A DEVELOPMENT OF A SET OF FUNCTIONS ANALOGOUS TO
THE TRIGONOMETRIC AND THE HYPERBOLIC FUNCTIONS

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CHAPTER I

INTRODUCTION

The purpose of this paper is to define and develop a set of functions of an area in such a manner as to be analogous to the trigonometric and the hyperbolic functions. If we represent the circle with equation $x^2 + y^2 = 1$ by means of a parameter $t$ in the form $x = \cos t$, $y = \sin t$, we can interpret the quantity $t$ as twice the area of the circular sector corresponding to that angle, the area being reckoned positive or negative according as the angle is positive or negative.¹

Fig. 1

¹Courant, Differential and Integral Calculus, p. 188.
The hyperbolic functions can be defined in a like manner, where the quantity \( t \) is twice the area of the hyperbolic sector of the curve \( x^2 - y^2 = 1 \) as shown shaded in Fig. 2.

A set of six functions will be defined in terms of an area obtained from the curve \( x^4 + y^4 = 1 \), in such a manner that they will be analogous to the trigonometric and the hyperbolic functions. Furthermore some of the properties of these functions will be developed.

If \( p(a,b) \) is a point on the curve \( x^4 + y^4 = 1 \), then the measure of the angle \( \theta \) between the positive x-axis and the ray from the origin through \( p(a,b) \) is defined to be twice the area bounded by the positive x-axis, the curve \( x^4 + y^4 = 1 \) and the segment Op. If the area is taken in such a manner

\(^2\text{Ibid.}\)
that it has a boundary point, \((x,y)\), on the curve \(x^4 + y^4 = 1\) such that \((x,y)\) is in the first quadrant and \(y < |b|\) then \(\theta\) is defined to be positive. If the area is taken such that it has a boundary point, \((x,y)\) in the fourth quadrant on the curve \(x^4 + y^4 = 1\) and \(-y < |b|\) then the measure of the angle \(\theta\) is defined to be negative.

If the point \((a,b)\) is at \((1,0)\) the angle \(\theta\) is defined to be 0 or \(\pm 2\pi\) where \(T\) is \(\int_{-1}^{1} (1-x^4)^{1/4} dx\). If \((a,b)\) is the point \((-1,0)\), then \(\theta = T\) if the region considered is above the \(x\)-axis and \(\theta = -T\) if the region considered is below the \(x\)-axis.

If \((a,b)\) is a point on the curve \(x^4 + y^4 = 1\), let:

\(\sinp \theta\) be the value of \(y\) at the point \((a,b)\).

\(\cosp \theta\) be the value of \(x\) at the point \((a,b)\).

\(\tanp \theta = \frac{\sinp \theta}{\cosp \theta}\)  \(\secp \theta = \frac{1}{\cosp \theta}\)

\(\cotp \theta = \frac{\cosp \theta}{\sinp \theta}\)  \(\cosp \theta = \frac{1}{\sinp \theta}\).
Sinp $\theta$ is to be read as $p$-sin $\theta$, cosp $\theta$ is to be read as $p$-cosine $\theta$, tanp $\theta$ is to be read as $p$-tangent $\theta$, cotp $\theta$ is to be read as $p$-cotangent $\theta$, sec$p$ $\theta$ is to be read as $p$-secant $\theta$, and cosp $\theta$ is to be read as $p$-cosecant $\theta$.

There are certain fundamental relationships between these functions that are analogous to the fundamental trigonometric identities. Since $\tan p \theta = \frac{\sin p \theta}{\cos p \theta}$, and $\tan p \theta = \frac{\cos p \theta}{\sin p \theta}$, it follows that $\tan p \theta = \frac{1}{\cos p \theta}$. Since $x^4 + y^4 = 1$ and $x = \sin p \theta$ and $y = \cos p \theta$, then $\sin p^4 \theta + \cos p^4 \theta = 1$.

By dividing the above equation by $\cos p^4 \theta$ we obtain $\frac{\sin p^4 \theta}{\cos p^4 \theta} + 1 = \frac{1}{\cos p^4 \theta}$. Hence $\tan p^4 \theta + 1 = \sec p^4 \theta$. In a similar manner we can show that $\cot p^4 \theta + 1 = \csc p^4 \theta$.

Also there are certain reduction formulas that are similar to the reduction formulas for the trigonometric functions.
Since the equation $x^4 + y^4 = 1$ is unchanged if we replace $x$ by $-x$ or $y$ by $-y$ or interchange $x$ and $y$, the curve is symmetrical about the $y$-axis, the $x$-axis, the origin, the line $y = x$, and the line $y = -x$. Therefore a segment in any quadrant of the curve $x^4 + y^4 = 1$ has the same area as any corresponding segment in any of the other quadrants.

