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HAMILTONIAN APPROACH TO THE MAGNETOSTATIC EQUILIBRIUM PROBLEM

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ABSTRACT

The purpose of this paper is to investigate the classical scalar-pressure magnetostatic equilibrium problem for non-symmetric configurations in the framework of a Hamiltonian approach. Requiring that the equilibrium admits locally, in a suitable subdomain, a family of nested toroidal magnetic surfaces, the Hamiltonian equations describing the magnetic flux lines in such a subdomain are obtained for general curvilinear coordinate systems. The properties of such Hamiltonian system are investigated. A representation of the magnetic field in terms of arbitrary general curvilinear coordinates is thus obtained. Its basic feature is that the magnetic field must fulfill suitable periodicity constraints to be imposed on arbitrary rational magnetic surfaces for general non-symmetric toroidal equilibria, i.e., it is quasi-symmetric. Implications for the existence of magnetostatic equilibria are pointed out. In particular, it is proven that a generalized equilibrium equation exists for such quasi-symmetric equilibria, which extends the Grad-Shafranov equation to fully three-dimensional configurations. As an application, the case is considered of quasi-helical equilibria, i.e., displaying a magnetic field magnitude depending on the poloidal (χ) and toroidal (ϑ) angles only in terms of $\alpha = \chi - N\vartheta$, with N an arbitrary integer.

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I - INTRODUCTION

The search for magnetostatic equilibria, i.e. of static magnetic configurations, with magnetic shear, non-vanishing pressure gradient and exhibiting nested magnetic surfaces, which are solutions of the so-called **scalar-pressure magnetostatic equilibrium problem**, in particular toroidal equilibria, has been the subject of theoretical investigation since the early days of theoretical research in plasma physics¹⁻⁵. The interest in this topic is obvious since its solution is of fundamental relevance for achieving the magnetostatic confinement of plasmas in experimental and reactor grade toroidal devices.

Grad in a number of papers^{1,5,6} raised the question of whether non-symmetric solutions (**non-symmetric equilibria**) exist to such a problem, concluding⁵ that it is "very unlikely that there exists a general class of toroidal equilibria" with toroidal nested magnetic surfaces and smooth pressure profiles, his conclusion being based on the existence of a periodicity constraint on the equilibrium magnetic field which he claimed to be not generally met in non-symmetric equilibria. The constraint considered by Grad, and earlier pointed out by Kruskal and Kulsrud² and Newcomb³ as a sufficient condition for the solubility of a suitable "magnetic differential equation", arises from the physical condition that the current density be solenoidal as a consequence of Ampere's law. Violation of this constraint would imply that on non-symmetric rational magnetic surfaces the parallel current density would diverge, producing effectively a sheet with infinite electric charge.

The issue was later reassessed by Lo Surdo⁷, who found "unlikely" the existence of non-symmetric solutions, and was confirmed by Grad himself in his last paper on the topic⁶, where he addressed the problem of the physical consequences of non-existence of non-symmetric solutions. This topic, certainly not an academic one, is closely related to the description of transport in the presence of magnetic islands, a problem now well understood in its basic aspects^{8,9}.

Actual rigorous existence proofs of solutions for the **scalar-pressure magnetostatic equilibrium problem** have been reached up to now only for two classes of so-called symmetric configurations, i.e., for equilibria exhibiting one ignorable coordinate, namely axial, translational or helical symmetry, (Edenstrasser¹⁰) or equilibria which are invariant with respect to specular reflection (plane-symmetric; Lortz¹¹).

Several approaches to this problem have been made with some success. Perhaps the earliest and most successful one was the introduction of the "stellarator expansion" developed by Greene and Johnson¹². In this work an asymptotic equilibrium solution was obtained by introducing an expansion in the inverse aspect-ratio and treating the magnetic field as constant and axial in the lowest order, with higher order corrections introduced to account for toroidicity, plasma pressure,

distorsion from axisymmetry, etc. It was shown that imposition of the periodicity condition over the toroidal angle in second order led to a unique specification of the lowest-order solution. In particular, it was proven that a sufficient number of constants of integration could be introduced in each order to satisfy the periodicity conditions in higher order.

Subsequently, other asymptotic solutions, based on the assumption of the nonaxisymmetric component of the field being small, were proposed by Kovrizhnikh and Shchepetov¹³ and Mercier¹⁴, utilizing an expansion about the magnetic axis, and by Pustovitov et al.¹⁵, incorporating the expansion into a numerical inverse-solution method, in which the position $\mathbf{r}$ is evaluated in terms of the magnetic coordinates [i.e., $\mathbf{r}=\mathbf{r}(\psi_p, \vartheta, \chi)$]. Codes based on these approaches, give equilibrium solutions that are in good agreement with each other and provide force-free field models that agree with Biot-Savart calculations using coil currents.

Another approach that has made considerable progress has been to search for quasi-helical [quasi-symmetric] equilibria, in which the magnitude of the magnetic field is a function of two variables,

$$B=B(\psi_p, \alpha) \quad (1)$$

with $\alpha=\chi-N\theta$, χ , θ , respectively, the poloidal and toroidal angles, and N an arbitrary integer. Numerical experiments conducted by Nührenberg and Zille¹⁶ have given evidence supporting the existence of such non-axisymmetric solutions. These results have been recently confirmed by Nührenberg¹⁷ and Merkel et al.¹⁸.

