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INFORMAL REPORT

Demonstration of No Feasibility of a Crystalline Beam in a Betatron Magnet

Alessandro G. Ruggiero

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AGS Department

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Brookhaven National Laboratory
Associated Universities, Inc.
Upton, Long Island, New York 11973
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*Abstract **

This technical report investigates the feasibility of a Crystalline Beam in a weak-focussing storage ring like a Betatron. At the same time the curvature effect due to the bending magnet is also investigated. As a special case, the example of a circular one-dimensional string of electrically charged particles is examined. It is found that the motion of the particles is unstable due to the dependence of the precession movement with respect to each other on their radial displacement. That is a form of negative-mass instability which can be avoided with an alternating-focussing structure corresponding to a transition energy above the energy of the particles.

The calculation of the particle-particle interaction as well as of the forces due to the external magnetic field is done directly in the laboratory frame. The retarded potential expressions are used at this purpose.

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1. Introduction

One of the most important issue in the study of Crystalline Beams [1-3] is the determination of the best configuration of the storage ring where the formation and detection of the state should occur. It was originally conceived that the best configuration would require a weak focussing structure like the one obtained in a Betatron. Recently, this issue was addressed in a paper by Wei, Li and Sessler [4] and found that a Crystalline Beam cannot be possibly be stable in a Betatron. The method used by these Authors is based on a non-obvious transformation of general relativity to the beam rotating frame. This paper deals with the same issue, and in particular discusses the special case of a Circular String of electrically charged particles. The method used applies directly to the laboratory frame and involves the correct calculation of the retarded potentials to describe the particle-particle interaction. It is found that though the Circular String is an Equilibrium Configuration compatible with a weak-focussing structure, it is nevertheless unstable against azimuthal perturbations, due to a sort of negative-mass instability which can be avoided with a storage ring having an alternating-focussing structure.

To simplify the calculations and to determine more directly the consequences of the use of a Betatron as the storage ring, we have limited our study to the case of a one-dimensional Circular String of electrically charged particles, which is defined in the following Section 2. The equations of motion of each particle are given exactly in Section 3. There are two contributions to the total force: the particle-particle interaction and the bending and focussing forces due to the magnetic field distribution in the Betatron Magnet. That the Circular String is indeed an Equilibrium Configuration is determined in Section 4. The exact calculation of the particle interaction is shown in Section 5. The calculation is done directly in the laboratory frame by using the correct retarded potentials. The stability of motion of the particles in the Circular String is investigated in Sections 6, where it is found that the motion in a simple Betatron Magnet is indeed unstable.

The instability is caused by the coupling between the longitudinal (azimuthal) and radial components of the motion due to the curvature of the trajectories. This is analyzed in Section 7. It is the equivalent of a negative-mass instability since the particle energy is above the corresponding transition energy of the storage ring. The instability can be avoided by using a storage ring with alternating-gradient magnets, thus raising the transition energy above the energy of the particles. An example of such storage ring made of combined-function magnets is given in Section 8. The field index in each magnet is very large and oscillates in sign from magnet to magnet.

2. The Circular String

According to the procedure [5] established to investigate the properties of a Crystalline Beam, we shall follow the following three steps:

- Identification of an Equilibrium Configuration
- Determination of the Confinement Conditions
- Determination of the Stability Conditions

In particular, we like to address the issue of the feasibility of a Crystalline Beam in a circular device made of a weak-focussing structure as a Betatron Magnet.

We shall consider, as a very simple and special case, a circular beam of ions all with the same electric charge Qe and mass at rest m_0 . The ions are moving on the same circular trajectory of radius R , with the same angular frequency ω_0 , and are equally spaced. An example is shown in Figure 1. The particles are lying on the same plane at $z = 0$. Aside from the azimuthal motion, constant and uniform, along the azimuthal coordinate θ , there is no radial motion; that is, particles remain on the circular radius $r = R$.

Fig. 1. A Circular String as a Crystalline Beam

We want to prove that the Circular String of particles defined above is indeed an Equilibrium Configuration for which we can subsequently determine the Confinement and Stability Conditions. We shall also prove that such a configuration circulating in a Betatron Magnet is unstable. The instability is caused by the dependence of the precession movement of the particles with respect to each other on their radial displacement. It is the equivalent of the negative-mass instability. Indeed, similarly, the instability can be avoided by choosing a storage ring with alternating-focussing gradients with the corresponding transition energy above the energy of the particles.

3. The Equations of Motion

The equations of motion of a particle can be written in vectorial form

$$d\mathbf{p}/dt = Qe\mathbf{E} + Qe(\mathbf{v}/c) \times \mathbf{B} \quad (3.1)$$

where $\mathbf{p}$ is the ion momentum, $\mathbf{E}$ the total electric field, $\mathbf{B}$ the total magnetic field and $\mathbf{v}$ the ion velocity

$$\mathbf{p} = m_0 \gamma \mathbf{v} = m \mathbf{v} \quad (3.2)$$

with γ the energy relativistic factor.

