SUPERDIFFUSION

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ABSTRACT

We are developing a new approach to the problem of anomalous transport, which is based on understanding of the phase space topological properties of incomplete dynamical chaos of magnetic surfaces and particles. General concepts are reviewed, self-similar behavior has been established for a force-free model, and the presence of superdiffusion in toroidal geometry has been confirmed for appropriate conditions. Manipulation with control parameters can be effectively used to change the rate of diffusion and characteristic exponent of the transport. As an additional example of the anomalous transport, we demonstrate chaotic instability induced by computer simulation, which is important in preparing of codes and simulation programs.
CONTENTS

- GENERAL DESCRIPTION
TECHNICAL REPORT
SUPERDIFFUSION

This report consists of a general description, Part 1, Part 2, and Part 3. Part 1 includes detailed description of computer simulation of one particle dynamics in toroidal geometry and introducional material to the problem of chaotic dynamics induced by computer simulation. Doctoral student, Qing Li, took part in the preparation of Part 1. Part 2 includes publications on the general theory of anomalous transport from dynamical point of view and on the discovery of self-similarity in a model of field-lines chaos. There are review article from Nature, article from Physical Review E, and an abstract from the Sherwood Meeting. Part 3 consists of the theory of chaos induced by simulation. This material has been accepted for publication in Communications on Pure and Applied Mathematics.

Part of the results were obtained in collaboration with scientists from the Courant Institute, the former Soviet Union, and Israel.

General Description

Our new approach to the anomalous transport problem is based on the study of magnetic and/or particle chaos from the dynamical point of view. It was obtained that intermittency of the dynamical chaos induces new statistics with competing trappings and flights in the phase space of a system, which reveals in superdiffusion and anomalous transport. There is a strong similarity between magnetic surfaces chaotic destruction, streamlines chaos of fluids, and chaotic behavior of advective particles. Our previous results in chaotic advection and Levy flights theory have been used in developing of the superdiffusion of magnetic field-lines in toroidal geometry.

Eventually it should be clear that character of diffusion, and especially of anomalous diffusion, depends entirely on the phase space topology when the dynamical chaos dominates. Topological characteristics of dynamical systems is a new powerful tool which has not been used yet effectively to study the problem of anomalous transport in toroidal plasmas. Stochastic webs is an important category of the “topological kit” for the phase space of a system. Presence of the stochastic webs can drastically change the phase space topology, bringing to the acceleration of transport processes. This phenomenon will be proved in Part 1 for toroidal geometry. Part 2 presents a review article from Nature on new kind of kinetic processes named “strange kinetics” in the article. Superdiffusion is an example of strange kinetics. The second article in Part 2 is evaluation of self-similar properties of chaotic dynamics when the transport is anomalous. We consider this article as a starting point for the future project on the anomalous kinetics in the SOL of tokamak systems. Part 3 is a beginning of new activity in an evaluation of the level of errorness that causes by the discretization procedure in computer simulation (see also the Appendix in Part 1). The important
comment to this subject is that the anomalous diffusion phenomenon persists in the chaotic wandering induced by a discretization procedure of differential equations and a roundoff procedure.
SUPERDIFFUSION

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1 Introduction

The problem of anomalous transport in fusion devices consists of several distinct elements. One crucial subject is the destruction of magnetic surfaces and the induced transport. This report examines a new mechanism for anomalous transport caused by the chaotic destruction of magnetic surfaces.

The possibility of chaotic magnetic surfaces destruction is not new [1] and many of its aspects were studied before. However, the effects of the new mechanism of strong destruction, leading to the acceleration of transport, have not been examined adequately. Under certain conditions, this mechanism causes superdiffusion - very fast diffusional processes produced by so-called stochastic jets. Anomalous fast evolution of particles or of magnetic field lines may appear as a result of the existence of stochastic webs, which were discovered recently [2,3].

A stochastic web is a connected network of channels in space along which the diffusion of particles or field lines occur. The appearance of a web is typically a large scale field pattern and has some symmetry properties.