The generality of these results has been put into question by Garren and Boozer^{19,20}, who have objected that quasi-helical solutions cannot exist close to the magnetic axis, whereas they might exist on isolated magnetic surfaces. Their approach consists of an expansion about the magnetic axis, essentially in terms of a generalized inverse aspect-ratio and the adoption of an inverse method to construct a Taylor-Fourier expansion of the position vector $\mathbf{r}(\psi_p, \vartheta, \chi)$, as well as of all relevant physical quantities, which relies on the assumption of analyticity; namely that they are physical observables, and the request of suitable smoothness. Their proof is based on a counting argument on the number of independent Taylor-Fourier coefficients, which, at third order in the expansion in terms of the generalized inverse aspect-ratio, are found to be less than the number of independent equations to be satisfied.

Although equilibrium solutions have only been obtained for models with special symmetry, or with current sheets on isolated magnetic surfaces (weak solutions), or in asymptotic calculations, there is considerable experimental evidence that good magnetostatic confinement is achieved in fully three-dimensional stellarator configurations. This suggests that, at least in some

suitable sense, quasi-symmetric [or possibly non-symmetric] magnetostatic equilibria exhibiting nested magnetic surfaces do indeed exist.

The purpose of this paper is to contribute to the understanding of the physical assumptions and basic mathematical aspects of the magnetostatic equilibrium problem. In particular, we investigate in detail the periodicity constraints on arbitrary non-symmetric magnetostatic equilibria placed by the assumed existence of nested magnetic surfaces and analyze their consequences on the admissible equilibrium magnetic field.

An essential feature of the approach is that a Hamiltonian formulation is utilized for describing the magnetic field lines, which allows us to cast the well known scalar-pressure magnetostatic equilibrium problem in the framework of a perspicuous formalism. This, in turn, permits us to determine in a straightforward way the fundamental periodicity constraints as well as a general representation of the magnetic field. The formulation utilizes general canonical curvilinear coordinates, so that its possible application to actual equilibrium problems appears to be greatly facilitated and the interpretation of the basic equations is not obscured by particular choices of the curvilinear coordinate systems which can thus be defined, for best convenience, "a posteriori".

A property of the representation is that the magnetic field is quasi-symmetric, when expressed in general canonical coordinates, thanks to the periodicity constraints imposed for general non-symmetric toroidal equilibria on rational magnetic surfaces. As a fundamental consequence, it is proven that a generalized equilibrium equation can be obtained for arbitrary quasi-symmetric equilibrium magnetic fields, which extends the Grad-Shafranov equation to fully three-dimensional configurations.

The scheme of the presentation is as follows. Thus, requiring that the equilibrium admits locally, in a suitable subdomain, a family of nested toroidal magnetic surfaces, the Hamiltonian equations describing the magnetic flux lines in such a subdomain are obtained for general curvilinear coordinate systems [Sec.II]. The properties of such a Hamiltonian system are investigated, showing that the Hamiltonian system is integrable for magnetostatic equilibria with nested magnetic surfaces. In particular, representations in terms of the customary magnetic coordinates, such as Hamada⁴ and Boozer coordinates²¹, can readily be obtained by introducing suitable extended canonical transformations.

In Section III, the periodicity constraints placed on the magnetostatic equilibrium by the solubility condition of suitable magnetic differential equations are obtained in general curvilinear coordinates. A detailed analysis shows that two different periodicity constraints exist, which are related to the condition that the current density be solenoidal and to the magnetostatic equilibrium equation itself.

The consequences of the scalar-pressure magnetostatic equilibrium equation are investigated [Sec.IV]. A general representation of the magnetic field is thus obtained for general non-symmetric equilibria, under the assumption of smoothness of all relevant quantities and the requirement of suitable periodicity constraints on rational magnetic surfaces. The basic feature of such a representation is that the magnetic field is quasi-symmetric, in the sense that, denoting by J the Jacobian of the curvilinear coordinate system, the quantity JB^2 depends only on two coordinates. An important consequence [Sec.V] is that a generalized equilibrium equation, extending the well known Grad-Shafranov equation to fully three-dimensional equilibria, can be obtained.

Finally, as an application of the Hamiltonian formalism, a general representation is obtained for quasi-helical equilibrium magnetic fields, which identically fulfills all the required periodicity constraints (Sec.VI).

II - HAMILTONIAN REPRESENTATION FOR THE MAGNETIC FIELD

This section is included for the sake of establishing a suitable *canonical coordinate system* and to give its relationship to customary coordinate systems, as well as for completeness and to introduce notations used throughout the paper. Here we point out some of the essential properties of the Hamilton system associated with the magnetic field lines for equilibria admitting nested magnetic surfaces ($\psi_P = \text{const}$), which are useful to define a general curvilinear coordinate system, to be related to a suitable set of extended canonical variables. In particular, under the assumption that suitable magnetic surfaces exist, the Hamiltonian system is integrable.