Referring to Fig. 5, if $\theta$ is a positive angle between $\frac{T}{2}$ and $T$ and $p(a, b)$ is the point of intersection of the terminal side of $\theta$ and $x^4 + y^4 = 1$, let $p'(a', b')$ be the point on $x^4 + y^4 = 1$ where $a' = -a$ and $b' = b$. Let $B$ be the angle whose sides are the positive $x$-axis and the ray $op'$. Because of the symmetry of $x^4 + y^4 = 1$ the angle $B$ is twice the area of the region in the second quadrant which is bounded by the negative $x$-axis, the ray $op$ and $x^4 + y^4 = 1$. Hence $B = T-\theta$. Now we have $\sin p = \sin p = \sin p(T-\theta)$. It may be shown that this is true for any angle $|\theta| \leq 2T$. 
The following formulas will be stated without derivation.
If angles are between \(-2\pi\) and \(2\pi\), then:

\[
\begin{align*}
\sin p - \theta &= -\sin p \theta, \\
\sin (T-\theta) &= \sin p \theta, \\
\cosp (T-\theta) &= -\cosp \theta, \\
\cosp (T+\theta) &= -\cosp \theta, \\
\tan p (T-\theta) &= -\tan \theta, \\
\cosp (T+\theta) &= -\cosp \theta, \\
\sin (T+\theta) &= -\sin \theta, \\
\cosp (T+\theta) &= -\cosp \theta, \\
\tan (T+\theta) &= \tan \theta, \\
\cosp (\frac{T-\theta}{2}) &= -\cosp \theta, \\
\cosp (\frac{T+\theta}{2}) &= -\cosp \theta, \\
\tan (\frac{T}{2}-\theta) &= cot p \theta, \\
\cosp (\frac{T}{2}-\theta) &= -\cosp \theta, \\
\tan (\frac{T}{2}+\theta) &= cot p \theta, \\
\cosp (\frac{T}{2}+\theta) &= -\cosp \theta.
\end{align*}
\]

In the development of the properties of the \(p\)-functions certain theorems will be needed. The following theorems will be stated and used without proof.

**Theorem 1.** A series \(\sum_{p=1}^{\infty} Ap\) in which limit \(\frac{A_{n+1}}{A_n} = 1\),

\(\frac{A_{n+1}}{A_n}\) can be reduced to the form \(\frac{1}{\frac{f(n)}{n}}\). Furthermore if \(\lim_{n \to \infty} f(n) > 1\) then the series is convergent and if the \(\lim_{n \to \infty} f(n) < 1\) then the series is divergent. \([2, p. 528]\)\(^3\)

\(^3\)Numbers appearing in brackets with the letter \(p\) in this chapter will refer to a book in the bibliography and the corresponding page number of each.
Theorem 2. If a power series in \( x \) converges for a value \( x = c \), it converges absolutely for every value \( x \) such that \( |x| < c \), and the convergence is uniform in every interval \( |x| \leq n \), where \( n \) is any positive number less than \( c \). \([1, \text{p. 392}]\)

Theorem 3. If \( G_n(x) \) is continuous for \( n = 1, 2, 3, \ldots \) and \( \sum_{n=1}^{\infty} G_n(x) \) converges uniformly to \( f(x) \) on an interval \([p, q]\) and \( b \) and \( x \) are in the interval \([p, q]\) then \( \sum_{n=1}^{\infty} \int_{q}^{x} G_n(x) \, dx \) converges uniformly with respect to \( x \) to \( \int_{b}^{x} f(x) \, dx \) for each \( b \). \([1, \text{p. 394}]\)

Theorem 4. Every function represented by a power series can be differentiated as often as we please within the interval of convergence, and the differentiation can be performed term by term. \([1, \text{p. 402}]\)

Theorem 5. If in the interval \( a < x < b \) the function \( f(x) \) is differentiable and in that interval either \( f'(x) < 0 \) everywhere or else \( f'(x) > 0 \) everywhere, then the inverse function \( g(y) \) also possesses a derivative at every point of its interval of definition, and between the derivatives of the given function \( y = f(x) \) and that of the inverse function \( x = g(y) \) there exists for corresponding values of \( x \) and \( y \) the relationship \( f'(x) \cdot g'(y) = 1 \), which also can be written in the form of \( \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \). \([1, \text{p. 145}]\)
Theorem 6. If \( f(x) \) and \( g(x) \) are both differentiable then:

1. \( P(x) = f(x) - g(x) \) is differentiable and
   \[ P'(x) = f'(x) - g'(x). \]

2. \( P(x) = \frac{f(x)}{g(x)} \) is differentiable provided \( g(x) \neq 0 \)
   and \( P'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g(x)^2} \). \([1, \text{p. 137}]\)
CHAPTER II

DERIVATIVES OF THE P-FUNCTIONS

If \(|\theta| < \frac{\pi}{2}\), then \(\theta\) can be expressed as an integral in the following manner:

\[
\theta = 2\int_0^b (1 - y^4)^{\frac{1}{2}} \, dy - b (1-b^4)^{\frac{1}{2}}.
\]

Now \((1-y^4)^{\frac{1}{2}}\) may be expanded by the binomial expansion:

\[
(1-y^4)^{\frac{1}{2}} = 1 - \frac{y^4}{2} - \frac{3y^8}{2^22!} - \cdots - \frac{3\cdot7\cdots(4n-9)y^{4n-1}}{2^{2n-2}(n-1)!}.
\]

Testing the series for convergence by the ratio test we find that \(\frac{A_{n+1}}{A_n} = \frac{(4n-5)y^4}{4n}\) and \(\lim_{n \to \infty} \frac{(4n-5)y^4}{4n} = y^4\).
Hence the series is convergent when \( y^4 < 1 \). When \( y^4 = 1 \),
\[
\lim_{n \to \infty} \frac{A_n}{\frac{1}{n}} = 1
\]
and further testing is required. When \( y = 1 \),
\[
\frac{A_n}{\frac{1}{n}} = \frac{5n}{4n-5} \rightarrow \frac{1}{1} = \frac{5n}{4n-5}
\]
and \( \lim_{n \to \infty} \frac{5n}{4n-5} = \frac{5}{4} > 1 \). Therefore by theorem 1 the series
is convergent for \( y^4 = 1 \) and hence for \( y^4 \leq 1 \). Since the
series in question is a power series convergent for \( y^4 \leq 1 \),
it is uniformly convergent, by theorem 2, and by theorem 3
may be integrated term by term over the range -1 to 1. The
point \( p(a, b) \) has a maximum possible \( y \) value of 1, therefore
the function can be integrated over the range of 0 to \( b \).

