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*Relativistic Collision Rate Calculations
for Electron-Air Interactions*

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RELATIVISTIC COLLISION RATE CALCULATIONS FOR ELECTRON- AIR INTERACTIONS

by

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ABSTRACT

The most recent data available on differential cross sections for electron-air interactions are used to calculate the avalanche, momentum transfer, and energy loss rates that enter into the fluid equations. Data for the important elastic, inelastic, and ionizing processes are generally available out to electron energies of 1-10 keV. Prescriptions for extending these cross sections to the relativistic regime are presented. The angular dependence of the cross sections is included where data are available as is the doubly differential cross section for ionizing collisions. The collision rates are computed by taking moments of the Boltzmann collision integrals with the assumption that the electron momentum distribution function is given by the Jüttner distribution function which satisfies the relativistic H-theorem and which reduces to the familiar Maxwellian velocity distribution in the nonrelativistic regime. The distribution function is parameterized in terms of the electron density, mean momentum, and thermal energy and the rates are therefore computed on a two dimensional grid as a function of mean kinetic energy and thermal energy.

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1 Introduction

Detailed analyses of electron transport in air are generally based on a kinetic treatment that incorporates the dominant cross sections for electron-air interactions either in the Boltzmann collision integrals or in the statistical techniques used in Monte Carlo calculations. Kinetic theory has been used extensively to study transport parameters obtained from steady-state swarm experiments and, in some cases, to adjust the unmeasured cross sections of certain inelastic processes to bring theory and experiment into better agreement.[1, 2, 3, 4, 5, 6, 7] The strong departure of the electron velocity distribution function from Maxwellian in the energy regime where inelastic processes are important is also well documented.[1, 2, 8, 9, 10] While these studies point to the importance of microscopic effects in characterizing the details of electron transport in air and other gas mixtures, the application of kinetic calculations to problems that involve large velocity, spatial and temporal domains is not possible with existing computing capabilities. In addition, for many problems of interest, it is sufficient to treat the electron transport by means of a fluid analysis in which the velocity distribution function is assumed to be Maxwellian. For example, the propagation of high-power microwave pulses through the earth's atmosphere and the simulation of microwave breakdown experiments are both nonlinear problems that necessitate solution of electron transport equations in conjunction with Maxwell's equations over a large velocity range, for many resolved wave periods, and over large spatial dimensions. The latter considerations preclude the use of a detailed kinetic treatment for the entire parameter range of interest. Furthermore, comparisons of kinetic and fluid calculations for isolated cases indicate that fluid theory is capable of characterizing both the microwave transmission of air over a broad energy range and the details of electron transport over a limited energy range to a high degree of accuracy. In addition to the modeling of microwave breakdown[11, 12, 13] and high-power microwave propagation[14], fluid theory has also been applied successfully to a number of ionospheric problems involving electron transport, namely, the development of basic ionospheric models, the study of electromagnetic wave propagation through the ionosphere, the development of various plasma instabilities (e.g., Rayleigh-Taylor type instabilities) in the ionosphere, and the simulation of active experiments (these problems have a vast literature associated with them and are generally described in various books[15, 16, 17]).

In this paper, the most recent data available on differential cross sections for electron-air interactions are used to calculate the avalanche, momentum transfer, and energy loss rates that enter into the fluid equations. Data for the important elastic, inelastic, and ionizing processes are generally available out to electron energies of 1-10 keV. Prescriptions for extending these cross sections to the relativistic regime are presented. The angular dependence of the cross sections is included where data are available as is the doubly differential cross section for ionizing collisions. The collision rates are computed by taking moments of the Boltzmann collision integrals with the assumption that the electron momentum distribution function is given by the Jüttner distribution function which satisfies the relativistic H-theorem and which reduces to the familiar Maxwellian velocity distribution in the nonrelativistic regime. The distribution function is parameterized in terms of the electron density, mean momentum, and thermal energy and the rates are therefore computed on a two-dimensional grid as a function of mean kinetic energy and thermal energy.

The Boltzmann collision integrals for the relativistic regime are derived in Section 2 following the nonrelativistic treatments of Holstein and Chapman and Cowling. Relativistic scattering cross sections are discussed in Section 3, while Section 4 presents expressions

for the particle, momentum, and energy transfer rates that were derived using differential scattering cross sections. Section 5 explains some of the computational methods that were used to calculate the collision rates, and Section 6 discusses the results of the rate calculations. Appendix A presents the derivation of the relativistic collision rates while Appendix B contains a series of two-dimensional plots of the rates as a function of electron mean kinetic energy for fixed thermal energy and as a function of electron mean thermal energy for fixed kinetic energy.

2 Boltzmann Collision Integrals

The Boltzmann collision integrals represent the time rate of change of a particle distribution caused by collisions. In the present treatment we calculate the rate of change of the electron momentum distribution $(\partial f / \partial t)|_{coll}$ resulting from elastic, inelastic, and ionizing collisions of electrons with neutral, nonexcited gas molecules. The probability for these collisions is specified in terms of differential cross sections obtained from experiment or derived from the combination of swarm measurements and kinetic theory. Presented here is the derivation of the relativistic collision integrals following the nonrelativistic treatments given by Holstein[18] and Chapman and Cowling[19].

To calculate $(\partial f / \partial t)|_{coll}$ we consider a volume $d\Upsilon dV = dp_x dp_y dp_z dx dy dz$ about a point (p_x, p_y, p_z, x, y, z) . The change in the number of electrons in $d\Upsilon dV$ during the time dt because of collisions with neutral gas molecules is

$$\left. \frac{\partial f}{\partial t} \right|_{coll} d\Upsilon dV dt = (\Gamma_{in} - \Gamma_{out}) d\Upsilon dV dt \quad ,$$

where $\Gamma_{in} d\Upsilon dV dt$ and $\Gamma_{out} d\Upsilon dV dt$ are the number of electrons respectively entering and leaving $d\Upsilon dV$ during dt . In the following we derive Γ_{in} and Γ_{out} for elastic, inelastic, and ionizing collisions using the geometry of Figure 1.

2.1 Elastic Collisions

Consider a collision between an electron of mass m and a gas molecule of mass M that is initially at rest. A direct collision is one in which an electron with initial momentum in the volume $d\Upsilon$ about (p_x, p_y, p_z) collides with a gas molecule (Figure 2). The total number of electrons leaving the volume element $d\Upsilon dV = dp_x dp_y dp_z dx dy dz$ at (p_x, p_y, p_z, x, y, z) in time dt in such a collision is equal to the number of electrons in the volume $d\Upsilon dV$, $f(\vec{p}) d\Upsilon dV$, multiplied by the probability of an electron leaving the element

$$\Gamma_{out|el} dt d\Upsilon dV = N \beta c f(\vec{p}) d\Upsilon dV dt \int_{\Omega'} \sigma_{el}(p, \psi) d\Omega' \quad . \quad (1)$$

Here c is the speed of light, N is the density of gas molecules, β is the mean electron velocity in units of the speed of light, f is the electron momentum distribution function, $p = \text{momentum}/mc$ is the normalized electron momentum, $\sigma_{el}(p, \psi)$ is the differential cross section for elastic scattering, and $d\Omega'$ is the solid angle into which the electron is scattered as defined in the center of mass frame. Note that $d\Omega' = d(\cos \psi) d\alpha = |J| d(\cos \theta') d\phi'$,

Figure 1: The coordinate system for the collision between an electron with initial momentum $\vec{p}$ and final momentum $\vec{p}'$ and a neutral gas molecule at rest. The angle α is measured from the direction $\hat{p} \times (\hat{z} \times \hat{p})$.

where the Jacobian

$$J = \begin{vmatrix} \partial \cos \psi / \partial \cos \theta' & \partial \alpha / \partial \cos \theta' \\ \partial \cos \psi / \partial \phi' & \partial \alpha / \partial \phi' \end{vmatrix} = 1 \quad .$$

Thus the solid angle $d\Omega'$ defined in the center of mass frame is equal in magnitude to the equivalent solid angle defined in the lab frame.

An inverse collision, shown in Figure 2, is one between an electron and a gas molecule such that the final electron momentum is in the volume $d\Upsilon$ about (p_x, p_y, p_z) . The probability that an electron in the volume $d\Upsilon' dV$ is scattered into the element $d\Upsilon$ in time dt is

$$N \beta' c \sigma_{el}(p', \psi) d\Omega dt \delta(p - g(p', \psi)) dp \quad ,$$

where the term $\delta(p - g(p', \psi)) dp$ is needed so that only specific types of collisions are included. In this case we wish to count only those inverse collisions that satisfy the conservation of energy for elastic collisions. The total number of electrons entering $d\Upsilon dV$ in time dt from all other elements $d\Upsilon' dV$ as a result of elastic collisions is

$$\int_{\Upsilon'} f(\vec{p}') N \beta' c \sigma_{el}(p', \psi) d\Omega dt \delta(p - g(p', \psi)) dp d\Upsilon' dV \quad .$$

The increment $d\Upsilon'$ can be expressed in polar coordinates as $d\Upsilon' = p'^2 d\Omega' dp'$. Trans-

Figure 2: A direct collision is one in which the *initial* momentum of the particle is in a volume $d\Upsilon$ about (p_x, p_y, p_z) , whereas an inverse encounter is one in which the *final* momentum of the particle is in the volume $d\Upsilon$ about (p_x, p_y, p_z) .

forming the integral over p' using

$$dp' = \frac{\partial p'}{\partial g} dg$$

and integrating over g yields the expression for the total number of electrons entering $d\Upsilon dV$ in time dt because of transitions from $d\Upsilon' dV$:

$$\Gamma_{in|el} dt d\Upsilon dV = dt d\Upsilon dV \int_{\Omega'} f(\vec{p}') \left(\frac{p'}{p} \right)^2 N \beta' c \sigma_{el}(p', \psi) \frac{\partial p'}{\partial p} d\Omega' . \quad (2)$$

Here we have rewritten the volume $p^2 dp d\Omega$ as $d\Upsilon$ and we require that the electron momenta p and p' satisfy the conservation of energy for elastic collisions.

The difference between Eq. (2) and Eq. (1) is the relativistic Boltzmann collision integral for elastic collisions:

$$\Gamma_{in} - \Gamma_{out|el} = Nc \int_{\Omega'} \left[f(\mu', p') \left(\frac{p'}{p} \right)^2 \beta' \sigma_{el}(p', \psi) \frac{\partial p'}{\partial p} - f(\mu, p) \beta \sigma_{el}(p, \psi) \right] d\Omega' , \quad (3)$$

where

$$\begin{aligned} \mu' &= \mu \cos \psi + (1 - \mu^2)^{\frac{1}{2}} \sin \psi \sin \alpha , \text{ and} \\ d\Omega' &= d(\cos \psi) d\alpha . \end{aligned}$$

Note that μ' is the cosine of the angle θ' (Figure 1) between the electron velocity after collision and the arbitrarily chosen $\hat{z}$ axis. Similarly, μ is the cosine of the angle θ (Figure 1) between the electron velocity before collision and the $\hat{z}$ axis. The solid angle $d\Omega' = d(\cos \theta') d\phi'$ has been expressed in terms of the coordinates ψ and α .

2.2 Inelastic Collisions

The derivation of the relativistic Boltzmann collision integral for inelastic collisions $\Gamma_{in} - \Gamma_{out|in}$ follows the same logic as the derivation for elastic collisions. However, in this case the electron momenta p and p' are restricted to those that satisfy the conservation of energy for inelastic collisions. The inelastic contribution to the Boltzmann collision integral is

$$\Gamma_{in} - \Gamma_{out|in} = \sum_h N_h c \int \left[f(\mu', p') \sigma_h(p', \psi) \frac{p'^2 \beta'}{p^2} \frac{\partial p'}{\partial p} - \beta f(\mu, p) \sigma_h(p, \psi) \right] d\Omega' , \quad (4)$$

where $\sigma_h(p, \psi)$ is the differential cross section for the excitation of level h , and Σ_h denotes the summation over all levels.

2.3 Ionizing Collisions

In deriving the Boltzmann collision integral for ionizing collisions we must include the contribution made by secondary electrons and their energy distribution. In this case we make use of the differential cross section $\sigma_I(\varepsilon, \varepsilon_s, \psi)$ which measures the probability that an incident electron of energy ε will undergo an ionizing collision with a neutral molecule, be scattered through an angle ψ , and release a secondary electron of energy ε_s . Thus the number of electrons that leave the volume $d\Upsilon dV$ in time dt after a particular ionizing collision is

$$\begin{aligned} \Gamma_{out|I} d\Upsilon dV dt &= \left(\text{No. of electrons in } d\Upsilon dV \right) \times \left(\text{Probability of an electron leaving } d\Upsilon dV \text{ because of an ionizing collision} \right) \\ &= d\Upsilon dV dt N \beta c f(\vec{p}) \int_{\Omega'} \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \sigma_I(\varepsilon, \varepsilon_s, \psi) d\Omega' d\varepsilon_s, \end{aligned} \quad (5)$$

where ε_I is the ionization threshold energy and ε_s is the secondary electron energy. Note that in general $\varepsilon_i = \varepsilon_f + \varepsilon_I + \varepsilon_s$ for initial energy ε_i and final energy ε_f .