As a further consequence it follows that any other arbitrary coordinate system preserving the Hamiltonian character of the magnetic flux lines can be simply obtained by an arbitrary extended canonical transformation. Thanks to the arbitrariness of the angle canonical coordinates, it follows that they could, in particular, be identified with customary coordinate systems used in equilibrium codes, such as cylindrical and toroidal coordinates, or with coordinate systems for which the magnetic flux lines are straight, i.e., the Hamada⁴ and Boozer coordinates²¹.

The Hamiltonian character of the magnetic flux lines, in particular for scalar-pressure equilibria with nested magnetic surfaces, is well known²² and can be recognized in an elementary way casting the relevant vector fields in terms of still arbitrary curvilinear coordinates (y^1, y^2, y^3) , which will be assumed to be defined in a suitable bounded subdomain Γ of $\mathbb{R}^3$. Thus, denoting by $\hat{e}^i = \nabla y^i$ and $\hat{e}_i = \partial \mathbf{r} / \partial y^i$ the dual basis vectors, i.e., respectively the contravariant and covariant bases, which are related through the identities $\hat{e}_i = \varepsilon_{i(j)(k)} J (\hat{e}^j \otimes \hat{e}^k)$ [$i=1,2,3$, with $\varepsilon_{i(j)(k)}$ the Ricci-Levi Civita tensor and J the Jacobian $J = |\nabla y^1 \times \nabla y^2 \cdot \nabla y^3|^{-1}$], there follow immediately for an arbitrary vector field $\underline{C}(\mathbf{r})$, on the euclidian space $\mathbb{R}^3$, the covariant and contravariant representations $\underline{C} = C_i \hat{e}^i$ and $\underline{C} = C^{i\hat{e}}_i$, related in terms of the transformations $C^i = g^{ik} C_k$ and its inverse $C_i = g_{ik} C^k$ [where $g^{ik} = \nabla y^i \cdot \nabla y^j$, $g_{ik} = \hat{e}_i \cdot \hat{e}_k$, with g^{ik} the metric tensor in the space of the coordinates (y^1, y^2, y^3)].

It follows that denoting $\underline{A} = \underline{A}(\mathbf{r})$ the vector potential of the magnetic field $\underline{B}$, with $\underline{B} = \nabla \times \underline{A}$, in terms of the contravariant representation of $\underline{A} = A_i \hat{e}^i$, and requiring $A_1(\mathbf{r}) = \partial G(\mathbf{r}) / \partial y^1$, with $G(\mathbf{r})$ and $\underline{A}$ suitably smooth scalar and vector fields, one obtains:

$$\underline{A}(\mathbf{r}) = \nabla G + \psi_P(\mathbf{r}) \nabla y^2 - \psi_T(\mathbf{r}) \nabla y^3. \quad (2)$$

Hence, the magnetic field $\underline{B}$ can be represented as:

$$\underline{B} = \nabla \psi_P \times \left\{ \nabla y^2 - \frac{\partial \psi_T}{\partial \psi_P} \nabla y^3 \right\} - \frac{\partial \psi_T}{\partial y^2} \nabla y^2 \times \nabla y^3, \quad (3)$$

in terms of which the covariant and contravariant representations of $\underline{B}$ can readily be expressed. Thus, if y^3 is defined in such a way that, at in some subdomain of Γ , for each flux line there exists a one-to-one mapping between y^3 and the flux line and, furthermore, $\psi_P(\mathbf{r})$ [or $\psi_T(\mathbf{r})$] can be identified with one of the curvilinear coordinates [for definiteness y^1], it follows²² that the equations of the magnetic flux lines are Hamiltonian equations with Hamiltonian $\psi_T = \psi_T(\psi_P, y^2, y^3)$, i.e., they read:

$$\frac{dy^1}{dy^3} = \frac{\underline{B} \cdot \nabla y^1}{\underline{B} \cdot \nabla y^3} = - \frac{\partial \psi_T}{\partial y^2}, \quad (4)$$

$$\frac{dy^2}{dy^3} = \frac{\underline{B} \cdot \nabla y^2}{\underline{B} \cdot \nabla y^3} = \frac{\partial \psi_T}{\partial y^1}. \quad (5)$$

If, additionally, we require that the surfaces $\psi_P(\mathbf{r}) = \text{const.}$ are magnetic surfaces (ψ_P -surfaces), i.e., $\underline{B} \cdot \nabla \psi_P = 0$, it follows that $B^1 \equiv 0$ on such magnetic surfaces and hence the equations of the corresponding magnetic flux lines yield simply $\psi_T = \psi_T(\psi_P, y^3)$, i.e., an *integrable Hamiltonian system*, admitting ψ_P as a first integral. Consequently, one recovers the well known result that the magnetic field can be written in terms of the Clebsh potentials $\{\psi_P, Q\}$

$$\underline{B} = \nabla \psi_P \times \nabla Q(\psi_P, y^2, y^3) \quad (6)$$

where $Q \equiv Q(\psi_P, y^2, y^3)$ reads

$$Q(\psi_P, y^2, y^3) \equiv y^2 - y_o^2 - \int_{y_o}^{y^3} dy^3 \frac{\partial}{\partial \psi_P} \psi_T(\psi_P, y^3). \quad (7)$$