2 Formulation

Without loss of generality, we shall focus on a magnetic field with toroidal symmetry. Naturally, concentric circular toroidal coordinates (or, simply toroidal coordinates) are used. Fig.1 illustrates the relation between Cartesian and toroidal coordinates.
\begin{align*}
x &= (R_0 + r \cos \theta) \sin \zeta, \\
y &= (R_0 + r \cos \theta) \cos \zeta, \\
z &= r \sin \zeta,
\end{align*}

where \( r, \theta, \zeta \), respectively, are the radial, poloidal and toroidal coordinates. Throughout this article, whenever it is necessary, we shall take \( R_0 = 4 \) and the toroidal region in which the particle may travel is bounded by \( r = R_0 \).

The choice of the unperturbed potential was inspired by that of the standard model [6]. We have modified it in a way which makes the final equation area preserving:

\begin{equation}
\tilde{A}_{\ast} = \frac{B_0 r}{2(1 + \frac{r}{R_0} \cos \theta)} \tilde{e}_\theta - \frac{B_0}{R_0 + r \cos \theta} \int^r r' dr' \frac{q(r')}{{q}(r')} \tilde{e}_\zeta, \tag{1}
\end{equation}

where safety factor \( q(r) \) is a dimensionless function of \( r \) alone

\begin{equation}
q(r) = q_0 \left[ 1 + \left( \frac{r}{r_0} \right)^2 \right]^\frac{1}{2} \tag{2}
\end{equation}

In addition to \( \tilde{A}_{\ast} \), we shall consider a perturbation:

\begin{equation}
\delta \tilde{A} = \varepsilon A_0(r) \cos(2s \theta) \cos(m \zeta) \tilde{e}_\theta, \tag{3}
\end{equation}

where \( s \) and \( m \) are typically integers and \( A_0(r) \), also a function of \( r \) exclusively, takes the form

\begin{equation}
A_0(r) = \frac{r}{1 + | \frac{r}{r_A} |^{\gamma_1}} B_0. \tag{4}
\end{equation}

Using the constants \( \gamma_1, r_A \), we can manipulate the space location of perturbation. The aim to do this is to consider the influence of local perturbation in the inner part of the plasma bulk rather than perturbation in the separatrix vicinity. For sufficiently large \( \gamma_1 \), asymptotically,

\begin{align*}
A_0 &\approx 0, r \gg r_A; \\
A_0 &\approx r B_0, r \ll r_A.
\end{align*}
Hence, the value of $r_A$ is pertinent to the domain of the perturbation and the effect on its change will be studied. Note that the factor 2 in Eq. (4) is chosen conveniently so that $\delta \vec{A}(r, \theta) = \delta \vec{A}(-r, \theta + \pi)$, i.e., $\delta \vec{A}$ is a single-value function of points of the 3D space.

We are able now to compute the magnetic field $\vec{B} = B_r \hat{e}_r + B_\theta \hat{e}_\theta + B_\zeta \hat{e}_\zeta$ explicitly:

$$B_r = \frac{\epsilon m r \cos(2\theta) \sin(m \zeta)}{(R_0 + r \cos \theta)(1 + |\frac{r}{r_A}|^\gamma)} B_0,$$

$$B_\theta = \frac{B_0 r}{(R_0 + r \cos \theta) q(r)},$$

$$B_\zeta = \frac{B_0 R_0}{R_0 + r \cos \theta} \left[1 - \frac{r \cos \theta}{2(R_0 + r \cos \theta)}\right]$$

$$+ \frac{\epsilon \cos(2\theta) \cos(m \zeta) [\gamma_1 + (2 - \gamma_1)(1 + |\frac{r}{r_A}|^\gamma)]}{(1 + |\frac{r}{r_A}|^\gamma)^2} B_0.$$

Magnetic field lines are defined by

$$l_r \frac{dr}{B_r} = l_\theta \frac{d\theta}{B_\theta} = l_\zeta \frac{d\zeta}{B_\zeta} \equiv dt,$$

where $l_r, l_\theta, l_\zeta$ are the scale factors of the coordinate system:

$$l_r = 1,$$

$$l_\theta = r,$$

$$l_\zeta = R_0 + r \cos \theta.$$

Therefore, the following differential equations will dictate the structure of the field lines (or, to the 0th order approximation, motion of particles):

$$\frac{dr}{dt} = \frac{\epsilon m r \cos(2\theta) \sin(m \zeta)}{(R_0 + r \cos \theta)(1 + |\frac{r}{r_A}|^\gamma)} B_0,$$

