MAGNETICALLY CONTROLLED DEPOSITION OF METALS USING GAS PLASMA

Quarterly Progress Report
October-December 1994

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Work Performed Under A U.S. Department of Energy Grant to the University of Idaho
DE-FEO7-93ID3220
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Abstract

This document reports the status of grant DE-FE07-93ID3220 for the October-December 1994 quarter.

The objective of the grant is to develop a method of spraying materials on a substrate in a controlled manner to eliminate the waste inherent in present plating processes. The process under consideration is magnetically controlled plasma spraying. The project continues to be on schedule. The field equations have been developed and were reported in the April-June 1994 Progress Report. The equations for the external magnetic field were reported in the July-September 1994 progress report.

The field equations have been cast in a format that allows solution using Finite Element (FE) techniques. The development of the computer code that will allow evaluation of the proposed technique and design of an experiment to prove the proposed process is underway.

Background

Thin layers of secondary material are plated on substrates either by plating or spraying processes. Plating operations produce large amounts of hazardous liquid waste. Spraying, while one of the less waste intensive methods, produces "over spray" which is waste that is a result of uncontrolled nature of the spray stream. In many cases the over spray produces a hazardous waste.

Spray coating is a mature process with many uses. Material can be deposited utilizing spraying technology in three basic ways: "Flame spraying", direct spraying of molten metals and/or plasma spraying. This project is directed at controlling the plasma spraying process and thereby minimizing the waste generated in that process. The proposed process will utilize a standard plasma spray gun with the addition of magnetic fields to focus and control the plasma.

In order to keep development cost at a minimum, the project was organized in phases. The first and current phase involves developing an analytical model that will prove the concept and be used to design a prototype. Analyzing the process and using the analysis has the potential to generate significant hardware cost savings.

The April-June 1994 quarterly report described the equations that were developed as well as the
selected numerical procedure that would be used. The numerical technique chosen is Finite Elements (FE), Method of Weighted Residuals (MWR), using the Galerkin approach. Several distinct advantages of the finite element scheme such as natural boundary conditions and the use of lower order equations led to the decision to use finite elements.

The Field Equations

The field equations are repeated here for convenience.

Conservation of Mass

The conservation of mass equation in vector form is:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \]  \hspace{1cm} (1)

Where:

\( \rho \) is fluid density  
\( t \) is time  
\( \vec{v} \) is the vector velocity

Conservation of Momentum

Conservation of momentum is a vector equation that contains three components, and hence represents three equations:

\[ \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) + \nabla \cdot F + \rho \vec{b} \]  \hspace{1cm} (2)

Where:

\( T \) is the stress Tensor (a tensor is represented in the equation with an under bar).
\( \rho \) is the body force term. Gravity is being neglected, and the only body force remaining is that which results from the magnetic field acting on the plasma.

This magnetic field generated force is described by:

\[ \rho \vec{b} = \frac{1}{\mu_0} \left[ (\nabla \times \vec{B}) \times \vec{B} \right] \]  \hspace{1cm} (3)

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\( \mu_0 \) is the magnetic permitivity, a constant, and 
\( \mathbf{B} \) is the magnetic flux density vector, represented with an over bar in the equation.

**The Energy Equation**

Conservation of energy is a scalar equation:

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} + \nabla \mathbf{u} \right) - \mathbf{T} : \mathbf{D} - \nabla \cdot \mathbf{q}'' - \mathbf{q}''' = 0
\]  
(4)

Where:
- \( u \) is internal energy,
- \( \mathbf{D} \) is the rate of deformation tensor. The tensor contraction of \( \mathbf{T} \) and \( \mathbf{D} \) represents the compression work performed on or by the fluid. This term is very small and will be neglected for most calculations,
- \( \mathbf{q}'' \) is heat flux, and
- \( \mathbf{q}''' \) is the internal heat generation term.

Making use of the definition of enthalpy, \( h = u + p/\rho \) and utilizing, \( C_p = \frac{\partial h}{\partial T} \), the energy equation can be combined with conservation of mass to form:

\[
\rho C_p \left( \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi \right) = -\nabla \cdot \mathbf{q}'' - \mathbf{q}'''/C_p
\]  
(5)

Where:
- \( \phi \) is temperature, and
- \( C_p \) is the specific heat at constant pressure

This development assumes that the compression work and the energy generated as a result of viscous effects is small compared with convection and advection.

**The Magnetic Field**

The magnetic field is represented by a combination of Faraday’s and Ohm’s Laws:

\[
\frac{\partial \mathbf{B}}{\partial t} - \frac{1}{\sigma \mu_0} \nabla^2 \mathbf{B} - (\nabla \cdot \mathbf{v}) \mathbf{B} = 0
\]  
(6)

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Where:

\( \sigma \)  
Is the electric conductivity. In the case of a plasma which is not embedded in a strong external magnetic field, the conductivity is a scalar. More complex magnetic field configurations require a tensor conductivity, but initial computations will be made with a scalar conductivity. The effect of tensor conductivity will be evaluated later.