Let us now introduce the assumption that the ψ_P -surfaces define, at least locally in some subdomain of Γ , a family of regular nested tori $\{\psi_P\}$, and, for definiteness, let us identify the curvilinear coordinates $y^2 \equiv \vartheta$, $y^3 \equiv \chi$, so that, locally, they are respectively angle-like toroidal and poloidal curvilinear coordinates defined, in the usual way³, on each torus. More precisely, on each toroidal surface $\psi_P(\mathbf{r}) = \text{const.}$, $y^2 \equiv \vartheta$ and $y^3 \equiv \chi$ are defined respectively as angle-like coordinates on arbitrary (regular) closed curves performing single loops respectively in the

toroidal (the long way) and poloidal (the short way) "directions". A number of elementary properties are immediately implied. In particular, since both $\mathbf{B}$ and the Hamiltonian $\psi_T = \psi_T(\psi_P, \chi)$ are physical observables, i.e. are single-valued both in ϑ and χ , it follows that the function $Q(\psi_P, \vartheta, \chi)$ is not generally an observable [however, evidently, $\nabla Q - |\nabla \psi_P|^{-2} \hat{e}^1 \hat{e}^1 \cdot \nabla Q$ must be single-valued]. Notice that $Q(\psi_P, \vartheta, \chi)$ can be written in the equivalent form

$$Q(\psi_P, \vartheta, \chi) \equiv Q(\psi_P, \vartheta, \chi; \vartheta_0, \chi_0) \equiv \vartheta - \vartheta_0 - (\chi - \chi_0) s(\psi_P, \chi) \quad (8)$$

where $s(\psi_P, \chi) \equiv s(\psi_P, \chi, \chi_0)$ is denoted as the "local rotational transform factor":

$$s(\psi_P, \chi) = (\chi - \chi_0)^{-1} \int_{\chi_0}^{\chi} d\chi \frac{\partial \psi_T}{\partial \psi_P}(\psi_P, \chi). \quad (9)$$

Thus, $s(\psi_P, \chi)$ in general depends on χ and is not an observable. However, the limit

$$\lim_{\chi \rightarrow \infty} 2\pi s(\psi_P, \chi) = \iota(\psi_P), \quad (10)$$

exists and is unique, with $\iota(\psi_P)$ being the rotational transform. It follows, furthermore, that all magnetic flux lines are non-singular, i.e., on each ψ_P -surface the initial value problem associated with the Hamiltonian equations (4) and (5) has a unique solution. A magnetic flux line crossing the point (ϑ_0, χ_0) on a given ψ_P -surface is, therefore, uniquely defined by the equation

$Q(\psi_P, \vartheta, \chi; \vartheta_0, \chi_0) = 0$. The coordinates $(\psi_P, \vartheta, \chi)$ are a subset of the extended canonical variables $(\psi_P, \vartheta, \chi, -\psi_T)$; thus, they shall be denoted hereon as a *canonical coordinate system*. This implies that any other coordinate system preserving the Hamiltonian character of the previous equations can be obtained by an arbitrary extended canonical transformation.

In particular, thanks to the arbitrariness of the canonical coordinates ϑ and χ , they can eventually be identified with coordinate systems used frequently in equilibrium codes, such as cylindrical and toroidal coordinates. As a further fundamental consequence, it follows that it is immediate to establish the connection with customary curvilinear coordinate systems, for which the magnetic flux lines are straight, i.e., the Hamada⁴ and Boozer coordinates²¹. They can be obtained, first, by introducing the canonical transformation to the coordinates $(\psi_P, \tilde{\vartheta}, \chi)$ for which the local rotational transform factor is constant, i.e.:

$$\vartheta \rightarrow \tilde{\vartheta} = \vartheta - (\chi - \chi_0) \tilde{s}(\psi_P, \chi), \quad (11)$$

where $\tilde{s}(\psi_P, \chi) = s(\psi_P, \chi) - \tilde{s}(\psi_P)$, and

$$\bar{s}(\psi_P) = 1/2\pi \int_0^{2\pi} d\chi \frac{\partial \psi_T}{\partial \psi_P}(\psi_P, \chi). \quad (12)$$

Then defining the transformed Hamiltonian

$$\bar{\psi}_T \equiv \bar{\psi}_T(\psi_P) = 1/2\pi \int_0^{2\pi} d\chi \psi_T(\psi_P, \chi) \quad (13)$$

the equations of the magnetic flux lines become

$$\frac{d\psi_P}{d\chi} = 0, \quad (14)$$

$$\frac{d\vartheta}{d\chi} = \frac{\partial \bar{\psi}_T}{\partial \psi_P}(\psi_P) \equiv \bar{s}(\psi_P). \quad (15)$$

It follows that the equation of the magnetic flux line belonging to $(\vartheta_0 \equiv \vartheta_0, \chi_0)$ is

$$\tilde{Q}(\psi_P, \vartheta, \chi; \vartheta_0, \chi_0) \equiv Q(\psi_P, \vartheta(\psi_P, \vartheta, \chi, \chi_0), \chi; \vartheta_0, \chi_0) = \vartheta - \vartheta_0 - (\chi - \chi_0)\bar{s}(\psi_P) = 0. \quad (16)$$

and, expressed in terms of the canonical coordinates $(\psi_P, \vartheta, \chi)$, the magnetic field becomes

$$\underline{B} = \nabla \psi_P \times \nabla \tilde{Q}(\psi_P, \vartheta, \chi). \quad (17)$$

Hence, the Hamiltonian system is integrable and (ψ_P, ϑ) are its action-angle variables. Consequently, on each ψ_P -surface the trajectories [i.e., the magnetic flux lines] are either closed or envelop ergodically the surface. In particular, this implies the well known consequence that if $\iota(\psi_P)/2\pi$ is a rational number on a given ψ_P -surface, then all magnetic flux lines belonging to that surface are closed.