$$\frac{d\theta}{dt} = \frac{1}{q_0(R_0 + r \cos \theta)[1 + (\frac{\gamma}{\gamma_0})^2]^{\frac{1}{\gamma}}},$$

$$\frac{d\zeta}{dt}.$$
\[ \frac{d\zeta}{dt} = \frac{B_0 R_0}{(R_0 + r \cos \theta)^2} \left[ 1 - \frac{r \cos \theta}{2(R_0 + r \cos \theta)} \right] \]

\[ + \frac{\varepsilon \cos(2\alpha \theta) \cos(m \zeta)[\gamma_1 + (2 - \gamma_1)(1 + |\frac{\zeta}{r_A}|^n)]}{(R_0 + r \cos \theta)(1 + |\frac{\zeta}{r_A}|^n)^2} B_0. \]

These equations are a typical case of a 3D flow. By applying the Poincaré sectioning method, we shall proceed to study its dynamical properties as well as diffusional behavior of particles (field lines) in the subsequent sections.

3 Dynamical Properties

In this section, we demonstrate results of numerical simulations at different parameter values. All figures shown (unless otherwise stated) are Poincaré sectioning planes at \( \zeta = 0.7854(\approx \frac{\pi}{4}), \text{mod}(2\pi) \). Of course this choice of the plane is purely arbitrary. Standard 4th order Runge-Kutta method was used with double precision. On average, it takes 4000-5000 Runge-Kutta steps (or, approximately, 20-25 units in real time) between two Poincaré sections.

Preliminary investigations (Fig. 2) were performed in the order of increasing \( \varepsilon = 0.0, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6 \). The case \( \varepsilon = 0 \) actually shows the advantage of using the toroidal coordinates, i.e., the trivial simplicity of the equation of the magnetic surfaces, \( r = r_0 \).

As \( \varepsilon \) increases, regions of instability occur and the chaotic area increases with \( \varepsilon \). For high perturbations, the central stable region become surrounded by the chaotic sea in which strong mixing of orbits occurs. One can also understand the fact that in all figures, points tend to be denser toward the left. Obviously, Eq.(6) show that when \( \theta \sim \pi, \frac{d\zeta}{dt} \) approaches maximum so that it takes shorter time for the particle to wander between two sectioning points and thus giving rise to smaller increases (or decreases) in \( r \) and \( \theta \).

Fig.3 displays some fine structures of (h) of Fig.2. Notice the phenomenon of sticking close to fixed points caused by a relatively small Lyapunov exponent in that region. Stickiness occurs in the vicinity of small sub-islands (see (c) and (d)) which belong to a higher order resonance. The stickiness reveals in almost regular (elliptic) sets of points around the resonance sub-islands.
One of the key issues we wanted to study is the occurrence of stochastic web because the existence of the web is directly relevant to the long range transportation of particles even at weak perturbation and we have not seen such behavior in Figs. 2 and 3.

The breakthrough lies in the nature of stability of the central toroidal axis \((r=0)\). The fact that the origin is a fixed point is trivial (since \(\frac{dr}{dt}|_{r=0} = 0\)). However, its nature of stability deserves some careful examination. To the lowest in \(r\) and \(\varepsilon\), we have, from Eq.(6),

\[
\frac{d\ln r}{d\theta} = \varepsilon \cos(2s\theta)\sin(m\zeta),
\]

\[
= \frac{\varepsilon m}{2}[\sin(2s\theta + m\zeta) - \sin(2s\theta - m\zeta)],
\]

and

\[
\frac{d\zeta}{d\theta} = q_0,
\]

or,

\[
\zeta = q_0\theta + \theta_0. \tag{8}
\]

After integration, we see that near the origin,

\[
r = r_0e^{[-\frac{\varepsilon m}{2}(\frac{1}{2s + mq_0}\cos(2s\theta + m\zeta) - \frac{1}{2s - mq_0}\cos(2s\theta - m\zeta))]}}. \tag{9}
\]

Apparently, once either of the conditions

\[
mq_0 = \pm 2s \tag{10}
\]

is satisfied, the origin becomes highly unstable whatever the value of \(\varepsilon\). Hence, Eq.(10) is the resonance condition.