Vector manipulation of the right hand side of this equation provides a form that will be used in the numerical analysis:

\[
\frac{\partial \vec{B}}{\partial t} - \frac{1}{\sigma \mu_0} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) \tag{7}
\]

Where the variables in the equation are the same as those shown above.

The Numerical Formation and Solution Technique

As was reported previously a computer code is being developed to solve equations (1), (2), (5), and (7) using Finite Element analysis. Finite Elements have many advantages over Finite Differences as was described in the April-June Progress Report. The code is being developed using the Method of Weighted Residuals and the Galerkin method.

The finite element technique, method of weighted residuals recognizes that in any numerical technique the results are not exact, and if equations (1), (2), (5), and (7) were rearranged so that all terms were moved to one side of the equality sign, the numerical technique will result in a residual, \( R \), rather than exactly zero:

\[
R_{\text{mass}} = -\frac{\partial \rho}{\partial t} \vec{\nabla} \cdot \vec{\rho} \vec{v} \tag{8}
\]

\[
R_{\text{momentum}} = \rho \left( \frac{\partial \vec{v}}{\partial t} - \vec{\nabla} \cdot \vec{\rho} \vec{v} \right) - \vec{\nabla} \cdot \vec{I} - \rho \vec{b} \tag{9}
\]

\[
R_{\text{energy}} = \rho c_p \left( \frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla} \phi \right) - \vec{\nabla} \cdot (\vec{q}' - \vec{q}^w) \tag{10}
\]

\[
R_{\text{Mag Flux Den.}} = -\frac{\partial \vec{B}}{\partial t} - \frac{1}{\sigma \mu_0} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) \tag{11}
\]

The method of weighted residuals by use of a weighing factor, \( W \), requires the integral of the weighing factor and the residual (one for each of the above equations, (8), (9), (10), and (11),
over the calculational domain to be zero:

$$\int_{vol} W R \, dVol = 0$$

Equations (8), (9), (10), and (11) can then be recast in the Method of Weighted residuals integral, i.e. equation (12). Gauss's divergence theorem is applied to the last three of these equations to provide "natural boundary" conditions as well as to reduce the order of the energy equation. Additionally two constitutive equations are introduced, Fourier's Law, equation (13), and a Newtonian fluid with Stokes condition, equation (14):

$$\bar{q} = -k \bar{\nabla} \phi$$
$$\bar{\tau} = -\frac{2}{3} \mu (\bar{\nabla} \cdot \bar{\nabla}) I + 2 \mu \left( \frac{1}{2} \bar{\nabla} \bar{\nabla} + \frac{1}{2} \bar{\nabla} \bar{\nabla} \right) \cdot P \, I$$

Where:
- $k$ is the thermal conductivity, and
- $\mu$ is the viscosity.

The resulting four field equations are then cast in matrix notation which is convenient for coding in a computer model:

$$\int_{vol} \left[ W \frac{\partial \rho}{\partial t} + W \rho \bar{\nabla} \cdot \bar{\nabla} + W \bar{\nabla} \cdot \bar{\nabla} \rho \right] dvol = 0$$
$$\int_{vol} \left[ \rho W \frac{\partial \bar{\nabla}}{\partial t} + \rho W \bar{\nabla} \cdot (\bar{\nabla} \bar{\nabla}) - \frac{2}{3} \mu (\bar{\nabla} \cdot \bar{\nabla})(\bar{\nabla} W) + \mu (\bar{\nabla} \bar{\nabla}) \cdot (\bar{\nabla} \bar{\nabla}) \ight] dvol = 0$$
$$\int_{vol} \left[ \rho C_p W \frac{\partial \phi}{\partial t} + \rho C_p W (\bar{\nabla} \cdot \bar{\nabla} \phi) + k (\bar{\nabla} W - \bar{\nabla} \bar{\nabla}) \cdot W \bar{\nabla} \right] dvol = 0$$

Where:
- $\bar{\nabla} = \rho \bar{\nabla} \cdot (\bar{\nabla} x \bar{\nabla} x \bar{\nabla})^T$, $\bar{\tau}_n$ is the surface shear, and $q_n$ is input heat flux.

The superscript, T, is used to denote the transpose of a vector.
The plasma spray problem is axisymmetric in nature and as such can be treated and solved in two dimensions as a first approximation. Three dimensional effects will be evaluated later. While equations (15), (16), (17), and (18) were simplified into two dimensions in both Cartesian and cylindrical coordinates. Only the cylindrical solution is presented here. The axis of symmetry (z) is chosen as the center line of the plasma spray gun. The equation set now consist of six equations. Two scalar field equations - conservation of mass and conservation of energy, two vector filed equations - conservation of momentum, and the time rate of change of magnetic flux density.

