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2D Deterministic Radiation Transport with the Discontinuous Finite Element Method

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Introduction

In what follows we give a complete description of the analytic and discretized equations for 2D deterministic radiation transport. This computational model has been checked against a wide variety of analytic test problems and found to give excellent results. We make extensive use of the discontinuous finite element method. A good introduction to the method may be found in reference 1.

Section 1 - Discretization of the Radiation Transport Equation

The analytic transport equation for cylindrically symmetric problems is

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mathcal{L}I - \rho\kappa(I_p(T_e) - I) = 0, \quad (1)$$

where

$$\mathcal{L} = \cos\theta \frac{\partial}{\partial Z} + \sin\theta \cos\gamma \frac{\partial}{\partial R} - \frac{\sin\theta \sin\gamma}{R} \frac{\partial}{\partial \gamma}, \quad (1a)$$

or equivalently, in explicit energy conserving form,

$$\mathcal{L} = \cos\theta \frac{\partial}{\partial Z} + \frac{\sin\theta \cos\gamma}{R} \frac{\partial}{\partial R} R - \frac{\sin\theta}{R} \frac{\partial}{\partial \gamma} \sin\gamma, \quad (1b)$$

$$\cos\theta = \hat{k} \cdot \hat{Z}, \quad \sin\theta \cos\gamma = \hat{k} \cdot \hat{R},$$

$I_p(T_e)$ is the Planck distribution function and $\hat{k}$ is the photon direction. We use fully implicit time differencing to obtain

$$\frac{I' - I}{c\Delta t} + \mathcal{L}I' - \rho\kappa(I_p(T_e') - I') = 0. \quad (2a)$$

where $I(I')$ is the beginning (end) of the time step radiation intensity and T_e' is the end of the time step electron temperature. We can write this as an algebraically equivalent steady state equation for I' ,

$$\mathcal{L}I' - \eta(S - I') = 0, \quad (2b)$$

where

$$\eta = \rho\kappa + \frac{1}{c\Delta t}, \quad S = \frac{I_p(T_e') + I \frac{1}{c\rho\kappa\Delta t}}{1 + \frac{1}{c\rho\kappa\Delta t}}.$$

Equation (1) has no derivatives with respect to θ so we have independent equations for each value of θ . We divide the interval $0 < \theta < \pi$ into n_θ (n_θ is always even) equal zones whose boundaries are given by

$$\theta_\mu = \pi(1 - \frac{(\mu-1)}{n_\theta}), \mu = 1, 2, 3, \dots, (n_\theta + 1), \quad (3)$$

and in zone μ we take the average value of $\cos\theta$ to be

$$\alpha_\mu = \left\{ \begin{array}{c} + \\ - \end{array} \right\} \sqrt{\frac{\cos^2\theta_{\mu+1} + \cos\theta_{\mu+1} \cos\theta_\mu + \cos^2\theta_\mu}{3}} \cdot \left\{ \begin{array}{c} \mu > \frac{n_\theta}{2} \\ \mu \leq \frac{n_\theta}{2} \end{array} \right\}. \quad (4)$$

We take the average value of $\sin\theta$ to be

$$\beta_\mu = \sqrt{1 - \alpha_\mu^2}. \quad (4a)$$

and the zone weight to be

$$\Delta_\mu = \cos \theta_{\mu+1} - \cos \theta_\mu. \quad (4b)$$

These choices ensure that

$$\int_{-1}^{+1} d\cos \theta = \sum_{\mu=1}^{n_\theta} \Delta_\mu = 2, \quad (4c)$$

$$\int_{-1}^{+1} \cos \theta d\cos \theta = \sum_{\mu=1}^{n_\theta} \alpha_\mu \Delta_\mu = 0, \quad (4d)$$

$$\int_{-1}^{+1} \cos^2 \theta d\cos \theta = \sum_{\mu=1}^{n_\theta} \alpha_\mu^2 \Delta_\mu = \frac{2}{3}, \quad (4e)$$

$$\int_{-1}^{+1} \cos \theta \sin \theta d\cos \theta = \sum_{\mu=1}^{n_\theta} \alpha_\mu \beta_\mu \Delta_\mu = 0, \quad (4f)$$

$$\int_{-1}^{+1} \sin^2 \theta d\cos \theta = \sum_{\mu=1}^{n_\theta} \beta_\mu^2 \Delta_\mu = \frac{4}{3}. \quad (4g)$$

We shall see that these relations must be satisfied in order that the discretized equations behave correctly in the thermal wave limit. For each α_μ value we choose a set of γ zone boundary values,

$$\gamma_{\mu\nu} = \frac{(\nu-1)\pi}{n_\gamma^\mu}, \nu = 1, 2, 3, \dots, (n_\gamma^\mu + 1), \quad (5)$$

where

$$n_\gamma^\mu = \begin{cases} 2\mu, & \mu \leq \frac{n_\theta}{2}; \\ 2(n_\theta - \mu + 1), & \mu > \frac{n_\theta}{2}, \end{cases} \quad (5a)$$

and we choose the zone weight to be

$$(\Delta\gamma)_{\mu\nu} = \frac{\pi}{n_\gamma^\mu}. \quad (5b)$$

$0 \leq \gamma \leq \pi$ (rather than $0 \leq \gamma \leq 2\pi$) because in cylindrical symmetry the intensity is invariant under $\gamma \rightarrow -\gamma$.

For our discontinuous finite element basis set in (R, Z, γ) space we choose zero order elements in γ space

$$\psi_{\mu\nu}(\gamma) = \begin{cases} 1, & \text{if } \gamma_{\mu\nu} \leq \gamma \leq \gamma_{\mu(\nu+1)} \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

and first order elements in (R, Z) space. We assume that the (R, Z) domain has been covered with a triangular mesh and choose basis elements $\phi_{ij}(R, Z)$, where ϕ_{ij} is 0 outside the i 'th triangle, is linear in R and Z inside the i 'th triangle and is 1 at the j 'th corner of

the i 'th triangle and 0 at the other two corners. Further details about the ϕ_{ij} basis elements may be found in appendix A. We then expand the radiation intensity for each α_μ value

$$I'_\mu(R, Z, \gamma) = \sum_i \sum_{j=1}^3 \sum_\nu I'_{ij\mu\nu} \phi_{ij}(R, Z) \psi_{\mu\nu}(\gamma), \quad (7)$$

Since our representation of I' is only zero'th order in γ we may use a zero'th order approximation for the cos and a first order approximation for the sin function in $\mathcal{L}$. Thus we approximate our operator $\mathcal{L}$ by

$$\mathcal{L} = \alpha_\mu \frac{\partial}{\partial Z} + \beta_\mu \text{cs}_\mu(\gamma) \frac{\partial}{\partial R} - \frac{\beta_\mu \text{sn}_\mu(\gamma)}{R} \frac{\partial}{\partial \gamma}, \quad (8)$$

where

$$\begin{aligned} \text{cs}_\mu(\gamma) &= \text{cs}_{\mu\nu}, \quad \gamma_{\mu\nu} \leq \gamma \leq \gamma_{\mu(\nu+1)}, \\ \text{sn}_\mu(\gamma) &= (\gamma - \gamma_{\mu\nu}) \text{cs}_{\mu 1}, \quad \gamma_{\mu 1} \leq \gamma \leq \gamma_{\mu 2}, \\ \text{sn}_\mu(\gamma) &= \sum_{\sigma=1}^{\nu-1} (\Delta\gamma)_{\mu\sigma} \text{cs}_{\mu\sigma} + (\gamma - \gamma_{\mu\nu}) \text{cs}_{\mu\nu}, \quad \gamma_{\mu\nu} \leq \gamma \leq \gamma_{\mu(\nu+1)}, \\ \text{sn}_\mu(\gamma_{\mu 1}) &= \text{sn}_\mu(0) = \text{sn}_\mu(\gamma_{\mu(n_\gamma^\mu+1)}) = \text{sn}_\mu(\pi) = 0, \end{aligned} \quad (8a)$$

since we shall choose $\text{cs}_{\mu\nu}$ so that $\sum_{\nu=1}^{n_\gamma^\mu} (\Delta\gamma)_{\mu\nu} \text{cs}_{\mu\nu} = 0$. To assure energy conservation, we have chosen $\text{sn}_\mu(\gamma)$ so that

$$\text{cs}_\mu(\gamma) = \frac{d\text{sn}_\mu(\gamma)}{d\gamma}.$$

This ensures that the operator, $\mathcal{L}$, in equation (8) is equivalent to its explicitly energy conserving form

$$\mathcal{L} = \alpha_\mu \frac{\partial}{\partial Z} + \frac{\beta_\mu \text{cs}_\mu(\gamma)}{R} \frac{\partial}{\partial R} R - \frac{\beta_\mu}{R} \frac{\partial}{\partial \gamma} \text{sn}_\mu(\gamma). \quad (8b)$$