\[
\theta = -b(1-b^4)^{\frac{3}{4}} + 2 \int_0^b \left[ y - \frac{y^5}{5 \cdot 2^2} - \frac{3y^9}{9 \cdot 2^4} - \ldots \right] dy.
\]

Integrating term by term we have
\[
\theta = b(1-b^4)^{\frac{3}{4}} + 2 \left[ y - \frac{y^5}{5 \cdot 2^2} - \frac{3y^9}{9 \cdot 2^4} - \ldots \right]^{b}_0
\]

\[
\frac{3 \cdot 7 \cdot \ldots \cdot (4n-3)y^{4n-3}}{(4n-3) \cdot 2^{2n-2} \cdot (n-1)!} \ldots
\]
\[
\theta = -b(1-b^{4})^{\frac{1}{4}} + 2b - \frac{2b^5}{5\cdot 2^2} - \frac{2\cdot 3b^9}{9\cdot 4\cdot 2^4} - \cdots
\]
\[
\frac{2\cdot 3\cdot 7\cdots (4n-9)b^{4n-3}}{4(n-3)2^{2n-2}(n-1)!} - \cdots
\]

Now \(-b(1-b^{4})^{\frac{1}{4}}\) may be expanded in a like manner. The power series expansion of \(b(1-b^{4})^{\frac{1}{4}}\) may be shown to be convergent for \(|b| < 1\).

\[
\theta = \left[ -b + \frac{b^5}{2^2} + \frac{3b^9}{2^4\cdot 2^1} + \cdots + \frac{3\cdot 7\cdots (4n-9)b^{4n-3}}{2^{2n-2}(n-1)!} + \cdots \right]
\]
\[
\left[ 2b - \frac{2b^5}{2^2} - \frac{2\cdot 3b^9}{2^4\cdot 2^1} - \cdots + \frac{2\cdot 3\cdot 7\cdots (4n-9)b^{4n-3}}{2^{2n-2}(n-1)!} - \cdots \right].
\]

Since the above series are power series, uniformly convergent for \(b^{4} \leq 1\) they may be added term by term.

\[
\theta = b + \frac{3b^5}{5\cdot 2^2} + \frac{3\cdot 7b^9}{9\cdot 4\cdot 2^4} + \cdots + \frac{3\cdot 7\cdots (4n-5)b^{4n-3}}{4(n-3)2^{2n-2}(n-1)!} + \cdots
\]

Since the above series was obtained by the addition of two power series, each convergent for \(b^{4} \leq 1\), it is convergent.
for \( b^4 \leq 1 \). Since \( b \) is a fixed value of \( y \), replacing \( b \) by \( y \) results in a function of \( y \) where \( |y| \leq 1 \).

\[
\theta = y + \frac{3y^5}{5 \cdot 2^2} + \frac{3 \cdot 7y^9}{9 \cdot 2^4 \cdot 2!} + \cdots + \frac{3 \cdot 7 \cdots (4n-5) y^{4n-3}}{(4n-3) 2^{2n-2} (n-1)!} + \cdots
\]

Since the point \((a, b)\) was limited to the first and fourth quadrants the angle \( \theta \) is also limited such that \( |\theta| \leq \frac{\pi}{2} \).

The prefix arc will be used to denote the inverse of a \( p \)-function. Since \( \sin p \theta = y \), then \( \text{arc} \sin p y = \theta \), and

\[
\text{Arc} \sin p y = y + \frac{3y^5}{5 \cdot 2^2} + \frac{3 \cdot 7y^9}{9 \cdot 2^4 \cdot 2!} + \cdots + \frac{3 \cdot 7 \cdots (4n-5) y^{4n-3}}{(4n-3) 2^{2n-2} (n-1)!} + \cdots
\]

where \( |y| \leq 1 \) and \( |\theta| \leq \frac{\pi}{2} \).

Since \( \text{arc} \sin p y \) is equal to a uniformly convergent power series, then by theorem \( \frac{1}{4} \) \( \text{arc} \sin p y \) is differentiable, and the series may be differentiated term by term.

\[
\frac{d}{dy} \text{arc} \sin p y = 1 + \frac{3y^4}{2 \cdot 2^2} + \frac{3 \cdot 7y^8}{2^4 2!} + \cdots + \frac{3 \cdot 7 \cdots (4n-5) y^{4n-4}}{2^{2n-2} (n-1)!} + \cdots
\]
The above series is the power series expansion of \((1-y^l)^{-\frac{3}{2}}\),
then \(\frac{d}{d\theta} \arcsin p y = (1-y^l)^{\frac{3}{2}}\). Now \(\lim_{n \to \infty} \frac{A_n + 1}{A_n} = \lim_{n \to \infty} \frac{\ln n - 1}{\ln n} \cdot y^l = y^l\). The series is therefore convergent when \(y^l < 1\). When \(y^l = 1\),

\[
\frac{A_n + 1}{A_n} = \frac{\ln n - 1}{n} = \frac{1}{1 + \frac{\ln n - 1}{n}}
\]

and \(\lim_{n \to \infty} \frac{n - 1}{\ln n} = \frac{1}{4} < 1\). The series is divergent when \(y^l = 1\).