Conservation of Mass

\[ \int_{Area} [W_r \frac{\partial p}{\partial t} + \rho W_r (\frac{\partial v_r}{\partial t} + v_r) + \rho W_{rv} \frac{\partial v_r}{\partial z} + \rho W_{rz} \frac{\partial v_r}{\partial z} + \rho W_{rz} \frac{\partial v_r}{\partial r}] \, d\text{Area} = 0 \]  \hspace{1cm} (19)

Conservation of Momentum

**Axial Component**

\[ \int_{Area} \left[ \rho W_r \frac{\partial v_z}{\partial t} + \rho W_{rv} \frac{\partial v_z}{\partial r} + \rho W_{rz} \frac{\partial v_z}{\partial z} - \frac{2}{3} \mu (r \frac{\partial v_r}{\partial r} + v_r) \frac{\partial W}{\partial z} - \frac{2}{3} \mu \frac{\partial v_z}{\partial z} \frac{\partial W}{\partial r} 
+ \frac{\mu_r}{\partial r} \frac{\partial W}{\partial z} \right] \, d\text{Area} 
- \int_{\text{Line}} W_t T_z^a \, d\text{Line} = 0 \]  \hspace{1cm} (20)

**where:**

\[ r \rho b_z \frac{r}{\mu_0} \left( \frac{\partial B_z}{\partial r} - \frac{\partial B_r}{\partial z} \right) B_r \]  \hspace{1cm} (21)

**Radial Component**

\[ \int_{\text{Area}} \left[ \rho W_r \frac{\partial v_r}{\partial t} + \rho W_{rv} \frac{\partial v_r}{\partial r} + \rho W_{rz} \frac{\partial v_r}{\partial z} - \frac{2}{3} \mu (r \frac{\partial v_r}{\partial r} + v_r) \frac{\partial W}{\partial r} - \frac{2}{3} \mu \frac{\partial v_z}{\partial z} \frac{\partial W}{\partial r} 
+ 2 \mu_r \frac{\partial W}{\partial r} \frac{\partial v_r}{\partial r} + \mu_r \frac{\partial W}{\partial z} \frac{\partial v_r}{\partial z} + \mu_r \frac{\partial W}{\partial z} \frac{\partial v_r}{\partial z} - P_r \frac{\partial W}{\partial r} - W_r b_z \right] \, d\text{Area} 
- \int_{\text{Line}} W_r T_z^a \, d\text{Line} = 0 \]  \hspace{1cm} (22)

**where:**

\[ r \rho b_z \frac{r}{\mu_0} \left( \frac{\partial B_z}{\partial z} - \frac{\partial B_r}{\partial r} \right) B_z \]  \hspace{1cm} (23)

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Conservation of Energy

\[ \int_{\text{area}} \left[ \rho C_p r \frac{\partial W}{\partial t} + \rho C_p W r v_r \frac{\partial \Phi}{\partial r} + \rho C_p W r v_z \frac{\partial \Phi}{\partial z} - W r q'' + k r \frac{\partial W}{\partial r} \right] \, d\text{area} - \int_{\text{Line}} W r q_n \, d\text{Line} = 0 \]  

(24)

Magnetic Flux Density

Axial Component

\[ \int_{\text{area}} \left[ W r \frac{\partial B_z}{\partial t} - \frac{r}{\sigma \mu_0} \left( \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \frac{\partial W}{\partial z} - W r \frac{\partial (v_z B_r)}{\partial r} - W v_r B_z \right] \, d\text{area} = 0 \]  

(25)

Radial Component

\[ \int_{\text{area}} \left[ W r \frac{\partial B_r}{\partial t} + \frac{r}{\sigma \mu_0} \left( \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \frac{\partial W}{\partial z} - W r \frac{\partial (v_z B_r)}{\partial r} - W v_r B_z \right] \, d\text{area} = 0 \]  

(26)

Discretization

Two type of elements are being used in the computer simulation, triangle and quadrilaterals. The triangular element contains 6 nodes, and the quadrilateral contains 9.

Equations (19) through (26), the six field equations contain six variables: \( v_z, v_r, \rho, \Phi, B_r, \text{ and } B_z \). All of the variables are evaluated at each node in the element with the exception \( \rho \) which is evaluated on at the corner nodes (see Figure 1.) The value of each variable in the element is determined combining the value if that variable at the node using the "shape function":

\[ u = \sum_{i=1}^{m} N_i(r,z) \, u_i \]  

(27)

Where:

- \( u \) is any of the variables, i.e. \( v_z, v_r, \rho, \Phi, B_r, \text{ and } B_z \),
- \( N_i \) is the shape function,
- \( u_i \) is the value of the variable \( u \) at node \( I \), and
- \( nn \) is the number of nodes in the element.
The shape functions (or interpolation functions) and the weighing factor described in equation (12) must be such that the \((k-1)\)th derivative is continuous where \(k\) is the order of differentiation of governing differential equations. Our use of the Gauss's theorem reduced the order of the field equations so that only the first derivative of the shape function must be continuous:

\[
\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{m} \frac{\partial N_i}{\partial x} u_i \quad \text{and} \quad \frac{\partial u}{\partial y} \cdot \sum_{i=1}^{m} \frac{\partial N_i}{\partial y} u_i
\]  

(28) & (29)

The Galerkin approximation sets the Weighing Factor \((W\) in Equation 12) and the shape function \((N_i\) in equation 27) to the same function.