The electric and magnetic fields are the sum of the particle-particle interaction, denoted in the following with the subindex "sc", and of the external fields, denoted with the subindex "ext", that is

$$\mathbf{E} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{sc}} \quad (3.3)$$

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{sc}} \quad (3.4)$$

Let us define

$$\mathbf{F}_{\text{sc}} = Qe \mathbf{E}_{\text{sc}} + Qe (\mathbf{v} / c) \times \mathbf{B}_{\text{sc}} \quad (3.5)$$

as the force acting on the particle from the interaction with all other particles.

We assume there are no external electric fields, that is

$$\mathbf{E}_{\text{ext}} = 0 \quad (3.6)$$

The external magnetic field $\mathbf{B}_{\text{ext}}$, which provides bending and focussing simultaneously, can be derived from a vector potential $\mathbf{A}$

$$\mathbf{B}_{\text{ext}} = \text{rot } \mathbf{A} \quad (3.7)$$

where

$$\text{div } \mathbf{A} = 0 \quad (3.8)$$

$$\nabla^2 \mathbf{A} = 0 \quad (3.9)$$

A Betatron Magnet is described by a vector potential which has only the azimuthal component non vanishing, that is $A_r = A_z = 0$ and $A_\theta = A(r, z)$. The field is stationary, so that there is no explicit dependence with time. We derive

$$B_r = -\partial A / \partial z \quad (3.10)$$

$$B_\theta = 0 \quad (3.11)$$

$$B_z = (1 / r) \partial (r A) / \partial r \quad (3.12)$$

The actual form of $A(r, z)$ is determined by solving a second-order partial differential equation as shown in Appendix A.

In cylindrical coordinates the components of the momentum vector $\mathbf{p}$ are

$$p_r = m \, dr / dt = m \, v_r \quad (3.13)$$

$$p_\theta = m \, r \, d\theta / dt = m \, v_\theta \quad (3.14)$$

$$p_z = m \, dz / dt = m \, v_z \quad (3.15)$$

where v_r , v_θ and v_z are the components of the velocity vector $\mathbf{v}$.

The following relations between the components of the vector $\mathbf{p}$ and those of the vector $d\mathbf{p}/dt$ hold

$$(d\mathbf{p}/dt)_r = dp_r/dt - p_\theta d\theta/dt \quad (3.16)$$

$$(d\mathbf{p}/dt)_\theta = dp_\theta/dt + p_r d\theta/dt \quad (3.17)$$

$$(d\mathbf{p}/dt)_z = dp_z/dt \quad (3.18)$$

Thus the three components of the vectorial Eq. (3.1) can be written as

$$dp_r/dt - p_\theta d\theta/dt = F_r + Qe(v_\theta/c)B_z \quad (3.19)$$

$$\begin{aligned} dp_\theta/dt + p_r d\theta/dt &= F_\theta + Qe(v_z/c)B_r \\ &\quad - Qe(v_r/c)B_z \end{aligned} \quad (3.20)$$

$$dp_z/dt = F_z - Qe(v_\theta/c)B_r \quad (3.21)$$

where F_r , F_θ and F_z are the components of the vector $\mathbf{F}_{sc}$ given by Eq. (3.5).

4. The Equilibrium Configuration of the Circular String

The Circular String is defined by letting:

$$\begin{aligned} r &= R, \text{ constant} \\ dr/dt &= 0 \\ d\theta/dt &= \omega_0, \text{ constant} \\ z &= dz/dt = 0 \end{aligned}$$

It derives that γ is also a constant, the velocity components $v_r = v_z = 0$ and that $v_\theta = \omega_0 R = v$ is a constant. We have also $p_r = p_z = 0$ and that the angular momentum $p_\theta = p$ is a constant of motion.

It is sufficient and necessary that the following conditions are satisfied at the radius $r = R$ on the $z = 0$ plane. From Eq. (3.19)

$$Qe B_\theta R = pc + R F_r / \beta \quad (4.1)$$

where $B_z = -B_\theta$ is the value of the bending field on the $z = 0$ plane at the radius $r = R$, and $\beta = v/c$. From Eq. (3.20), it is required that

$$F_\theta = 0 \quad (4.2)$$

Finally, from Eq. (3.21), we derive the condition

$$F_z = Qe \beta B_r \quad (4.3)$$

Eq. (4.1) is the usual Lorentz condition where the centrifugal force is balanced by the Lorentz force. A repulsive force, due to the particle-particle interaction, appears and has to be taken into account as well. Since the beam has a perfect cylindrical arrangement, from symmetry considerations, it can be determined that F_r is constant and has the same value for all particles. Thus the imposition of Eq. (4.1) simply requires that the guiding field is to be increased to balance also the repulsive particle-particle forces. For typical beam parameters this contribution is small, and in the following we shall often neglect it.