Four different resonance situations were studied, the results of which are demonstrated in Figs. 4-7. In all these instances, one can conclude that however weak the perturbation is, destructions of magnetic surfaces can occur and particles can travel along the web through distances of order 1. Practically, of course, this makes problems for plasma confinement if
there is not enough shear in the central part of plasma. As perturbation grows stronger, local instability around the central axis will merge with the stochastic sea and strong diffusion can occur.

In Fig.4 we display the chaotic behavior of particles for the second order resonance as a function of decreasing $\epsilon$. Strong chaos in the central part of plasma becomes isolated for very small $\epsilon$ and superdiffusion disappears. The phenomena of sticking can be observed in Fig.4(c), which could delay the plasma decay. In Fig.5(d), we again observe the phenomenon of sticking for the 5th order resonance and in this case sticking occurs at the boundary of the stochastic sea. Fig.6(b), with higher resolution manifests detailed structures of the sticky region.

Fig.7 provides two tests for the nonlinear resonance condition of Eq.(10). Both $(m, q_0, s)$ combinations (each with 2 perturbation intensities), namely,

$$m = 1, \quad 2s = 2, \quad q_0 = 2;$$
$$m = 4, \quad 2s = 2, \quad q_0 = 0.5,$$

satisfy the condition and sure enough we have resonance. Adjustment of the resonance condition Eq.(10) can be done by choosing a corresponding value of $m$ for different values of safety factors $q_0$. Strong chaos exists for not too small perturbation $\epsilon = 0.1$, and diffusion disappears for the fairly small values of $\epsilon = 0.02$. Note the nontrivial dependence of the order of resonance (i.e., the number of resonance branches) on the value of $2s$. It can be proven analytically [3] that under resonance, $2s$ exactly equals the number of the same signature branches observed at weak perturbation.

Since it is clear already that nonlinear resonance creates local instability at weak perturbation, we now ask what kind of role resonance plays at high perturbation. Resonance, as such, should enhance diffusion. Figures 8 and 9 confirm exactly this intuition. All at high perturbation ($\epsilon = 0.4$), Fig.8(a) satisfies the resonance condition while Fig.8(b),(c),(d) don't. One can tell even at a casual glance that the area of the stochastic sea is considerably larger in Fig.8(a) while those of the rest are about the same size. By the same token, since all pictures of Fig.(9) are exclusively non-resonant, it is well-understood that their stochastic sea should have approximately the same size. Increasing of $q_0$ from 9(a) to 9(d) enhances the inner area of diffusion because $q_0$ approaches its resonant value $q_0 = 2$. 

6
The size of the stochastic sea also depends on a number of other factors. Among them is the scope of effect of the perturbation characterized by \( r_A \) (as mentioned in Section 2). In Fig. (10), all are resonance cases and one can find that the area of the stochastic sea increases with the value of \( r_A \) which is consistent with our analysis in the previous section. Similarly, if we reduce \( \gamma_1 \) (Fig. 11), we are literally allowing perturbation to extend beyond the boundary \( r = r_A \). In this case, the area of the stochastic sea should also increase.

Up to now, we have assumed that the perturbation we have to deal with is "mono-chromatic", i.e., has only one harmonic component. In the real world, we have to consider perturbations with different Fourier components. It is in this spirit that we investigated some simple cases that are illustrated in Fig. 12. The originally non-diffusive (or, at least slowly diffusive) resonance is now subject to interactions with many harmonics of the perturbation so that in Eq. (6) now there are more terms on the right hand side:

\[
\begin{align*}
\frac{dr}{dt} &= \frac{\varepsilon r \sum_m m \cos(2s\theta) \sin(m\zeta)}{(R_0 + r \cos\theta)(1 + \left| \frac{r}{r_A} \right|^n) B_0}, \\
\frac{d\theta}{dt} &= \frac{1}{q_0(R_0 + r \cos\theta)[1 + (R_0/r_0)^2]^{\frac{1}{2}}}, \\
\frac{d\zeta}{dt} &= \frac{B_0 R_0}{(R_0 + r \cos\theta)^2 \left[ 1 - \frac{r \cos\theta}{2(R_0 + r \cos\theta)} \right]} \left[ \varepsilon [\gamma_1 + (2 - \gamma_1)(1 + \left| \frac{r}{r_A} \right|^n)] \sum_m \cos(2s\theta) \cos(m\zeta) \right] \\
&+ \frac{\varepsilon [\gamma_1 + (2 - \gamma_1)(1 + \left| \frac{r}{r_A} \right|^n)] \sum_m \cos(2s\theta) \cos(m\zeta)}{(R_0 + r \cos\theta)(1 + \left| \frac{r}{r_A} \right|^n)^2} B_0.
\end{align*}
\]