The finite element method is an integral method based on a computational process which is typically composed of quadrilateral and triangular elements with either straight or curved sides. We have chosen to use straight sided elements for both the triangular and quadrilateral elements with linear shape functions for the triangles and isoparametric shape functions for the quadrilaterals.

Velocities, temperature, and magnetic flux density are evaluated using either 6 node triangular elements of 9 quadrilateral elements. Density is evaluated using either 3 node triangular elements or 4 node quadrilaterals. It is necessary to use mixed interpolation for the Navier-Stokes equation since equal order interpolation for velocity and pressure (density in our case) causes a singular (and unsolvable) set of equations (Reference 1).

The 6 and 9 node shape functions are denoted in further development by \(N\), and the 4 and 3 node shape functions are denoted by \(M\).

Casting the equations in terms of each element and summing each elements contribution results in equations (30) through (33). The appropriate change of nomenclature has been made, with the subscript \(i\) referring to a shape factor and \(j\) to the weighing factor.

**Figure 1**, Nodal arrangement used for various elements and variables
Conservation of Mass

\[ \sum_{i=1}^{m} \int_{\text{Elem Area}} [M_j(n_{i}r_{i}) \sum_{k=1}^{m} \frac{\partial p_{i}}{\partial t} + (M_j(n_{i}r_{i}) \sum_{k=1}^{m} M_{k}p_{i}) \sum_{k=1}^{m} \frac{\partial n_{i}v_{i}}{\partial z} + \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} \frac{\partial M_{k}}{\partial z} \rho_{i} \sum_{k=1}^{m} \frac{\partial M_{k}}{\partial r} \rho_{i}] \, dz \, dr = 0 \]  (30)

Conservation of Momentum

Axial (z)

\[ \sum_{i=1}^{m} \int_{\text{Elem Area}} [n_{i} \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} M_{k}p_{i} \sum_{k=1}^{m} \frac{\partial v_{z}}{\partial t} + n_{i} \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} M_{k}p_{i} \sum_{k=1}^{m} \frac{\partial n_{i}v_{z}}{\partial z} + \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} \frac{\partial M_{k}}{\partial z} \rho_{i} \sum_{k=1}^{m} \frac{\partial M_{k}}{\partial r} \rho_{i}] \, dz \, dr = 0 \]  (31)

Radial (r)

\[ \sum_{i=1}^{m} \int_{\text{Elem Area}} [n_{i} \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} M_{k}p_{i} \sum_{k=1}^{m} \frac{\partial v_{i}}{\partial t} + n_{i} \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} M_{k}p_{i} \sum_{k=1}^{m} \frac{\partial n_{i}v_{i}}{\partial z} + \sum_{k=1}^{m} n_{i}v_{i} \sum_{k=1}^{m} \frac{\partial M_{k}}{\partial z} \rho_{i} \sum_{k=1}^{m} \frac{\partial M_{k}}{\partial r} \rho_{i}] \, dz \, dr = 0 \]  (32)
Conservation of Energy

\[
\sum_{j=1}^{n_e} \int_{\text{ElemArea}} \left[ \left( \frac{C_p N_j}{\sum_{i=1}^{n_n} N_j r_i} \sum_{k=1}^{n_n} M_{k,j} \phi \frac{\partial \phi}{\partial t} \right) + N_j C_p \sum_{i=1}^{n_n} N_j r_i \sum_{k=1}^{n_n} M_{k,j} \phi \sum_{i=1}^{n_n} N_j r_i \sum_{k=1}^{n_n} K_{z,k} \frac{\partial N_j \phi}{\partial z} \right. \\
+ \frac{1}{n_n} \sum_{k=1}^{n_n} \frac{\partial N_j}{\partial t} \phi_i - N_j \sum_{i=1}^{n_n} N_j r_i \sum_{k=1}^{n_n} N_j \phi_i + k \sum_{i=1}^{n_n} N_j r_i \left( \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial z} \right) \phi_i \\
\left. + \frac{1}{n_n} \sum_{k=1}^{n_n} \frac{\partial N_j}{\partial t} \phi_i \right] \text{d line} = 0
\] (33)