As we shall see, we require

$$\sum_{\nu=1}^{n_\gamma^\mu} (\Delta\gamma)_{\mu\nu} (\text{cs}_{\mu\nu})^2 = \frac{\pi}{2} = \int_0^\pi d\gamma \cos^2(\gamma), \quad (9a)$$

in order that the discretized equations behave correctly in the thermal wave limit. Therefore we choose $\text{cs}_{\mu\nu}$ so that

$$(\Delta\gamma)_{\mu\nu} (\text{cs}_{\mu\nu})^2 = \int_{\gamma_{\mu\nu}}^{\gamma_{\mu(\nu+1)}} d\gamma (\text{cs}_\mu(\gamma))^2 = \int_{\gamma_{\mu\nu}}^{\gamma_{\mu(\nu+1)}} d\gamma \cos^2(\gamma),$$

or

$$\text{cs}_{\mu\nu} = \left\{ \begin{array}{c} + \\ - \end{array} \right\} \sqrt{\frac{1}{2} + \left(\frac{n_\gamma^\mu}{2\pi}\right) \sin\left(\frac{\pi}{n_\gamma^\mu}\right) \cos\left(\frac{(2\nu-1)\pi}{n_\gamma^\mu}\right)}, \quad \left\{ \begin{array}{c} \nu \leq \frac{n_\gamma^\mu}{2} \\ \nu > \frac{n_\gamma^\mu}{2} \end{array} \right\}. \quad (9b)$$

Finally, we multiply equation (2), with $\mathcal{L}$ given by equation (8) and I'_μ given by equation (7), by our basis functions, $\phi_{ij}\psi_{\mu\nu}$, integrate and obtain our discontinuous finite element discretized transport equation

$$\begin{aligned}
0 = & \int_{\Delta_i} R dR dZ \int_{\gamma_{\mu\nu}}^{\gamma_{\mu(\nu+1)}} d\gamma \phi_{ij}(R, Z) \psi_{\mu\nu}(\gamma) (\mathcal{L}I'_\mu - \eta(S - I'_\mu)) \\
& + \int_{\gamma_{\mu\nu}}^{\gamma_{\mu(\nu+1)}} d\gamma \int_{\partial_- \Delta_i} d\ell R \phi_{ij}(R, Z) (\alpha_\mu \hat{n}_Z + \beta_\mu \text{cs}_\mu(\gamma) \hat{n}_R) (I'_{\mu\tilde{i}} - I'_{\mu i}) \\
& - \beta_\mu \text{sn}_\mu(\gamma_{\mu(\nu+1)}) \int_{\Delta_i} R dR dZ \phi_{ij}(R, Z) (I'_{\mu(\nu+1)}(R, Z) - I'_{\mu\nu}(R, Z)),
\end{aligned} \tag{10}$$

where $(\hat{n}_Z, \hat{n}_R)$ is the outward directed unit normal on the boundary of triangle i , $\tilde{i}$ is the triangle on the other side of the boundary, and $\partial_- \Delta_i$ is the inflow boundary of the i 'th triangle, i.e. that part of the boundary where

$$(\alpha_\mu \hat{n}_Z + \beta_\mu \text{cs}_{\mu\nu} \hat{n}_R) < 0.$$

$(I'_{\mu\tilde{i}} - I'_{\mu i})$ is the jump in I'_μ across the triangle boundary and $(I'_{\mu(\nu+1)}(R, Z) - I'_{\mu\nu}(R, Z))$ is the jump in I'_μ across the γ zone boundary. The first double integral is over the interior of the i 'th triangle and the ν 'th γ zone. The second double integral comes from the Dirac δ function due to the gradient of the discontinuous jump in I' at the triangle boundary. The last integral comes from the Dirac δ function due to the $\frac{\partial}{\partial \gamma}$ of the discontinuous jump in I' at the γ zone boundary. In accordance with the discontinuous finite element algorithm (see reference 1) these Dirac δ function terms are always 100% included in the upwind zone's equation and 0% included in the downwind zone's equation. This gives the method the best possible stability properties, and its ability to propagate discontinuities along characteristics with minimal Gibbs oscillation, while still retaining the higher order accuracy of a finite element method.

Evaluating these integrals using the identities given in appendix A and assuming that η is constant within each triangle, equation (10) becomes for each discrete θ_μ value, triangle i , and γ zone ν ,

$$\sum_{j'=1}^3 M_{jj'}^{i\mu\nu} I'_{ij'\mu\nu} = Y_j^{i\mu\nu}, \tag{11}$$

where

$$M = Mrb + Mt + Mtb + Ma$$

and

$$Y = Yrb + Ytb + Ye.$$

The angular redistribution γ zone boundary terms are

$$(Mrb)_{jj'}^{i\mu\nu} = \beta_\mu \text{sn}_\mu(\gamma_{\mu(\nu+1)}) D_i C_{jj'}, \tag{12a}$$

where D_i is twice the area of triangle i (see appendix A),

$$C_{jj'} = \frac{1}{24} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

and

$$(Yrb)_j^{i\mu\nu} = \beta_\mu \text{sn}_\mu(\gamma_{\mu(\nu+1)}) D_i \sum_{j'=1}^3 C_{jj'} I'_{ij'\mu(\nu+1)}. \quad (12b)$$

The spatial transport interior term is

$$(Mt)_{jj'}^{i\mu\nu} = -(\Delta\gamma)_{\mu\nu} \left(\sum_{k=1}^3 C_{jk} R_{ik} \right) (\alpha_\mu dS_{ij'}^Z + \beta_\mu \text{cs}_{\mu\nu} dS_{ij'}^R), \quad (13)$$

where R_{ik} is the R coordinate at vertex k of triangle i , and $(dS_{ij'}^Z, dS_{ij'}^R)$ is the outward directed unit normal of the side of triangle i opposite vertex j times the length of that side.

The spatial transport zone boundary matrix is

$$(Mtb)_{jj'}^{i\mu\nu} = -(\Delta\gamma)_{\mu\nu} \sum_{\substack{k=1 \\ k \neq j \\ k \neq j'}}^3 \epsilon(k) (\alpha_\mu dS_{ik}^Z + \beta_\mu \text{cs}_{\mu\nu} dS_{ik}^R) F_{jkn(j',k)}^i, \quad (14a)$$

where

$$\epsilon(k) = \begin{cases} 0, & \text{if the side of triangle } i \text{ opposite vertex } k \text{ is an outflow side;} \\ 1, & \text{if the side of triangle } i \text{ opposite vertex } k \text{ is an inflow side,} \end{cases}$$

$$n(j', k) = (j' - k) \bmod 3,$$

and

$$F_{jkn}^i = \frac{1}{12} \begin{pmatrix} 0 & \begin{pmatrix} R_3+R_1 \\ R_3+3R_1 \end{pmatrix} & \begin{pmatrix} 3R_1+R_2 \\ R_1+R_2 \end{pmatrix} \\ \begin{pmatrix} 3R_2+R_3 \\ R_2+R_3 \end{pmatrix} & 0 & \begin{pmatrix} R_1+R_2 \\ R_1+3R_2 \end{pmatrix} \\ \begin{pmatrix} R_2+R_3 \\ R_2+3R_3 \end{pmatrix} & \begin{pmatrix} 3R_3+R_1 \\ R_3+R_1 \end{pmatrix} & 0 \end{pmatrix},$$

where $j = 1, 2, 3$ is the row, $k = 1, 2, 3$ is the column, $n = 1, 2$ is the column vector element, and for ease of presentation we have suppressed the i subscript of R_{ik} . The spatial transport zone boundary vector is

$$(Ytb)_j^{i\mu\nu} = -(\Delta\gamma)_{\mu\nu} \sum_{j'=1}^3 \sum_{\substack{k=1 \\ k \neq j \\ k \neq j'}}^3 \epsilon(k) (\alpha_\mu dS_{ik}^Z + \beta_\mu \text{cs}_{\mu\nu} dS_{ik}^R) F_{jkn(j',k)}^i I'_{ij'\mu\nu}, \quad (14b)$$

where $\tilde{ij}'$ is the triangle and the vertex of that triangle just across side $\vec{dS}_{ik}$ from ij' .