Now \(\frac{d}{dy} \arcsin p y = \frac{1}{(1-y^l)^{\frac{3}{2}}}\) and since \(\theta = \arcsin p y\) for

\[|\theta| < T, \quad \frac{d}{dy} \arcsin p y = \frac{1}{(1-y^l)^{\frac{3}{2}}} \]

is defined for \(y^l < 1\) then \(\frac{d\theta}{dy} < 0\) and by theorem 5, \(\frac{dy}{d\theta} = 1\). Then \(\frac{dy}{d\theta} = (1-y^l)^{\frac{3}{2}}\). Since \(\sin p \theta = y\), then \(\frac{d}{d\theta} \sin p \theta = \frac{d}{dy} \arcsin p y = (1-y^l)^{\frac{3}{2}}\). Replacing \(y\) by \(\sin p \theta\) we find

\[
\sin p \theta = \cos^3 p \theta\]

for \(|\theta| < T\). Since \(\cos^3 p \theta + \sin^3 p \theta = 1\), then \(\cos p \theta = (1-\sin^3 p \theta)^{\frac{1}{2}}\) and \(\cos p \theta\) has a derivative at each value of \(\theta\) for which \(\sin p \theta\) has a derivative.

From \(\sin^4 \theta + \cos^4 \theta = 1\), we have \(4 \sin^3 \theta \cos^3 \theta + 4 \cos^3 \theta \frac{d}{d\theta} \cos p \theta = 0\) and \(\frac{d}{d\theta} \cos p \theta = -\sin^3 p \theta\).

Since in the equation \(\theta = \arcsin p y\), \(\theta\) was restricted to positive angles terminating in the first quadrant, or to negative angles terminating in the fourth quadrant, and
since \( \frac{d}{d\theta} \sinp \theta \) was developed from \( \frac{d}{d\theta} \) arc \( \sinp y \), then our derivations of \( \frac{d}{d\theta} \sinp \theta = \cosp^3 \theta \) and \( \frac{d}{d\theta} \cosp \theta = -\sinp^3 \theta \) are restricted to \( |\theta| < \frac{T}{2} \). We will now remove this restriction. If \( \frac{T}{2} < \theta < T \) then by a reduction formula \( \sinp \theta = \cos(T-\theta) \), and \( 0 < T-\theta < T \). We now have: \( \frac{d}{d\theta} \sinp \theta = \cosp(T-\theta) \), \( \frac{d}{d\theta} \cosp \theta = -\sinp(T-\theta) \).

In a similar manner it may be shown by the use of a reduction formula that if \( -2T < \theta < 2T \), \( \frac{d}{d\theta} \sinp \theta = \cosp \theta \). Since \( \cosp \theta \) has a derivative for each value of \( \theta \) that \( \frac{d}{d\theta} \sinp \theta \) exists, then \( \frac{d}{d\theta} \cosp \theta = -\sinp^3 \theta \) for \( -2T < \theta < 2T \).

We may find the derivatives of the other \( p \)-function in the following manners. Since \( \sinp \theta \) and \( \cosp \theta \) have derivatives then by theorem 6, \( \tanp \theta = \frac{\sinp \theta}{\cosp \theta} \) has a derivative if \( \cosp \theta \neq 0 \). Since \( \tanp \theta = \frac{\sinp \theta}{\cosp \theta} \) then,

\[
\frac{d}{d\theta} \tanp \theta = \frac{\cosp \theta \cosp^3 \theta - \sinp \theta (-\sinp^3 \theta)}{\cosp^2 \theta} = \frac{\cosp^4 \theta + \sinp^4 \theta}{\cosp^2 \theta} = \secp^2 \theta.
\]

In a like manner we can show that \( \frac{d}{d\theta} \cotp \theta = -\cosp^2 \theta \).

Since \( \secp \theta = \frac{1}{\cosp \theta} \), \( \secp \theta \) has a derivative when \( \cosp \theta \neq 0 \), and \( \frac{d}{d\theta} \secp \theta = \frac{\sinp^3 \theta}{\cosp^2 \theta} = \sinp \theta \tanp^2 \theta \). In a like manner we can show that \( \frac{d}{d\theta} \cscp \theta = -\cosp \theta \cotp^2 \theta \).
In order to limit the inverse functions of the
p-functions to single valued functions, we will make the
following definitions for the principal values of the in-
verse functions. Let the inverse p-functions be restricted
to the following ranges of values.

\[ \begin{align*}
\theta &= \text{arc} \sin p x, \quad 0 \leq \theta \leq \frac{\pi}{2}; \\
\theta &= \text{arc} \cos p x, \quad 0 \leq \theta \leq \frac{\pi}{2}; \\
\theta &= \text{arc} \tan p x, \quad 0 < \theta < \frac{\pi}{2}; \\
\theta &= \text{arc} \cot p x, \quad 0 < \theta < \frac{\pi}{2}; \\
\theta &= \text{arc} \sec p x, \quad 0 \leq \theta \leq \frac{\pi}{2}; \\
\theta &= \text{arc} \csc p x, \quad 0 \leq \theta \leq \frac{\pi}{2}.
\end{align*} \]

It is of interest to note that the ranges of \( \theta \) in \( \theta = \text{arc} \sec p x \) and \( \theta = \text{arc} \csc p x \) do not agree with the cor-
responding ranges in the definitions of principal values of
the inverse trigonometric functions. Now for the above
ranges of values of \( \theta \), each of the p-functions of \( \theta \) satisfy
the hypothesis of theorem 5.

