then the matrix product of A and X is T(X).

Proof. Suppose X is any element of E^n, then it follows that X: $[x_1, x_2, \ldots, x_m]$ or $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$.

By Theorem 4.4 and following remarks there exists a set $e_1, e_2, \ldots, e_n \epsilon E^n$ which spans E^n. It follows by Theorem 4.9 that $X = x_1e_1 + x_2e_2 + \ldots + x_ne_n$. Since $T(X) = T[x_1e_1 + x_2e_2 + \ldots + x_ne_n]$ and since T is a linear transformation by the hypothesis then $T(X) = x_1T(e_1) + x_2T(e_2) + \ldots + x_nT(e_n)$.

Define $T(e_i) = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ \cdot \\ a_{ni} \end{bmatrix}$, for $i = 1, 2, \ldots, n$. It follows that $T(X) = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{nn} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$, and part

(1) of the theorem holds.

For proof of part (2) suppose A is any n by n matrix of real numbers, $X \epsilon E^n$, $Y \epsilon E^n$, $k \epsilon R$, and $T(X)$ is the product of the matrix A and the matrix X. It follows that

$$T(X \quad \diamond \; Y) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} (x_1 + y_1) \\ (x_2 + y_2) \\ \vdots \\ (x_n + y_n) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & x_1 & + a_{12} & x_2 & + \dots + & a_{1n} & x_n \\ a_{21} & x_1 & + a_{22} & x_2 & + \dots + & a_{2n} & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & x_1 & + a_{n2} & x_2 & + \dots + & a_{nn} & x_n \end{bmatrix}$$

$$+ \begin{bmatrix} a_{11} & y_1 & + a_{12} & y_2 & + \dots + a_{1n} & y_n \\ a_{21} & y_1 & + a_{22} & y_2 & + \dots + a_{2n} & y_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & y_1 & + a_{n2} & y_2 & + \dots + a_{nn} & y_n \end{bmatrix}$$

$= T(X) + T(Y)$. Also, it follows that

$$T(k \diamond X) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}$$

$$= \begin{bmatrix} ka_{11} & x_1 & + ka_{12} & x_2 & + \dots + ka_{1n} & x_n \\ ka_{21} & x_1 & + ka_{22} & x_2 & + \dots + ka_{2n} & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ ka_{n1} & x_1 & + ka_{n2} & x_2 & + \dots + ka_{nn} & x_n \end{bmatrix}$$

$= kT(X)$. Hence, by Definition 4.8 it follows that T is a linear transformation from E^n to E^n.

Consider the following as an application of the preceding two theorems. Suppose $T_1, T_2, \dots, T_n$ is any basis set for E^n and $S_1, S_2, \dots, S_n$ is any ordered set of points of E^n. Then there exists one and only one n by n matrix A of real numbers such that if $1 \leq i \leq n$, then S_i is the matrix product of A and the one-column matrix T_i.

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