The number of electrons that enter the volume $d\Upsilon dV$ in time dt as a result of a particular ionizing collision is

$$\begin{aligned} \Gamma_{in|I} d\Upsilon dV dt &= \left(\text{No. of electrons in } d\Upsilon' dV \right) \times \left(\text{Probability of an electron entering } d\Upsilon dV \text{ because of an ionizing collision} \right) \\ &= dV dt \int_{\Omega'} \int_{\varepsilon'} N \beta' c f(\vec{p}') \sigma_I(\varepsilon', \varepsilon, \psi) \delta(p - g(p', \psi)) dp d\Upsilon' d\varepsilon' d\Omega'. \end{aligned}$$

Using $d\Upsilon = p^2 dp d\Omega$ and $d\Upsilon' = p'^2 dp' d\Omega'$ as was done in the derivation of the elastic collision integral and integrating over dg yields

$$\Gamma_{in|I} d\Upsilon dV dt = d\Upsilon dV dt N c \int \beta' f(\vec{p}') \sigma_I(\varepsilon', \varepsilon, \psi) \left(\frac{p'}{p} \right)^2 \frac{\partial p'}{\partial p} d\varepsilon' d\Omega'. \quad (6)$$

In this case the momenta (and energies) are restricted to those that satisfy the law of energy conservation for an electron of energy ε' making an ionizing collision in such a way that an electron is sent into the energy range $\varepsilon + d\varepsilon$. This leads to the expression

$$\frac{\partial p'}{\partial p} \approx \frac{\beta}{\beta'}. \quad (7)$$

Substituting Eq. (7) in Eq. (6), subtracting Eq. (5), and summing over all ionizing collisions gives

$$\begin{aligned} \Gamma_{in} - \Gamma_{out|ion} &= \sum_I N_I c \int d\Omega' \left[\int_0^\infty d\varepsilon' f(\mu', p') \sigma_I(\varepsilon', \varepsilon, \psi) \frac{p'^2 \beta'}{p^2} \left(\frac{\beta}{\beta'} \right) \right. \\ &\quad \left. - \beta f(\mu, p) \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \sigma_I(\varepsilon, \varepsilon_s, \psi) d\varepsilon_s \right]. \end{aligned} \quad (8)$$

3 Relativistic Cross Sections

Relativistic and nonrelativistic cross-section expressions for this calculation were taken from Murphy,[20] while cross-section data were taken from Phelps[21], Phelps and Pitchford[22], Rapp and Englander-Golden[23], and Opal, Beaty, and Peterson[24, 25] as cited in Murphy. In this section, we summarize Murphy's cross sections for the readers' convenience and discuss adjustments we made to those expressions.

The differential cross sections given in Murphy have the form

$$\sigma(\varepsilon, \mu_\psi) = A(\varepsilon, \mu_\psi) \frac{\sigma^T(\varepsilon)}{4\pi} ,$$

where $\sigma^T(\varepsilon)$ is the total collision cross section as a function of incident electron energy, and $A(\varepsilon, \mu_\psi)$ is the cross-section angular dependence such that

$$\int_{-1}^1 A(\varepsilon, \mu_\psi) d\mu_\psi = 2.0 ,$$

where μ_ψ is the cosine of the scattering angle ψ . In the following, we present analytic expressions for the total cross section and the cross-section angular dependence for elastic, inelastic, and ionizing collisions.

3.1 Elastic Collisions

The total cross section as a function of electron energy for elastic collisions with electron energies above 10 keV is given by Murphy as

$$\sigma^T(\varepsilon) = \frac{2\pi z^2 e^4}{\beta^2 p^2 \eta (\eta + 1) (mc^2)^2} , \quad (9)$$

$$\eta = \frac{\chi_0^2}{2} \left[1 + 4\alpha z \chi_0 \left(\frac{1 - \beta^2}{\beta} \log \chi_0 + \frac{0.231}{\beta} + 1.448\beta \right) \right] , \text{ and} \quad (10)$$

$$\chi_0 = \frac{\hbar \tilde{\mu}}{p} \frac{z^{1/3}}{0.885 a_0 mc} . \quad (11)$$

Here z is the average atomic number of air, $\beta = v/c$ is the electron velocity, p is the normalized electron momentum, α is the fine-structure constant, a_0 is the Bohr radius of the electron, and $\tilde{\mu}$ is a constant taken to be 0.635. For cross-section data with electron energies below 10 keV, Murphy cites the data of Phelps[21] and Phelps and Pitchford[22]. These cross sections are used in our relativistic rate calculation without adjustment. A plot of the total elastic cross section as a function of energy is shown in Figure 3.

The angular contribution to the elastic cross section comes from the form

$$A(\varepsilon, \mu_\psi) = \frac{4N\eta_1(1 + \eta_1)}{(1 - \mu_\psi + 2\eta_1)^2} + \frac{4(1 - N)\eta_2(1 + \eta_2)}{(1 + \mu_\psi + 2\eta_2)^2} ,$$

where

$$\begin{aligned} \eta_1 &= 5.77\varepsilon^{-1.377} , \text{ and} \\ \eta_2 &= 2.64\varepsilon^{-0.8964} , \end{aligned}$$

and the energy dependent parameter N is given in Table 1. In the high energy limit this angular dependence becomes a delta function that is peaked in the forward direction.

Figure 3: Total cross section as a function of energy for elastic collisions between an electron and N_2 and O_2 molecules.

3.2 Inelastic Collisions

The total cross sections for vibrational, rotational, and most electronic excitation collisions are again taken from the data of Phelps[21] and Phelps and Pitchford[22]. For Rydberg states we use Murphy's modified fit to the total cross sections given in Banks and Kockarts,

$$\sigma^T(\epsilon) = (q_0 c_0 f_0 / W^2) [1 - (W/\epsilon)^\gamma]^\nu (W/\epsilon)^\Omega ,$$

where $q_0 = 6.51 \times 10^{-14} \text{eV}^2 \text{cm}^2$ and W is the threshold energy for excitation in eV. The cross-section fit parameters are listed in Table 2.

Many of the inelastic cross sections used in our relativistic calculation are not specified or measured above a certain energy and it was necessary to extend the values to higher energy in the following way. At energies above 1000 eV, the rotational cross sections are considered to be negligible. Nonzero vibrational and electron excitation cross sections above 1000 eV are obtained from the relativistic expression for ionization cross sections given by Schram *et al.*[26] and adapted for the inelastic process. This expression can be written as

$$\sigma^T(\epsilon) = \frac{4\pi a_0^2 R}{\frac{1}{2}mc^2} M_I^2 \frac{\gamma^2}{\gamma^2 - 1} [\ln(C_i \frac{1}{2}mc^2) - 1 + \ln(\gamma^2 - 1) + \frac{1}{\gamma^2}] , \quad (12)$$

where we have taken C_i to be $4.0/U$ and U is the excitation energy. Here a_0 is the first Bohr radius of hydrogen, R is the Rydberg energy, M_I^2 is a constant determined by the particular excitation, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the normal relativistic factor. In our implementation of this expression, Eq. (12) is evaluated at the highest nonrelativistic

Table 1: The parameter N for elastic angular dependent cross sections.

Energy (eV)	N
5	0.67
10	0.60
15	0.60
20	0.75
30	0.80
50	0.85
100	0.95
500	1.00
1000	1.00
10000	1.00

Table 2. Cross-section fit parameters for Rydberg electronic excitation total cross sections.

Gas	W	$c_0 f_0$	Ω	ν	γ
N ₂	13.75	2.66	0.75	3.0	1.0
O ₂	13.5	2.77	0.75	3.0	1.0

energy ε_{nr} for which cross-section data are available. Then the cross section at higher energies is evaluated as

$$\sigma^T(\varepsilon_r) = \sigma^T(\varepsilon_{nr})^{\dagger} \frac{\text{tab}}{\text{tab0}} , \quad (13)$$

where

$$\text{tab0} = \frac{\sigma^T(\varepsilon_{nr})}{4\pi a_0^2 R M_I^2 / \frac{1}{2} mc^2} ,$$

and

$$\text{tab} = \frac{\sigma^T(\varepsilon_r)}{4\pi a_0^2 R M_I^2 / \frac{1}{2} mc^2} .$$

The $\dagger$ indicates the cross section obtained from data. Note that the accuracy of the cross sections above 1000 eV obtained through this scheme depends on the accuracy of the cross sections below 1000 eV. Equation (13) allows us to calculate the high-energy cross sections without knowing values for M_I^2 . A plot of the total inelastic cross section for scattering off of N₂ and O₂ is shown in Figure 4 as a function of energy.

The cross-section angular function for rotational and vibrational excitations given by Murphy

$$A(\varepsilon, \mu_\psi) = \frac{15}{14} (1 - 3\mu_\psi^2 + \frac{14}{3}\mu_\psi^4)$$

Figure 4: Total cross section as a function of energy for inelastic collisions between an electron and N_2 and O_2 molecules.

is used for electron energies below 400 eV. For electronic excitations, the following fits were used below 400 eV:

$$\begin{aligned}
 N_2 \quad A^3\Sigma_u^+ : \quad A(\varepsilon, \mu_\psi) &= 0.207 + \frac{0.138}{0.25 + (\frac{\pi}{2} - \psi)^2} + 3.244e^{1.5(\psi - \pi)} , \\
 N_2 \quad B^3\Pi_g : \quad A(\varepsilon, \mu_\psi) &= \frac{1.374}{0.8 + (\frac{3\pi}{4} - \psi)^2} , \\
 N_2 \quad a^1\Pi_g : \quad A(\varepsilon, \mu_\psi) &= 0.2947 + 9.5286e^{-2.4\psi} , \text{ and} \\
 N_2 \quad C^3\Pi_u : \quad A(\varepsilon, \mu_\psi) &= \frac{8.2712}{4.0 + (\pi - \psi)^2} - 0.3102 .
 \end{aligned}$$

The angular behavior of the cross sections for the Rydberg states are given for energies below 400 eV by

$$A(\varepsilon, \mu_\psi) = 1.0 + \sum_{i=1}^5 c_i P_i(\mu_\psi) ,$$

where

$$c_1 = c_2 = \frac{\varepsilon^2}{2500 + \varepsilon^2} ,$$

and

$$c_3 = c_4 = c_5 = \frac{\varepsilon^2}{3500 + \varepsilon^2} .$$

For inelastic collisions with electron energies above 400 eV, we use the function $A(\varepsilon, \mu_\psi)$ given for the angular dependence of the elastic cross sections. Inelastic collisions that have not been mentioned here are considered to be isotropic.

3.3 Ionizing Collisions

Collision cross sections for ionizing collisions with incident electron energies below 15 keV are taken from Rapp and Englander-Golden[23] and Opal, Beaty, and Peterson[24, 25] as tabulated in Murphy. Murphy cites the following relativistic expression for ionizing collisions with air molecules in which the incident electron energy is above 15 keV:

$$\sigma^T(\varepsilon) = 5.94 \times 10^{-20} \frac{\gamma^2}{\gamma^2 - 1} [8.68 + \log(\gamma^2 - 1) + \frac{1}{\gamma^2}] .$$

This expression was obtained by adjusting Longmire and Longley's[27] version of Bethe's[28] relativistic cross-section expression to fit smoothly with the nonrelativistic cross-section data. The fact that Bethe's expression does not match the nonrelativistic data at 15 keV suggests that some inaccuracy may exist in the nonrelativistic data or in Bethe's formula in this energy regime. This problem is left for future study. We have extracted the individual ionization cross sections for N_2 and O_2 from Murphy's expression to give the following formulas:

$$N_2 : \quad \sigma^T(\varepsilon) = 5.647 \times 10^{-20} \frac{\gamma^2}{\gamma^2 - 1} [8.79 + \log(\gamma^2 - 1) + \frac{1}{\gamma^2}] , \text{ and}$$

$$O_2 : \quad \sigma^T(\varepsilon) = 6.967 \times 10^{-20} \frac{\gamma^2}{\gamma^2 - 1} [8.51 + \log(\gamma^2 - 1) + \frac{1}{\gamma^2}] .$$

Plots of the total cross section for ionizing collisions between an incident electron and N_2 and O_2 molecules are given in Figures 5 and 6 respectively.

The angular dependence of the ionizing collision cross sections is separated into two parts,

$$A(\varepsilon, \mu_\psi) = B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) ,$$

where ε_S is the secondary electron energy. Murphy gives $B(\varepsilon, \varepsilon_S)$ as

$$B(\varepsilon, \varepsilon_S) = \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \begin{cases} \frac{1}{1 + \left(\frac{\varepsilon_S}{a}\right)^{2.1}} & \varepsilon_S \leq \frac{\varepsilon - \varepsilon_I}{2} \\ \frac{1}{1 + \left(\frac{\varepsilon - \varepsilon_I - \varepsilon_S}{a}\right)^{2.1}} & \frac{\varepsilon - \varepsilon_I}{2} < \varepsilon_S \leq \varepsilon - \varepsilon_I , \end{cases}$$

where ε_I is the ionization threshold energy. For nitrogen, $a = 13.0$ eV and $\varepsilon_I = 15.6$ eV; for oxygen, $a = 17.25$ eV and $\varepsilon_I = 12.6$ eV. The normalization factor for the function $B(\varepsilon, \varepsilon_S)$ arises from the integral

$$\int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} = a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right) .$$

The power 2.1 fits the energy dependent form of the data and leads to a normalization slightly different from one.