The absorption matrix is

$$(Ma)_{jj'}^{i\mu\nu} = \eta_i D_i(\Delta\gamma)_{\mu\nu} \left(\sum_{k=1}^3 C_{jk} R_{ik} \right) \delta_{jj'}, \quad (15a)$$

where

$$\delta_{jj'} = \begin{cases} 1, & \text{if } j = j'; \\ 0, & \text{otherwise,} \end{cases}$$

An exact evaluation of the absorption term in equation (10) yields

$$(Ma)_{jj'}^{i\mu\nu} = \eta_i D_i(\Delta\gamma)_{\mu\nu} (RM)_{jj'},$$

where

$$(RM)_{jj'} = \frac{1}{120} \begin{pmatrix} 6R_1 + 2(R_2 + R_3) & 2(R_1 + R_2) + R_3 & 2(R_1 + R_3) + R_2 \\ 2(R_1 + R_2) + R_3 & 6R_2 + 2(R_1 + R_3) & 2(R_2 + R_3) + R_1 \\ 2(R_1 + R_3) + R_2 & 2(R_2 + R_3) + R_1 & 2(R_1 + R_2) + 6R_3 \end{pmatrix}.$$

We have approximated (RM) (or "lumped" it in finite element jargon) by taking the sum of each row and putting it all in the diagonal element. Thus we obtain

$$\begin{aligned} (RM)_{jj'} &\approx \frac{1}{24} \begin{pmatrix} 2R_1 + R_2 + R_3 & 0 & 0 \\ 0 & R_1 + 2R_2 + R_3 & 0 \\ 0 & 0 & R_1 + R_2 + 2R_3 \end{pmatrix} \\ &= \left(\sum_{k=1}^3 C_{jk} R_{ik} \right) \delta_{jj'}. \end{aligned}$$

This preserves algebraic energy conservation. It makes the discrete approximation be not as high order accurate in (R, Z) space, but it makes it more robust when the spatial zoning is coarse (which is most often the case in real applications) and it makes the thermal wave limit easier to analyze and get right. Similarly the emission vector is approximated by its "lumped" form

$$(Ye)_j^{i\mu\nu} = \eta_i D_i(\Delta\gamma)_{\mu\nu} \left(\sum_{k=1}^3 C_{jk} R_{ik} \right) S_{ij\mu\nu}. \quad (15b)$$

Note that the source function, $S_{ij\mu\nu}$, depends on the photon direction angles because it includes a term,

$$\frac{I \frac{1}{c\rho\kappa\Delta t}}{1 + \frac{1}{c\rho\kappa\Delta t}},$$

which comes from the fully implicit differencing of the time dependence (see equation (2b)).

We note that the equations for each discrete θ_μ value are completely independent. the equations for each γ zone ν depend only on the intensities in the $(\nu + 1)$ zone and the equations for each triangle i depend only on the intensities in the adjacent triangles where

the common side is an inflow side for triangle i . For each μ, ν this determines a partial ordering of the triangles

$$i \prec i' \iff \begin{array}{l} i \text{ and } i' \text{ share a common side which is an outflow side for } i \\ \text{and therefore an inflow side for } i'. \end{array}$$

It is shown in reference 1 that there are no cycles in this partial ordering and therefore it is always possible to find a total ordering of the triangles which is consistent with the partial ordering. To find this total ordering we use the topological sort algorithm given in reference 2. Thus by ordering our unknowns $I'_{ij\mu\nu}$ so that $\nu = n_\gamma^\mu, n_\gamma^\mu - 1, \dots, 1$ and i is in the order which is consistent with the partial ordering of the triangles, our linear system, equation (11), becomes block lower triangular with three dimensional blocks and if we solve our equations in this order we only have to solve a three dimensional linear system at each step.

Section 2 - Discretization of the Electron Coupling Equation

Paired with equation (1) is the electron temperature equation

$$C_V \frac{\partial T_e}{\partial t} + \int_0^\infty d\omega \int_{-1}^{+1} \frac{d\cos\theta}{2} \int_0^\pi \frac{d\gamma}{\pi} (c\rho\kappa)(I_p(T_e) - I) = 0, \quad (16)$$

where C_V is the electron specific heat. T_e is the electron temperature, we have explicitly introduced the frequency (ω) dependence of I , and the units of I and $I_p(T_e)$ are chosen so that

$$2\pi \int R dR dZ \int_0^\infty d\omega \int_{-1}^{+1} \frac{d\cos\theta}{2} \int_0^\pi \frac{d\gamma}{\pi} I$$

is the total energy in the radiation field. For simplicity of exposition we shall continue to suppress the frequency dependence of I , since the explicit inclusion of the frequency dependence is straightforward and presents no particular problems. When we wish to explicitly show the frequency dependence, we divide frequency space into bins whose boundaries are

$$\omega_n, n = 1, 2, 3, \dots, n_\omega + 1,$$

and whose bin weights are

$$\Delta\omega_n = \omega_{n+1} - \omega_n,$$

and the radiation intensity acquires one more index and becomes $I_{ijn\mu\nu}$.

To discretize equation (16) we use fully implicit time differencing to obtain

$$C_V \frac{(T'_e - T_e)}{\Delta t} + \int_0^\infty d\omega \int_{-1}^{+1} \frac{d\cos\theta}{2} \int_0^\pi \frac{d\gamma}{\pi} (c\rho\kappa)(I_p(T'_e) - I') = 0, \quad (17)$$

and in equations (2a) and (17) we approximate $I_p(T'_e)$ by its linear fit

$$I_p(T'_e) = I_p(T_e) + (T'_e - T_e) \frac{dI_p}{dT_e}.$$

This linearized fully implicit treatment of the radiation - electron coupling removes the stability time step restriction that comes with explicit coupling. We assume that C_V is constant within each triangle and that T_e is linear within each triangle and continuous across triangle boundaries. Thus

$$T_e(R, Z) = \sum T_\ell^e \phi_\ell(R, Z),$$

where T_ℓ^e is the value of $T_e(R, Z)$ at the ℓ th node, and

$$\phi_\ell(R, Z) = \sum_{(i,j)=\ell} \phi_{i,j}(R, Z),$$

where the sum is over all vertices j of all triangles i which are coincident with node ℓ . It is this linear representation of T_e which allows us to get the thermal wave limit right.

Previous deterministic radiation transport codes, that used a zero'th order representation of T_e , have had incorrect thermal wave limits. We choose a continuous finite element representation for T_e , rather than the discontinuous finite element representation we use for I , because in the thermal wave limit we want to obtain the standard continuous finite element discretization of the diffusion equation. If we had chosen a discontinuous finite element representation for T_e this would introduce a whole set of additional T_e modes which vary rapidly across a single triangle and might not be rapidly damped in time. Some preliminary study indicates that these extra modes probably are damped, but the analysis of the thermal wave limit is enormously simplified if we use a continuous finite element representation for T_e , and we simply avoid the extra modes that might go unstable and cause problems.