The relationship of the canonical coordinate system $(\psi_P, \vartheta, \chi)$ with Hamada coordinates⁴, [denoted by (h^1, h^2, h^3)], and which are such that their Jacobian, i.e., the Jacobian of the transformation $\underline{h} \rightarrow (h^1, h^2, h^3)$ is $J(h^1, h^2, h^3) = 1$, is elementary. For this purpose, let us define the extended canonical transformation of the type $(\psi_P, \vartheta, \chi) \rightarrow (h^1, h^2, h^3)$, with

$$h^1 = \psi_P, \quad (19)$$

$$h^2 = \vartheta + \bar{s}(\psi_P)C(\psi_P, \vartheta, \chi), \quad (20)$$

$$h^3 = \chi + C(\psi_P, \vartheta, \chi), \quad (21)$$

with $C(\psi_P, \vartheta, \chi)$ an arbitrary smooth function, periodic in χ with period 2π , defining a smooth one-to-one mapping. Thus, letting $h_0^1 = h_0^2 = 0$ and $Q^h(h^1, h^2, h^3) \equiv \vartheta - \chi \bar{s}(\psi_P)$, it follows that

$$Q^h(h^1, h^2, h^3) = h^2 \cdot h^3 \bar{s}(h^1), \quad (23)$$

and hence the magnetic field in terms of the variables h^i ($i=1,3$) reads

$$\underline{\mathbf{B}} = \nabla h^1 \times \nabla Q^h(h^1, h^2, h^3). \quad (23)$$

Then, $C(\psi_P, \vartheta, \chi)$ can be defined to yield $J(h^1, h^2, h^3) = 1$. In fact there results

$$J(h^1, h^2, h^3) = J_1(\psi_P, \vartheta, \chi) / \{1 - \frac{\partial}{\partial \chi} C(\psi_P, \vartheta, \chi)\} \quad (24)$$

where:

$$J_1(\psi_P, \vartheta, \chi) = \frac{1}{|\nabla \psi_P \times \nabla \vartheta \cdot \nabla \chi|}, \quad (25)$$

implying:

$$C(\psi_P, \vartheta, \chi) = \int^\chi d\chi \{1 - J_1(\psi_P, \vartheta, \chi)\} + C_0(\psi_P, \vartheta, \chi_0). \quad (26)$$

In a similar fashion, requiring that $J_1(\psi_P, \vartheta, \chi) = f(\psi_P)/B^2$, with a suitable function $f(\psi_P)$, one immediately recovers Boozer coordinates²¹.

Finally, the previous results can be used to obtain the covariant components of $\underline{\mathbf{B}}$ with respect to these coordinate systems. Thus, for example, considering the case in which the ψ_P -surfaces form a family of nested tori, there follows for $\underline{\mathbf{B}}$ the covariant representation

$$\begin{aligned} \underline{\mathbf{B}} = JB^2 \nabla \chi + \bar{a}(\psi_P, \vartheta, \chi) \{ \nabla \vartheta - s(\psi_P, \chi) \nabla \chi \} + \\ + \bar{c}(\psi_P, \vartheta, \chi) \nabla \psi_P, \end{aligned} \quad (27)$$

This representation, and the fact that magnetic flux lines are characterized by the two invariants [Clebsch potentials] ψ_P and Q , suggests that another convenient canonical coordinate system is given by (ψ_P, Q, χ) . Notice, in fact, that, provided $s(\psi_P, \chi)$ is strictly positive (or negative), the mapping $(\psi_P, \vartheta, \chi) \rightarrow (\psi_P, Q, \chi)$ is one-to-one in the restriction for which $\chi, \vartheta \in [0, 2\pi]$ [see Appendix]; hence, under the same condition $\underline{\mathbf{r}} \rightarrow (\psi_P, Q, \chi)$ is one-to-one too with Jacobian $J(\psi_P, \vartheta, \chi) = J(\psi_P, Q, \chi)$. Thus, letting $a(\psi_P, Q, \chi) \equiv \bar{a}(\psi_P, Q, \chi) - \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi), \chi)$ and $c(\psi_P, Q, \chi) \equiv \bar{c}(\psi_P, Q, \chi) - \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi), \chi)$, the covariant representations of $\underline{\mathbf{B}}$ in terms of the curvilinear coordinates

(ψ_P, Q, χ) :

$$\underline{\mathbf{B}} = JB^2 \nabla \chi + a(\psi_P, Q, \chi) \nabla Q + \{c(\psi_P, Q, \chi) + a(\psi_P, Q, \chi) \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi)\} \nabla \psi_P, \quad (28)$$

implying that indeed $\underline{\mathbf{B}} \cdot \underline{\mathbf{B}} = B^2$. Moreover, $a(\psi_P, Q, \chi)$ and $c(\psi_P, Q, \chi)$ are observables to be

determined by imposing the conditions that $\underline{B} \cdot \nabla Q = \underline{B} \cdot \nabla \psi_P = 0$, yielding

$$a(\psi_P, Q, \chi) = J \{ g^{13} \nabla Q \cdot \nabla \psi_P - g^{11} \nabla Q \cdot \nabla \chi \}, \quad (29)$$

$$c(\psi_P, Q, \chi) = J \{ \nabla Q \cdot \nabla \psi_P \nabla Q \cdot \nabla \chi - |\nabla Q|^2 g^{13} \} - a(\psi_P, Q, \chi) \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi), \quad (30)$$

and g^{ij} ($i, j=1, 3$) are the elements of the metric tensor corresponding to the coordinates $(\psi_P, \vartheta, \chi)$.