Because only a limited amount of data are available for the angular dependence of the ionization cross section, Murphy splits the function $\mathcal{A}$ into two terms: one term for secondary electron energies less than half the energy of incident electrons, and another

Figure 5: Total cross section as a function of energy for ionizing collisions between an electron and N_2 molecules.

Figure 6: Total cross section as a function of energy for ionizing collisions between an electron and O_2 molecules.

term for secondary electron energies greater than half the incident electron energy. For secondary electrons with energies less than half the incident energy, the function is

$$\mathcal{A}(\varepsilon_S, \mu_\psi) = N_1 + N_2 e^{-\frac{(\psi-c)^2}{w^2}} , \quad (14)$$

where N_1 , N_2 , c , and w are energy-dependent parameters given in Table 3 and derived to fit the available data. For secondary electrons with energies greater than half the incident energy, Murphy uses the angular fit for elastic scattering given in Section 5.1. Because of the large electron energies involved in our relativistic calculation, we use Eq. (14) alone for secondary electron energies less than 200 eV. For higher secondary electron energies, we interpolate between the result from Eq. (14) and the result of the elastic scattering fit. Note that the function $\mathcal{A}(\varepsilon_S, \mu_\psi)$ is normalized to 2.0.

Table 3. Energy-dependent parameters for calculating the angular term $\mathcal{A}$ for ionizing collisions.

ε (eV)	N_1	N_2	c	w
4.13	1.0928	-0.1366	1.80	1.0
20.1	0.7682	0.5122	0.70	0.9
48.5	0.5988	0.8982	0.80	0.8
98.0	0.4546	1.3636	0.80	0.7
196.0	0.3140	2.1020	0.90	0.5

4 Relativistic Collision Rates

The relativistic collision (transfer) rates are obtained by taking appropriate moments of the Boltzmann collision integral, and summing contributions from elastic, inelastic, and ionizing collisions, Eqs. (3), (4), and (8) respectively. The particle production rate R_c , the momentum transfer rate R_m , the thermal energy loss rate R_E , and the total energy loss rate R_L are

$$R_c(\beta_o, T') n_e = \int [\Gamma_{in} - \Gamma_{out}] d^3p , \quad (15)$$

$$R_m(\beta_o, T') n_e \langle \vec{p} \rangle = - \int [\Gamma_{in} - \Gamma_{out}] \vec{p} d^3p , \quad (16)$$

$$R_E(\beta_o, T') n_e \langle T' \rangle = \int [\Gamma_{in} - \Gamma_{out}] \left(\sqrt{\vec{p}^2 + 1} - 1 \right) mc^2 d^3p , \text{ and } (17)$$

$$R_L(\beta_o, T') n_e \langle \gamma - 1 \rangle mc^2 = \int [\Gamma_{in} - \Gamma_{out}] \left(\sqrt{\vec{p}^2 + 1} - 1 \right) mc^2 d^3p , \quad (18)$$

where $\Gamma_{in} - \Gamma_{out}$ is the sum of the elastic, inelastic, and ionizing collision terms. Here n_e and $\langle \vec{p} \rangle$ are the electron density and normalized mean momentum, respectively. The total electron energy is $\langle \gamma - 1 \rangle mc^2$ and $\langle T' \rangle$ is the mean electron thermal energy (the relativistic kinetic energy in the reference frame that is moving at the mean electron

velocity β_o). The momentum $\tilde{p}$ is defined as the electron momentum in this same reference frame. The energy $(\sqrt{\tilde{p}^2 + 1} - 1)mc^2$ is transformed to the laboratory frame as

$$(\sqrt{\tilde{p}^2 + 1} - 1) = (\gamma_o \sqrt{p^2 + 1} - \gamma_o \beta_o p \mu - 1) \quad .$$

Note that whereas R_L takes into account mean energy loss to the neutral gas molecules, R_E accounts for the thermal energy losses to the gas and energy losses caused by redirected electron momentum.

In the calculations of relativistic collision rates discussed here we make use of the relativistic equivalent of the Maxwellian velocity distribution

$$f(\mu, p) = C \exp \left[-\frac{3mc^2}{2T'} \gamma_o \left(\sqrt{p^2 + 1} - \beta_o p \mu \right) \right] \quad ,$$

where $\vec{\beta}_o$ is the mean electron velocity, and

$$\gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}} \quad .$$

The constant C is

$$C = \frac{3mc^2}{8\pi T'} \frac{n_e}{\gamma_o K_2 \left(\frac{3mc^2}{2T'} \right)} \quad ,$$

where K_2 is the second-order modified Bessel function of the second kind. In these equations, the mean thermal energy T' is assumed equal to $\frac{3}{2}k\tilde{T}$, where k is Boltzmann's constant and $\tilde{T}$ is the mean electron temperature. This momentum distribution function, called the Jüttner distribution, satisfies the relativistic H-theorem for the case when no external forces are applied to the system[29, 30]. It reduces in the nonrelativistic limit to the familiar Maxwellian velocity distribution.

In this section, we present certain simplifications to the angular integrals necessary to obtain the final expressions for the transfer rates. Detailed derivations of the transfer rates are presented in Appendix A while the final expressions are given in Section 4.2.

4.1 Angular Integrals

The angular integrals that need to be evaluated in order to calculate the rates of Eqs. (15), (16), and (17) have the forms

$$\mathcal{I}_n = \int f(p, \mu) \sigma(p, \mu_\psi) \mu^n d\mu_\psi d\alpha d\mu d\varphi \quad ,$$

and

$$\mathcal{I}'_n = \int f(p', \mu') \sigma(p', \mu_\psi) (\mu)^n d\mu_\psi d\alpha d\mu d\varphi \quad ,$$

where $\mu_\psi = \cos \psi$ and $n = 0, 1$.

In the previous relativistic collision rate calculation, the cross sections σ were considered to be independent of angle, and the angular integrals could be evaluated analytically. For the case of angular dependent cross sections, the integral over μ_ψ must be evaluated numerically. The other angular integrals can be evaluated analytically provided the integrals over μ in $\mathcal{I}'_n$ can be converted to integrals over μ' . The conversion of these angular

integrals, identical to the nonrelativistic case given by Roussel-Dupré¹, is presented in this section.

It will be useful to first determine expressions for μ and $\frac{\partial \mu}{\partial \mu'}$ in terms of μ_ψ and μ' . From the expression

$$\mu' = \mu \cos \psi + (1 - \mu^2)^{\frac{1}{2}} \sin \psi \sin \alpha \quad ,$$

we find

$$\mu = \frac{\mu' \mu_\psi}{\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha} \pm \frac{(1 - \mu_\psi^2)^{\frac{1}{2}} \sin \alpha \sqrt{\mu_\psi^2 - \mu'^2 + (1 - \mu_\psi^2) \sin^2 \alpha}}{\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha} \quad .$$

The derivative is

$$\frac{\partial \mu}{\partial \mu'} = \frac{\mu_\psi}{\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha} \mp \frac{(1 - \mu_\psi^2)^{\frac{1}{2}} \mu' \sin \alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha)(\mu_\psi^2 - \mu'^2 + (1 - \mu_\psi^2) \sin^2 \alpha)^{\frac{1}{2}}} \quad .$$

First consider the integral

$$\begin{aligned} \mathcal{I}'_0 &= \int f(p', \mu') \sigma(p', \mu_\psi) d\mu_\psi d\alpha d\mu d\varphi = \int \frac{\partial \mu}{\partial \mu'} d\mu' \int d\mu_\psi \int d\alpha \int d\varphi f(p', \mu') \sigma(p', \mu_\psi) \\ &= 2\pi \int d\mu' \int d\mu_\psi f(p', \mu') \sigma(p', \mu_\psi) \int \frac{\partial \mu}{\partial \mu'} d\alpha \quad . \end{aligned}$$

Using the expression for $\frac{\partial \mu}{\partial \mu'}$, we obtain

$$\begin{aligned} \int \frac{\partial \mu}{\partial \mu'} d\alpha &= \mu_\psi \int_0^{2\pi} \frac{d\alpha}{\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha} \\ &\mp (1 - \mu_\psi^2)^{\frac{1}{2}} \mu' \int_0^{2\pi} \frac{\sin \alpha d\alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha) \sqrt{\mu_\psi^2 - \mu'^2 + (1 - \mu_\psi^2) \sin^2 \alpha}} \quad . \end{aligned}$$

The first part of the integral is

$$4\mu_\psi \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\mu_\psi^2 \cos^2 \alpha + \sin^2 \alpha} = 4\mu_\psi \frac{\pi}{2\mu_\psi} = 2\pi \quad .$$

The second part is

$$\begin{aligned} \mp (1 - \mu_\psi^2)^{\frac{1}{2}} \mu' \left\{ \int_0^\pi \frac{\sin \alpha d\alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha) \sqrt{\mu_\psi^2 - \mu'^2 + (1 - \mu_\psi^2) \sin^2 \alpha}} \right. \\ \left. - \int_0^\pi \frac{\sin \alpha' d\alpha'}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha') \sqrt{\mu_\psi^2 - \mu'^2 + (1 - \mu_\psi^2) \sin^2 \alpha'}} \right\} = 0 \quad , \end{aligned}$$

where $\alpha' = \alpha - \pi$. Therefore

$$\mathcal{I}'_0 = \int f(p', \mu') \sigma(p', \mu_\psi) d\mu_\psi d\alpha d\mu d\varphi = 4\pi^2 \int d\mu' \int d\mu_\psi f(p', \mu') \sigma(p', \mu_\psi) \quad . \quad (19)$$

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¹R. A. Roussel-Dupré, private communication.

Next consider the integral

$$\begin{aligned}
\mathcal{I}'_1 &= \int \mu f(p', \mu') \sigma(p', \mu_\psi) d\mu_\psi d\alpha d\mu d\varphi \\
&= \int \mu \frac{\partial \mu}{\partial \mu'} d\mu' \int d\mu_\psi \int d\alpha \int d\varphi f(p', \mu') \sigma(p', \mu_\psi) \\
&= 2\pi \int d\mu' \int d\mu_\psi f(p', \mu') \sigma(p', \mu_\psi) \int \frac{\partial \mu}{\partial \mu'} \mu d\alpha .
\end{aligned}$$

Using the expressions for $\frac{\partial \mu}{\partial \mu'}$ and μ we obtain

$$\int \frac{\partial \mu}{\partial \mu'} \mu d\alpha = \int_0^{2\pi} \frac{\mu' \mu_\psi^2 d\alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha)^2} - \int_0^{2\pi} \frac{(1 - \mu_\psi^2) \mu' \sin^2 \alpha d\alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha)^2} .$$

The first half of the integral is

$$\begin{aligned}
4\mu' \mu_\psi^2 \int_0^{\pi/2} \frac{d\alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha)^2} \\
= 4\mu' \mu_\psi^2 \frac{-\partial}{\partial a} \int_0^{\pi/2} \frac{d\alpha}{(a \cos^2 \alpha + (a + b) \sin^2 \alpha)} = \frac{\pi \mu' (1 + \mu_\psi^2)}{\mu_\psi} ,
\end{aligned}$$

where $a = \mu_\psi^2$ and $b = 1 - \mu_\psi^2$. The final result was obtained from Gradshteyn and Ryzhik[31].