Several new effects can occur due to multiharmonic perturbation in Eqs. (11). It can be many resonant terms and can be chaos if these resonances overlap. For a few harmonics in perturbation it can be that nonresonant harmonics enhance the process of diffusion by the resonant harmonics. Another to the simple overlapping mechanism of the diffusion enhancing is that the inner web merges the otherwise unconnected outer stochastic sea. Due to encouragement of diffusion by the interactions, the stable regions tend to shrink with the increase of the number of interactions. This is more apparent in Fig. 13. The only difference between corresponding pictures in
Figs. 12 and 13 is their difference in $\gamma$.

4 Diffusion and Superdiffusion

The previous section provides a rather static view of the phase portrait of the differential Eq.(6). This section will discuss the evolution of magnetic surfaces (or, roughly, particle orbits).

In order to have a better understanding of how unstable orbits within the stochastic sea are, we need to calculate the so called Lyapunov exponent. In Fig.14, we give two examples to carry out such computations. The parameters of the two cases correspond to Fig.13(c), (d) respectively. Two initially close conditions were chosen in each case. As time goes on, the two orbits become more and more separated and as the two figures show, rapid exponential separation occurs within the first 30-40 periods (or, $t \cong 1000$ in real time). After that their distance more or less stabilizes and saturates. The Lyapunov exponent $\lambda$, characteristic of the initial rapid divergence, is defined as

$$\lambda = \frac{\ln d - \ln d_0}{t - t_0},$$  \hspace{1cm} (12)$$

where $d, d_0$ are the distances between the two orbits at moments $t, t_0$.

The Lyapunov exponents in Figs.14 are both close to 0.01. Under strong perturbation, we can observe stronger instability which can trigger superdiffusion. Fig.15 demonstrates such effect under strong perturbation ($\epsilon = 0.4$). A group of 100 close initial conditions were chosen and the square of the average distance of the points is plotted against real time. In Fig.15(a), the regular three-dimensional distance was used while in Fig.15(b), we are merely concerned with the transverse evolution of the points. In our toroidal system, it is the latter that is directly relevant to confinement.

The pictures are self-explanatory. Under normal diffusion, we should observe that the square of the distance evolve linearly with slope 1. However, this is not the case. Strong local instability at the beginning of the motion provides the seed for superdiffusion. As shown in Fig.15(b), diffusion almost occurred at the boundary of the stochastic sea shortly after $t=0$ and after dispersion of the points was completed, we observed a rapid
saturation of diffusion which indicates that the boundary had been reached. Due to the rapid saturation, the power law of the diffusion process becomes obscured.

5 Conclusion

The confinement attempts in fusion devices are greatly complicated by the fast transport due to perturbation. In general, the greater the perturbation, the greater the instability of magnetic surfaces (and thus, the particle orbits). When resonance conditions are satisfied and nonlinearity is small, material magnetic surface destruction occurs even against weak perturbation and long-range (as compared with the dimension of the device) transport of particles can occur. The intensity and scope of diffusion are strongly related to the intensity and form of perturbation as well as what Fourier components make up of the perturbation. Strong local instability in the vicinity of the center of the device provides the seed to superdiffusion which usually leads to rapid saturation apart from the origin.

Appendix: Lack of Accuracy Leads to Computational Chaos

There is yet one other aspect of vital importance that has been explored far from adequately, that is, justification of the precision of the calculations carried out in the study. It turns out that the accuracy study itself plays such a crucial role in studying chaos and diffusion that it should deserve to be treated rigorously in a separate report. What is provided in this appendix is an introduction on this topic and a suggestion for future explorations.

For simplicity, we shall adopt a one-dimensional model and study the two-dimensional phase space. The following equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = -\varepsilon \sin(kx - \Omega t)$$

(13)
describes the motion of a non-linear oscillator perturbed by a plane travelling wave. In the fourth order resonance situation (i.e., $\omega = \Omega/4$), again we use 4th order Runge-Kutta method to trace orbits in phase space. In all examples shown, phase points are sampled at $\Omega t = 10n\pi$. Often, inaccuracy can be introduced by using single-precision calculations inherent in the computer system instead of adhering to its double precision capabilities. Figs 16 and 17 give two comparisons between double and single precision calculations for two initial conditions. From this numerical analysis, we can be alerted that what we normally see as islands for more accurate computations may turn out to resemble the stochastic sea under less accurate calculations. We shall refer to this seemingly chaotic situation as computational chaos.