Thus, multiplying equation (17) by $\Delta t \phi_\ell(R, Z)$, integrating $2\pi \int R dR dZ$, and "lumping" as we did in equations (15a) and (15b), we obtain our discretized electron coupling equation,

$$\begin{aligned}
0 = 2\pi \sum_{(i,j)=\ell} (C_{Vi} D_i (\sum_{k=1}^3 C_{jk} R_{ik})) (T_\ell^{e'} - T_\ell^e) & \quad (18) \\
+ 2\pi c \Delta t \sum_{(i,j)=\ell} \sum_{n=1}^{n_\omega} \Delta \omega_n (\rho_i \kappa_i D_i (\sum_{k=1}^3 C_{jk} R_{ik})) & \\
((I_p(T_\ell^e) + (T_\ell^{e'} - T_\ell^e) \frac{dI_p}{dT_\ell^e}) - \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \sum_{\nu=1}^{n_\gamma^\mu} \frac{(\Delta \gamma)_{\mu\nu}}{\pi} I'_{ijn\mu\nu}). &
\end{aligned}$$

Section 3 - Thermal Wave Limit of the Discretized Equations

The thermal wave limit is the limit in which the coupling coefficient $\rho\kappa$ becomes very large. In this limit the $\frac{\partial I}{\partial t}$ term in the radiation transport equation may be neglected and we have equation (2b) with $\eta = \rho\kappa$, $S = I_p(T_e')$ and $I' = I$, i.e. we have the true steady state radiation transport equation. For the electron coupling equation we have equation (16). The standard method of investigating the thermal wave limit is to let $\eta \rightarrow \frac{\eta}{\epsilon}$, where ϵ is assumed to be very small, and then solve for the radiation intensity as a power series in ϵ ,

$$I = \sum_{n=0}^{\infty} \epsilon^n I^n. \quad (19)$$

From the analytic equation (2b) in steady state form we obtain, to 0'th order,

$$I^0 = S, \quad (20a)$$

to 1'st order,

$$I^1 = -\frac{1}{\eta} \mathcal{L} S, \quad (20b)$$

and to 2'nd order,

$$I^2 = -\frac{1}{\eta} \mathcal{L} I^1 = \frac{1}{\eta} \mathcal{L} \frac{1}{\eta} \mathcal{L} S. \quad (20c)$$

Substituting these into equation (16) and integrating over photon direction, we obtain,

$$C_V \frac{\partial T_e}{\partial t} - \frac{c}{3} \int_0^\infty d\omega \left(\frac{\partial}{\partial Z} \frac{\epsilon}{\eta} \frac{\partial}{\partial Z} + \frac{1}{R} \frac{\partial}{\partial R} \frac{R\epsilon}{\eta} \frac{\partial}{\partial R} \right) I_p(T_e) = 0, \quad (21)$$

which is the correct cylindrically symmetric thermal wave equation.

If our discretized equations (11) and (18) are to behave correctly in the strong coupling limit, then they too must approach a finite element discretized version of equation (21) as $\epsilon \rightarrow 0$. Discretizing equation (21) in the same way we discretized equation (18) and using the relations given in Appendix A, we find,

$$\begin{aligned} 0 = 2\pi \sum_{(i,j)=\ell} (C_{Vi} D_i (\sum_{k=1}^3 C_{jk} R_{ik})) (T_\ell^{\epsilon'} - T_\ell^\epsilon) \\ + 2\pi c \Delta t \sum_{n=1}^{n_\omega} \Delta \omega_n \sum_{(i,j)=\ell} \frac{\epsilon \sum_{j'=1}^3 R_{ij'}}{18 \eta_i D_i} \sum_{j''=1}^3 (dS_{ij}^Z dS_{ij''}^Z + dS_{ij}^R dS_{ij''}^R) S_{ij'',} \end{aligned} \quad (21a)$$

is the desired $\epsilon \rightarrow 0$ limit of equations (11) and (18). To see that we obtain this limit, we let $\eta \rightarrow \frac{\eta}{\epsilon}$ in equations (11) and (18), and solve equation (11) for the radiation intensity as a power series in ϵ . We find, to 0'th order,

$$I_{ij\mu\nu}^0 = S_{ij} = S_\ell = I_p(T_\ell^\epsilon) + (T_\ell^{\epsilon'} - T_\ell^\epsilon) \frac{dI_p}{dT_\ell^\epsilon}, \quad (22a)$$

where ℓ is the node which is coincident with vertex j of triangle i . Note that $I_{ij\mu\nu}^0$ doesn't depend on μ or ν . We find to 1'st order,

$$I_{ij\mu\nu}^1 = \frac{1}{\eta_i D_i} \sum_{j'=1}^3 (\alpha_\mu dS_{ij'}^Z + \beta_\mu cs_{\mu\nu} dS_{ij'}^R) S_{ij'}, \quad (22b)$$

where since $I_{ij\mu\nu}^0$ doesn't depend on μ or ν , terms (12a) and (12b) cancel and do not contribute, and since $I_{ij\mu\nu}^0$ is continuous across triangle boundaries, terms (14a) and (14b) cancel and do not contribute. Note that $I_{ij\mu\nu}^1$ doesn't depend on j . We find to 2'nd order,

$$\begin{aligned} I_{ij\mu\nu}^2 = & \frac{1}{\eta_i D_i (\Delta\gamma)_{\mu\nu} (\sum_{k=1}^3 C_{jk} R_{ik})} \left(\beta_\mu^2 \text{sn}_\mu(\gamma_{\mu(\nu+1)}) \frac{(cs_{\mu(\nu+1)} - cs_{\mu\nu})}{6\eta_i} \sum_{j'=1}^3 dS_{ij'}^R S_{ij'} \right. \\ & \left. + (\Delta\gamma)_{\mu\nu} \sum_{j'=1}^3 F_{jj'}^i \epsilon(j') (\alpha_\mu dS_{ij'}^Z + \beta_\mu cs_{\mu\nu} dS_{ij'}^R) \right) \\ & \left(\frac{1}{\eta_i D_i} \sum_{j''=1}^3 (\alpha_\mu dS_{ij''}^Z + \beta_\mu cs_{\mu\nu} dS_{ij''}^R) S_{ij''} - \frac{1}{\eta_i D_i} \sum_{j''=1}^3 (\alpha_\mu dS_{\tilde{i}j''}^Z + \beta_\mu cs_{\mu\nu} dS_{\tilde{i}j''}^R) S_{\tilde{i}j''} \right). \quad (22c) \end{aligned}$$

where $F_{jj'}^i = \sum_{n=1}^2 F_{jj'n}^i$, and as in equation (14b), $\tilde{i}$ is the triangle just across side $dS_{ij'}$ from triangle i .

Substituting (22a), (22b) and (22c) for $I_{ij\mu\nu}$ in equation (18) we find that the $I_{ij\mu\nu}^0$ term exactly cancels the S_ℓ term, the sum over photon directions of $I_{ij\mu\nu}^1$ is zero because of equations (4d) and (9), and so,

$$2\pi \sum_{(i,j)=\ell} (C_{Vi} D_i (\sum_{k=1}^3 C_{jk} R_{ik})) (T_\ell^{e'} - T_\ell^e) = 2\pi c \Delta t \epsilon \sum_{n=1}^{n_\omega} \Delta\omega_n P_{\ell n}, \quad (23)$$

where,

$$P_{\ell n} = \sum_{(i,j)=\ell} (\eta_i D_i (\sum_{k=1}^3 C_{jk} R_{ik})) \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \sum_{\nu=1}^{n_\gamma^\mu} \frac{(\Delta\gamma)_{\mu\nu}}{\pi} I_{ij\mu\nu}^2.$$

The first part of $I_{ij\mu\nu}^2$ gives

$$P_{\ell n}^1 = \left(\sum_{(i,j)=\ell} \frac{1}{6\eta_i} \sum_{j'=1}^3 dS_{ij'}^R S_{ij'} \right) \left(\sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu \beta_\mu^2}{2\pi} \sum_{\nu=1}^{n_\gamma^\mu} \text{sn}_\mu(\gamma_{\mu(\nu+1)}) (cs_{\mu(\nu+1)} - cs_{\mu\nu}) \right).$$

Equations (8a) and (9a) imply that

$$\begin{aligned} \sum_{\nu=1}^{n_\gamma^\mu} \text{sn}_\mu(\gamma_{\mu(\nu+1)}) (cs_{\mu(\nu+1)} - cs_{\mu\nu}) &= \sum_{\nu=1}^{n_\gamma^\mu} (\text{sn}_\mu(\gamma_{\mu\nu}) - \text{sn}_\mu(\gamma_{\mu(\nu+1)})) cs_{\mu\nu} = \\ &= - \sum_{\nu=1}^{n_\gamma^\mu} (\Delta\gamma)_{\mu\nu} (cs_{\mu\nu})^2 = -\frac{\pi}{2}, \end{aligned}$$

and equation (4g) implies that

$$\sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu \beta_\mu^2}{2\pi} = \frac{2}{3\pi}.$$

Therefore,

$$P_{\ell n}^1 = -\frac{1}{18} \left(\sum_{(i,j)=\ell} \frac{1}{\eta_i} \sum_{j'=1}^3 dS_{ij'}^R S_{ij'} \right).$$