The second part of the integral is

$$\begin{aligned}
-4(1 - \mu_\psi^2) \mu' \int_0^{\pi/2} \frac{\sin^2 \alpha d\alpha}{(\mu_\psi^2 + (1 - \mu_\psi^2) \sin^2 \alpha)^2} \\
= -4(1 - \mu_\psi^2) \mu' \frac{-\partial}{\partial b} \int_0^{\pi/2} \frac{\sin^2 \alpha d\alpha}{(a \cos^2 \alpha + (a + b) \sin^2 \alpha)} = -\frac{\pi(1 - \mu_\psi^2) \mu'}{\mu_\psi} .
\end{aligned}$$

Combining these results gives

$$\mathcal{I}'_1 = \int \mu f(p', \mu') \sigma(p', \mu_\psi) d\mu_\psi d\alpha d\mu d\varphi = 4\pi^2 \int d\mu' \mu' \int d\mu_\psi \mu_\psi f(p', \mu') \sigma(p', \mu_\psi) . \quad (20)$$

4.2 Summary of Collision Rates

The derivation of the relativistic collision rates is presented in Appendix A. Combining the elastic, inelastic, and ionization rate results of that appendix, we have to order m/M :

$$\begin{aligned}
R_{c|io}(\beta_o, T') n_e &= 4\pi^2 \mathcal{C} \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \sum_I N_{Ic} \int dp p \beta \\
&\times \left[e^{-\frac{3mc^2}{2T'} \gamma_o [\sqrt{p^2+1} - \beta_o p]} - e^{-\frac{3mc^2}{2T'} \gamma_o [\sqrt{p^2+1} + \beta_o p]} \right] \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_S \sigma_I(\epsilon, \epsilon_S, \mu_\psi) .
\end{aligned} \quad (21)$$

$$\begin{aligned}
R_m(\beta_o, T') n_e | \vec{p} | &= -4\pi^2 \mathcal{C} \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) c \int dp p^2 \\
&\times \left[\left(1 - \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2+1} - \beta_o p)} + \left(1 + \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2+1} + \beta_o p)} \right] \\
&\times \left[N \int d\mu_\psi (\mu_\psi - 1) \sigma_e(p, \mu_\psi) \left(\beta + \frac{m}{M} p \mu_\psi \right) \right. \\
&+ \sum_h N_h \left(\int d\mu_\psi (\mu_\psi - 1) \sigma_h(p, \mu_\psi) \left(\beta + \frac{m}{M} p \mu_\psi \right) - \frac{T_h}{mc^2} \frac{1}{p} \int d\mu_\psi \mu_\psi \sigma_h(p, \mu_\psi) \right) \\
&+ \sum_I N_I \frac{1}{p} \left(\beta \int d\mu_\psi \mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S - \frac{\epsilon_I}{mc^2} \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right. \\
&\left. \left. - \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \frac{\epsilon_S}{mc^2} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S - \frac{1}{2} \left(\frac{m}{M} + \frac{1}{\gamma} \right) \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S^2 \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right) \right]
\end{aligned} \tag{22}$$

$$\begin{aligned}
R_E(\beta_o, T') n_e < T' > &= 4\pi^2 \mathcal{C} \left(\frac{2T'}{3} \right) c \left\{ \int dp p^2 \right. \\
&\times \left[\left(1 - \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2+1} - \beta_o p)} + \left(1 + \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2+1} + \beta_o p)} \right] \\
&\times \left[N \int d\mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) \left(\beta + \frac{m}{M} p \mu_\psi \right) \right. \\
&- \sum_h N_h \left(\int d\mu_\psi (\mu_\psi - 1) \sigma_h(p, \mu_\psi) \left(\beta + \frac{m}{M} p \mu_\psi \right) - \frac{T_h}{mc^2} \frac{1}{p} \int d\mu_\psi \mu_\psi \sigma_h(p, \mu_\psi) \right) \\
&- \sum_I N_I \frac{1}{p} \left(\beta \int d\mu_\psi \mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S - \frac{\epsilon_I}{mc^2} \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right. \\
&- \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \frac{\epsilon_S}{mc^2} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S - \frac{1}{2} \left(\frac{m}{M} + \frac{1}{\gamma} \right) \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S^2 \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \left. \right) \left. \right] \\
&+ \int dp p^2 \beta \left[e^{-\frac{3m\epsilon^2}{2T'} \gamma_o [\sqrt{p^2+1} + \beta_o p]} - e^{-\frac{3m\epsilon^2}{2T'} \gamma_o [\sqrt{p^2+1} - \beta_o p]} \right] \\
&\left[-N \frac{m}{M} \frac{p}{\beta_o} \int d\mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) - \sum_h N_h \left(\frac{T_h}{\beta_o p m c^2} \int d\mu_\psi \sigma_h(p, \mu_\psi) \right. \right. \\
&+ \left. \frac{m}{M} \frac{p}{\beta_o} \int d\mu_\psi (1 - \mu_\psi) \sigma_h(p, \mu_\psi) \right) \\
&+ \sum_I N_I \frac{1}{\beta_o p} \left(\left(1 - \frac{1}{\gamma_o} - \frac{\epsilon_I}{mc^2} \right) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right. \\
&\left. \left. - \frac{1}{2} \frac{m}{M} \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S^2 \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right) \right] \left. \right\} .
\end{aligned} \tag{23}$$

$$\begin{aligned}
R_L(\beta_o, T') n_e < \gamma - 1 > mc^2 = 4\pi^2 C \left(\frac{2T'}{3\gamma_o\beta_o} \right) c \int dp p^2 \beta \\
\left[e^{-\frac{3m\epsilon^2}{2T'}} \gamma_o [\sqrt{p^2+1} + \beta_o p] - e^{-\frac{3m\epsilon^2}{2T'}} \gamma_o [\sqrt{p^2+1} - \beta_o p] \right] \left[-N \frac{m}{M} p \int d\mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) \right. \\
- \sum_h N_h \left(\frac{T_h}{pmc^2} \int d\mu_\psi \sigma_h(p, \mu_\psi) + \frac{m}{M} p \int d\mu_\psi (1 - \mu_\psi) \sigma_h(p, \mu_\psi) \right) \\
+ \sum_I N_I \frac{1}{p} \left(-\frac{\epsilon_I}{mc^2} \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_s, \mu_\psi) d\epsilon_s \right. \\
\left. \left. - \frac{1}{2} \frac{m}{M} \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_s^2 \sigma_I(\epsilon, \epsilon_s, \mu_\psi) d\epsilon_s \right) \right] .
\end{aligned} \tag{24}$$

5 Computational Methods

The solution of the integrals of Eqs. (21) – (23) using the cross sections outlined in Section 3 yields the relativistic collision rates as a function of electron mean kinetic energy and electron thermal energy. The accuracy of these collision rates depends in part on the numerical methods used to obtain values for the cross-section integrals and the final integrals over momentum. Presented here is a discussion of these numerical calculations.

5.1 Cross Section

The cross-section integrals for the collision rate calculation are needed over a wide range of energies. Unfortunately the large computation time necessary to calculate cross-section integrals at each energy where they are needed is prohibitive. In order to obtain accurate integrals in a reasonable amount of time, the integrals are calculated differently for electron energies above and below 3.0 MeV. For the low-energy portion of this interval, integrals are calculated for a fixed set of energies and stored in a table. As cross-section integrals are needed for a particular energy in this lower range, they are interpolated from the table. High-energy cross-section integrals are calculated for a particular energy as they are needed.

The energy grid for the calculation of the lower energy cross-sections integrals is divided in three parts between 0 eV and about 35 MeV (the energy equivalent of .9999 times the speed of light). The integrals for a particular grid point in the lower range are calculated as an average of cross-section integrals between adjacent grid points. The range from 0 to 1 eV is evenly divided in velocity by 17 points. Between 1 eV and the rest energy of an electron (mc^2), 600 points evenly divide the log-velocity range. Lastly, the log-velocity range between mc^2 and .9999c is evenly divided by 15 points. This irregular grid for the calculation of cross-section integrals allows the structure of the cross sections at low and intermediate energies to be resolved yet provides an adequate distribution of cross section integrals at high energies where small changes in velocity result in much larger changes in energy.

Although some of the cross-section integrals needed to compute the relativistic collision rates can be evaluated analytically, others must be approximated. Presented here are the methods used for evaluating the integrals of the relativistic cross sections for elastic, inelastic, and ionizing collisions.

5.1.1 Elastic Collisions

The angular integrals of the elastic cross section given in Section 3.1 are evaluated analytically and used in the calculation of rates for elastic, inelastic, and ionizing collision rates. These angular integrals are

$$\int d\mu_\psi A(\varepsilon, \mu_\psi) = 2 \quad ,$$

$$\int d\mu_\psi \mu_\psi A(\varepsilon, \mu_\psi) = 4N\eta_1(1+\eta_1) \left\{ \left[\frac{1}{2} \frac{1+2\eta_1}{\eta_1(\eta_1+1)} \right] + \log \left(\frac{\eta_1}{(1+\eta_1)} \right) \right\} \quad (25)$$

$$-4(1-N)\eta_2(1+\eta_2) \left\{ \left[\frac{1}{2} \frac{1+2\eta_2}{\eta_2(\eta_2+1)} \right] + \log \left(\frac{\eta_2}{(1+\eta_2)} \right) \right\} \quad ,$$

and

$$\int d\mu_\psi \mu_\psi^2 A(\varepsilon, \mu_\psi) = 8N\eta_1(1+\eta_1) \left\{ 1 + \left[\frac{1}{4} \frac{(1+2\eta_1)^2}{\eta_1(1+\eta_1)} \right] + (1+2\eta_1) \log \left(\frac{\eta_1}{(1+\eta_1)} \right) \right\} \quad (26)$$

$$+8(1-N)\eta_2(1+\eta_2) \left\{ 1 + \left[\frac{1}{4} \frac{(1+2\eta_2)^2}{\eta_2(1+\eta_2)} \right] + (1+2\eta_2) \log \left(\frac{\eta_2}{(1+\eta_2)} \right) \right\} \quad ,$$

where $A(\varepsilon, \mu_\psi)$, N , η_1 , and η_2 are defined in Section 3.

5.1.2 Inelastic Collisions

Similar angular integrals need to be evaluated for inelastic collision rates. However, in this case we use different angular cross-section functions $A(\varepsilon, \mu_\psi)$, depending on the energy of the incident electrons. Each of these angular functions obeys the standard normalization $\int d\mu_\psi A(\varepsilon, \mu_\psi) = 2$.

For incident electron energies below 400 eV, Gaussian integration is used to evaluate the remaining angular integrals. That is,

$$\int_{-1}^1 d\mu_\psi \mu_\psi^l A(\varepsilon, \mu_\psi) = \int_{-1}^1 d\mu f_l(\mu) = \sum_{i=1}^n w_i f_l(\mu_i) \quad ,$$

where w_i are Gaussian weight factors, and μ_i are the zeros of the Legendre polynomials[32]. Here $f_l(\mu) = \mu_\psi^l A(\varepsilon, \mu_\psi)$, $l = 1, 2$ is calculated using the appropriate angular cross-section functions $A(\varepsilon, \mu_\psi)$ given in Section 3. Above 400 eV, the angular cross-section function is taken to be the elastic angular function, and the angular integration result given in the previous section is used.

5.1.3 Ionizing Collisions

To calculate the particle, momentum, and energy transfer rates for ionizing collisions, the following integrals are needed:

$$\int d\mu_\psi \int_0^{\frac{\varepsilon-\varepsilon_I}{2}} \sigma_I d\varepsilon_S = \frac{\sigma^T(\varepsilon)}{4\pi} \int d\mu_\psi \int_0^{\frac{\varepsilon-\varepsilon_I}{2}} B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \quad , \quad (27)$$

$$\int d\mu_\psi \mu_\psi \int_0^{\frac{\varepsilon-\varepsilon_I}{2}} p_S \sigma_I d\varepsilon_S = \frac{\sigma^T(\varepsilon)}{4\pi} \int d\mu_\psi \mu_\psi \int_0^{\frac{\varepsilon-\varepsilon_I}{2}} p_S B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \quad , \quad (28)$$

$$\int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon-\varepsilon_I}{2}} p_S^2 \sigma_I d\varepsilon_S = \frac{\sigma^T(\varepsilon)}{4\pi} \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon-\varepsilon_I}{2}} r_S^2 B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \quad , \quad (29)$$

and

$$\int d\mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S}{mc^2} \sigma_I d\varepsilon_S = \frac{\sigma^T(\varepsilon)}{4\pi} \int d\mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S}{mc^2} B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S, \quad (30)$$

where ε_S is the secondary electron energy, and $\mathcal{A}$ and B are given in Section 3. These integrals are calculated numerically in our code and entered into a table. For fixed ε_S , $\int d\mu_\psi \mathcal{A}(\varepsilon_S, \mu_\psi) = 2$. Below 200 eV, $\int d\mu_\psi \mu_\psi^l \mathcal{A}(\varepsilon_S, \mu_\psi)$ is calculated by Gaussian integration with $f = \mu_\psi^l \mathcal{A}$. Above 200 eV we interpolate between the Gaussian integration result and the result for elastic collisions. The integration over the secondary electron energy is a simple sum with the energy increment $d\varepsilon_S$ determined by the grid on which the cross sections are calculated.

The direct calculation of cross-section integrals for energies above 3.0 MeV adds significantly to the run time of our code. To shorten this run time, approximations for the integrals, Eqs. (28 – 30), are used.