Fig. 18 offers to give us an idea of just how quickly computational chaos comes to dominate the picture. Fig. 18(b) shows the initial evolvement of phase orbits of three initially close conditions one of which evolves into the four islands in Fig. 18(a). Figs. 18(c) and 18(d) are sketched with a different but clearer scale to compare detailed differences between double and single precision calculations for the three conditions.

As a conclusion drawn from Figs. 16-18, in studying phase portraits of non-linear dynamical equations, double precision is heavily favored in numerical analysis against single precision. In times of real need, the quality of the double precision feature of the computer system should also be examined. The choice of precision features certainly provides one way to tune the accuracy of numerical analysis. There is, in our case, at least one other approach and it turns out to be a continuous tuning. We may, in turn, increase the Runge-Kutta step continuously and see how the inaccuracy in integration can affect our view of the phase portrait.

The result is astounding, as displayed in Fig. 19. Four increased steps were applied to the case of Fig. 18(a) to produce Fig. 19(a)-(d). Though aesthetically exquisite, the patterns observed provide a distorted view of the real situation in which stable islands should emerge. The chaotic regions also mark the appearance of attractors which are rather astonishing for our area preserving dynamics.

To conclude, computational chaos and "computational transport" may become a crucial concern depending on different situations and deserves more research efforts in the future.
References


Figure Captions

Figure 1
Toroidal coordinate system \((r, \theta, \zeta)\) vs Cartesian coordinate system \((x, y, z)\).

Figure 2
Phase portraits in the order of increasing \(\epsilon\) for \(R_0 = 4, m = 1, 2s = 2, q_0 = 1, r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = .8, \delta t = 0.005\).
\(N\): number of Runge-Kutta steps; \(NP\): number of Poincaré sections.
(a). \(\epsilon = 0.0, N = 7438216, NP = 2000;\)
(b). \(\epsilon = 0.1, N = 15304015, NP = 4000;\)
(c). \(\epsilon = 0.15, N = 27141779, NP = 6400;\)
(d). \(\epsilon = 0.2, N = 27361406, NP = 7000;\)
(e). \(\epsilon = 0.3, N = 33342636, NP = 7000;\)
(f). \(\epsilon = 0.4, N = 28566566, NP = 7000;\)
(g). \(\epsilon = 0.5, N = 30073237, NP = 7400;\)
(h). \(\epsilon = 0.6, N = 73311462, NP = 19200;\)

Figure 3
Fine structures for (h) of Fig.2. Examples of the phenomenon of sticking in the vicinity of fixed points.

Figure 4
Second order resonance with decreasing \(\epsilon\). \(R_0 = 4, m = 2, 2s = 2, q_0 = 1, r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = .8, \delta t = 0.005\).
\(N\): number of Runge-Kutta steps; \(NP\): number of Poincaré sections.
(a). \(\epsilon = 0.2, N = 29812734, NP = 6800;\)
(b). \(\epsilon = 0.1, N = 31296551, NP = 7000;\)
(c). \(\epsilon = 0.08, N = 60519837, NP = 13000;\)
(d). $\varepsilon = 0.06, N = 28246683, NP = 6200$

**Figure 5**
Sixth order resonance with decreasing $\varepsilon$. $R_0 = 4, m = 6, 2s = 6, q_0 = 1, r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = 0.8, \delta t = 0.005$.

$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.

(a). $\varepsilon = 0.1, N = 22780452, NP = 5200$;
(b). $\varepsilon = 0.08, N = 31871547, NP = 7200$;
(c). $\varepsilon = 0.04, N = 31756310, NP = 7200$;
(d). $\varepsilon = 0.02, N = 50450161, NP = 11200$;

**Figure 6**
Fine structures for (d) of Fig.5.

**Figure 7**
Testing the nonlinear resonance condition. $R_0 = 4r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = 0.8, \delta t = 0.005$.

$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.