Note that the second term in the coefficient of $\nabla \psi_P$ is not an observable; however, it cancels exactly a corresponding contribution in the term proportional to ∇Q , thus making $\underline{B}$ an observable, as expected.

In summary, in this section various possible canonical coordinate systems have been pointed out which are useful for obtaining covariant and contravariant representations for the magnetic field. The connection with coordinate systems previously used in the literature has been clearly displayed in terms of suitable extended canonical transformations.

In the sequel the general canonical coordinate system (ψ_P, Q, χ) will be used to obtain a representation of the scalar-pressure magnetostatic equilibrium equation and to investigate its related solubility conditions.

III - THE MAGNETOSTATIC EQUILIBRIUM PROBLEM : PERIODICITY CONSTRAINTS

Here the fundamental periodicity constraints for the scalar-pressure magnetostatic equation are pointed out. A careful analysis shows that, when making use of the canonical coordinate system (ψ_p, Q, ϑ) , two different periodicity constraints are determined, which are related both to the condition that the current density be solenoidal, implied by Ampere's law, and to a further solubility condition stemming from the previous covariant representation Eq.(27), as well as the magnetostatic equilibrium equation. Their physical interpretation is elementary and appears determined by the condition that on rational magnetic surfaces both a suitable "effective" length and the average energy density carried by a given closed field line are physical observables [i.e., they are the same for all closed field lines belonging to the same rational magnetic surface].

In addition, we anticipate that the existence of such periodicity constraints is fundamental for determining the representation of the equilibrium magnetic field in arbitrary non-symmetric configurations, to be determined in the next section.

For this purpose let us first assume that the magnetic field $\underline{B}(\underline{r})$ fulfills Ampere's law in Γ , i.e.:

$$\nabla \times \underline{B}(\underline{r}) = \frac{4\pi}{c} \underline{J}(\underline{r}) \quad (31)$$

with current density $\underline{J}(\underline{r}) = J_{\parallel}(\underline{r}) \underline{b} + \underline{J}_{\perp}(\underline{r})$, with $\underline{b} = \underline{B}/B$ and $\underline{J}_{\perp}(\underline{r})$ determined by the momentum balance equation,

$$\underline{J}_{\perp}(\underline{r}) = c \frac{1}{B^2} \nabla \times \left\{ \underline{B} \cdot \left[\nabla \cdot \underline{P} + \sum_s M_s N_s \nabla \cdot \underline{V} \underline{V} \right] \right\}, \quad (32)$$

where $N_s \underline{V}$ and $\underline{P}$ are respectively the number density, the center-of-mass velocity and the pressure tensor, i.e., $N_s = \int d\underline{v} f_s(\underline{r}, \underline{v})$, $N_s \underline{V} = \int d\underline{v} \underline{v} f_s(\underline{r}, \underline{v})$ and $\underline{P} = \sum_s M_s \int d\underline{v} (\underline{v} - \underline{V})(\underline{v} - \underline{V}) f_s(\underline{r}, \underline{v})$.

Then, introducing a Larmor radius expansion [with $\epsilon = r_L/L \ll 1$], requiring that $\underline{V} = O(\epsilon)$, $\underline{P} = p \underline{I} (1 + O(\epsilon))$, with $p = 3 \sum_s N_s T_s$ the scalar kinetic pressure, the current density $\underline{J}_{\perp}(\underline{r})$ reduces in lowest order to the diamagnetic current density

$$\underline{J}_{\perp}(\underline{r}) = -c \frac{\nabla p \times \underline{B}}{B^2}. \quad (33)$$

Here the kinetic pressure $p(\underline{r})$ as well as the magnetic field $\underline{B}$ are assumed to be suitably smooth scalar and vector fields [i.e., they are $C^{(k)}(\Gamma)$ with $k \geq 1$]. In particular, $\nabla p(\underline{r})$ is defined and

non-vanishing in a bounded toroidal domain $\Gamma \subset \mathbb{R}^3$; moreover the set Γ and $p(\mathbf{r})$ are defined so that $p(\mathbf{r})$ is constant on its boundary $\partial\Gamma$. Then it follows that, in Γ , $\underline{\mathbf{B}}$ fulfills the well known static scalar-pressure magnetostatic equilibrium equation, i.e.