The second part of $I_{ijn\mu\nu}^2$ gives

$$\begin{aligned} P_{\ell n}^2 &= \sum_{(i,j)=\ell} \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \sum_{\nu=1}^{n_\gamma} \frac{(\Delta\gamma)_{\mu\nu}}{\pi} \sum_{j'=1}^3 F_{jj'}^i \epsilon(j') (\alpha_\mu dS_{ij'}^Z + \beta_\mu c s_{\mu\nu} dS_{ij'}^R) \\ &\quad \left(\frac{1}{\eta_i D_i} \sum_{j''=1}^3 (\alpha_\mu dS_{ij''}^Z + \beta_\mu c s_{\mu\nu} dS_{ij''}^R) S_{ij''} - \frac{1}{\eta_{\tilde{i}} D_{\tilde{i}}} \sum_{j''=1}^3 (\alpha_\mu dS_{\tilde{i}j''}^Z + \beta_\mu c s_{\mu\nu} dS_{\tilde{i}j''}^R) S_{\tilde{i}j''} \right). \end{aligned}$$

We may eliminate $\epsilon(j')$ in the previous equation by observing that the triangle $\tilde{i}$ terms make the same contribution for outflow boundaries of triangle i as do the triangle i terms for inflow boundaries of triangle i . Thus we have,

$$\begin{aligned} P_{\ell n}^2 &= \sum_{(i,j)=\ell} \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \sum_{\nu=1}^{n_\gamma} \frac{(\Delta\gamma)_{\mu\nu}}{\pi} \sum_{j'=1}^3 F_{jj'}^i (\alpha_\mu dS_{ij'}^Z + \beta_\mu c s_{\mu\nu} dS_{ij'}^R) \\ &\quad \frac{1}{\eta_i D_i} \sum_{j''=1}^3 (\alpha_\mu dS_{ij''}^Z + \beta_\mu c s_{\mu\nu} dS_{ij''}^R) S_{ij''}. \end{aligned}$$

Now we do the sums over μ and ν , and using equations (4e), (4g), (9a) and (9b) we find.

$$P_{\ell n}^2 = \sum_{(i,j)=\ell} \frac{1}{3\eta_i D_i} \sum_{j'=1}^3 F_{jj'}^i \sum_{j''=1}^3 (dS_{ij'}^Z dS_{ij''}^Z + dS_{ij'}^R dS_{ij''}^R) S_{ij''}.$$

From Appendix A it follows that,

$$\sum_{j=1}^3 dS_{ij}^R S_{ij} = -\frac{1}{D_i} \sum_{j'=1}^3 R_{ij'} \sum_{j=1}^3 (dS_{ij'}^Z dS_{ij}^Z + dS_{ij'}^R dS_{ij}^R) S_{ij},$$

so that,

$$P_{\ell n}^1 = \sum_{(i,j)=\ell} \frac{1}{3\eta_i D_i} \sum_{j'=1}^3 \frac{R_{ij'}}{6} \sum_{j''=1}^3 (dS_{ij'}^Z dS_{ij''}^Z + dS_{ij'}^R dS_{ij''}^R) S_{ij''}.$$

Combining P^1 and P^2 and using the fact that,

$$\sum_{j=1}^3 \vec{dS}_{ij} = 0,$$

we obtain,

$$P_{\ell n} = - \sum_{(i,j)=\ell} \frac{\sum_{j'=1}^3 R_{ij'}}{18\eta_i D_i} \sum_{j''=1}^3 (dS_{ij}^Z dS_{ij''}^Z + dS_{ij}^R dS_{ij''}^R) S_{ij''}.$$

But substituting this expression for $P_{\ell n}$ in equation (23) gives exactly equation (21a) so we have shown that our discretized transport equation does have the correct thermal wave limit.

Section 4 - Boundary Conditions and Energy Conservation

In the analytic theory of the radiation transport equation it is shown that the correct boundary condition consists of specifying the intensity at each boundary point for every photon ray direction which is entering the problem domain from outside the domain. The boundary intensity for rays which are leaving the problem domain is determined when we solve the transport equation. Similarly, in our discretized scheme, for each problem domain boundary line segment we specify the intensity at the node at each end of the line segment, for those rays where

$$(\alpha_\mu \hat{n}_Z + \beta_\mu \cos \mu_\nu \hat{n}_R) < 0,$$

where $(\hat{n}_Z, \hat{n}_R)$ is the outward directed unit normal on the problem domain boundary line segment, and the intensity of rays for which

$$(\alpha_\mu \hat{n}_Z + \beta_\mu \cos \mu_\nu \hat{n}_R) > 0,$$

is determined when we solve our discretized equations. Referring to our discretized equation (14b) we see that if triangle i is adjacent to the problem domain boundary and the side of triangle i opposite vertex k is a boundary line segment, then $I'_{ij'\mu\nu}$ is now taken to be the specified boundary line segment intensity at node ij' .

To study the energy conservation properties of our discretized scheme, we start from equation (10), multiply it by 2π and divide it by π to get the volume and γ bin normalization right, and sum it:

- a) over all triangles, i , in the problem domain,
- b) over the three vertices, j , of each triangle,
- c) over all photon direction θ bins, μ , weighted by the bin weight, $\frac{\Delta_\mu}{2}$,
- d) over all photon direction γ bins, ν .

Thus we obtain:

$$2\pi \int_{\mathcal{D}} R dR dZ \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \int_0^\pi \frac{d\gamma}{\pi} \eta(S - I'_\mu) = \quad (24)$$

$$\sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \int_0^\pi \frac{d\gamma}{\pi} \left(\int_{\partial_+ \mathcal{D}} d\ell 2\pi R (\alpha_\mu \hat{n}_Z + \beta_\mu \cos \mu_\nu (\gamma) \hat{n}_R) I'_\mu + \int_{\partial_- \mathcal{D}} d\ell 2\pi R (\alpha_\mu \hat{n}_Z + \beta_\mu \cos \mu_\nu (\gamma) \hat{n}_R) \tilde{I}'_\mu \right).$$

where,

$(\hat{n}_Z, \hat{n}_R)$ is the outward directed unit normal on the boundary of the problem domain.

$\int_{\mathcal{D}}$ is the integral over the whole problem domain,

$\int_{\partial_- \mathcal{D}}$ is the integral over the inflow boundary of the problem domain, i.e. that part of the boundary where $(\alpha_\mu \hat{n}_Z + \beta_\mu \cos \mu_\nu \hat{n}_R) < 0$,

$\int_{\partial_+ \mathcal{D}}$ is the integral over the outflow boundary of the problem domain, i.e. that part of the boundary where $(\alpha_\mu \hat{n}_Z + \beta_\mu \cos \mu_\nu \hat{n}_R) > 0$,

$\tilde{I}'_\mu$ is the specified boundary value intensity along the inflow boundary.

We take the left hand side of equation (24) to be the total energy coupled to the electrons and the change in the energy present in the radiation field (if we are doing a time dependent

problem and S and η are as defined in equation (2b)). We take the first term of the right hand side of equation (24) to be the total energy escaping out the problem boundary, and the second term to be the total energy entering the problem boundary from sources outside. With these definitions, equation (24) shows that our discretized scheme conserves energy exactly. Evaluating the integrals in equation (24) and "lumping" the matrix elements for the electron coupling energy as we did in equations (15a) and (15b) we obtain our final statement of energy conservation,

$$\begin{aligned}
2 \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \sum_{\nu=1}^{n_\gamma^\mu} \sum_{i,j} ((Ye)_j^{i\mu\nu} - \sum_{j'=1}^3 (Ma)_{jj'}^{i\mu\nu} I'_{ij'\mu\nu}) = \quad (25) \\
2 \sum_{\mu=1}^{n_\theta} \frac{\Delta_\mu}{2} \sum_{\nu=1}^{n_\gamma^\mu} (\Delta\gamma)_{\mu\nu} \\
\left(\sum_{\ell+} (\alpha_\mu dS_\ell^Z + \beta_\mu cs_{\mu\nu} dS_\ell^R) \frac{(2R_{\ell 1} I'_{\ell 1 \mu\nu} + R_{\ell 2} I'_{\ell 1 \mu\nu} + R_{\ell 1} I'_{\ell 2 \mu\nu} + 2R_{\ell 2} I'_{\ell 2 \mu\nu})}{6} \right. \\
\left. + \sum_{\ell-} (\alpha_\mu dS_\ell^Z + \beta_\mu cs_{\mu\nu} dS_\ell^R) \frac{(2R_{\ell 1} \tilde{I}'_{\ell 1 \mu\nu} + R_{\ell 2} \tilde{I}'_{\ell 1 \mu\nu} + R_{\ell 1} \tilde{I}'_{\ell 2 \mu\nu} + 2R_{\ell 2} \tilde{I}'_{\ell 2 \mu\nu})}{6} \right).
\end{aligned}$$

where,

$\sum_{\ell+}$ is a sum over the line segments ℓ on the outflow boundary of the problem domain,

$\sum_{\ell-}$ is a sum over the line segments ℓ on the inflow boundary of the problem domain.