(a). $m = 1, 2s = 2, q_0 = 2, \varepsilon = 0.1, N = 40703837, NP = 9000$;
(b). $m = 1, 2s = 2, q_0 = 2, \varepsilon = 0.02, N = 41572618, NP = 9800$;
(c). $m = 4, 2s = 2, q_0 = 0.5, \varepsilon = 0.1, N = 33945205, NP = 7400$;
(d). $m = 4, 2s = 2, q_0 = 0.5, \varepsilon = 0.02, N = 40930366, NP = 8600$;

**Figure 8**
Area of stochastic sea and resonance: $\varepsilon = 0.4, R_0 = 4, m = 2, 2s = 2, r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = 0.8, \delta t = 0.005$.

$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.

(a). resonance: $q_0 = 1, N = 24478578, NP = 5800$;
(b). non-resonance: $q_0 = 1.2, N = 29817032, NP = 6800$;
(c). non-resonance: $q_0 = 1.3, N = 31040999, NP = 7800$;
(d). non-resonance: $q_0 = 1.4, N = 27117497, NP = 6200$;
Figure 9
Area of stochastic sea and non-resonance: $\epsilon = 0.4, R_0 = 4, m = 1.2s = 2, r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = 0.8, \delta t = 0.005$.
$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.
(a). non-resonance: $q_0 = 1, N = 28656566, NP = 7000$;
(b). non-resonance: $q_0 = 1.2, N = 29253828, NP = 7000$;
(c). non-resonance: $q_0 = 1.3, N = 29477430, NP = 7000$;
(d). non-resonance: $q_0 = 1.4, N = 32728010, NP = 7600$;

Figure 10
Area of stochastic sea (resonance) and area of perturbation (1) $\epsilon = 0.3, R_0 = 4, m = 2.2s = 2, q_0 = 1, r_0 = 1, \gamma = 2, \gamma_1 = 4, \delta t = 0.005$.
$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.
(a). $r_A = 0.8, N = 29948663, NP = 6800$;
(b). $r_A = 1.2, N = 25130277, NP = 6600$;
(c). $r_A = 1.6, N = 32125504, NP = 8800$;

Figure 11
Area of stochastic sea (resonance) and area of perturbation (2) $\epsilon = 0.3, R_0 = 4, m = 2.2s = 2, q_0 = 1, r_0 = 1, \gamma = 4, r_A = 0.8, \delta t = 0.005$.
$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.
(a). $\gamma_1 = 4, N = 29229765, NP = 6800$;
(b). $\gamma_1 = 3, N = 28468041, NP = 6800$;
(c). $\gamma_1 = 2, N = 27614225, NP = 6800$;
(d). $\gamma_1 = 1, N = 28945672, NP = 6800$;

Figure 12
Multi-harmonic interactions and diffusion (1). $\epsilon = 0.08, R_0 = 4, 2s = 2, q_0 = 1, r_0 = 1, \gamma = 4, \gamma_1 = 4, r_A = 0.8, \delta t = 0.005$.
$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.
(a). $m = 2; N = 60519837, NP = 13000$;
(b). $m = 1.2; N = 37067624, NP = 8000$;
(c). $m = 1, 2, 3; N = 30236524, NP = 7000$
(d). $m = 1, 2, 3, 4; N = 40003844, NP = 7600$

**Figure 13**
Multi-harmonic interactions and diffusion (2). $\varepsilon = 0.08, R_0 = 4.2s = 2, q_0 = 1, r_0 = 1, \gamma = 1, \gamma_1 = 4, r_A = .8, \delta t = 0.005$
$N$: number of Runge-Kutta steps; $NP$: number of Poincaré sections.
(a). $m = 2; N = 60484381, NP = 13800$
(b). $m = 1, 2; N = 45542368, NP = 9800$
(c). $m = 1, 2, 3; N = 45036343, NP = 9800$
(d). $m = 1, 2, 3, 4; N = 42200474, NP = 9200$