Using the same symmetry considerations, it is easily seen that $F_z = 0$. Thus, in order to satisfy Eq. (4.3), we need $B_r = 0$ on the $z = 0$ plane and at the radius $r = R$. A magnetic field with these properties is indeed the one which exists in a Betatron Magnet as demonstrated in Appendix A.

We have to deal finally with the condition set by Eq. (4.2). We give below a formal prove that this condition is indeed satisfied for the Circular String. We shall calculate explicitly the components of the interaction force in the following section. With this, we would have thus completed the first task of demonstrating that this simple Crystalline Beam is indeed an Equilibrium Configuration. At the same time we shall have also demonstrated that the Confinement Condition is given by Eq. (4.1) in conjunction to the field distribution of a pure Betatron Magnet.

Suppose that at a given instant all the particles are ideally located and equally spaced on the same circular trajectory of radius R . Moreover, assume that, also at the same instant, particles have only an azimuthal component of the velocity and move with the same angular rotation frequency. From symmetry considerations, then, if there is a non-vanishing azimuthal component of the force due to the particle-particle interaction, this component F_θ has to be the same at that instant to all particles; that is, at the same instant all particles would suffer the same variation of the angular momentum and thus the whole system would increase or decrease its total angular momentum. This violates the principle of conservation of the angular momentum since we are dealing with a conservative isolated system where there is only an external, stationary magnetic field distribution. It derives therefore that $F_\theta = 0$ identically.

5. Calculation of the Interaction Forces at the Equilibrium

The field distribution generated by a point-like charged particle of arbitrary motion is given by J. D. Jackson in *Classical Electrodynamics* [6]. There are two terms: the "velocity fields", which are independent of acceleration, and "acceleration fields", which depend linearly with time variation of the velocity vector $d\mathbf{v}/dt$. The second term is usually very small, especially for $\beta \ll 1$. It causes energy and motion variations over long periods of time. We are interested to determine the particle motion over a relatively shorter period of time, therefore we shall neglect and ignore the "acceleration fields". Moreover retaining only the "velocity fields" insures that the system is indeed conservative and that principles like the conservation of total angular momentum do apply.

Let us denote with $\mathbf{E}_{ij}$ and $\mathbf{B}_{ij}$ the fields generated by the j -th particle on the i -th parti-

cle of the Circular String. Let $\mathbf{n}_{ij}$ be the unit vector directed from the j-th particle to the i-th particle, and R_{ij} the distance between the two particles. Then according to Jackson,

$$\mathbf{B}_{ij} = [\mathbf{n}_{ij} \times \mathbf{E}_{ij}]_{\text{ret}} \quad (5.1)$$

$$\mathbf{E}_{ij} = (Qe / \gamma^2) [(\mathbf{n}_{ij} - \beta_j) / (1 - \beta_j \cdot \mathbf{n}_{ij})^3 R_{ij}^2]_{\text{ret}} \quad (5.2)$$

where γ is the same to all particles and $\beta_j = \mathbf{c} \mathbf{v}_j$ is the velocity of the j-th particle. The square brackets with subscript "ret" mean that the quantity in the brackets is to be evaluated at the retarded time

$$\tau = t - R_{ij} / c \quad (5.3)$$

A more detailed and complete derivation of this result is given by R. Becker in *Theorie der Elektrizitat* [7]. Becker proves that

$$[(\mathbf{n}_{ij} - \beta_j) R_{ij}]_{\text{ret}} = \mathbf{r}_{ij} \quad (5.4)$$

where $\mathbf{r}_{ij}$ is the actual distance vector between the two particles at same time t , directed from the j-th toward the i-th particle. It is also seen that

$$[(1 - \beta_j \cdot \mathbf{n}_{ij}) R_{ij}]_{\text{ret}}^2 = \mathbf{r}_{ij}^2 - (\mathbf{r}_{ij} \times \beta_j)^2 = s_{ij}^2 \quad (5.5)$$

where the quantity at the right-hand side is estimated at the same instant t .

Thus

$$\mathbf{E}_{ij} = Qe \mathbf{r}_{ij} / \gamma^2 s_{ij}^3 \quad (5.6)$$

Let us apply now these equations to estimate the field that the particles in the Circular String experience. Again, from symmetry considerations, the field is constant and has the same value for all particles. In particular, we can estimate the field acting on the $i = 0$ particle when it is located at $\theta = 0$ at the time $t = 0$. The distance to the j-th particle is

$$r_{0j} = 2 R \sin | \theta_j / 2 | \quad (5.7)$$

where θ_j is the angular separation between the two particles. The vector $\mathbf{r}_{0j}$ has the following components: $2 R \sin^2 (\theta_j / 2)$ radially, and $- R \sin \theta_j$ azimuthally. It is also seen that

$$(\mathbf{r}_{0j} \times \beta_j)^2 = 4 \beta^2 R^2 \sin^4 (\theta_j / 2) \quad (5.8)$$

and

$$s_{0j} = (2 R \sin | \theta_j / 2 |) \sqrt{ 1 - \beta^2 \sin^2 (\theta_j / 2) } \quad (5.9)$$

Since for every particle at the angular distance θ_j there is another one at $-\theta_j$, it is immediately seen that the electric field component $E_\theta = 0$ whereas

$$E_r = \frac{Qe}{2 R^2 \gamma^2} \sum_j \frac{1}{[1 - \beta^2 \sin^2 (\theta_j / 2)]^{3/2} \sin (\theta_j / 2)} \quad (5.10)$$

where the summation is done only over those particles with $0 < \theta_j < \pi$.