Magnetic Flux Density

Axial (z)

\[
\sum_{j=1}^{n_e} \int_{\text{ElemArea}} N_j \left( \sum_{i=1}^{n_n} N_j r_i \right) \frac{\partial B_z}{\partial t} + \frac{1}{\mu_0 \sigma} \left( \sum_{i=1}^{n_n} N_j r_i \right) \sum_{i=1}^{n_n} \frac{\partial N_j B_{z_i}}{\partial r} - \sum_{k=1}^{n_n} \frac{\partial N_j B_{z_k}}{\partial r} \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial r} \\
+ N_j \left( \sum_{i=1}^{n_n} N_j r_i \right) \left( \sum_{k=1}^{n_n} N_j \phi_{z_k} \right) \sum_{i=1}^{n_n} \frac{\partial N_j B_{z_i}}{\partial z} - \sum_{k=1}^{n_n} \frac{\partial N_j B_{z_k}}{\partial z} \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial z} \\
+ \sum_{k=1}^{n_n} \frac{\partial N_j B_{z_k}}{\partial r} \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial r} \frac{\partial N_j}{\partial z} - N_j \left( \sum_{i=1}^{n_n} N_j r_i \right) \left( \sum_{i=1}^{n_n} N_j \phi_{z_i} \right) \text{d dz} = 0
\] (34)

Radial (r)

\[
\sum_{j=1}^{n_e} \int_{\text{ElemArea}} \left[ N_j \left( \sum_{i=1}^{n_n} N_j r_i \right) \frac{\partial B_r}{\partial t} + \frac{1}{\mu_0 \sigma} \left( \sum_{i=1}^{n_n} N_j r_i \right) \sum_{i=1}^{n_n} \frac{\partial N_j B_{r_i}}{\partial r} - \sum_{k=1}^{n_n} \frac{\partial N_j B_{r_k}}{\partial r} \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial r} \\
+ N_j \left( \sum_{i=1}^{n_n} N_j r_i \right) \left( \sum_{k=1}^{n_n} N_j \phi_{z_k} \right) \sum_{i=1}^{n_n} \frac{\partial N_j B_{z_i}}{\partial z} - \sum_{k=1}^{n_n} \frac{\partial N_j B_{z_k}}{\partial z} \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial z} \\
+ \sum_{k=1}^{n_n} \frac{\partial N_j B_{z_k}}{\partial r} \sum_{i=1}^{n_n} \frac{\partial N_j}{\partial r} \frac{\partial N_j}{\partial z} \right] \text{dr dz} = 0
\] (35)

Where:

- \( n_e \) is the number of elements in the calculation domain
- \( n_n \) is the number of nodes in an element (9 for a quadrilateral or 6 for a triangle)
- \( n_{nnn} \) is the number of nodes in an element (4 for a quadrilateral of 3 for a triangle)

If the time dependant terms are eliminated i.e. a steady state solution the field equation set, equations (30), (31), (32), (33), (34), and (35) can be expressed as:

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Where:

- \( C \) is a matrix of coefficients
- \( u \) is a vector consisting of all the independent variables
- \( b \) is a vector
- \( p \) is 1, 2, 3, \ldots, number of independent variables, and
- \( q \) is 1, 2, 3, \ldots, number of independent variables.

The \( C \) matrix is a collection of all of the variable coefficients and \( u \) is the vector containing all of the independent variables. The number of independent variables per node is 6 for the corner nodes and 5 for the interior nodes. The total number of independent variables is then the number of nodes multiplied by the variables to be evaluated at that node. The calculation process in the finite element code involves proceeding one element at a time and collecting the contributions to the coefficient for each node, and in this way to construct the entire \( C \) matrix and \( b \) vector.

One node's contribution to the variables in one element is represented by the Matrix \( C \) such that:

\[
[C_{pq}] = \sum_{k=1}^{n_a} [C_{ij}]_k
\]

\[
[C_{ij}]_k = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}
\]

\[
[u_q] = \sum_{k=1}^{n_a} [u_i]_k
\]

and

\[
[b_p] = \sum_{k=1}^{n_a} [b_i]_k
\]
The contributions i.e. $C_i$'s and $b_i$'s are shown in Appendix C. The $C$ matrix and the $b$ vector are constructed by collecting the contributions from one element at a time. The resultant matrices contain each of the variables for each calculational location. The $u$ vector contains the independent variables for which we seek a solution. In a given element the $u$ vector is:

$$u_1 = v_z, \quad u_2 = \rho, \quad u_3 = v_r, \quad u_4 = \phi, \quad u_5 = B_r, \quad u_6 = B_z$$

**Numerical Integration**

The contributions to the $C$ matrix as present in Appendix C are integral equations. The integration of these contributions is carried out in two different ways. Contributions from quadrilateral elements are integrated using Gauss Quadrature. The contributions from triangular elements is obtained using a closed form technique.

The Gauss Quadrature method is a highly accurate numerical approximation that involves transforming the problem into a normalized curvilinear coordinates, and summing weighted values of a function for specific points. In two dimensions the function $F(\xi, \eta)$ for a Gauss Quadrature is:

$$\int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta) d\xi d\eta = \sum_{k=1}^{n_\xi} \sum_{l=1}^{n_\eta} W_l(\xi) W_k(\eta) F(\xi, \eta)$$

$$0 \leq W_l(\xi) \leq 1, \quad 0 \leq W_k(\eta) \leq 1$$

Where the $W$'s are the weights at the Gauss points and $n_\xi$ and $n_\eta$ are the number of Gauss points in the intervals $-1 \leq \xi \leq +1$ and $-1 \leq \eta \leq +1$. For a polynomial function of order $m$, Gauss quadrature provides an exact integration with $(n+1)/2$ Gauss points. The integration here uses 3 Gauss points as shown in Table 1.