The derivatives of the inverse p-functions may be ob-
tained in the following manners. Let \( \theta = \text{arc} \sin p u \) where \( u \)
is a differentiable function of \( x \), then \( u = \sin p \theta \) and \( \theta =
\frac{d}{dx} \text{arc} \sin p u, \)

\[
\frac{du}{d\theta} = \cos^3 \theta,
\]

\[
\frac{d\theta}{du} = \frac{1}{\cos^3 \theta},
\]

\[
\frac{d\theta}{dx} = \frac{1}{\cos^3 \theta} \frac{du}{dx} = \frac{1}{(1-\sin^2 \theta)^\frac{3}{2}} \frac{du}{dx},
\]

and \( \frac{d\theta}{dx} = \frac{d}{dx} \text{arc} \sin p u = \frac{1}{(1-u^2)^\frac{3}{2}} \frac{du}{dx} \).
Let $\theta = \text{arc tanp } u$ where $u$ is a differentiable function of $x$, then $u = \text{tanp } \theta$ and $\frac{d\theta}{dx} = \frac{d}{dx} \text{arc tanp } u$.

\[
\frac{du}{d\theta} = \sec^2 \theta,
\]

\[
\frac{d\theta}{du} = \frac{1}{\sec^2 \theta}.
\]

\[
\frac{d\theta}{dx} = \frac{1}{\sec^2 \theta} \frac{du}{dx} = \frac{1}{\sqrt{1 + \tan^{1/4} u}} \frac{du}{dx}
\]

and $\frac{d\theta}{dx} = \frac{d}{dx} \text{arc tanp } u = \frac{1}{\sqrt{1 + u^{1/4}}} \frac{du}{dx}$.

In a similar manner it may be shown that:

\[
\frac{d}{dx} \text{arc cosp } u = -\frac{1}{(1-u^{1/4})^{3/4}} \frac{du}{dx}
\]

\[
\frac{d}{dx} \text{arc cotp } u = -\frac{1}{\sqrt{1+u^{1/4}}} \frac{du}{dx}
\]

\[
\frac{d}{dx} \text{arc secp } u = \frac{u}{(u^{1/4}-1)^{3/4}} \frac{du}{dx}
\]

\[
\frac{d}{dx} \text{arc cosp } u = \frac{-u}{(u^{1/4}-1)^{3/4}} \frac{du}{dx}
\]
CHAPTER III

INTEGRATION OF THE ρ-FUNCTIONS

The following integrals may be evaluated by using a substitution of the ρ-functions analogous to the trigonometric substitutions for evaluating integrals containing expressions of the form \( \sqrt{a^2-x^2}, \sqrt{x^2-a^2}, \) and \( \sqrt{a^2+x^2} \).

\[
\int \frac{dx}{(\frac{a^4-x^4}{x^4})^{\frac{3}{2}}} = \frac{1}{a^2} \text{ arc } \sinp \frac{x}{a} + c \quad |x| < |a|.
\]

\[
\int \frac{-dx}{(\frac{a^4-x^4}{x^4})^{\frac{3}{2}}} = \frac{1}{a^2} \text{ arc } \cosp \frac{x}{a} + c \quad |x| < |a|.
\]

\[
\int \frac{dx}{\sqrt{a^4 x^4}} = \frac{1}{a} \text{ arc } \tanp \frac{x}{a} + c.
\]

\[
\int \frac{-dx}{\sqrt{a^4 x^4}} = \frac{1}{a} \text{ arc } \cotp \frac{x}{a} + c.
\]

\[
\int \frac{xdx}{(\frac{a^4-x^4}{x^4})^{\frac{3}{2}}} = \frac{1}{a} \text{ arc } \secp \frac{x}{a} + c \quad |x| > |a|.
\]

\[
\int \frac{-xdx}{(\frac{a^4-x^4}{x^4})^{\frac{3}{2}}} = \frac{1}{a} \text{ arc } \cscp \frac{x}{a} + c \quad |x| > |a|.
\]

Since each of the ρ-functions has a derivative at every point where they are defined, they are continuous.
and are integrable over any interval where they are defined. Also each of the principal valued inverse p-functions has a derivative at each point of definition with the possible exception of two points. For example arc sinp x, arc cosp x, arc secp x, and arc cosep x do not have derivatives at \( x = \pm 1 \) but each of these is continuous at \( x = \pm 1 \). All the others have derivatives over any interval in which they are defined.

In order to determine \( \int \sinp \theta \, d\theta \), let \( \sinp^{1/2} \theta = v^2 \cos^2 \theta \). Then \( d\theta = \frac{dv}{\sinp \theta \sqrt{4 + v^4}} \), and \( V = \sinp \theta \tanp \theta \).

\[
\int \sinp \theta \, d\theta = \int \frac{dv}{\sqrt{4 + v^4}}
\]

\[
= -\frac{1}{\sqrt{2}} \arctanp \frac{v}{\sqrt{2}} + c
\]

\[
= -\frac{1}{\sqrt{2}} \arctanp \frac{\sinp \theta \tanp \theta}{\sqrt{2}} + c.
\]

It may be shown in a similar manner that \( \int \cosp \theta \, d\theta = \frac{1}{\sqrt{2}} \arccotp \cosp \theta \cotp \theta + c. \)