It remains to estimate the components of the magnetic field $\mathbf{B}_{0j}$. According to Becker

$$[\mathbf{n}_{ij} \times \mathbf{E}_{ij}]_{\text{ret}} = \mathbf{B}_j \times \mathbf{E}_{ij} \quad (5.11)$$

where the quantity at the right-hand side is estimated at the instant t . Since the two vectors $\mathbf{B}_j$ and $\mathbf{E}_{ij}$ are on the same plane ($z = 0$), the magnetic field $\mathbf{B}_{ij}$ is directed along the z -axis and perpendicular to the $z=0$ plane which contains the Cylindrical String of ions. Since the velocity of any particle is directed along the azimuthal direction θ , the cross product between $\mathbf{B}_i$ and $\mathbf{B}_{ij}$ has only a radial component. In summary from Eq. (3.5)

$$F_r = Qe (1 - \beta^2) E_r = \bar{F}_r \quad (5.12)$$

where E_r is given by Eq. (5.10) and $F_\theta = F_z = 0$. This completes the prove that the beam made of a Circular String is indeed an Equilibrium Configuration consistent with the field distribution in a Betatron Magnet.

6. Determination of Stability of the Circular String

We shall apply a perturbation method where the motion of a single particle is perturbed by a small amount, whereas the motion of all other particles remains unchanged. The perturbation is described by small displacements

$$\begin{aligned} r &= R + x \\ \theta &= \omega_0 t + q \\ z &= z \end{aligned} \quad (6.1)$$

where x , Rq and z are smaller than the unperturbed particle-particle separation $2\pi R/N$ and N is the total number of particles of the String. Let us take as the perturbed particle the one which would be located at $\theta = 0$ at the time $t = 0$. Linearization of the expressions of the components of the force resulting from the particle-particle interaction gives

$$\begin{aligned} F_r &= \bar{F}_r + k_r x - Qe \beta^2 E_r (dq / \omega_0 dt) \\ F_\theta &= -k_\theta q + Qe \beta^2 E_r (dx / v dt) \\ F_z &= k_z z \end{aligned} \quad (6.2)$$

where $\bar{F}_r$ is given by Eq. (5.12) and E_r is given by Eq. (5.10). The constant k_r , k_θ and k_z are positive quantities which are evaluated in Appendix B. In the same Appendix it is shown that a radial perturbation of amount x , to first order, gives rise only to a radial component of the electric field. The same applies to the azimuthal perturbation q which gives rise only to an azimuthal component, and to the axial perturbation of amount z which is accompanied only by a vertical component of the electric field. There are nevertheless mixed contributions due to the magnetic field component of the particle-particle interaction, which are the last terms at the right-hand side of Eqs. (6.2) for F_r and F_θ .

To first order, the energy relativistic factor γ is constant, whereas

$$p_\theta \sim p [1 + (dq / \omega_0 dt) + x / R] \quad (6.3)$$

and

$$p_\theta d\theta / dt \sim p \omega_0 [1 + 2 (dq / \omega_0 dt) + x / R] \quad (6.4)$$

Taking into account the confinement condition given by Eq. (4.1), where F_r is replaced by F_r from Eq. (5.12), the equations of motion Eqs. (3.19-21) become

$$d^2 x / dt^2 + \Omega_H^2 x = v (\omega_0 x / R + dq / dt) \quad (6.5)$$

$$d^2 q / dt^2 + (\omega_0 / R) dx / dt = - \omega_0 k_\theta q / p \quad (6.6)$$

$$d^2 z / dt^2 + \Omega_V^2 z = 0 \quad (6.7)$$

where

$$\Omega_H^2 = Qe \beta B_0 (1 - n) / mR - k_r / m \quad (6.8)$$

and

$$\Omega_V^2 = Qe \beta B_0 n / mR - k_z / m \quad (6.9)$$

In deriving these equations we have made use of the linearized expressions of the field components in a Betatron Magnet, Eqs. (A15, 16).

The equations of motion (6.5-7) are very general. They apply to the motion of a charged particle in a combined function magnet described by the field index n , under the action of other moving particles in a Circular String Configuration. The axial motion can be treated independently from that occurring on the $z = 0$ plane. Solving Eq. (6.7) gives a stable oscillatory solution as long as $\Omega_V^2 > 0$, that is, from Eq. (6.9),

$$n > R^2 k_z / \beta pc \quad (6.10)$$

For typical beam parameters, this condition is satisfied for positive and very large values of the field index n .