<table>
<thead>
<tr>
<th>Location of Gauss Points</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-(0.6)^{16}$</td>
<td>5/9</td>
</tr>
<tr>
<td>0</td>
<td>8/9</td>
</tr>
<tr>
<td>$(0.6)^{16}$</td>
<td>5/9</td>
</tr>
</tbody>
</table>

**TABLE 1. Location of points and weights used in Gauss Quadrature**

The use of Gauss Quadrature requires transformation of the elements from the physical coordinate system to a normalized computational coordinate system i.e. from $r,z$ to $\xi, \eta$. (Figure 2)

The derivatives must also be converted to this computational space. The process involves using
the chain rule to obtain the derivative of the shape factor in the new computational space and generating a matrix that is used to accomplish this transformation. The resulting transformation matrix is called the Jacobean:

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{bmatrix} - J
\begin{bmatrix}
\frac{\partial z}{\partial \xi} & \frac{\partial r}{\partial \xi} \\
\frac{\partial z}{\partial \eta} & \frac{\partial r}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial N_i}{\partial z} \\
\frac{\partial N_i}{\partial r}
\end{bmatrix} = J
\begin{bmatrix}
\frac{\partial N_i}{\partial z} \\
\frac{\partial N_i}{\partial r}
\end{bmatrix}
\]  

(42)

Where \( J \) is used to signify the Jacobean. It is also necessary to convert back to the physical coordinate system from the computational system. This is accomplished by utilizing the inverse of the Jacobean, \( J^{-1} \).

The interpolation or shape functions for \( r \) and \( z \) are defined as:

\[
\begin{align*}
    z &= \sum_{i=1}^{n_n} N_i \, z_i  \\
    r &= \sum_{i=1}^{n_n} N_i \, r_i
\end{align*}
\]  

(43) & (44)

Where \( n_n \) is the number of nodes in the element being considered. The Jacobean is then obtained using (46), (47), and (48) as:

\[
J = \begin{bmatrix}
\sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \xi} z_i & \sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \eta} r_i \\
\sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \eta} z_i & \sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \xi} r_i
\end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\
J_{21} & J_{22} \end{bmatrix}
\]

(45)
The inverse of the Jacobean, \( J^{-1} \), found from:

\[
J^{-1} = \frac{1}{\text{Det } J} \left[ \sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \eta} r_i - \sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \xi} z_i \right] \left[ \sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \eta} z_i \sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \xi} r_i \right]^{-1}
\]

Where: \( \text{Det } J = (\sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \xi}) (\sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \eta}) - (\sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \eta}) (\sum_{i=1}^{n_n} \frac{\partial N_i}{\partial \xi}) \)

\( \text{Det } J \) is the determinate of \( J \). The area integral is also transformed using the determinate of \( J \) (\( dx \ dy = \text{det } J \ d\xi \ d\eta \)). Now all of the elements have been defined that allow the transformation of the integral equations that form the contributions to the coefficient matrix \( C \), as shown in the following sample:

\[
\int \int_{\text{Element Area}} \begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{bmatrix} \ dx \ dy = \int \int_{-1,1} J^{-1} \begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{bmatrix} \ Det \ J \ d\xi \ d\eta
\]

Using the same techniques the line integral that results from the application of Gauss Theorem to produce natural boundary conditions is evaluated. Using \( F \) as an arbitrary function the conversion of the line integral becomes:

\[
\int_{\text{Line}} F \ ds = \int_{-1}^{1} F \sqrt{[J_{11}(\xi,\eta=-1)] + [J_{12}(\xi,\eta=-1)]} \ d\xi
\]

Now all the equations are in a form that will allow the use of Gauss Quadrature for integration.

**Integration of Triangular Elements**

It is possible to evaluate the integrals in the contribution equations that make up coefficient matrix in closed form when the element is a triangle. Figure 3 shows both the three node triangle that is being used for density, and the six triangular elements that are being used for velocity, temperature, and magnetic flux density.
Independent variables represented by $u$ which is a function of the shape function and the variable at the node point. Equation (50) is the equation for any of the independent variables for a 3 node triangular element.

$$u = \sum_{i=1}^{3} N_i(z,r) u_i$$  \hspace{1cm} (50)

*Where:*

$$N_i = \frac{a_i \cdot b_i \cdot c_i}{2\Delta}$$  \hspace{1cm} (51)

and

$$a_1 = z_2 r_3 - z_3 r_2 \quad b_1 = r_2 - r_3 \quad c_1 = z_3 - z_2$$
$$a_2 = z_3 r_1 - z_1 r_3 \quad b_2 = r_3 - r_1 \quad c_2 = z_1 - z_3$$
$$a_3 = z_1 r_2 - z_2 r_1 \quad b_3 = r_1 - r_2 \quad c_3 = z_2 - z_1$$

$$\Delta = (\text{Area of the Element}) = \frac{1}{2} \begin{vmatrix} 1 & z_1 & r_1 \\ 1 & z_2 & r_2 \\ 1 & z_3 & r_3 \end{vmatrix}$$  \hspace{1cm} (52)