Also \( \int \tanp \theta \, d\theta = \frac{1}{4} \int \frac{\sinp \theta \cosp^3 \theta}{\cosp^4 \theta} \, d\theta \),

\[
= \frac{1}{4} \int \frac{\sinp \theta \cosp^3 \theta}{1 - \sinp^4 \theta}
\]
\[
\int \frac{2 \sin \theta \cos^3 \theta}{1 - \sin^2 \theta} \, d\theta + \frac{1}{4} \int \frac{2 \sin \theta \cos^3 \theta}{1 + \sin^2 \theta} \, d\theta,
\]

\[
= -\frac{1}{4} \ln(1 - \sin^2 \theta) + \frac{1}{4} \ln(1 + \sin^2 \theta) + c,
\]

\[
= \frac{1}{4} \ln \left( \frac{1 + \sin^2 \theta}{1 - \sin^2 \theta} \right) + c.
\]

We may obtain \( \int \tan \theta \, d\theta \) in another form by use of the integrating factor \( \frac{\tan^2 \theta + \sec^2 \theta}{\tan^2 \theta + \sec^2 \theta} \).

\[
\int \tan \theta \, d\theta = \int \frac{\tan \theta (\tan^2 \theta + \sec^2 \theta)}{\tan^2 \theta + \sec^2 \theta} \, d\theta,
\]

\[
= \frac{1}{2} \int \frac{2 \tan \theta \sec^2 \theta + 2 \tan^3 \theta}{\tan^2 \theta + \sec^2 \theta} \, d\theta,
\]

which is in the form of \( \frac{du}{u} \). Hence

\[
\int \tan \theta \, d\theta = \frac{1}{2} \ln(\tan^2 \theta + \sec^2 \theta) + c.
\]

In a similar manner it may be shown that,

\[
\int \cot \theta \, d\theta = -\frac{1}{4} \ln \left( \frac{1 + \cos^2 \theta}{1 - \cos^2 \theta} \right) + c.
\]

\[
= -\frac{1}{2} \ln(\cot^2 \theta + \csc^2 \theta) + c.
\]

In order to determine the integral of \( \sec \theta \), we multiply the integrand by \( \frac{\cos^3 \theta}{\cos^3 \theta} \).
\[
\int \sec \theta \, d\theta = \int \frac{\sec \theta \, \csc^2 \theta}{\csc^2 \theta} \, d\theta,
\]

\[
= \int \frac{\csc^2 \theta}{\csc^2 \theta} \, d\theta = \int \frac{\csc^2 \theta}{1 - \sin^2 \theta} \, d\theta,
\]

\[
= \frac{1}{2} \int \frac{\csc^2 \theta}{1 - \sin^2 \theta} \, d\theta + \frac{1}{2} \int \frac{\csc^2 \theta}{1 + \sin^2 \theta} \, d\theta,
\]

\[= \frac{1}{2} \arctan \sin \theta + \frac{1}{2} \arctan \sin \theta + c.\]

In a similar manner it may be shown that,

\[
\int \csc \theta \, d\theta = -\frac{1}{2} \arctan \csc \theta - \frac{1}{2} \arctan \csc \theta + c.
\]

The inverse p-functions are continuous if we restrict them to their principal values. In order to determine 
\[
\int \arcsin p \, x \, dx \text{ let } f(x) \text{ be such that } \int \arcsin p \, x \, dx = f(x) + x \arcsin p \, x, \text{ where } f(x) \text{ is to be determined. By differentiating we have,}
\]

\[\arcsin p \, x = \frac{d}{dx} f(x) + \frac{x}{(1-x^2)^{\frac{3}{2}}} \arcsin p \, x,\]

\[\frac{d}{dx} f(x) = \frac{-x}{(1-x^2)^{\frac{3}{2}}},\]

\[f(x) = \int \frac{-x \, dx}{(1-x^2)^{\frac{3}{2}}}.\]

Let \( x = \sin p \, u, \) then \( dx = \csc^3 u \, du, \) and,

\[\int \frac{-x \, dx}{(1-x^2)^{\frac{3}{2}}} = \int -\sin p \, u \, du = -\frac{1}{\sqrt{2}} \arctan p \, \tan p \, u \, \sin p \, u + c.\]
\[
\int \frac{-x \, dx}{(1-x^4)} = -\frac{1}{\sqrt{2}} \arctan \frac{x^2}{\sqrt{2}} + \frac{x}{\sqrt{2}} (1-x^4)^{\frac{3}{2}}.
\]

and \[
\int \arcsin x \, dx = -\frac{1}{\sqrt{2}} \arctan \frac{x^2}{\sqrt{2}} + x \arcsin x + c.
\]

In a similar manner we can show that
\[
\int \arccos x \, dx = \frac{1}{\sqrt{2}} \arctan \frac{x^2}{\sqrt{2}} + x \arccos x + c.
\]

In order to determine \(\int \arctan x \, dx\) let \(f(x)\) be such that \(\int \arctan x \, dx = f(x) + x \arctan x\) where \(f(x)\) is to be determined. By differentiating we have,

\[
\arctan x = \frac{d}{dx} f(x) + \frac{x}{\sqrt{1 + x^4}} + \arctan x,
\]

\[
\frac{d}{dx} f(x) = \frac{-x}{\sqrt{1 + x^4}},
\]

\[
f(x) = \int \frac{-x \, dx}{\sqrt{1 + x^4}}.
\]