7. Radial-Azimuthal Coupling. Negative Mass Instability.

The first two equations of motion (6.5 and 6) show that the components of motion along the azimuthal and radial directions are coupled. The coupling can be expressed by a common angular frequency Ω_q . We can try the solution

$$x = a \cos \Omega_q t$$

$$q = b \sin \Omega_q t \quad (7.1)$$

When these are inserted into Eqs. (6.5 and 6), we find the following system of linear equations

$$\begin{aligned} (\Omega_0^2 - \Omega_H^2 + \omega_0^2) a + v \Omega_0 b &= 0 \\ v \Omega_0 a / R^2 + (\Omega_0^2 - \omega_0 k_\theta / p) b &= 0 \end{aligned} \quad (7.2)$$

In order for this system to have solution, the associated determinant of the coefficients has to vanish leading to the following equation for the angular oscillation frequency Ω_0

$$(\Omega_0^2 - \Omega_H^2 + \omega_0^2)(\Omega_0^2 - \omega_0 k_\theta / p) - \omega_0^2 \Omega_0^2 = 0 \quad (7.3)$$

or

$$\Omega_0^4 - M \Omega_0^2 + (\Omega_H^2 - \omega_0^2) \omega_0 k_\theta / p = 0 \quad (7.4)$$

where

$$M = \Omega_H^2 + \omega_0 k_\theta / p \quad (7.5)$$

Solving Eq. (7.4) for Ω_0^2 gives

$$2 \Omega_0^2 = M \pm \sqrt{M^2 - 4(\Omega_H^2 - \omega_0^2) \omega_0 k_\theta / p} \quad (7.6)$$

For the motion to be stable $\Omega_0^2 > 0$ is required. This in turns requires that the following two conditions are satisfied simultaneously

$$\Omega_H^2 > \omega_0^2 \quad (7.7)$$

and

$$M > 0 \quad (7.8)$$

Eq. (7.7) in particular yields the following condition for the field index

$$1 - n > 1 + R^2 k_r / \beta p c \quad (7.9)$$

In Appendix C it is shown that $k_r \sim k_z$. Thus, in order to satisfy Eq. (7.9) the field index n has to take very large negative values. It is not possible that the conditions of Eqs. (6.10) and (7.9) can be satisfied simultaneously. It derives that the Equilibrium Configuration of a Circular String is not a Stable Configuration that can be realized in a simple Beta-tron Magnet. This seems to be in agreement with the results of Wei, Li and Sessler [4].

8. Storage Ring with Alternating Focussing Gradient

Define the positive quantity, with the help of Eq. (C10), where $r_0 = 1.535 \times 10^{-18}$ m is the classical proton radius and A is the mass number of a particle,

$$\begin{aligned}
n_0 &= R^2 k_z / \beta p c \\
&= Q^2 r_0 g_0 N^3 / 4 \pi^3 A R \beta^2 \gamma^5
\end{aligned} \tag{8.1}$$

Then the stability condition for the axial component of the motion is, from Eq. (6.10),

$$n > n_0 \tag{8.2}$$

whereas the stability condition for the motion on the $z = 0$ plane is, from Eq. (7.9),

$$n < -n_0 \tag{8.3}$$

This suggests that a solution can be found with an alternating gradient storage ring where the field index n takes very large values and changes sign periodically.

We shall apply these results to the same example discussed in ref.[5] where a beam of ions partially stripped is injected into a small storage ring from the Tandem-ALPI complex of the *Laboratori Nazionali di Legnaro* [3]. The beam has the following characteristics:

Charge State, Q	51	
Mass Number, A	197	
Rest Energy	931.15	MeV/u
Magnetic Rigidity	1.2	T - m
Kinetic Energy	4.59	MeV/u
β	0.09893	
Bending Field, B_0	3	kGauss
Bending Radius, R	4	m

The injector complex can provide an average beam current of $3 \mu\text{A}$ -particle with an emittance of about $1 \pi \text{ mm mrad}$ and a momentum spread $\Delta p/p \sim 0.01\%$.

From Eq. (8.1) we derive that the critical value of the field index n_0 for the beam stability of this example is

$$n_0 = 4.9 \times 10^{-18} N^3 \tag{8.4}$$

It is seen that already for $N = 10^7$ particles, the field index gets a too large value. Not only this may be very difficult to achieve in practice, but also it causes concern about the stability of the storage ring lattice in the limit of a single particle.