For Eq. (30), the integral over $d\mu_\psi$ simply integrates to 2, whereas the integral over $d\varepsilon_S$ can be approximated as follows:

$$\begin{aligned} & \int d\mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S}{mc^2} B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \\ &= 2 \left[\frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \frac{1}{mc^2} \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \right] \\ &\approx 2 \left[\frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \frac{1}{mc^2} \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \right]. \end{aligned}$$

Completing the integration yields the result

$$\int d\mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S}{mc^2} B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S = \frac{a}{\arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right) mc^2} \log \left(1 + \left(\frac{\varepsilon - \varepsilon_I}{2a} \right)^2 \right). \quad (31)$$

The expression for Eq. (28) is not so neat. The angular integral is complicated by the factor μ_ψ , and the secondary energy integral by the momentum p_S . We divide this integral into four separate terms and use the following approximations:

$$\varepsilon_S \ll mc^2 \implies p_S = \sqrt{\left(\frac{\varepsilon_S}{mc^2} + 1\right)^2 - 1} \approx \sqrt{\frac{2\varepsilon_S}{mc^2}},$$

$$\varepsilon_S \gg a \implies 1 + \left(\frac{\varepsilon_S}{a}\right)^2 \approx \left(\frac{\varepsilon_S}{a}\right)^2, \text{ and}$$

$$\varepsilon_S \gg mc^2 \implies p_S = \frac{\varepsilon_S}{mc^2} \sqrt{1 + \frac{2mc^2}{\varepsilon_S}} \approx \frac{\varepsilon_S}{mc^2} + 1.$$

If we approximate the exponent in the denominator as 2.0, the integral is

$$\begin{aligned}
& \int d\mu_\psi \mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} p_S B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \\
&= \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi \mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \mathcal{A}(\varepsilon_S, \mu_\psi) \frac{p_S d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \\
&\approx \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi \mu_\psi \int_0^{500} \mathcal{A}(\varepsilon_S, \mu_\psi) \frac{p_S d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \quad (32)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi \mu_\psi \int_{500}^{\frac{mc^2}{10}} \mathcal{A}(\varepsilon_S, \mu_\psi) \frac{\sqrt{\frac{2\varepsilon_S}{mc^2}} d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \quad (33)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi \mu_\psi \int_{\frac{mc^2}{10}}^{\frac{\varepsilon - \varepsilon_I}{2}} \mathcal{A}(\varepsilon_S, \mu_\psi) \frac{\sqrt{\frac{2\varepsilon_S}{mc^2}} \sqrt{\frac{\varepsilon_S}{2mc^2} + 1}}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} d\varepsilon_S \quad (34)
\end{aligned}$$

Integral (32) is evaluated numerically with a small effect on the run time of the code. The remaining two terms are approximated by evaluating the function $\mathcal{A}(\varepsilon_S, \mu_\psi)$ at the lower limits of the respective secondary energy integrals. The angular integral may then be separated from the energy integral. Integral (33) is integrated to give

$$\begin{aligned}
& \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi \mu_\psi \int_{500}^{\frac{mc^2}{10}} \mathcal{A}(\varepsilon_S, \mu_\psi) \frac{\sqrt{\frac{2\varepsilon_S}{mc^2}} d\varepsilon_S}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \quad (35) \\
& \approx \frac{\sqrt{a} \int d\mu_\psi \mu_\psi \mathcal{A}(500, \mu_\psi)}{\arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right) \sqrt{mc^2}} \left[-\ln\left(\frac{mc^2/10a + \sqrt{2mc^2/10a + 1}}{(1 + (mc^2/10a)^2)^{1/2}}\right) \right. \\
& \quad \left. + \arctan\frac{\sqrt{2mc^2/10a}}{1 - mc^2/10a} + \ln\left(\frac{500/a + \sqrt{1000/a + 1}}{(1 + (500/a)^2)^{1/2}}\right) - \arctan\frac{\sqrt{1000/a}}{1 - 500/a} \right] .
\end{aligned}$$

Integral (34) is found to be

$$\begin{aligned}
& \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi \mu_\psi \int_{\frac{mc^2}{10}}^{\frac{\varepsilon - \varepsilon_I}{2}} \mathcal{A}(\varepsilon_S, \mu_\psi) \frac{\sqrt{\frac{2\varepsilon_S}{mc^2}} \sqrt{\frac{\varepsilon_S}{2mc^2} + 1}}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} d\varepsilon_S \quad (36) \\
& = \frac{2a \int d\mu_\psi \mu_\psi \mathcal{A}(mc^2/10, \mu_\psi)}{mc^2 \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \left[-\frac{\sqrt{1 + (\varepsilon - \varepsilon_I)/4mc^2}}{\sqrt{(\varepsilon - \varepsilon_I)/4mc^2}} \right. \\
& \quad \left. + \ln\left(2\sqrt{(\varepsilon - \varepsilon_I)/4mc^2} + 2\sqrt{1 + (\varepsilon - \varepsilon_I)/4mc^2}\right) + \frac{\sqrt{1 + 1/20}}{\sqrt{1/20}} \right. \\
& \quad \left. - \ln\left(2\sqrt{1/20} + 2\sqrt{1 + 1/20}\right) \right] .
\end{aligned}$$

Equation (29) can be divided into the following two parts:

$$\begin{aligned} \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} p_S^2 B(\varepsilon, \varepsilon_S) \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S &= \frac{1}{a \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right)} \int d\mu_\psi (1 - \mu_\psi^2) \\ &\times \left\{ \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S^2 / (mc^2)^2}{1 + \left(\frac{\varepsilon_S}{a}\right)^{2.1}} \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S + \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S / mc^2}{1 + \left(\frac{\varepsilon_S}{a}\right)^{2.1}} \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \right\}, \end{aligned}$$

where the momentum is expressed as $p_S^2 = \varepsilon_S^2 / (mc^2)^2 + 2\varepsilon_S / mc^2$. The first part is approximately

$$\begin{aligned} \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S^2 / (mc^2)^2}{1 + \left(\frac{\varepsilon_S}{a}\right)^{2.1}} \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S &\approx \int_0^{500} \frac{\varepsilon_S^2 / (mc^2)^2}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \\ &+ \frac{a^3}{(mc^2)^2} \mathcal{A}(\varepsilon_{500}, \mu_\psi) \left[\frac{\varepsilon - \varepsilon_I}{2a} - \frac{500}{a} + \arctan\left(\frac{500}{a}\right) - \arctan\left(\frac{\varepsilon - \varepsilon_I}{2a}\right) \right]. \end{aligned} \quad (37)$$

Similarly the second part can be approximated as

$$\begin{aligned} \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \frac{\varepsilon_S / mc^2}{1 + \left(\frac{\varepsilon_S}{a}\right)^{2.1}} \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S &\approx \int_0^{500} \frac{\varepsilon_S / mc^2}{1 + \left(\frac{\varepsilon_S}{a}\right)^2} \mathcal{A}(\varepsilon_S, \mu_\psi) d\varepsilon_S \\ &+ \frac{a^2}{2mc^2} \mathcal{A}(\varepsilon_{500}, \mu_\psi) \log \left\{ \left(\frac{\varepsilon - \varepsilon_I}{2a} + 1 \right) / \left(\frac{500}{a} + 1 \right) \right\}. \end{aligned} \quad (38)$$

The remaining integrals on the right sides of Eqs. (37) and (38) are evaluated numerically.

5.2 Collision Rates

Despite the complexity of Eqs. (21) – (23), the momentum integration of these equations is straightforward. Nevertheless a few points are worth mentioning.

The upper limit to the momentum for the integrals is determined using the assumption

$$\frac{f(1, p) e^{\frac{3mc^2}{2T'}}}{C} = e^{-\frac{3mc^2}{2T'} [\gamma_o (\sqrt{p^2 + 1} - \beta_o p) - 1]} = e^{-16} \quad (39)$$

and solving for p . The integration grid over momentum is obtained by dividing the velocity range (not momentum) by the desired number of points and transforming to a momentum grid. Whereas this scheme results in an evenly spaced velocity grid, it results in a momentum grid that is weighted toward a larger number of points at lower energy, where they are needed.

At relativistic energies, the integration grid is sparse and more points are needed. In our code, the step size in the relativistic range is chosen from the smallest of three values. These values are based on the maximum derivative of the momentum $p = \gamma\beta$, the relativistic cross section

$$\sigma^T(\varepsilon) = \frac{4\pi a_0^2 R}{\frac{1}{2}mc^2} M_I^2 \frac{\gamma^2}{\gamma^2 - 1} \left[\ln(C_i \frac{1}{2}mc^2) - 1 + \ln(\gamma^2 - 1) + \frac{1}{\gamma^2} \right],$$

and the relativistic momentum distribution function

$$f(\mu, p) = A \exp \left[-\frac{3mc^2}{2T'} \gamma_o \left(\sqrt{p^2 + 1} - \beta_o p \mu \right) \right].$$

6 Results

The ionization, momentum transfer, energy transfer, and total energy loss rates resulting from the solution of Eqs. (21)–(24) for elastic, inelastic, and ionizing collisions are plotted in Appendix B. Our calculation of these rates is the first electron-air collision rate calculation that has been carried out to high energy as a function of both the thermal velocity and mean velocity of the electron distribution function. The need for this two-dimensional analysis is made apparent by the comparison of the momentum transfer rate curves in Figures B-6 and B-7. In Figure B-6 the vibrational excitation peak at about 2 eV is washed out because of the effect of the distribution of electron thermal energies. However, in Figure B-7, where the thermal energy was fixed at the low value of $7.1(10^{-3})$ eV, the peak at 2 eV is very sharp. This washing out of features can be seen in several other plots of rates as a function of thermal energy (Figures B-1, B-11, and B-16).

The behavior of the ionization rate at high energy (above .5 MeV) is due to the relativistic effects on the electric field of the incident electron. When the incident electron reaches relativistic energies, the electric field perpendicular to the direction of motion of the electron increases by the relativistic factor γ . This increased field results in an increase in the ionization rate at relativistic energies greater than 1.5 MeV (Figures B-1, B-2, and B-5).

A modification of Bethe's[28] relativistic cross-section expression is used in our code. These modifications were made so that the nonrelativistic cross-section data and the relativistic cross-section formula fit smoothly together. However, if both data and formula were accurate, no modifications would be necessary. This difficulty will be left for future study.

7 Summary

A new relativistic fluid code, RELRATE, was developed to calculate collision rates for use in modeling the propagation of high-power microwaves through the atmosphere. Expressions for particle, momentum, thermal energy transfer, and total energy loss rates were derived from moments of the relativistic Boltzmann collision integrals. These rates were calculated for elastic, inelastic, and ionizing collisions between electrons and N_2 and O_2 molecules using differential cross sections. The details of the relativistic collision integral derivation, expressions for relativistic cross sections, the relativistic collision rate derivation, and methods of computation of the rates are presented in this report.

The derivation of the relativistic Boltzmann collision integrals, presented in Section 2, was based on the nonrelativistic treatments of Holstein[18] and of Chapman and Cowling[19]. The Boltzmann collision integrals represent the time rate of change of the electron momentum distribution resulting from collisions. They are expressed in terms of the difference between the number of electrons entering and leaving a particular volume after scattering from neutral, nonexcited gas molecules. The relativistic forms for the collision integrals simplify to the expressions of Holstein and Roussel-Dupré² in the nonrelativistic limit.

The relativistic and nonrelativistic cross sections for electron scattering from N_2 and O_2 molecules, used in the rate calculation, are based on the cross sections compiled by Murphy[20] and those given by Phelps[21] and Phelps and Pitchford[22]. In Section 3, analytic expressions for the total cross section and the cross-section angular dependence for elastic, inelastic, and ionizing collisions were presented. Adjustments we made to those

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²R. A. Roussel-Dupré, private communication.

expressions and our sources for cross-section data were also discussed.

The relativistic collision rates, presented in Section 4, were obtained from the solution of Eqs. (15)–(17) using the expressions for the relativistic collision integrals and the relativistic equivalent of the Maxwellian distribution function. The derivation of these expressions was based on the derivation of nonrelativistic rates by Roussel-Dupré and was presented in Appendix A. The differences between the relativistic and nonrelativistic rates can be attributed to the different forms for the energy conservation laws in the two cases and to the relativistic form of the equilibrium distribution function. In the nonrelativistic limit, the expressions for the relativistic collision rates reduce to those of Roussel-Dupré.

The accuracy of the relativistic collision rate calculation depends in part upon the computational methods that are used. Presented in Section 5 was a discussion of the methods used in our calculation to evaluate cross-section integrals and integrals over momentum. The results of the rate calculation are discussed in Section 6 whereas two-dimensional plots of the electron-air collision rates calculated by RELRATE can be found in Appendix B.

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Appendix A

A Collision Rate Derivations

A.1 Elastic Collisions

The relativistic elastic collision rates with angular dependent cross sections are calculated from Eqs. (15)-(17) using the results given in Eqs. (19) and (20).

A.1.1 Particle Production Rate

Because elastic collisions do not increase or decrease the number of particles in a volume of phase space, this particle production rate should be zero. Using the expression for $\mathcal{I}'_0$, we have

$$\begin{aligned}
 R_{c|el}(\beta_o, T') n_e &= Nc \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\
 &\quad \times [f(\mu', p') \frac{p'^2}{p^2} \beta' \frac{\partial p'}{\partial p} \sigma_e(p', \mu_\psi) - \beta f(\mu, p) \sigma_e(p, \mu_\psi)] \\
 &= 4\pi^2 Nc \left\{ \int d\mu' \int d\mu_\psi \int dp \frac{\partial p'}{\partial p} p'^2 \beta' f(\mu', p') \sigma_e(p', \mu_\psi) \right. \\
 &\quad \left. - \int d\mu \int d\mu_\psi \int dp p^2 \beta f(\mu, p) \sigma_e(p, \mu_\psi) \right\} \\
 &= 4\pi^2 Nc \left\{ \int d\mu' \int d\mu_\psi \int dp' p'^2 \beta' f(\mu', p') \sigma_e(p', \mu_\psi) \right. \\
 &\quad \left. - \int d\mu \int d\mu_\psi \int dp p^2 \beta f(\mu, p) \sigma_e(p, \mu_\psi) \right\} = 0,
 \end{aligned}$$

where $\mu_\psi = \cos \psi$, and $dp' = dp \frac{\partial p'}{\partial p}$.