Let \(x = \tan u\), then \(\sec^2 u = \sqrt{1 + x^4}\), and \(dx = \sec^2 u \, du\),

\[
\int \frac{-x \, dx}{\sqrt{1 + x^4}} = \int -\tan u \, du
\]

\[
= -\frac{1}{2} \ln(\tan^2 u + \sec^2 u) + c
\]

\[
= -\frac{1}{2} \ln(x^2 + \sqrt{1 + x^4}) + c,
\]

and \(\int \arctan x \, dx = -\frac{1}{2} \ln(x^2 + \sqrt{1 + x^4}) + x \arctan x + c\).
In a similar manner we can show that

\[ \int \text{arc cotp } x \, dx = \frac{1}{2} \ln \left( x^2 + \sqrt{1 + x^4} \right) + x \text{ arc cotp } x + c. \]

In order to determine \( \int \text{arc secp } x \, dx \) let \( f(x) \) be such that \( \int \text{arc secp } x \, dx = f(x) + x \text{ arc secp } x \) where \( f(x) \) is to be determined. By differentiating we have,

\[
\text{arc secp } x = \frac{d}{dx} f(x) + \frac{x^2}{(x^4-1)^{\frac{3}{2}}} + \text{arc secp } x,
\]

\[
\frac{d}{dx} f(x) = -\frac{x^2}{(x^4-1)},
\]

\[
f(x) = \int \frac{-x^2 \, dx}{(x^4-1)}.
\]

Let \( x = \text{secp } u \), then \( \sin p u = (x^4-1)^{\frac{3}{4}} \), and \( dx = (x^4-1)^{\frac{1}{4}} \, du \),

\[
\int \frac{-x^2 \, dx}{(x^4-1)} = \int -\text{secp } u \, du
\]

\[ = -\frac{1}{2} \text{ arc tanh } \sin p u - \frac{1}{2} \text{ arc tan } \sin p u + c\]

\[ = -\frac{1}{2} \text{ arc tanh } \left( \frac{x^4-1}{x} \right)^{\frac{3}{4}} - \frac{1}{2} \text{ arc tan } \left( \frac{x^4-1}{x} \right)^{\frac{1}{4}} + c,
\]

and \( \int \text{arc secp } x \, dx = -\frac{1}{2} \text{ arc tanh } \left( \frac{x^4-1}{x} \right)^{\frac{3}{4}} - \frac{1}{2} \text{ arc tan } \left( \frac{x^4-1}{x} \right)^{\frac{1}{4}} + x \text{ arc secp } x + c. \)

In a similar manner we can show that

\[ \int \text{arc cs cp } x \, dx = \frac{1}{2} \text{ arc tanh } \left( \frac{x^4-1}{x} \right)^{\frac{3}{4}} - \frac{1}{2} \text{ arc tan } \left( \frac{x^4-1}{x} \right)^{\frac{1}{4}} + x \text{ arc cs cp } x + c. \]
TABLE 1

TABLE OF DERIVATIVES

1. \( \frac{d}{dx} \sin p \theta = \cos p^3 \theta \)

2. \( \frac{d}{dx} \cos p \theta = - \sin p^3 \theta \)

3. \( \frac{d}{dx} \tan p \theta = \sec p^2 \theta \)

4. \( \frac{d}{dx} \cot p \theta = - \csc p^2 \theta \)

5. \( \frac{d}{dx} \sec p \theta = \sin p \theta \tan p^2 \theta \)

6. \( \frac{d}{dx} \csc p \theta = - \cos p \theta \cot p^2 \theta \)

7. \( \frac{d}{dx} \arcsin p u = \frac{1}{\sqrt{1-u^4}} \cdot \frac{du}{dx} \)

8. \( \frac{d}{dx} \arccos p u = - \frac{1}{\sqrt{1-u^4}} \cdot \frac{du}{dx} \)

9. \( \frac{d}{dx} \arctan p u = \frac{1}{\sqrt{1+u^4}} \cdot \frac{du}{dx} \)

10. \( \frac{d}{dx} \arccot p u = \frac{-1}{\sqrt{1+u^4}} \cdot \frac{du}{dx} \)

11. \( \frac{d}{dx} \arcsec p u = \frac{u}{(u^4-1)^{\frac{3}{2}}} \cdot \frac{du}{dx} \)

12. \( \frac{d}{dx} \arccsc p u = \frac{-u}{(u^4-1)^{\frac{3}{2}}} \cdot \frac{du}{dx} \)
### TABLE 2

**TABLE OF INTEGRALS**

1. \[ \int \sin \theta \, d\theta = \frac{\sqrt{2}}{2} \arctan \left( \frac{\sqrt{2}}{2} \sin \theta \right) + c \]

2. \[ \int \sin^2 \theta \, d\theta = - \frac{1}{2} \left[ \arctan \left( 1 + \sqrt{2} \tan \theta \right) - \arctan \left( 1 - \sqrt{2} \tan \theta \right) \right] + c \]

3. \[ \int \sin^3 \theta \, d\theta = - \cos \theta + c \]

4. \[ \int \sin^4 \theta \, d\theta = \frac{1}{2} \left[ \theta - \sin \theta \cos \theta \right] + c \]

5. \[ \int \cos \theta \, d\theta = \frac{1}{2} \left[ \arcsin \left( \sqrt{2} \cos \theta \right) \right] + c \]

6. \[ \int \cos^2 \theta \, d\theta = \frac{1}{2} \left[ \arcsin \left( \sqrt{2} \cos \theta \right) - \arcsin \left( - \sqrt{2} \cos \theta \right) \right] + c \]