We shall assume that the storage ring is made of n_p superperiods, each with a pair of combined function magnets of alternating gradient sign, that is having field index $-n$ and $+n$ respectively. Even neglecting the presence of drift spaces, the motion of a single particle in the lattice is stable if the following condition is satisfied

$$|\cos \zeta \cosh \zeta| < 1 \tag{8.5}$$

where

$$\zeta = \alpha \sqrt{n} \quad (8.6)$$

and $\alpha = \pi / n_p$ is the bending angle per magnet. The stability condition (8.5) can be satisfied by choosing

$$\zeta \sim (2h + 1) \pi / 2 \quad (8.7)$$

where h is an integer. This condition is equivalent of setting a betatron phase-advance per superperiod to about 90° .

Eqs. (8.6 and 8.7) combined together set a relation between the periodicity of the storage ring and the beam intensity since the critical field index depends with the number N of particles according to Eq.(8.4), and one requires $n > n_0$,

$$n_p \sim 4.4 \times 10^{-9} N^{3/2} / (2h + 1) \quad (8.8)$$

Small values of h are desirable, since for larger values the stability interval reduces considerably. It is then seen that a large periodicity is needed. For instance for $N = 10^7$ particles $n_p \sim 140 / (2h + 1)$, and a periodicity $n_p = 8$ can be obtained by closing $h = 8$.

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10. References

- [1] J. P. Schiffer and A. Rahman, "Feasibility of a Crystalline Condensed State in Cooled Ion Beams of a Storage Ring", PHY-5121-ME-88, Feb. 1988, Argonne National Laboratory.
- [2] Proceedings of the Workshop on Crystalline Ion Beams, Werheim, Germany, Oct. 1988. GSI-89-10 Report.
- [3] CRYSTAL. A Storage Ring for Ion Beam Crystallization Studies. LNL-INFN (REP) 59/92 (1992). *Laboratori Nazionali di Legnaro*, Via Romea 4, I-35020 Legnaro (Padova), Italy
- [4] J. Wei, X. P. Li, A. Sessler, "Crystalline Beam Ground State", BNL-52381, June 1993.
- [5] A. G. Ruggiero, "Confinement and Stability of a Crystalline Beam", BNL-49090 (Informal Report). May 1993.
- [6] J. D. Jackson, *Classical Electrodynamics*, second edition. John Wiley & sons. 1975
- [7] R. Becker, *Teoria della Eletticit*, Vol. II, translated by M. Ageno. Sansoni Edizioni Scientifiche, Firenze, 1950.

Appendix A: Field Distribution in a Betatron Magnet

The actual form of $A(r, z)$ is determined by solving the second-order partial differential equation

$$(1/r) \partial (r \partial A / \partial r) / \partial r + \partial^2 A / \partial z^2 - A / r^2 = 0 \quad (A1)$$

which can be solved with the method of variable separation by letting $A(r, z) = R(r) Z(z)$. Eq. (A1) splits in the following two ordinary differential equations

$$d^2 Z / dz^2 + D^2 Z = 0 \quad (A2)$$

$$(1/r) d (r d R / dr) / dr - (D^2 + 1/r^2) R = 0 \quad (A3)$$

where D is a constant introduced during the separation of the variables. Both of these equations have well-known solutions involving Trigonometric and Bessel functions. We require that the axial field component B_z has a symmetry around the $z = 0$ plane; thus

$$Z(z) \sim \cos(Dz) \quad (A4)$$

In order to focus the motion of a charged particle, we need also that the axial field increases with the radial field. If the particle moves away, it will encounter a field of larger magnitude and it will be brought back. On the other hand, if the particle moves to the inside of a reference orbit, it will encounter a lower field which will push it outside. All cases can be derived by letting D be either a real or a purely imaginary constant. Thus

$$R(r) \sim I_1(Dr) \quad (A5)$$

where I_1 is the modified Bessel function of the first kind and order 1. Thus in conclusion

$$A(r, z) = C I_1(Dr) \cos(Dz) \quad (A6)$$

where C is another constant of integration.

From Eqs. (3.10) and (3.12) we obtain

$$B_r = CD I_1(Dr) \sin(Dz) \quad (A7)$$

$$B_z = CD I_0(Dr) \cos(Dz) \quad (A8)$$

Let $B_z = -B_0$ on the $z = 0$ plane at the radius $r = R$, then

$$B_r = -B_0 I_1(Dr) \sin(Dz) / I_0(DR) \quad (A9)$$

$$B_z = -B_0 I_0(Dr) \cos(Dz) / I_0(DR) \quad (A10)$$

This is the correct and most general expression of the field distribution in a Combined-Function Magnet, in particular a Betatron Magnet.

Let us perform an expansion of the two field components around $z = 0$ and $r = R$, and let us retain only the linear terms. We obtain

$$B_r = - B_0 I_1(D R) D z / I_0(D R) + \dots \quad (A11)$$

$$B_z = - B_0 [1 + I_1(D R) D x / I_0(D R)] + \dots \quad (A12)$$

where $x = r - R$.