$$(\nabla \times \underline{\mathbf{B}}(\mathbf{r})) \times \underline{\mathbf{B}}(\mathbf{r}) = 4\pi \nabla p(\mathbf{r}), \quad (34)$$

supplemented by boundary conditions for $\underline{\mathbf{B}}$ on $\partial\Gamma$, to be suitably specified [i.e., to define a wellposed problem]. Notice, however, that if the pressure profile is sufficiently flat, which may occur, for example, sufficiently near the magnetic axis, or at some other rational magnetic surface, to leading order in ε the equilibrium equation may become in such a domain, the force-free equilibrium equation

$$(\nabla \times \underline{\mathbf{B}}(\mathbf{r})) \times \underline{\mathbf{B}}(\mathbf{r}) = 0. \quad (35)$$

Let us assume, in particular, that $p=p(\psi_P)$ [or, respectively, $p=p(\psi_T)$] be strictly monotonic in Γ where the ψ_P -surfaces [ψ_T -surfaces] form a family of nested tori and, moreover, that the magnetic shear never vanishes [i.e., $\frac{\partial}{\partial \psi_P} s(\psi_P) \neq 0$] in the same subdomain. Then, using the

representation given by Eq.(27) for the magnetic field and taking the scalar product of Eq.(49) with $\nabla \chi \times \nabla \psi_P$, yields

$$\frac{\partial a}{\partial \chi}(\psi_P, Q(\psi_P, \vartheta, \chi), \chi) = \frac{\partial}{\partial Q} (JB^2). \quad (36)$$

Now, let us consider an arbitrary magnetic surface ψ_P and let

$$\bar{s}(\psi_P) = (2\pi)^{-1} \int_0^{2\pi} d\chi \frac{\partial \psi_T}{\partial \psi_P}(\psi_P, \chi) = m/n$$

with $m \in \mathbb{N}, n \in \mathbb{N}_0$. Since all observables must be periodic along a closed magnetic flux line, there results for the observable $a(\psi_P, Q(\psi_P, \vartheta, \chi), \chi)$ the condition [see Appendix]

$$\frac{\partial a}{\partial \chi}(\psi_P, Q(\psi_P, \vartheta + 2\pi m, \chi + 2\pi n), \chi + 2\pi n) = \frac{\partial a}{\partial \chi}(\psi_P, Q(\psi_P, \vartheta, \chi), \chi) \quad (37)$$

for arbitrary integers m, n . It is implied that the solubility condition

$$\int_0^{2\pi m} d\chi \frac{\partial}{\partial Q} JB^2 = \frac{\partial}{\partial Q} \int_0^{2\pi m} d\chi JB^2 = 0. \quad (38)$$

must be fulfilled in order that Eq.(36) admits a single-valued solution on an arbitrary rational magnetic surface. To obtain a physical interpretation of this constraint, it is sufficient to note that

$$\frac{\partial}{\partial Q} \int_0^{2\pi m} d\chi JB^2 = 8\pi \frac{\partial}{\partial Q} \int_0^{2\pi m} d\chi J \left\{ \frac{1}{8\pi} B^2 + p \right\} = 0 \quad (39)$$

where $E = \frac{1}{8\pi} B^2 + p$ is the total energy density; hence, closed field lines on the same rational magnetic surface must carry the same mean energy density. Eq.(38) implies, therefore, that

$$\int_0^{2\pi m} d\chi \ J \left\{ \frac{1}{8\pi} B^2 + p \right\} \text{ is an observable, i.e. it is independent of } Q. \text{ Analogously, since Ampere's law}$$

implies for $\underline{J}(\underline{r})$ the constraint

$$\nabla \cdot \underline{J}(\underline{r}) = 0, \quad (40)$$

there follows the well known solubility condition^{1,2,3}

$$\int_0^{2\pi m} d\chi \ J \nabla \cdot \underline{J}_\perp = 0. \quad (41)$$

It is important to stress that this solubility condition has been invoked in the past by Harold Grad to assess that non-symmetric equilibrium are "unlikely"^{3,4}. His argument was essentially that $\underline{J}_\perp$ may not fulfill the periodicity constraint (41) and hence $J_\parallel$ is not generally defined, i.e., is not single-valued. Eq.(41) can in fact be readily written in terms of the coordinates $(\psi_P, \vartheta, \chi)$;

Eq.(40), thus, implies

$$\nabla \cdot \underline{J}_\perp = \frac{1}{J} \frac{\partial}{\partial \chi} \{ J \nabla \chi \cdot \underline{J}_\perp \} + \frac{1}{J} \frac{\partial}{\partial Q} \{ J \nabla Q \cdot \underline{J}_\perp \} = \frac{1}{J} \frac{\partial}{\partial \chi} \{ J \nabla \chi \cdot \underline{J}_\perp \} - c \frac{\partial p}{\partial \psi_P} \frac{1}{J} \frac{\partial}{\partial Q} J. \quad (42)$$

Hence it follows that

$$\frac{1}{J} \frac{\partial}{\partial \chi} \{ J_\parallel / B \} = - \frac{1}{J} \frac{\partial}{\partial \chi} \{ J \nabla \chi \cdot \underline{J}_\perp \} + \frac{c \partial p}{J \partial \psi_P} \frac{\partial}{\partial Q} \{ J \}, \quad (43)$$

which, letting $p = p(\psi_P)$, yields the periodicity constraint on rational magnetic surfaces

$$[\tilde{s}(\psi_P) = m/n, m \in \mathbb{N}, n \in \mathbb{N}_0]$$

$$\frac{\partial p}{\partial \psi_P} \int_0^{2\pi m} d\chi \ \frac{\partial}{\partial Q} J = \frac{\partial p}{\partial \psi_P} \frac{\partial}{\partial Q} \int_0^{2\pi m} d\chi \ J = 0. \quad (44)$$