$R_{\ell 1}$ and $R_{\ell 2}$ are the cylindrical radii at the two ends of line segment ℓ ,

$I'_{\ell 1 \mu\nu}$ and $I'_{\ell 2 \mu\nu}$ are the radiation intensities in the triangle adjacent to line segment ℓ , at the two ends of line segment ℓ ,

$\tilde{I}'_{\ell 1 \mu\nu}$ and $\tilde{I}'_{\ell 2 \mu\nu}$ are the specified boundary value radiation intensities at the two ends of line segment ℓ ,

and (dS_ℓ^Z, dS_ℓ^R) is a vector whose magnitude is the length of line segment ℓ and whose direction is the outward (from the problem domain) directed unit normal to line segment ℓ .

Section 5 - Implicit Electron Temperature Coupling Iteration

To advance the electron temperature and the radiation intensity each time step, we need to solve the electron coupling equation, equation (18), and the radiation transport equation, equation (11), with $S_{ij\mu\nu}$ in equation (15b) given by

$$S_{ij\mu\nu} = \frac{(I_p(T_\ell^e) + (T_\ell^{e'} - T_\ell^e) \frac{dI_p}{dT_\ell^e}) + I_{ij\mu\nu} \frac{1}{c\rho_i\kappa_i\Delta t}}{1 + \frac{1}{c\rho_i\kappa_i\Delta t}}, \quad (26a)$$

where, ℓ is the node coincident with vertex j of triangle i . These equations must be simultaneously solved for I' and T_e' , in order to avoid the severe time step restriction associated with explicit differencing of the electron temperature. To do this we use a variant of the splitting scheme given in reference 3. We begin by solving equation (11) for $I'_{ij\mu\nu}$, with $S_{ij\mu\nu}$ as given above, keeping the dependence of $I'_{ij\mu\nu}$ on all the $T_\ell^{e'}$ upwind of it explicitly. Thus we end up with a solution of the form,

$$I'_{ij\mu\nu} = I_{ij\mu\nu}^{0'} + \sum_{\ell} I_{ij\mu\nu}^{\ell'} (T_\ell^{e'} - T_\ell^e), \quad (26b)$$

where $\sum_{\ell}$ is a sum over all nodes in the problem domain which are upwind of $I'_{ij\mu\nu}$. If we substitute this into equation (18) we obtain a dense linear set of equations for $T_\ell^{e'}$ whose dimension is the number of nodes in the problem domain. In principle we could solve this equation for the $T_\ell^{e'}$ and then substitute the answer into equation (26b) to find the $I'_{ij\mu\nu}$. In practice, it is much faster to solve iteratively. Assume that we are given a current best guess for $T_\ell^{e'}$ (usually we use T_ℓ^e for the initial guess). In equation (26b) we replace $T_\ell^{e'}$ by the current best guess value for every node ℓ except the node that is coincident with ij . For this node we keep the explicit dependence on $T_\ell^{e'}$. Substituting this into equation (18) we obtain independent equations for each node which we use to determine our new improved guess for $T_\ell^{e'}$ at that node. Since it is trivial to solve the independent equations for each node, if our iteration converges at all rapidly, we will get our answer much faster than by solving a dense linear system whose dimension is the number of nodes in the problem domain. In practice this iteration is found to converge quite rapidly on most problems and to converge eventually on all problems. Also, in the actual implementation, we never solve for $I'_{ij\mu\nu}$ in the form of equation (26b). Instead when we solve for the three $I'_{ij\mu\nu}$ in each triangle we keep explicit only the $T_\ell^{e'}$ dependence for the three nodes at the vertices of that triangle. Thus we obtain,

$$I'_{ij\mu\nu} = I_{ij\mu\nu}^{0'} + \sum_{\ell(i)} I_{ij\mu\nu}^{\ell'} (T_\ell^{e'} - T_\ell^e), \quad (27)$$

where $\sum_{\ell(i)}$ is the sum over the three nodes at the vertices of triangle i . This expression for $I'_{ij\mu\nu}$ enters the equations for adjacent triangles thru equation (14b) and when it does, we keep the explicit $T_\ell^{e'}$ dependence for the two nodes shared by both triangles and replace $T_\ell^{e'}$ with the current best guess for the third node which is not a vertex of the triangle being solved. Finally, after all expressions of the form of equation (27) have been obtained, we replace $T_\ell^{e'}$ by the current best guess value for every node ℓ except the node that is coincident with ij . This is algebraically equivalent to the iteration described above and is just a more efficient way of organizing it.

Appendix A - Linear elements on a Triangular Mesh

Let $(R_1, Z_1), (R_2, Z_2), (R_3, Z_3)$ be the (R, Z) coordinates of the vertices of the i 'th triangle. R is always the "y" axis, Z the "x" axis and the triangle vertices are always numbered going counterclockwise around the triangle. Let

$$D_i = \begin{vmatrix} Z_1 & Z_2 & Z_3 \\ R_1 & R_2 & R_3 \\ 1 & 1 & 1 \end{vmatrix},$$

which is twice the area of the triangle and is always positive. Then the linear basis elements are $\phi_{ij}(R, Z)$, where

$$\phi_{i1}(R, Z) = \frac{1}{D_i} \begin{vmatrix} Z & Z_2 & Z_3 \\ R & R_2 & R_3 \\ 1 & 1 & 1 \end{vmatrix},$$

$$\phi_{i2}(R, Z) = \frac{1}{D_i} \begin{vmatrix} Z_1 & Z & Z_3 \\ R_1 & R & R_3 \\ 1 & 1 & 1 \end{vmatrix},$$

$$\phi_{i3}(R, Z) = \frac{1}{D_i} \begin{vmatrix} Z_1 & Z_2 & Z \\ R_1 & R_2 & R \\ 1 & 1 & 1 \end{vmatrix},$$

inside the i 'th triangle and $\phi_{ij}(R, Z) = 0$ outside the i 'th triangle. We note that inside triangle i ,

$$1 = \sum_{j=1}^3 \phi_{ij}(R, Z),$$

$$R = \sum_{j=1}^3 R_{ij} \phi_{ij}(R, Z),$$

and

$$Z = \sum_{j=1}^3 Z_{ij} \phi_{ij}(R, Z).$$

To find the finite element discretization we need the integral over the i 'th triangle of various powers of the ϕ_{ij} . These can be found from the identity

$$\int dR dZ \phi_{i1}^{n_1} \phi_{i2}^{n_2} \phi_{i3}^{n_3} = D_i \frac{n_1! n_2! n_3!}{(n_1 + n_2 + n_3 + 2)!}.$$

We also need the gradient of the basis elements,

$$\vec{\nabla} \phi_{ij}(R, Z) = \frac{-\vec{\nabla} S_{ij}}{D_i},$$

where $\vec{dS}_{ij} = (dS_{ij}^Z, dS_{ij}^R)$ is a vector whose magnitude is the length of the side of the triangle opposite vertex j and whose direction is the outward normal from that side. We also need the corresponding relations for the basis elements on a line segment of length ℓ .

$$\phi_1(x) = \frac{x}{\ell} \text{ and } \phi_2(x) = 1 - \frac{x}{\ell}.$$

We have

$$\int_0^\ell dx \phi_1^{n_1} \phi_2^{n_2} = \ell \frac{n_1! n_2!}{(n_1 + n_2 + 1)!},$$

and

$$\frac{d\phi_1}{dx} = \frac{1}{\ell}, \quad \frac{d\phi_2}{dx} = \frac{-1}{\ell}.$$

Appendix B - 1D Slab and 1D Spherically Symmetric Radiation Transport

In what follows we shall treat the spherically symmetric case in detail. The slab symmetric case may be obtained from these formulas by simply letting the spherical radius, r , go to infinity, since a spherical shell at very large radius behaves just like a 1D slab.