The derivatives of $N_i$ with respect to $z$ and $r$ can be obtained from (40):

$$\frac{\partial N_i}{\partial z} = \frac{b_i}{2\Delta} \quad \text{and} \quad \frac{\partial N_i}{\partial r} = \frac{c_i}{2\Delta}$$  \hspace{1cm} (53) & (54)

The integrals encountered in the computer code can be solved directly:

$$\int \int_{\Delta} \frac{\partial N_i}{\partial z} dz dr - \frac{b_i \Delta}{2\Delta}$$  \hspace{1cm} and  \hspace{1cm} $$\int \int_{\Delta} \frac{\partial N_i}{\partial r} dz dr - \frac{a_i \Delta}{2\Delta}$$  \hspace{1cm} (55) & (56)
SUMMARY

A computer code based on the development described above is being checked-out. Check cases without magnetic fields/forces are currently in process. A body of work on the solutions to jet flow problems was generated about the time the turbo-jet engine was been developed. Exact solutions and test data from this jet flow work is being used to check the program prior to complicating the solution with magnetic fields. Problems in both incompressible (Ref. 3) and incompressible (Ref. 4) flow are being used to assure code accuracy.

References
The six equations being solved in the finite element program are arranged so that they take the form:

\[ C_{pq} \, u_q = b_p \quad (A1) \]

Where:
- \( C \) is a matrix of coefficients
- \( u \) is a vector consisting of all the independent variables
- \( b \) is a vector
- \( p \) is \( 1,2,3,... \), number of independent variables, and
- \( q \) is \( 1,2,3,... \), number of independent variables.

The \( C \) matrix is arranged such that each element in the matrix is the coefficient of one of the independent variables. In this case we are solving for six variables at each node: \( v, \rho, v, \phi, B_z, \) and \( B_r \). These variables become the \( u \) vector. This process results in collecting the "contributions" to the solution from each node. Which takes the form:

\[ [C_{pq}] = \sum_{k=1}^{n_0} [C_{ij}]_k \]

WHERE:

\[ [C_{ij}]_k = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix} \quad (A2) \]

\[ [u_q] = \sum_{k=1}^{n_0} [u_j]_k \quad and \quad [b_p] = \sum_{k=1}^{n_0} [b_j]_k \]

\[ [u] = \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{bmatrix} \quad and \quad [b] = \begin{bmatrix}b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6
\end{bmatrix} \quad (A4) \text{ and } (A5) \]
The integral equations for each of the C and b elements follows:

\[
C_{11} = \int_{\text{element}} [N_j^4(\sum_{k=1}^{9} M_k r_k) (\sum_{k=1}^{9} N_k r_k)(\sum_{k=1}^{9} N_k v_{zk}) - \frac{\partial N_i}{\partial z} + \frac{\partial N_i}{\partial r}] dr dz
\]

\[
C_{12} = 0
\]

\[
C_{13} = \int_{\text{element}} [\mu \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial z} - \frac{2}{3} \frac{\partial N_i}{\partial r} \left[ \sum_{k=1}^{9} N_k r_k \right] + N_j] dr dz
\]

\[
C_{14} = - \int_{\text{element}} \left[ R \left( \sum_{k=1}^{4} M_k p_k \right) \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial z} \right] dr dz
\]

\[
C_{15} = - \int_{\text{element}} \left[ \frac{1}{\mu_0} \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k B_{zk} \right) \frac{\partial N_i}{\partial r} \right] dr dz
\]

\[
C_{16} = - \int_{\text{element}} \left[ \frac{1}{\mu_0} \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k B_{zk} \right) \frac{\partial N_i}{\partial z} \right] dr dz
\]

\[
C_{21} = \int_{\text{element}} \left[ M \left( \sum_{k=1}^{9} M_k r_k \right) \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial z} \right] dr dz
\]

\[
C_{22} = \int_{\text{element}} \left[ M \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial M_k}{\partial z} + \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial M_k}{\partial r} \right] dr dz
\]

\[
C_{23} = \int_{\text{element}} \left[ M \left( \sum_{k=1}^{4} M_k p_k \right) \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial r} + N_j \right] dz dr
\]

\[
C_{24} = 0
\]

\[
C_{25} = 0
\]

\[
C_{26} = 0
\]