A.1.2 Momentum Transfer Rate

The elastic momentum transfer rate is found using $\mathcal{I}'_1$:

$$\begin{aligned}
 R_{m|el}(\beta_o, T') n_e \langle \vec{p} \rangle &= -Nc \int dp p^3 \int d\mu \mu \int d\phi \int d\mu_\psi \int d\alpha \\
 &\quad \times [f(\mu', p') \frac{p'^2}{p^2} \beta' \frac{\partial p'}{\partial p} \sigma_e(p', \mu_\psi) - \beta f(\mu, p) \sigma_e(p, \mu_\psi)] \hat{z} \\
 &= -4\pi^2 Nc \left\{ \int dp' p'^2 \beta' p \int d\mu' \mu' \int d\mu_\psi \mu_\psi f(\mu', p') \sigma_e(p', \mu_\psi) \right. \\
 &\quad \left. - \int dp p^3 \beta \int d\mu \mu \int d\mu_\psi f(\mu, p) \sigma_e(p, \mu_\psi) \right\} \hat{z},
 \end{aligned}$$

where only the $\hat{z}$, as defined in Figure 1, momentum contribution is nonzero. Using the conservation of energy for elastic collisions we can express the electron momentum

after collision, p , in terms of the electron momentum before collision, p' , the mass of the particles, and the scattering angle

$$p = p' - p' \sqrt{p'^2 + 1} \frac{m}{M} (1 - \cos \psi) \quad . \quad (\text{A} - 1)$$

The momentum transfer rate is then

$$\begin{aligned} R_{\text{m}|el}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 N c \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi (\mu_\psi - 1) \sigma_e(p, \mu_\psi) \\ &\quad + 4\pi^2 N c \frac{m}{M} \int dp p^4 \int d\mu \mu f(\mu, p) \int d\mu_\psi \mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) \quad , \end{aligned}$$

where we have used $p = \gamma\beta = (\sqrt{p^2 + 1})\beta$, and $|\vec{p}|$ indicates the magnitude of the mean momentum $\langle \vec{p} \rangle$. The integrals over μ and μ' have the same form as

$$\int_{-1}^1 d\mu \mu e^{B\mu} = \frac{\partial}{\partial B} \int_{-1}^1 d\mu e^{B\mu} = \frac{1}{B} \left[\left(1 - \frac{1}{B}\right) e^B + \left(1 + \frac{1}{B}\right) e^{-B} \right] \quad , \quad (\text{A} - 2)$$

where $B = \frac{3mc^2}{2T'} \gamma_o \beta_o p$. The final result is

$$\begin{aligned} R_{\text{m}|el}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 C N c \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \\ &\quad \times \left\{ \int dp p^2 \int d\mu_\psi (\mu_\psi - 1) \sigma_e(p, \mu_\psi) \left(\beta + \frac{m}{M} p \mu_\psi \right) \right. \\ &\quad \times \left[\left(1 - \frac{2T'}{3mc^2 \gamma_o \beta_o p}\right) e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} + \left(1 + \frac{2T'}{3mc^2 \gamma_o \beta_o p}\right) e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] \left. \right\} \quad . \end{aligned}$$

A.1.3 Thermal Energy Transfer Rate

The elastic thermal energy transfer rate is

$$\begin{aligned} R_{\text{E}|el}(\beta_o, T') n_e \langle T' \rangle &= N c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\ &\quad \times [f(\mu', p') \frac{p'^2}{p^2} \beta' \frac{\partial p'}{\partial p} \sigma_e(p', \mu_\psi) - \beta f(\mu, p) \sigma_e(p, \mu_\psi)] (\gamma_o \sqrt{p^2 + 1} - \gamma_o \beta_o p \mu - 1) m c^2 \quad . \end{aligned}$$

Using expressions for both T'_0 and T'_1 , and transforming the source term integral over dp to one over dp' yields

$$\begin{aligned} R_{\text{E}|el}(\beta_o, T') n_e \langle T' \rangle &= +4\pi^2 N c \left\{ \gamma_o m c^2 \int dp' p'^2 \beta' \sqrt{p'^2 + 1} \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_e(p', \mu_\psi) \right. \\ &\quad - \gamma_o \beta_o m c^2 \int dp' p'^2 \beta' p \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \sigma_e(p', \mu_\psi) \\ &\quad - m c^2 \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_e(p', \mu_\psi) - \gamma_o m c^2 \int dp p^3 \int d\mu f(\mu, p) \int d\mu_\psi \sigma_e(p, \mu_\psi) \\ &\quad \left. + \gamma_o \beta_o m c^2 \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \sigma_e(p, \mu_\psi) + m c^2 \int dp p^2 \beta \int d\mu f(\mu, p) \int d\mu_\psi \sigma_e(p, \mu_\psi) \right\} \quad . \end{aligned}$$

The electron energy after collision can be expressed in terms of the electron energy before collision, the mass of the particles, and the scattering angle using the conservation of energy for elastic collisions,

$$\sqrt{p^2 + 1} = \sqrt{p'^2 + 1} - p'^2 \frac{m}{M} (1 - \cos \psi) . \quad (\text{A-3})$$

Using this result and Eq. (A-1) yields

$$\begin{aligned} R_E|_{el}(\beta_o, T') n_e < T' > &= +4\pi^2 N c \\ &\left\{ \gamma_0 m c^2 \int dp' p'^3 \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_e(p', \mu_\psi) (1 - p' \beta' \frac{m}{M} (1 - \cos \mu_\psi)) \right. \\ &- \gamma_0 \beta_0 m c^2 \int dp' p'^3 \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \sigma_e(p', \mu_\psi) (\beta' - p' \frac{m}{M} (1 - \cos \mu_\psi)) \\ &- \gamma_0 m c^2 \int dp p^3 \int d\mu f(\mu, p) \int d\mu_\psi \sigma_e(p, \mu_\psi) + \gamma_0 \beta_0 m c^2 \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \sigma_e(p, \mu_\psi) \left. \right\} . \end{aligned}$$

Finally, using Eq. (A-2) and

$$\begin{aligned} \int_{-1}^1 d\mu' f(\mu', p') &= C e^{-\frac{3m c^2}{2T'} \gamma_0 \sqrt{p'^2 + 1}} \int_{-1}^1 d\mu' e^{-\frac{3m c^2}{2T'} \gamma_0 \beta_0 p' \mu'} \\ &= C \frac{2T'}{3m c^2 \gamma_0 \beta_0 p'} \left\{ e^{-\frac{3m c^2}{2T'} \gamma_0 [\sqrt{p'^2 + 1} - \beta_0 p']} - e^{-\frac{3m c^2}{2T'} \gamma_0 [\sqrt{p'^2 + 1} + \beta_0 p']} \right\} \end{aligned} \quad (\text{A-4})$$

yields the result

$$\begin{aligned} R_E|_{el}(\beta_o, T') n_e < T' > &= +4\pi^2 C N c \left(\frac{2T'}{3m c^2 \gamma_0 \beta_0} \right) \\ &\times \left\{ \gamma_0 \beta_0 m c^2 \int dp p^2 \int d\mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) (\beta + p \frac{m}{M} \mu_\psi) \right. \\ &\times \left[\left(1 - \frac{2T'}{3m c^2 \gamma_0 \beta_0 p} \right) e^{-\frac{3m c^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} - \beta_0 p)} + \left(1 + \frac{2T'}{3m c^2 \gamma_0 \beta_0 p} \right) e^{-\frac{3m c^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} + \beta_0 p)} \right] \\ &- \gamma_0 m c^2 \frac{m}{M} \int dp p^3 \beta \int d\mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) \left[e^{-\frac{3m c^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} - \beta_0 p)} - e^{-\frac{3m c^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} + \beta_0 p)} \right] \left. \right\} . \end{aligned} \quad (\text{A-5})$$

A.1.4 Total Energy Transfer Rate

The elastic total energy transfer rate is

$$\begin{aligned} R_L|_{el}(\beta_o, T') n_e < \gamma - 1 > m c^2 &= N c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\ &\times [f(\mu', p') \frac{p'^2}{p^2} \beta' \frac{\partial p'}{\partial p} \sigma_e(p', \mu_\psi) - \beta f(\mu, p) \sigma_e(p, \mu_\psi)] (\gamma - 1) m c^2 . \end{aligned}$$

Using expressions for both T'_0 and T'_1 , and transforming the source term integral over dp to one over dp' yields

$$\begin{aligned}
R_L|_{el}(\beta_o, T') n_e < \gamma - 1 > mc^2 &= +4\pi^2 N c m c^2 \\
&\times \left\{ \int dp' p'^2 \beta' (\sqrt{p'^2 + 1} - 1) \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_e(p', \mu_\psi) \right. \\
&\quad \left. - \int dp p^2 \beta (\sqrt{p^2 + 1} - 1) \int d\mu f(\mu, p) \int d\mu_\psi \sigma_e(p, \mu_\psi) \right\} .
\end{aligned}$$

Substituting Eq. A-3 in the first integral and canceling gives

$$\begin{aligned}
R_L|_{el}(\beta_o, T') n_e < \gamma - 1 > mc^2 &= +4\pi^2 \mathcal{C} N c m c^2 \frac{m}{M} \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \\
&\times \left\{ - \int dp p^3 \beta \int d\mu_\psi (1 - \mu_\psi) \sigma_e(p, \mu_\psi) \left[e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} - e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] \right\} .
\end{aligned} \tag{A-6}$$

In the expressions for the two energy loss rates R_L and R_E , Eqs. A-5 and A-6, the terms that are proportional to m/M represent the energy losses to the neutral gas. The remaining term in Eq. A-5 is due to the energy loss caused by redirected electron momentum.

A.2 Inelastic Collisions

The inelastic contribution to the collision rates with angular dependent cross sections is calculated from Eqs. (15) - (18) using the results given in Eqs. (19) and (20) and the Boltzman collision integral for inelastic collisions.

A.2.1 Particle Production Rate

Inelastic collisions do not increase or decrease the number of particles in a volume of phase space, and this particle production rate should be zero. From Eq. (15)

$$\begin{aligned}
R_C|_{in}(\beta_o, T') n_e &= \sum_h N_h c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\
&\quad \times [f(\mu', p') \sigma_h(p', \mu_\psi) \frac{p'^2 \beta'}{p^2} \frac{\partial p'}{\partial p} - \beta f(\mu, p) \sigma_h(p, \mu_\psi)] \\
&= 4\pi^2 \sum_h N_h c \left\{ \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_h(p', \mu_\psi) \right. \\
&\quad \left. - \int dp p^2 \beta \int d\mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) \right\} = 0 .
\end{aligned}$$

A.2.2 Momentum Transfer Rate

The inelastic momentum transfer rate is

$$\begin{aligned}
R_m|_{in}(\beta_o, T') n_e < \vec{p} > &= - \sum_h N_h c \int dp p^3 \int d\mu \mu \int d\phi \int d\mu_\psi \int d\alpha \\
&\quad \times [f(\mu', p') \sigma_h(p', \mu_\psi) \frac{p'^2 \beta'}{p^2} \frac{\partial p'}{\partial p} - \beta f(\mu, p) \sigma_h(p, \mu_\psi)] \hat{z} \\
&= -4\pi^2 \sum_h N_h c \left\{ \int dp' p'^2 \beta' p \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \sigma_h(p', \mu_\psi) \right. \\
&\quad \left. - \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) \right\} \hat{z} ,
\end{aligned}$$

where the $\hat{x}$ and $\hat{y}$ contributions are zero. The conservation of energy for inelastic collisions yields an expression for the electron momentum after collision in terms of the electron momentum before collision, the mass of the particles, the excitation energy, and the scattering angle

$$p = p' - \frac{\sqrt{p'^2 + 1}}{p'} \left[\frac{m}{M} p'^2 (1 - \cos \psi) + \frac{T_h}{mc^2} \right] . \quad (A - 7)$$

Using this expression and Eq. A-2 yields the final result

$$\begin{aligned}
R_m|_{in}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 C \sum_h N_h c \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \\
&\quad \int dp p \int d\mu_\psi \sigma_h(p, \mu_\psi) \left[p \left(\beta + \frac{m}{M} p \mu_\psi \right) (\mu_\psi - 1) - \frac{T_h}{mc^2} \mu_\psi \right] \\
&\quad \left[\left(1 - \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} + \left(1 + \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] .
\end{aligned}$$

A.2.3 Energy Transfer Rate

The inelastic energy transfer rate is

$$\begin{aligned}
R_E|_{in}(\beta_o, T') n_e < T' > &= \sum_h N_h c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\
&\quad \times [f(\mu', p') \frac{p'^2}{p^2} \beta' \frac{\partial p'}{\partial p} \sigma_h(p', \mu_\psi) - \beta f(\mu, p) \sigma_h(p, \mu_\psi)] (\gamma_o \sqrt{p^2 + 1} - \gamma_o \beta_o p \mu - 1) mc^2 .
\end{aligned}$$