We start with the analytic radiation transport equation,

$$\mathcal{L}I' - \eta(S - I') = 0, \quad (b1)$$

which is just like equation (2b) except that now,

$$\mathcal{L} = \mu \frac{\partial}{\partial r} + \frac{(1 - \mu^2)}{r} \frac{\partial}{\partial \mu},$$

where,

$$\mu = \hat{k} \cdot \hat{r}.$$

We assume that the problem domain,

$$r \leq r \leq \bar{r},$$

is divided into n_r (not necessarily equal spaced) bins whose boundary nodes are at

$$r = r_1 < r_2 < \dots < r_{n_r} < r_{n_r+1} = \bar{r},$$

and we divide the photon direction domain

$$-1 \leq \mu \leq 1,$$

into n_μ (n_μ must be even) equal spaced bins whose boundary nodes are at

$$-1 = \mu_1 < \mu_2 < \dots < 0 = \mu_{\frac{n_\mu}{2}+1} < \dots < \mu_{n_\mu} < \mu_{n_\mu+1} = 1.$$

We use linear discontinuous elements on both r and μ zones, so that

$$\phi_{i1}(r) = \begin{cases} \frac{r_{i+1}-r}{r_{i+1}-r_i} & , r_i \leq r \leq r_{i+1} \\ 0 & , \text{otherwise} \end{cases},$$

$$\phi_{i2}(r) = \begin{cases} \frac{r-r_i}{r_{i+1}-r_i} & , r_i \leq r \leq r_{i+1} \\ 0 & , \text{otherwise} \end{cases},$$

$$\psi_{\alpha 1}(\mu) = \begin{cases} \frac{\mu_{\alpha+1}-\mu}{\mu_{\alpha+1}-\mu_\alpha} & , \mu_\alpha \leq \mu \leq \mu_{\alpha+1} \\ 0 & , \text{otherwise} \end{cases},$$

$$\psi_{\alpha 2}(\mu) = \begin{cases} \frac{\mu-\mu_\alpha}{\mu_{\alpha+1}-\mu_\alpha} & , \mu_\alpha \leq \mu \leq \mu_{\alpha+1} \\ 0 & , \text{otherwise} \end{cases}.$$

and

$$I(r, \mu) = \sum_{i=1}^{n_r} \sum_{j=1}^2 \sum_{\alpha=1}^{n_\mu} \sum_{\beta=1}^2 I_{ij\alpha\beta} \phi_{ij}(r) \psi_{\alpha\beta}(\mu). \quad (b2)$$

Substituting equation (b2) into (b1), multiplying by a basis function $\phi_{ij}\psi_{\alpha\beta}$ and integrating, we obtain,

$$\begin{aligned} 0 = & \int_{r_i}^{r_{i+1}} r^2 dr \int_{\mu_\alpha}^{\mu_{\alpha+1}} d\mu \phi_{ij}(r) \psi_{\alpha\beta}(\mu) (\mathcal{L}I' - \eta(S - I')) \\ & + \int_{r_i}^{r_{i+1}} r dr (1 - \mu_\alpha^2) \phi_{ij}(r) \psi_{\alpha\beta}(\mu_\alpha) \sum_{j'=1}^2 (I_{ij'\alpha 1} - I_{ij'(\alpha-1)2}) \phi_{ij'}(r) \\ & + \int_{\mu_\alpha}^{\mu_{\alpha+1}} \mu d\mu \psi_{\alpha\beta}(\mu) \left\{ \begin{array}{ll} r_{i+1}^2 \phi_{ij}(r_{i+1}) \sum_{\beta'=1}^2 (I_{(i+1)1\alpha\beta'} - I_{i2\alpha\beta'}) \psi_{\alpha\beta'}(\mu) & , \alpha \leq \frac{n_\mu}{2} \\ r_i^2 \phi_{ij}(r_i) \sum_{\beta'=1}^2 (I_{i1\alpha\beta'} - I_{(i-1)2\alpha\beta'}) \psi_{\alpha\beta'}(\mu) & , \alpha > \frac{n_\mu}{2} \end{array} \right. \end{aligned} \quad (b3)$$

where as in the 2D case, the Dirac δ function terms are always 100% included in the upwind zone's equation and 0% included in the downwind zone's equation. In 1D spherical radiation transport, as we move along a characteristic, μ always increases, while r decreases when $\mu < 0$ and increases when $\mu > 0$.

Performing the integrals in equation (b3) and "lumping" the electron coupling terms. we obtain,

$$\sum_{j'=1}^2 \sum_{\beta'=1}^2 M_{j'\beta'}^{ij\alpha\beta} I_{ij'\alpha\beta'} = Y^{ij\alpha\beta}, \quad (b4)$$

where,

$$M = Mr + Mrb + Mt + Mtb + Ma$$

and

$$Y = Yrb + Ytb + Ye.$$

The angular redistribution interior terms are

$$(Mr)_{j'\beta'}^{ij\alpha\beta} = \frac{(\Delta r)_i}{24} (1 - (A\mu)_\beta^\alpha) (Br)_{jj'}^i (-1)^{\beta'}, \quad (b5)$$

where,

$$\begin{aligned} (A\mu)^\alpha &= \frac{1}{6} \left(\frac{3\mu_\alpha^2 + 2\mu_\alpha\mu_{(\alpha+1)} + \mu_{(\alpha+1)}^2}{\mu_\alpha^2 + 2\mu_\alpha\mu_{(\alpha+1)} + 3\mu_{(\alpha+1)}^2} \right), \\ (Br)^i &= \begin{pmatrix} 3r_i + r_{(i+1)} & r_i + r_{(i+1)} \\ r_i + r_{(i+1)} & r_i + 3r_{(i+1)} \end{pmatrix}, \end{aligned}$$

and

$$(\Delta r)_i = r_{(i+1)} - r_i.$$

The angular redistribution boundary terms are

$$(Mrb)_{j'\beta'}^{ij\alpha\beta} = \frac{(1 - \mu_\alpha^2)(\Delta r)_i}{12} (Br)_{jj'}^i \delta_{\beta 1} \delta_{\beta' 1}, \quad (b6)$$

and

$$(Yrb)_{j'\beta'}^{ij\alpha\beta} = \frac{(1 - \mu_\alpha^2)(\Delta r)_i}{12} \delta_{\beta 1} \sum_{j'=1}^2 (Br)_{jj'}^i I_{ij'(\alpha-1)2},$$

where δ is the Kronecker delta function.

The spatial transport interior terms are

$$(Mt)_{j'\beta'}^{ij\alpha\beta} = \frac{(\Delta\mu)_\alpha}{24} (Ar)_j^i (B\mu)_{\beta\beta'}^\alpha (-1)^{j'}, \quad (b7)$$

where,

$$(Ar)^i = \frac{1}{6} \left(3r_i^2 + 2r_i r_{(i+1)} + r_{(i+1)}^2 \right),$$

$$(B\mu)^\alpha = \begin{pmatrix} 3\mu_\alpha + \mu_{(\alpha+1)} & \mu_\alpha + \mu_{(\alpha+1)} \\ \mu_\alpha + \mu_{(\alpha+1)} & \mu_\alpha + 3\mu_{(\alpha+1)} \end{pmatrix},$$

and

$$(\Delta\mu)_\alpha = \mu_{(\alpha+1)} - \mu_\alpha.$$

The spatial transport boundary terms are

$$(Mtb)_{j'\beta'}^{ij\alpha\beta} = \frac{(\Delta\mu)_\alpha}{12} (B\mu)_{\beta\beta'}^\alpha \begin{cases} -r_{i+1}^2 \delta_{j2} \delta_{j'2} & , \alpha \leq \frac{n_\mu}{2} \\ r_i^2 \delta_{j1} \delta_{j'1} & , \alpha > \frac{n_\mu}{2} \end{cases}, \quad (b8)$$

and

$$(Ytb)_{j'\beta'}^{ij\alpha\beta} = \frac{(\Delta\mu)_\alpha}{12} \sum_{\beta'=1}^2 (B\mu)_{\beta\beta'}^\alpha \begin{cases} -r_{i+1}^2 \delta_{j2} I_{(i+1)1\alpha\beta'} & , \alpha \leq \frac{n_\mu}{2} \\ r_i^2 \delta_{j1} I_{(i-1)2\alpha\beta'} & , \alpha > \frac{n_\mu}{2} \end{cases}.$$

The electron coupling terms are

$$(Ma)_{j'\beta'}^{ij\alpha\beta} = \frac{\eta_i (\Delta r)_i (\Delta\mu)_\alpha}{4} (Ar)_j^i \delta_{jj'} C_{\beta\beta'}, \quad (b9)$$

where,

$$C = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and

$$(Ye)_{j'\beta'}^{ij\alpha\beta} = \frac{\eta_i (\Delta r)_i (\Delta\mu)_\alpha}{4} (Ar)_j^i \sum_{\beta'=1}^2 C_{\beta\beta'} S_{ij\alpha\beta'},$$

where we have "lumped" in r but not in μ space. As in equation (15b) the source function, $S_{ij\alpha\beta}$, depends on the photon direction angles because it includes a term which comes from the fully implicit differencing of the time dependence.