We introduce now the field index n as the relative variation of the axial field with radius

$$n = - (r / B_z) d B_z / d r |_{r=R} \quad (A13)$$

From Eq. (A12)

$$n = - R D I_1(D R) / I_0(D R) \quad (A14)$$

so that

$$B_r = B_0 n z / R \quad (A15)$$

$$B_z = - B_0 (1 - n x / R) \quad (A16)$$

which are the commonly used linearized expressions of the field components in a Combined-Function Magnet. In a Betatron Magnet the field index $0 < n < 1$.

Appendix B: Forces due to the Particle-Particle Interaction

Here we calculate the forces acting on the particle the motion of which has been perturbed by small amounts. Let x , q and z be the displacements due to the perturbation respectively along the radial, azimuthal and axial direction. We shall take the particle $i = 0$ which is crossing the location $\theta = 0$ at the instant $t = 0$. The electric field from the j -th particle E_{0j} is given by Eq. (5.6) where s_{0j} is obtained from Eq. (5.5) and r_{0j} is the vector with the following components

$$r_{0j} = [2 R \sin^2 ((\theta_j - q) / 2) + x, - R \sin (\theta_j - q), z] \quad (B1)$$

Here and in the following the first element refers to the radial, the second to the azimuthal and the last to the axial component. Let us notice that

$$B_j = [0, \beta, 0] \quad (B2)$$

and

$$r_{oj}^2 = [2 R \sin^2 ((\theta_j - q) / 2) + x]^2 + R^2 \sin^2 (\theta_j - q) + z^2 \quad (B3)$$

The perturbation is denoted by the vector

$$\mathbf{u} = [x, R q, z] = [u_1, u_2, u_3] \quad (B4)$$

We shall assume that each component is very small with respect to the unperturbed distance between neighboring particles. Let us expand the expression of Eq. (5.6) and retain only up to the linear terms with the elements of the perturbation. We obtain

$$\mathbf{E}_{oj} = \frac{Qe}{\gamma^2} \left\{ \frac{1}{s_{oj}^3} \Big|_0 [\mathbf{r}_{oj} + \sum_n \left(\frac{d \mathbf{r}_{oj}}{d u_n} - \frac{3}{s_{oj}} \frac{d s_{oj}}{d u_n} \mathbf{r}_{oj} \right) \Big|_0 u_n] \right\} \quad (B5)$$

where the vertical bar with the subindex zero means that the preceding quantity is to be evaluated with $u_1 = u_2 = u_3 = 0$. The total electric field is then obtained by summing over all the particles

$$\mathbf{E}_0 = \sum_{j \neq 0} \mathbf{E}_{oj} \quad (B6)$$

We have already seen that the first term at the right-hand side of Eq. (B5), after the summation, gives a vector with only the radial component E_r non-vanishing given by Eq. (5.10). The following quantity is a matrix

$$d \mathbf{r}_{oj} / d \mathbf{u} \Big|_0 = \begin{vmatrix} 1 & 0 & 0 \\ -\sin \theta_j & \cos \theta_j & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad (B7)$$

Noting that $s_{oj} \Big|_0$ given by Eq. (5.9) is an even function of θ_j , we find

$$\begin{aligned} \frac{Qe}{\gamma^2} \sum_{j \neq 0} \left(\frac{1}{s_{oj}^3} \sum_n \frac{d \mathbf{r}_{oj}}{d u_n} \Big|_0 u_n \right) &= \\ &= \frac{Qe}{\gamma^2} \sum_{j \neq 0} \frac{[x, R q \cos \theta_j, z]}{s_{oj}^3} \end{aligned} \quad (B8)$$

where s_{oj} is given again by Eq. (5.9). For the third and last term at the right-hand side of Eq. (B5) we note that $\mathbf{r}_{oj} \Big|_0$ has only the radial and azimuthal components non-vanishing

$$\mathbf{r}_{oj} \Big|_0 = [2 R \sin^2 (\theta_j / 2), -R \sin \theta_j, 0] \quad (B9)$$

Moreover from Eq. (5.5)

$$d s_{oj}^2 / d \mathbf{u} = 4 R (1 - \beta^2) [1, -\sin \theta_j, 0] \sin^2 (\theta_j / 2) +$$

$$- [0, R, 0] \sin(2\theta_j) \quad (B10)$$

so that we have

$$\begin{aligned} -3 \frac{Qe}{\gamma^2} \sum_{j \neq 0} \left. \frac{\mathbf{r}_{oj}}{s_{oj}^4} \right|_0 \left(\sum_n \frac{d s_{oj}}{d u_n} \right|_0 u_n \Big) &= \\ &= -6 \frac{Qe}{\gamma^4} \sum_{j \neq 0} \frac{[2 \times R^2 \sin^2(\theta_j/2), q R^3 \sin^2 \theta_j, 0]}{s_{oj}^5} \sin^2(\theta_j/2) + \\ &\quad - \frac{3 Qe}{2 \gamma^2} \sum_{j \neq 0} \frac{[0, q R^3 \sin \theta_j, 0]}{s_{oj}^5} \sin(2\theta_j) \end{aligned} \quad (B11)$$