Transforming the source term integral over dp to one over dp' and using the expressions for T'_0 and T'_1 leads to

$$\begin{aligned}
R_{E|in}(\beta_o, T') n_e < T' > = & +4\pi^2 \sum_h N_h c \left\{ \gamma_0 m c^2 \int dp' p'^2 \beta' \sqrt{p^2 + 1} \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_h(p', \mu_\psi) \right. \\
& - \gamma_0 \beta_0 m c^2 \int dp' p'^2 \beta' p \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \sigma_h(p', \mu_\psi) \\
& - m c^2 \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_h(p', \mu_\psi) - \gamma_0 m c^2 \int dp p^3 \int d\mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) \\
& \left. + \gamma_0 \beta_0 m c^2 \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) + m c^2 \int dp p^2 \beta \int d\mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) \right\} .
\end{aligned}$$

Using the conservation of energy for inelastic collisions, the electron energy after collision can be expressed in terms of the electron energy before collision, the mass of the particles, the excitation energy, and the scattering angle

$$\sqrt{p^2 + 1} = \sqrt{p'^2 + 1} - \frac{T_h}{m c^2} - p'^2 \frac{m}{M} (1 - \cos \psi) . \quad (A-8)$$

Using this result and Eq. (A-7) yields

$$\begin{aligned}
R_{E|in}(\beta_o, T') n_e < T' > = & +4\pi^2 \sum_h N_h c \\
& \left\{ \gamma_0 m c^2 \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_h(p', \mu_\psi) \left(\sqrt{p'^2 + 1} - \frac{T_h}{m c^2} - p'^2 \frac{m}{M} (1 - \mu_\psi) \right) \right. \\
& - \gamma_0 \beta_0 m c^2 \int dp' p'^2 \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \sigma_h(p', \mu_\psi) \left(p' \beta' - p'^2 \frac{m}{M} (1 - \mu_\psi) - \frac{T_h}{m c^2} \right) \\
& \left. - \gamma_0 m c^2 \int dp p^3 \int d\mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) + \gamma_0 \beta_0 m c^2 \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) \right\} .
\end{aligned}$$

Finally, using Eq. (A-2) and Eq. (A-4) yields

$$\begin{aligned}
R_{E|in}(\beta_o, T') n_e < T' > = & +4\pi^2 \sum_h N_h c \left\{ \gamma_0 m c^2 \int dp p \beta \int d\mu_\psi \sigma_h(p, \mu_\psi) \right. \\
& \times \left(-\frac{T_h}{m c^2} - p^2 \frac{m}{M} (1 - \mu_\psi) \right) \left[e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} - e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] \\
& - \gamma_0 \beta_0 m c^2 \int dp p \int d\mu_\psi \sigma_h(p, \mu_\psi) \left(p (\mu_\psi - 1) \left[\beta + \frac{m}{M} p \mu_\psi \right] - \frac{T_h}{m c^2} \mu_\psi \right) \\
& \times \left[\left(1 - \frac{2T'}{3m c^2 \gamma_o \beta_o p} \right) e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} + \left(1 + \frac{2T'}{3m c^2 \gamma_o \beta_o p} \right) e^{-\frac{3m\epsilon^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] \left. \right\} .
\end{aligned}$$

A.2.4 Total Energy Transfer Rate

The inelastic total energy transfer rate is

$$\begin{aligned}
R_{L|in}(\beta_o, T') n_e < \gamma - 1 > m c^2 = & \sum_h N_h c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\
& \times [f(\mu', p') \frac{p'^2}{p^2} \beta' \frac{\partial p'}{\partial p} \sigma_h(p', \mu_\psi) - \beta f(\mu, p) \sigma_h(p, \mu_\psi)] (\gamma - 1) m c^2 .
\end{aligned}$$

Using expressions for both I'_0 and I'_1 , and transforming the source term integral over dp to one over dp' yields

$$\begin{aligned} R_{L|el}(\beta_o, T') n_e < \gamma - 1 > mc^2 = & +4\pi^2 \sum_h N_h c mc^2 \\ & \times \left\{ \int dp' p'^2 \beta' (\sqrt{p^2 + 1} - 1) \int d\mu' f(\mu', p') \int d\mu_\psi \sigma_h(p', \mu_\psi) \right. \\ & \left. - \int dp p^2 \beta (\sqrt{p^2 + 1} - 1) \int d\mu f(\mu, p) \int d\mu_\psi \sigma_h(p, \mu_\psi) \right\} . \end{aligned}$$

Substituting Eq. A-8 in the first integral and canceling gives

$$\begin{aligned} R_{L|el}(\beta_o, T') n_e < \gamma - 1 > mc^2 = & -4\pi^2 C \sum_h N_h c mc^2 \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \int dp p \beta \int d\mu_\psi \sigma_h(p, \mu_\psi) \\ & \times \left(\frac{T_h}{mc^2} + p^2 \frac{m}{M} (1 - \mu_\psi) \right) \left[e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} - e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] . \end{aligned}$$

A.3 Ionizing Collisions

The ionization contribution to the collision rates can be obtained from Eqs. (15) - (17), the angular integrals Eqs. (19) and (20), and the Boltzman collision integral for ionizing collisions.

A.3.1 Particle Production Rate

For the particle production rate, we have

$$\begin{aligned} R_{c|io}(\beta_o, T') n_e = & \sum_I N_I c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha \\ & \times \left[\int_0^\infty f(\mu', p') \sigma_I(\epsilon', \epsilon, \mu_\psi) \frac{p'^2 \beta'}{p^2} \frac{\beta}{\beta'} d\epsilon' - \beta f(\mu, p) \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_s, \mu_\psi) d\epsilon_s \right] . \end{aligned}$$

Using the angular integral Eq. (19) and recognizing that

$$d\epsilon' = \beta' dp' mc^2 , \quad (A - 9)$$

and

$$\beta dp = d\epsilon / mc^2 , \quad (A - 10)$$

we can transform the integral to

$$\begin{aligned} R_{c|io}(\beta_o, T') n_e = & 4\pi^2 \sum_I N_I c \left\{ \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \int_0^{\epsilon' - \epsilon_I} d\epsilon \sigma_I(\epsilon', \epsilon, \mu_\psi) \right. \\ & \left. - \int dp p^2 \beta \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_s, \mu_\psi) d\epsilon_s \right\} . \end{aligned}$$

Finally, substituting Eq. (A-4) and using the fact that

$$\int_0^{\epsilon' - \epsilon_I} \sigma_I(\epsilon', \epsilon) d\epsilon = 2 \int_0^{\frac{\epsilon' - \epsilon_I}{2}} \sigma_I(\epsilon', \epsilon_{\mathbf{s}}) d\epsilon_{\mathbf{s}}$$

leads to

$$\begin{aligned} R_{\text{c}|io}(\beta_o, T') n_e &= 4\pi^2 \mathcal{C} \sum_I N_I c \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \int dp p \beta \int d\mu_{\psi} \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_{\mathbf{s}} \sigma_I(\epsilon, \epsilon_{\mathbf{s}}, \mu_{\psi}) \\ &\times \left[e^{-\frac{3m\epsilon^2}{2T'}} \gamma_o [\sqrt{p^2 + 1} - \beta_o p] - e^{-\frac{3m\epsilon^2}{2T'}} \gamma_o [\sqrt{p^2 + 1} + \beta_o p] \right] . \end{aligned} \quad (\text{A-11})$$

A.3.2 Momentum Transfer Rate

The ionization momentum transfer rate is

$$\begin{aligned} R_{\text{m}|io}(\beta_o, T') n_e < \vec{p} > &= - \sum_I N_I c \int dp p^3 \int d\mu_{\psi} \int d\phi \int d\mu_{\psi} \int d\alpha \\ &\times \left[\int_0^{\infty} f(\mu', p') \sigma_I(\epsilon', \epsilon, \mu_{\psi}) \frac{p'^2 \beta'}{p^2} \frac{\beta}{\beta'} d\epsilon' - \beta f(\mu, p) \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_{\mathbf{s}}, \mu_{\psi}) d\epsilon_{\mathbf{s}} \right] \hat{z} , \end{aligned}$$

where, as with elastic and inelastic momentum transfer rates, the $\hat{x}$ and $\hat{y}$ terms are zero. Using Eqs. (20), (A-9), and (A-10) to reduce the momentum transfer rate yields

$$\begin{aligned} R_{\text{m}|io}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 \sum_I N_I c \left\{ \right. \\ &\int dp' p'^2 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_{\psi} \mu_{\psi} \int_0^{\epsilon - \epsilon_I} d\epsilon p \sigma_I(\epsilon', \epsilon, \mu_{\psi}) \\ &\left. - \int dp p^3 \beta \int d\mu f(\mu, p) \int d\mu_{\psi} \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_{\mathbf{s}}, \mu_{\psi}) d\epsilon_{\mathbf{s}} \right\} . \end{aligned}$$

The first term in this expression is split in two parts yielding

$$\begin{aligned} R_{\text{m}|io}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 \sum_I N_I c \left\{ \right. \\ &\int dp' p'^2 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_{\psi} \mu_{\psi} \left[\int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_{\mathbf{s}} p_{\mathbf{s}} \sigma_I(\epsilon', \epsilon_{\mathbf{s}}, \mu_{\psi}) + \int_{\frac{\epsilon' - \epsilon_I}{2}}^{\epsilon' - \epsilon_I} d\epsilon p \sigma_I(\epsilon', \epsilon, \mu_{\psi}) \right] \\ &\left. - \int dp p^3 \beta \int d\mu f(\mu, p) \int d\mu_{\psi} \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_{\mathbf{s}}, \mu_{\psi}) d\epsilon_{\mathbf{s}} \right\} . \end{aligned}$$

Using the conservation of energy for ionizing collisions to obtain the expression

$$p \cos \psi \approx p' - \frac{\sqrt{p'^2 + 1}}{p'} \left\{ \frac{\epsilon_I}{mc^2} + \frac{\epsilon_{\mathbf{s}}}{mc^2} + \frac{1}{2} \frac{m}{M} p_{\mathbf{s}}^2 (1 - \cos^2 \psi_{\mathbf{s}}) \right\} - \frac{1}{2} \frac{p_{\mathbf{s}}^2}{p'} (1 - \cos^2 \psi_{\mathbf{s}}) , \quad (\text{A-12})$$

and

$$d\varepsilon = -d\varepsilon_{\mathbf{s}} \quad , \quad (\text{A} - 13)$$

the integral over $d\varepsilon$ can be transformed to one over $d\varepsilon_{\mathbf{s}}$

$$\begin{aligned} R_{m|io}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 \sum_I N_I c \left\{ \right. \\ &\int dp' p'^2 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} p_{\mathbf{s}} \sigma_I(\varepsilon', \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &+ \int dp' p'^3 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} \sigma_I(\varepsilon', \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &- \frac{\varepsilon_I}{mc^2} \int dp' p'^2 \beta' \frac{\sqrt{p'^2 + 1}}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} \sigma_I(\varepsilon', \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &- \int dp' p'^2 \beta' \frac{\sqrt{p'^2 + 1}}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} \frac{\varepsilon_{\mathbf{s}}}{mc^2} \sigma_I(\varepsilon', \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &- \frac{1}{2} \frac{m}{M} \int dp' p'^2 \beta' \frac{\sqrt{p'^2 + 1}}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} p_{\mathbf{s}}^2 \sigma_I(\varepsilon', \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &- \frac{1}{2} \int dp' p' \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} p_{\mathbf{s}}^2 \sigma_I(\varepsilon', \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &\left. - \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \sigma_I(\varepsilon, \varepsilon_{\mathbf{s}}, \mu_\psi) d\varepsilon_{\mathbf{s}} \right\} . \end{aligned}$$

Substituting Eq. (A-2) yields the final form

$$\begin{aligned} R_{m|io}(\beta_o, T') n_e |\vec{p}| &= -4\pi^2 C \sum_I N_I c \left(\frac{2T'}{3mc^2 \gamma_o \beta_o} \right) \int dp \int d\mu_\psi \left\{ p \beta \mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} p_{\mathbf{s}} \sigma_I(\varepsilon, \varepsilon_{\mathbf{s}}, \mu_\psi) \right. \\ &- \frac{\varepsilon_I}{mc^2} p \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} \sigma_I(\varepsilon, \varepsilon_{\mathbf{s}}, \mu_\psi) - p \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} \frac{\varepsilon_{\mathbf{s}}}{mc^2} \sigma_I(\varepsilon, \varepsilon_{\mathbf{s}}, \mu_\psi) \\ &\left. - \frac{1}{2} \frac{m}{M} p (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} p_{\mathbf{s}}^2 \sigma_I(\varepsilon, \varepsilon_{\mathbf{s}}, \mu_\psi) - \frac{1}{2} \beta (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} d\varepsilon_{\mathbf{s}} p_{\mathbf{s}}^2 \sigma_I(\varepsilon, \varepsilon_{\mathbf{s}}, \mu_\psi) \right\} \\ &\times \left[\left(1 - \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} - \beta_o p)} + \left(1 + \frac{2T'}{3mc^2 \gamma_o \beta_o p} \right) e^{-\frac{3mc^2}{2T'} \gamma_o (\sqrt{p^2 + 1} + \beta_o p)} \right] . \end{aligned}$$