**Figure 14**
Computation of the Lyapunov exponent.
$N$: number of Runge-Kutta steps; $NP$: number of points sampled.
(a). $\varepsilon = 0.08, R_0 = 4, m = 1, 2, 3, 4; 2s = 2, q_0 = 1, r_0 = 1, \gamma = 1, \gamma_1 = 4, r_A = 0.8, r_1 = 0.05, r_2 = r_1 + 10^{-6}, \theta_0 = 0, \zeta_0 = 0.7854(\approx \pi/4). N = 600000, NP = 3000, \delta t = 0.005$
(b). $\varepsilon = 0.08, R_0 = 4, m = 1, 2, 3; 2s = 2, q_0 = 1, r_0 = 1, \gamma = 1, \gamma_1 = 4, r_A = 0.8, r_1 = 0.05, r_2 = r_1 + 10^{-6}, \theta_0 = 0, \zeta_0 = 0.7854(\approx \pi/4). N = 600000, NP = 3000, \delta t = 0.005$

**Figure 15**
Diffusion. $\varepsilon = 0.4, R_0 = 4, m = 1, 2; 2s = 2, q_0 = 1, r_0 = 1, \gamma = 1, \gamma_1 = 4, r_A = 0.8$
$x_i(0) = 0.1 + 0.001 \cdot (0.05 + .1 \cdot \text{mod}(i, 10))$
y_i(0) = 0.0 + 0.001 \cdot (0.05 + .1 \cdot \text{mod}(i, 10))$
$\zeta = 0.25 \pi i = 1, 2, ..., 100$
$N$: number of Runge-Kutta steps; $NP$: number of points sampled.
$N = 600000, NP = 3000, \delta t = 0.005$
(a). Diffusion distance in real space vs real time.
(b). Transverse diffusion distance vs real time.
**Figure 16**
Comparison between single precision and double precision (1).
\( \varepsilon = 0.125, k = 15, \omega = \Omega/4, \Omega = 2\pi 5/8. \)
Initially, \( t = 0, x = 1.601866, \frac{dx}{dt} = 0; dt = 0.005. \)
All points sampled at \( \Omega t = 10n\pi, n = 0, 1, \ldots, 4096. \)
(a). Double precision;
(b). Single precision.

**Figure 17**
Comparison between single precision and double precision (2).
\( \varepsilon = 0.0125, k = 15, \omega = \Omega/4, \Omega = 2\pi 5/8. \)
Initially, \( t = 0, x = 1.601866, \frac{dx}{dt} = 0; dt = 0.005. \)
All points sampled at \( \Omega t = 10n\pi; n = 0, 1, \ldots, 2048. \)
(a). Double precision;
(b). Single precision.

**Figure 18**
Comparison between single precision and double precision (3).
\( \varepsilon = 0.0125, k = 15, \omega = \Omega/4, \Omega = 2\pi 5/8. \)
Initially, \( t = 0, \frac{dx}{dt} = 0; dt = 0.005. \)
All points sampled at \( \Omega t = 10n\pi. \)
(a). Double precision: \( t=0,x=1.58,n=0,1,\ldots,3000; \)
(b). Double precision: \( t=0,x=1.57,1.58,1.59,n=0,1,\ldots,128; \)
(c). Double precision: same as (b). different scale;
(d). Single precision: \( t=0,x=1.57,1.58,1.59,n=0,1,\ldots,128. \)

**Figure 19**
Computational chaos caused by inaccurate integration.
\( \varepsilon = 0.0125, k = 15, \omega = \Omega/4, \Omega = 2\pi 5/8. \)
Initially, \( t = 0, x = 1.58, \frac{dx}{dt} = 0. \)
All points sampled at \( \Omega t = 10n\pi. \)
(a). Double precision: \( dt=0.25,n=0,1,\ldots,10240; \)
(b). Double precision: $dt=0.32, n=0.1, \ldots, 3000;
(c). Double precision: $dt=0.5, n=0.1, \ldots, 3000;
(d). Double precision: $dt=1.0, n=0.1, \ldots, 3000.$
FIGURE 2
FIGURE 3
FIGURE 4
FIGURE 5

(a) 

(b) 

(c) 

(d)

-4 -2 0 2 4 -4 -2 0 2 4

-4 -2 0 2 4 -4 -2 0 2 4

-4 -2 0 2 4 -4 -2 0 2 4

-4 -2 0 2 4 -4 -2 0 2 4

-4 -2 0 2 4 -4 -2 0 2 4

-4 -2 0 2 4 -4 -2 0 2 4
FIGURE 7
FIGURE 9
FIGURE 10
FIGURE 16

FIGURE 17
FIGURE 18