We note that the equations for each $(i\alpha)$ zone depend only on the intensities in the $(i(\alpha-1))$ zone and in the $\left\{ \begin{array}{l} ((i+1)\alpha) \quad , \alpha \leq \frac{n_\mu}{2} \\ ((i-1)\alpha) \quad , \alpha > \frac{n_\mu}{2} \end{array} \right\}$ zone. Thus if we order our unknowns so that α is always increasing and i is $\left\{ \begin{array}{l} \text{decreasing} \quad , \alpha \leq \frac{n_\mu}{2} \\ \text{increasing} \quad , \alpha > \frac{n_\mu}{2} \end{array} \right\}$, our linear system, equation (b4), becomes block lower triangular with four dimensional blocks and if we solve our equations in this order we only have to solve a four dimensional linear system at each step.

As in section 2, we use continuous elements,

$$\phi_i(r) = \phi_{i1}(r) + \phi_{(i-1)2}(r),$$

to represent the electron temperature,

$$T_e(r) = \sum_{i=1}^{n_r+1} T_i^e \phi_i(r),$$

and our 1D spherical electron temperature equation is,

$$C_V \frac{(T_e' - T_e)}{\Delta t} + \int_0^\infty d\omega \int_{-1}^{+1} \frac{d\mu}{2} (c\rho\kappa)(I_p(T_e') - I') = 0.$$

Multiplying this by $\Delta t \phi_i(r)$, integrating $4\pi \int r^2 dr$, and "lumping" as we did in equations (b9), we obtain our discretized electron coupling equation,

$$\begin{aligned} 0 = & 2\pi(C_{V(i-1)}(\Delta r)_{i-1}(Ar)_2^{i-1} + C_{Vi}(\Delta r)_i(Ar)_1^i)(T_i^{e'} - T_i^e) \\ & + \pi c \Delta t \sum_{n=1}^{n_\omega} \Delta\omega_n \sum_{\alpha=1}^{n_\mu} (\Delta\mu)_\alpha \\ & \left((\rho\kappa)_{i-1}(\Delta r)_{i-1}(Ar)_2^{i-1} \left((I_p(T_i^e) + (T_i^{e'} - T_i^e) \frac{dI_p}{dT_i^e}) - \frac{(I'_{(i-1)2\alpha 1} + I'_{(i-1)2\alpha 2})}{2} \right) \right. \\ & \left. + (\rho\kappa)_i(\Delta r)_i(Ar)_1^i \left((I_p(T_i^e) + (T_i^{e'} - T_i^e) \frac{dI_p}{dT_i^e}) - \frac{(I'_{i1\alpha 1} + I'_{i1\alpha 2})}{2} \right) \right), \end{aligned} \quad (b10)$$

where C_V and $\rho\kappa$ are zone centered quantities.

To examine the thermal wave limit, as in section 3, we let $\eta \rightarrow \frac{\eta}{\epsilon}$, where ϵ is assumed to be very small, and then solve equation (b4) for the radiation intensity as a power series in ϵ . We find, to 0'th order only equations (b9) matter, and we obtain,

$$I_{i1\alpha\beta}^0 = I_{(i-1)2\alpha\beta}^0 = S_i = I_p(T_i^e) + (T_i^{e'} - T_i^e) \frac{dI_p}{dT_i^e}.$$

so I^0 is independent of α and β . We find, to 1'st order the terms from (b5), (b6) and (b8) all evaluate to zero, and only equations (b7) and (b9) matter. Solving these, we obtain,

$$I_{ij(\alpha-1)2}^1 = I_{ij\alpha 1}^1 = -\frac{(S_{i+1} - S_i)}{\eta_i(\Delta r)_i} \mu_\alpha.$$

so I^1 is independent of j . We find to 2'nd order, the terms from (b6) and (b7) all evaluate to zero, and only equations (b5), (b8) and (b9) matter. Solving these, we obtain,

$$I_{ij\alpha\beta}^2 = \frac{1}{\eta_i(\Delta r)_i(Ar)_j^i} \left(\frac{(Br)_j^i(S_{i+1} - S_i)}{6\eta_i} (1 - (C\mu)_\beta^\alpha) \right. \\ \left. + 2(C\mu)_\beta^\alpha \left\{ \begin{array}{ll} r_{i+1}^2 \delta_{j2} \left(\frac{S_{i+2} - S_{i+1}}{\eta_{i+1}(\Delta r)_{i+1}} - \frac{S_{i+1} - S_i}{\eta_i(\Delta r)_i} \right) & , \alpha \leq \frac{n_\mu}{2} \\ r_i^2 \delta_{j1} \left(\frac{S_{i+1} - S_i}{\eta_i(\Delta r)_i} - \frac{S_i - S_{i-1}}{\eta_{i-1}(\Delta r)_{i-1}} \right) & , \alpha > \frac{n_\mu}{2} \end{array} \right\} \right), \quad (b11)$$

where,

$$(Br)_j^i = \sum_{j'=1}^2 (Br)_{jj'}^i,$$

and,

$$(C\mu)^\alpha = \begin{pmatrix} 2(A\mu)_1^\alpha - (A\mu)_2^\alpha \\ -(A\mu)_1^\alpha + 2(A\mu)_2^\alpha \end{pmatrix}.$$

Substituting these expressions into equation (b10) we find that the $I_{ij\alpha\beta}^0$ terms exactly cancel the S_i terms, the sum over photon directions of the I^1 terms is zero, since

$$\sum_{\alpha=1}^{n_\mu} (\Delta\mu)_\alpha (\mu_\alpha + \mu_{\alpha+1}) = \sum_{\alpha=1}^{n_\mu} ((\mu_{\alpha+1})^2 - (\mu_\alpha)^2) = 1 - 1 = 0,$$

and so we have,

$$2\pi(C_{V(i-1)}(\Delta r)_{i-1}(Ar)_2^{i-1} + C_{Vi}(\Delta r)_i(Ar)_1^i)(T_i^{e'} - T_i^e) = \pi c \Delta t \epsilon \sum_{n=1}^{n_\omega} \Delta\omega_n P_{in}, \quad (b12)$$

where,

$$P_{in} = \sum_{\alpha=1}^{n_\mu} (\Delta\mu)_\alpha \\ \left(\eta_{i-1}(\Delta r)_{i-1}(Ar)_2^{i-1} \frac{(I_{(i-1)2\alpha 1}^2 + I_{(i-1)2\alpha 2}^2)}{2} + \eta_i(\Delta r)_i(Ar)_1^i \frac{(I_{i1\alpha 1}^2 + I_{i1\alpha 2}^2)}{2} \right).$$

Substituting the expression in equation (b11) for I^2 we obtain.

$$P_{in} = \frac{4}{3} \left((Dr)_i \frac{(S_{i+1} - S_i)}{\eta_i(\Delta r)_i} - (Dr)_{i-1} \frac{(S_i - S_{i-1})}{\eta_{i-1}(\Delta r)_{i-1}} \right).$$

where,

$$(Dr)_i = \frac{(r_i)^2 + r_i r_{i+1} + (r_{i+1})^2}{3},$$

and we used the fact that,

$$\sum_{\alpha=1}^{n_\mu} (\Delta\mu)_\alpha ((\mu_\alpha)^2 + \mu_\alpha \mu_{\alpha+1} + (\mu_{\alpha+1})^2) = \sum_{\alpha=1}^{n_\mu} ((\mu_{\alpha+1})^3 - (\mu_\alpha)^3) = 1^3 - (-1)^3 = 2.$$

Substituting this expression for P_{in} into equation (b12) we get exactly the finite element discretization of,

$$C_V \frac{\partial T_e}{\partial t} = \frac{c}{3} \int_0^\infty d\omega \left(\frac{1}{r^2} \frac{\partial}{\partial r} \frac{r^2 \epsilon}{\eta} \frac{\partial}{\partial r} \right) I_p(T_e),$$

which is the correct spherically symmetric thermal wave equation.

References

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