A.3.3 Energy Transfer Rate

The ionization energy transfer rate is

$$\begin{aligned} R_{E|io}(\beta_o, T') n_e < T' > &= \sum_I N_I c \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha (\gamma_0 \sqrt{p^2 + 1} - \gamma_0 \beta_o p \mu - 1) m c^2 \\ &\times \left[\int_0^\infty f(\mu', p') \sigma_I(\epsilon', \epsilon, \mu_\psi) \frac{p'^2 \beta'}{p^2} \frac{\beta}{\beta'} d\epsilon' - \beta f(\mu, p) \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_s, \mu_\psi) d\epsilon_s \right]. \end{aligned}$$

Using the angular integrals, Eqs. (19) and (20) and Eqs. (A-9) and (A-10), and splitting the integrals over $d\epsilon$ yields

$$\begin{aligned} R_{E|io}(\beta_o, T') n_e < T' > &= 4\pi^2 \sum_I N_I c m c^2 \left\{ \right. \\ &\gamma_0 \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \left[\int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sqrt{p_s^2 + 1} \sigma_I(\epsilon', \epsilon_s, \mu_\psi) + \int_{\frac{\epsilon' - \epsilon_I}{2}}^{\epsilon' - \epsilon_I} d\epsilon \sqrt{p^2 + 1} \sigma_I(\epsilon', \epsilon, \mu_\psi) \right] \\ &- \gamma_0 \beta_o \int dp' p'^2 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \left[\int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s p_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) + \int_{\frac{\epsilon' - \epsilon_I}{2}}^{\epsilon' - \epsilon_I} d\epsilon p \sigma_I(\epsilon', \epsilon, \mu_\psi) \right] \\ &- \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \left[\int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) + \int_{\frac{\epsilon' - \epsilon_I}{2}}^{\epsilon' - \epsilon_I} d\epsilon \sigma_I(\epsilon', \epsilon, \mu_\psi) \right] \\ &- \gamma_0 \int dp p^3 \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon, \epsilon_s, \mu_\psi) \\ &+ \gamma_0 \beta_o \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon, \epsilon_s, \mu_\psi) \\ &\left. + \int dp p^2 \beta \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon, \epsilon_s, \mu_\psi) \right\}. \end{aligned}$$

The remaining integral over $d\epsilon$ can be transformed to one over $d\epsilon_s$ using Eqs. (A-12) and (A-13) and the following statement of the conservation of energy

$$\sqrt{p^2 + 1} \approx \sqrt{p'^2 + 1} - \frac{\epsilon_I}{m c^2} - \frac{\epsilon_s}{m c^2} - \frac{m}{M} \frac{1}{2} p_s^2 (1 - \cos^2 \psi_s) . \quad (\text{A} - 14)$$

Then

$$\begin{aligned}
R_E|_{i_0}(\beta_0, T') n_e < T' > = 4\pi^2 \sum_I N_I c m c^2 \left\{ \right. \\
& \gamma_0 \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sqrt{p_s^2 + 1} \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& + \gamma_0 \int dp' p'^2 \beta' \sqrt{p'^2 + 1} \int d\mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& - \gamma_0 \frac{\epsilon_I}{m c^2} \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& - \gamma_0 \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \frac{\epsilon_s}{m c^2} \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& - \gamma_0 \frac{1}{2} \frac{m}{M} \int dp' p'^2 \beta' \int d\mu' f(\mu', p') \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s p_s^2 \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& - \gamma_0 \beta_0 \int dp' p'^2 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s p_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& - \gamma_0 \beta_0 \int dp' p'^3 \beta' \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& + \gamma_0 \beta_0 \frac{\epsilon_I}{m c^2} \int dp' p'^2 \beta' \frac{\sqrt{p'^2 + 1}}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& + \gamma_0 \beta_0 \int dp' p'^2 \beta' \frac{\sqrt{p'^2 + 1}}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s \frac{\epsilon_s}{m c^2} \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& + \gamma_0 \beta_0 \frac{1}{2} \frac{m}{M} \int dp' p'^2 \beta' \frac{\sqrt{p'^2 + 1}}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s p_s^2 \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& + \gamma_0 \beta_0 \frac{1}{2} \int dp' p'^2 \beta' \frac{1}{p'} \int d\mu' \mu' f(\mu', p') \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon' - \epsilon_I}{2}} d\epsilon_s p_s^2 \sigma_I(\epsilon', \epsilon_s, \mu_\psi) \\
& - \gamma_0 \int dp p^3 \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon, \epsilon_s, \mu_\psi) \\
& + \gamma_0 \beta_0 \int dp p^3 \beta \int d\mu \mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon, \epsilon_s, \mu_\psi) \\
& \left. - \int dp p^2 \beta \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} d\epsilon_s \sigma_I(\epsilon, \epsilon_s, \mu_\psi) \right\} .
\end{aligned}$$

Canceling and using Eqs. (A-2) and (A-4) yields

$$\begin{aligned}
R_E(\beta_o, T') n_e < T' > = 4\pi^2 \mathcal{C} \sum_I N_{Ic} \left(\frac{2T'}{3} \right) \left\{ - \int dp p \right. \\
& \left(\beta \int d\mu_\psi \mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S - \frac{\epsilon_I}{mc^2} \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right. \\
& - \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \frac{\epsilon_S}{mc^2} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S - \frac{1}{2} \left(\frac{m}{M} + \frac{1}{\gamma} \right) \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S^2 \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \\
& \times \left[\left(1 - \frac{2T'}{3mc^2 \gamma_0 \beta_0 p} \right) e^{-\frac{3mc^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} - \beta_0 p)} + \left(1 + \frac{2T'}{3mc^2 \gamma_0 \beta_0 p} \right) e^{-\frac{3mc^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} + \beta_0 p)} \right] \\
& + \frac{1}{\beta_0} \int dp p \beta \left[e^{-\frac{3mc^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} + \beta_0 p)} - e^{-\frac{3mc^2}{2T'} \gamma_0 (\sqrt{p^2 + 1} - \beta_0 p)} \right] \\
& \left(\left(1 - \frac{1}{\gamma_0} - \frac{\epsilon_I}{mc^2} \right) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right. \\
& \left. \left. - \frac{1}{2} \frac{m}{M} \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\epsilon - \epsilon_I}{2}} p_S^2 \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right) \right\} .
\end{aligned} \tag{A-15}$$

A.3.4 Total Energy Transfer Rate

The ionization contribution to the total energy transfer rate is

$$\begin{aligned}
R_{L|io}(\beta_o, T') n_e < \gamma - 1 > mc^2 &= \sum_I N_{Ic} \int dp p^2 \int d\mu \int d\phi \int d\mu_\psi \int d\alpha (\gamma - 1) mc^2 \\
&\times \left[\int_0^\infty f(\mu', p') \sigma_I(\epsilon', \epsilon, \mu_\psi) \frac{p'^2 \beta'}{p^2 \beta} d\epsilon' - \beta f(\mu, p) \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right] .
\end{aligned}$$

Using expressions for both $\mathcal{I}'_0$ and $\mathcal{I}'_1$, and transforming the source term integral over dp to one over dp' yields

$$\begin{aligned}
R_{L|io}(\beta_o, T') n_e < \gamma - 1 > mc^2 &= 4\pi^2 \sum_I N_{Ic} \left\{ \int dp' p'^2 \beta' \int d\mu f(\mu', p') \int d\mu_\psi \right. \\
&\times \left[\int_0^{\frac{\epsilon' - \epsilon_I}{2}} (\gamma_S - 1) mc^2 \sigma_I(\epsilon', \epsilon_S, \mu_\psi) d\epsilon_S + \int_{\frac{\epsilon' - \epsilon_I}{2}}^{\epsilon' - \epsilon_I} (\gamma - 1) mc^2 \sigma_I(\epsilon', \epsilon, \mu_\psi) d\epsilon \right] \\
&\left. - \int dp p^2 \beta (\gamma - 1) mc^2 \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\epsilon - \epsilon_I}{2}} \sigma_I(\epsilon, \epsilon_S, \mu_\psi) d\epsilon_S \right\} .
\end{aligned}$$

The integral over $d\varepsilon$ can be transformed to one over $d\varepsilon_s$ using Eqs. A-13 and A-14

$$\begin{aligned}
R_{L|io}(\beta_o, T') n_e < \gamma - 1 > mc^2 = 4\pi^2 \sum_I N_I c mc^2 \left\{ \right. \\
& \int dp' p'^2 \beta' \int d\mu f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} (\gamma_s - 1) \sigma_I(\varepsilon', \varepsilon_s, \mu_\psi) d\varepsilon_s \\
& + \int dp' p'^2 \beta' (\sqrt{p'^2 + 1} - 1) \int d\mu f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} \sigma_I(\varepsilon', \varepsilon, \mu_\psi) d\varepsilon \\
& - \frac{\varepsilon_I}{mc^2} \int dp' p'^2 \beta' \int d\mu f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} \sigma_I(\varepsilon', \varepsilon, \mu_\psi) d\varepsilon \\
& - \int dp' p'^2 \beta' \int d\mu f(\mu', p') \int d\mu_\psi \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} \frac{\varepsilon_s}{mc^2} \sigma_I(\varepsilon', \varepsilon, \mu_\psi) d\varepsilon \\
& - \frac{1}{2} \frac{m}{M} \int dp' p'^2 \beta' \int d\mu f(\mu', p') \int d\mu_\psi (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon' - \varepsilon_I}{2}} p_s^2 \sigma_I(\varepsilon', \varepsilon, \mu_\psi) d\varepsilon \\
& \left. - \int dp p^2 \beta (\gamma - 1) \int d\mu f(\mu, p) \int d\mu_\psi \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \sigma_I(\varepsilon, \varepsilon_s, \mu_\psi) d\varepsilon_s \right\} .
\end{aligned}$$

The final result, after canceling and integrating over $d\mu$, is

$$\begin{aligned}
R_{L|io}(\beta_o, T') n_e < \gamma - 1 > mc^2 = 4\pi^2 C \sum_I N_I c \left(\frac{2T'}{3\gamma_o \beta_o} \right) \int dp p \beta \int d\mu_\psi \\
& \left(- \frac{\varepsilon_I}{mc^2} \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} \sigma_I(\varepsilon, \varepsilon, \mu_\psi) d\varepsilon - \frac{1}{2} \frac{m}{M} (1 - \mu_\psi^2) \int_0^{\frac{\varepsilon - \varepsilon_I}{2}} p_s^2 \sigma_I(\varepsilon, \varepsilon, \mu_\psi) d\varepsilon \right) \\
& \times \left[\epsilon^{-\frac{3m\epsilon^2}{2T'}} \gamma_o [\sqrt{p'^2 + 1} + \beta_o p] - \epsilon^{-\frac{3m\epsilon^2}{2T'}} \gamma_o [\sqrt{p'^2 + 1} - \beta_o p] \right] .
\end{aligned}$$

Appendix B

B Collision Rate Plots

The following pages contain two-dimensional plots of the various electron-air collision rates calculated by the authors' relativistic collision rate code RELRATE. There are four sets of rate plots given in this appendix. Each set consists of four plots of a collision rate as a function of thermal energy for a set of fixed kinetic energies and one plot of the rate as a function of kinetic energy for mean thermal energy fixed at $7.11(10^{-3})$ eV. Figures B-1 through B-5 are plots of the ionization rate, Figures B-6 through B-10 show the momentum transfer rate, Figures B-11 through B-15 give the thermal energy transfer rate, and Figures B-16 through B-20 present the total energy loss rate.

Figure B-1. The particle production rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy and for mean electron kinetic energy fixed at $4.44(10^{-3})$ eV.

Figure B-2. The particle production rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of kinetic energy and for mean electron thermal energy fixed at $7.11(10^{-3})$ eV.

Figure B-3. The particle production rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 0 and 50 eV.

Figure B-4. The particle production rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 50 eV and 5 keV.

Figure B-5. The particle production rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 58.7 keV and 35.6 MeV.

Figure B-6. The momentum transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy and for mean electron kinetic energy fixed at $4.44(10^{-3})$ eV.

Figure B-7. The momentum transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of kinetic energy and for mean electron thermal energy fixed at $7.11(10^{-3})$ eV.

Figure B-8. The momentum transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 0 and 50 eV.

Figure B-9. The momentum transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 50 eV and 5 keV.

Figure B-10. The momentum transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 58.7 keV and 35.6 MeV.

Figure B-11. The thermal energy transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy and for mean electron kinetic energy fixed at $4.44(10^{-3})$ eV.

Figure B-12. The thermal energy transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of kinetic energy and for mean electron thermal energy fixed at $7.11(10^{-3})$ eV.

Figure B-13. The thermal energy transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 0 and 50 eV.

Figure B-14. The thermal energy transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 50 eV and 5 keV.

Figure B-15. The thermal energy transfer rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 58.7 keV and 35.6 MeV.

Figure B-16. The total energy loss rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy and for mean electron kinetic energy fixed at $4.44(10^{-3})$ eV.

Figure B-17. The total energy loss rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of kinetic energy and for mean electron thermal energy fixed at $7.11(10^{-3})$ eV.

Figure B-18. The total energy loss rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 0 and 50 eV.

Figure B-19. The total energy loss rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 50 eV and 5 keV.

Figure B-20. The total energy loss rate resulting from electron-neutral collisions, calculated by the FORTRAN code RELRATE, as a function of thermal energy for ten values of mean electron kinetic energy between 58.7 keV and 35.6 MeV.

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