7. \[ \int \cos^3 \theta \, d\theta = \sin \theta + c \]

8. \[ \int \cos^4 \theta \, d\theta = \frac{1}{2} \left[ \theta + \sin \theta \cos \theta \right] + c \]

9. \[ \int \tan \theta \, d\theta = \frac{1}{4} \ln \left( \frac{1 + \sin^2 \theta}{1 - \sin^2 \theta} \right) + c \]

\[ = \frac{1}{2} \ln (\tan^2 \theta + \sec^2 \theta) + c \]

\[ = \frac{1}{2} \arctanh \sin^2 \theta + c \]

10. \[ \int \tan^3 \theta \, d\theta = \frac{1}{2} \sec^2 \theta + c \]
11. \[ \int \tan^3 \theta \, d\theta = \frac{1}{3} (\tan \theta \sec^2 \theta - \theta) + c \]

12. \[ \int \cot^3 \theta \, d\theta = -\frac{1}{4} \ln \left( \frac{1 + \csc^2 \theta}{1 - \csc^2 \theta} \right) + c \]
   \[= -\frac{1}{2} \ln (\cot^2 \theta + \csc^2 \theta) + c \]
   \[= -\frac{1}{2} \tanh \csc^2 \theta + c \]

13. \[ \int \cot^3 \theta \, d\theta = -\frac{1}{2} \csc^2 \theta + c \]

14. \[ \int \cot^4 \theta \, d\theta = -\frac{1}{3} (\cot \theta \csc^2 \theta + \theta) + c \]

15. \[ \int \sec \theta \, d\theta = \frac{1}{2} \arctan \sin \theta + \frac{1}{2} \arctan \sin \theta + c \]
   \[= \frac{1}{4} \ln \left( \frac{1 + \sin \theta}{1 - \sin \theta} \right) + \frac{1}{3} \arctan \sin \theta + c \]

16. \[ \int \sec^2 \theta \, d\theta = \tan \theta + c \]

17. \[ \int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \cos \theta \, d\theta \]

18. \[ \int \sec^4 \theta \, d\theta = \frac{1}{3} (\tan \theta \sec^2 \theta \geq 0) + c \]

19. \[ \int \csc \theta \, d\theta = -\frac{1}{4} \ln \left( \frac{1 + \csc \theta}{1 - \csc \theta} \right) - \frac{1}{2} \arctan \csc \theta + c \]
   \[= -\frac{1}{2} \tanh \csc \theta + c \]
   \[= \frac{1}{2} \arctan \csc \theta + c \]
20. $\int \csc^2 \theta \, d\theta = - \cot \theta + c$

21. $\int \csc^3 \theta \, d\theta = - \frac{1}{2} \csc \theta \cot \theta +$ 

$\frac{1}{2} \int \sin \theta \, d\theta$

22. $\int \csc^4 \theta \, d\theta = - \frac{1}{3} (\cot \theta \csc^2 \theta - 2 \theta) + c$

23. $\int \sin \theta \cos \theta \, d\theta + \frac{1}{2} \arctan \tan^2 \theta + c$

24. $\int \sec \csc \theta \, d\theta = \ln \tan \theta + c$

25. $\int \arcsin x \, dx = - \frac{1}{\sqrt{2}} \arctan \frac{x^2}{x^2} +$

$\frac{x}{\sqrt{2}} \arcsin x + c$

26. $\int \arccos x \, dx = \frac{1}{\sqrt{2}} \arctan \frac{x^2}{x^2} +$

$x \arccos x + c$

27. $\int \arctan x \, dx = - \frac{1}{x} \ln(x^2 + \sqrt{1 + x^4}) +$

$x \arctan x + c$

28. $\int \arccot x \, dx = \frac{1}{2} \ln(x^2 + \sqrt{1 + x^4}) +$

$x \arccot x + c$

29. $\int \sec x \, dx = - \frac{1}{x^4} \ln \left( \frac{x + (x^4 - 1)^{1/4}}{x - (x^4 - 1)^{1/4}} \right) -$

$\frac{1}{2} \arctan \left( \frac{x^4 - 1}{x^4} \right)^{1/4} + x \sec x + c$
30. \[ \int \text{arc csc} \ x \ dx = \frac{1}{a} \ln \left( \frac{x + (x^4 - 1)^{1/4}}{x - (x^4 - 1)^{1/4}} \right) + \frac{1}{2} \arctan \left( \frac{x^4 - 1}{x^4} \right)^{1/2} + \frac{1}{2} x \ \text{arc csc} \ x + c \]

31. \[ \int \frac{dx}{(a^4 - x^4)^{3/2}} = \frac{1}{a^2} \arcsin \frac{x}{a} + c \]

32. \[ \int \frac{-dx}{(a^4 + x^4)^{3/2}} = \frac{1}{a^2} \arccos \frac{x}{a} + c \]

33. \[ \int \frac{dx}{\sqrt{a^4 + x^4}} = \frac{1}{a} \arctan \frac{x}{a} + c \]

34. \[ \int \frac{-dx}{\sqrt{a^4 + x^4}} = \frac{1}{a} \arccot \frac{x}{a} + c \]

35. \[ \int \frac{x \ dx}{(x^4 - a^4)^{3/2}} = \frac{1}{a} \arccsc \frac{x}{a} + c \]

36. \[ \int \frac{-x \ dx}{(x^4 - a^4)^{3/2}} = \frac{1}{a} \arcsec \frac{x}{a} + c \]
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