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\[ C_{31} = \int_{\text{element}} \left[ \mu \left( \sum_{k=1}^{9} N_k r_k \right) \left( \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial r} - \frac{2}{3} \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial z} \right) \right] \, dr \, dz \]

\[ C_{32} = 0 \]

\[ C_{33} = \int_{\text{element}} \left[ N_j \left( \sum_{k=1}^{4} N_k r_k \right) \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial N_i}{\partial z} + \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial N_i}{\partial r} \right] \right] \, dr \, dz \]

\[ + \frac{4}{3} \mu \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial r} \left( \sum_{j=1}^{9} N_j \right) + \mu \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial z} \right] \, dr \, dz \]

\[ C_{34} = -\int_{\text{element}} \left[ R \left( \sum_{k=1}^{4} M_k \rho_k \right) \left( \sum_{k=1}^{9} N_k r_k \right) \frac{\partial N_i}{\partial r} \right] \, dr \, dz \]

\[ C_{35} = -\int_{\text{element}} \left[ \frac{N_j}{\mu_0} \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k B_{zk} \right) \frac{\partial N_i}{\partial r} \right] \, dr \, dz \]

\[ C_{36} = -\int_{\text{element}} \left[ \frac{N_j}{\mu_0} \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k B_{zk} \right) \frac{\partial N_i}{\partial z} \right] \, dr \, dz \]

\[ C_{41} = 0 \]

\[ C_{42} = 0 \]

\[ C_{43} = 0 \]

\[ C_{44} = \int_{\text{element}} \left[ N_j C_k \left( \sum_{k=1}^{4} M_k \rho_k \right) \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial N_i}{\partial z} + \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial N_i}{\partial r} \right] \right] \, dr \, dz \]

\[ + \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial N_i}{\partial z} \right] \, dr \, dz \]

\[ C_{45} = 0 \]

\[ C_{46} = 0 \]

\[ C_{51} = -\int_{\text{element}} \left[ N_j \left( \sum_{k=1}^{9} N_k r_k \right) \left( \sum_{k=1}^{9} N_k B_{zk} \right) \frac{\partial N_i}{\partial r} \right] \, dr \, dz \]

\[ C_{52} = 0 \]

\[ C_{53} = 0 \]

\[ C_{54} = 0 \]

\[ C_{55} = \int_{\text{element}} \left[ \frac{1}{\sigma_0} \sum_{k=1}^{9} N_k r_k \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial r} \right] \, dr \, dz \]

\[ C_{56} = -\int_{\text{element}} \left[ \frac{1}{\sigma_0} \sum_{k=1}^{9} N_k r_k \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial r} + N_j \left( \sum_{k=1}^{9} N_k v_{zk} \right) \frac{\partial N_i}{\partial z} \right] \, dr \, dz \]

\[ + N_j \left( \sum_{k=1}^{9} N_k v_{zk} \right) N_i \right] \, dr \, dz \]

A4

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\[ C_{61} = \int_{\text{element}} [N_j \left( \sum_{k=1}^{g} N_k r_k \right) \left( \sum_{k=1}^{g} N_k B_{rk} \right) \frac{\partial N_i}{\partial z}] \, dr \, dz \]  
\[ C_{62} = 0 \]  
\[ C_{63} = 0 \]  
\[ C_{64} = 0 \]  
\[ C_{65} = - \int_{\text{element}} \left[ \frac{1}{\sigma \mu_0} \left( \sum_{k=1}^{g} N_k r_k \right) \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial r} \right] \, dr \, dz \]  
\[ C_{66} = \int_{\text{element}} \left[ \sum_{k=1}^{g} N_k r_k \left[ \frac{1}{\sigma \mu_0} \frac{\partial N_j}{\partial z} - \frac{\partial N_i}{\partial z} \right] + N_j \left( \sum_{k=1}^{g} N_k v_{zk} \right) \frac{\partial N_i}{\partial z} \right] \, dr \, dz \]  

\[ b_1 = \int_{\text{line at boundary}} [N_j \left( \sum_{k=1}^{g} N_k r_k \right) \sqrt{g} \frac{\partial n_x}{\partial x}] \, dl \, \text{line} - \text{Fixed } v_z \text{ Boundary} \]  
\[ b_2 = \text{Fixed } p \text{ Boundary} \]  
\[ b_3 = \int_{\text{line at boundary}} [N_j \left( \sum_{k=1}^{g} N_k r_k \right) \sqrt{g} \frac{\partial n_y}{\partial y}] \, dl \, \text{line} - \text{Fixed } v_t \text{ Boundary} \]  
\[ b_4 = - \int_{\text{line at boundary}} [N_j \left( \sum_{k=1}^{g} N_k r_k \right) \left( \sum_{k=1}^{g} N_k g_{nk} \right)] \, dl \, \text{line} - \text{Fixed } \phi \text{ Boundary} \]  
\[ b_5 = \text{Fixed } B_z \text{ boundary} \]  
\[ b_6 = \text{Fixed } B_z \text{ boundary} \]  

Where: \( l = 1, 2, 3, \text{ and } 4 \), \( m = 1, 2, 3, \text{ and } 4 \), and

Pressure is evaluated using the perfect gas law.