Combining Eq. (B8) with Eq. (B11) in Eq. (B5) and Eq. (44), adding the contribution from the magnetic field and inserting the result in Eq.s (3.5) and (6.2) give

$$k_z = \frac{Q^2 e^2}{4 \gamma^4 R^3} \sum_j \frac{1}{[1 - \beta^2 \sin^2(\theta_j/2)]^{3/2} \sin^3(\theta_j/2)} \quad (B12)$$

$$k_r = k_z - \frac{3 Q^2 e^2}{4 \gamma^6 R^3} \sum_j \frac{1}{[1 - \beta^2 \sin^2(\theta_j/2)]^{5/2} \sin |\theta_j/2|} \quad (B13)$$

and

$$\begin{aligned} k_\theta &= - \frac{Q^2 e^2}{4 \gamma^2 R^2} \sum_j \frac{\cos \theta_j}{[1 - \beta^2 \sin^2(\theta_j/2)]^{3/2} \sin^3 |\theta_j/2|} + \\ &\quad + \frac{3 Q^2 e^2}{8 \gamma^4 R^2} \sum_j \frac{\sin^2 \theta_j}{[1 - \beta^2 \sin^2(\theta_j/2)]^{5/2} \sin^3 |\theta_j/2|} + \\ &\quad + \frac{3 Q^2 e^2}{32 \gamma^2 R^2} \sum_j \frac{\sin \theta_j \sin(2\theta_j)}{[1 - \beta^2 \sin^2(\theta_j/2)]^{5/2} \sin^5 |\theta_j/2|} \end{aligned} \quad (B14)$$

The summations in Eq.s (B12-14) are done only counting those particles with $0 < \theta_j < \pi$.

To estimate the contribution from the magnetic field we notice the following. The contribution to the total force is represented by the term

$$\sum_{j \neq 0} (\mathbf{B}_0 + d\mathbf{u}/dt) \times (\mathbf{B}_j \times \mathbf{E}_{oj}) \quad (B15)$$

The first term in B_0 has already been included in the expressions of Eqs. (B12-14). The second term in $d\mathbf{u}/dt$, after linearization, gives the cross-terms at the r.h. side of the first two equations in Eq. (6.2).

Appendix C: Application to Non-Relativistic Beams

For non relativistic beams, β is small and the term associated with it at the denominator inside the summation of Eq. (5.10) can be ignored. Moreover, if the particles are equally spaced

$$\theta_j = \Delta j \quad (C1)$$

where Δ is the angular separation. If N is the total number of particles in the String

$$\Delta = 2\pi/N \quad (C2)$$

In the limit of large number N we can replace the summation of Eq. (5.10) with an integral

$$\sum_j 1/\sin(\theta_j/2) = (2/\Delta) \int_{\Delta/2}^{\pi/2} d\theta / \sin \theta \quad (C3)$$

The lower extreme of the integration interval corresponds to the $j=1$ particle. We have

$$\sum_j 1/\sin(\theta_j/2) = NL/\pi \quad (C4)$$

where

$$L = \log(2N/\pi) \quad (C5)$$

Typical values of L range between 10 and 30. Thus

$$E_r = QeNL / 2\pi R^2 \gamma^2 \quad (C6)$$

and

$$F_r = Q^2 e^2 NL / 2\pi R^2 \gamma^4 \quad (C7)$$

The summation of Eq. (B12) can also be calculated for a non-relativistic beam for which the term in β at the denominator can be ignored. For a large number N of particles, we expect most of the contribution to arise from the terms with low values of j . Thus, after replacing the *sine* function with its argument, we have

$$\begin{aligned} \sum_j 1/\sin^3(\theta_j/2) &= (8/\Delta^3) \sum_j (1/j^3) \\ &= 8g_0/\Delta^3 \end{aligned} \quad (C8)$$

where

$$g_0 = \sum_j (1/j^3) = 1.2 \quad (C9)$$

Thus

$$k_z = Q^2 e^2 g_0 N^3 / 4 \pi^3 R^3 \gamma^4 \quad (C10)$$

The summation of Eq. (B13) can be evaluated with the same procedure so that

$$k_r = k_z (1 - \eta) \quad (C11)$$

where

$$\eta = 3 \pi^2 L / g_0 \gamma^2 N^2 \quad (C12)$$

For typical beam parameters, $\eta \ll 1$ and $k_r \sim k_z$. With the same procedure it is also found

$$\begin{aligned} k_\theta &\sim (Q^2 e^2 / 2 R^2 \gamma^2) \sum_j 1 / \sin^3 |\theta_j / 2| \\ &= 2 R \gamma^2 k_z \end{aligned} \quad (C13)$$

Finally from Eqs. (C7), (C10) and (C12) we easily derive with the same assumptions

$$F_r = (2/3) \gamma^2 \eta R k_z \quad (C14)$$

Thus, in general,

$$F_r \ll R k_z \quad (C15)$$

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