Thus, if by assumption $\nabla p(\underline{r}) \neq 0$, implying $\frac{\partial p}{\partial \psi_P} \neq 0$, there must result

$$\frac{\partial}{\partial Q} \int_0^{2\pi m} d\chi \ J = 0. \quad (45)$$

This must be supplemented by the constraint (38) which arises due to the condition that the coefficient $a(\psi_P, Q, \chi)$ be a physical observable. Eq.(45) provides a relatively mild limitation on the coordinate system that can be used for the representation, together with Eq.(38) then providing a physical constraint.

The Jacobian $J \equiv J(\psi_P, Q, \chi)$ is quite free. Thus, for example, it can be defined either to fulfill identically Eq.(54), namely so that $\frac{\partial}{\partial Q}\{B^2 J\}=0$, or Eq.(45), namely imposing $\frac{\partial}{\partial Q}J=0$. The former case corresponds to Boozer coordinates²¹ and the latter to Hamada coordinates⁴. In the sequel, however, for greater generality we find it preferable to leave unspecified the form of the Jacobian J , and thus we shall invoke both constraint equations, i.e., Eqs.(38) and (45). Due to the periodicity of both J and JB^2 with respect to χ and ϑ , and the previous periodicity constraints, it follows that, provided both are C^∞ , they admit the Fourier representations:

$$J = \sum_{p, n=0}^{\infty} e^{ip\chi + in\vartheta} J_{pn}(\psi_P), \quad (46)$$

$$JB^2 = \sum_{p, n=0}^{\infty} e^{ip\chi + in\vartheta} g_{pn}(\psi_P), \quad (47)$$

where $J \equiv J(\psi_P, \vartheta, \chi)$, $B \equiv B(\psi_P, \vartheta, \chi)$, with $J_{0n}(\psi_P) = g_{0n}(\psi_P) = 0 \quad \forall n \in \mathbb{N}_0$ and $J_{00}(\psi_P), g_{00}(\psi_P) \neq 0$.

Thus, it is evident that a particular case corresponds to require:

$$J(\psi_P, \vartheta, \chi) = \sum_{n=-\infty}^{\infty} J_n(\psi_P, \chi - n\vartheta), \quad (48)$$

and

$$JB^2(\psi_P, \vartheta, \chi) = f_1(\psi_P, \chi), \quad (49)$$

with $f_1(\psi_P, \chi) > 0$, implying:

$$1/B^2 = f_1(\psi_P, \chi) \sum_{n=-\infty}^{\infty} J_n(\psi_P, \chi - n\vartheta). \quad (50)$$

A magnetic field fulfilling the periodicity constraints defined by Eqs.(38) and (45), and hence Eqs.(46) and (47), will be labeled hereon as *quasi-symmetric*.

It should be remarked that important physical limits exist on the validity of the static scalar-pressure magnetostatic equilibrium equation. In fact this equation does not describe correctly the magnetic field for sufficiently small spatial scale lengths, comparable for example to the ion Larmor radius. The previous periodicity constraints lose their meaning on rational magnetic surfaces in which each magnetic flux line covers sufficiently densely each surface in such a way that the minimum distance between a magnetic flux line and its images [after an integer number of toroidal loops] is comparable to the characteristic scale length of the physical feature that has been ignored. It is also possible to satisfy Eq.(44) by making $\partial p / \partial \psi_P = 0$ on the

lower-order rational magnetic surfaces where the scalar pressure model is applicable. In this case Eq.(38) only limits the choice of the coordinate system (the Jacobian) that can be used. In the sequel we shall investigate in detail the physical consequences of the periodicity constraints in the covariant representation of the magnetic field, which include as a special case the force-free equilibria.

IV - CONSEQUENCES OF THE MAGNETOSTATIC EQUILIBRIUM EQUATION: QUASI-SYMMETRIC MAGNETIC FIELDS

In this section the covariant representation of the magnetic field determined in Sect.II [Eq. (28)] is used to analyze in detail the magnetostatic equilibrium equation and the previous periodicity constraints [Eqs.(38) and (45)]. This permits to obtain a general representation of the magnetic field which *identically* satisfies the periodicity constraints when expressed in terms of the general canonical coordinate system (ψ_P, Q, ϑ) . This magnetic field is found to be *quasi-symmetric* when expressed in such general coordinates. This implies that there exists a non-unique set of canonical coordinates for which two of the covariants components of $\underline{B}$, i.e., JB^2 and a , have at least an ignorable coordinate (Q), i.e., they are symmetric with respect to Q .

In particular, the previous periodicity constraints are proven not to be independent when requiring that the magnetic field be an observable if it is expressed in terms of the covariant representation of $\underline{B}$ given with respect to the coordinates (ψ_P, Q, ϑ) . In fact, representing Eq.(34) in terms of the (ψ_P, Q, χ) , implies

$$\frac{1}{J} \nabla_{\psi_P} \left\{ \frac{\partial}{\partial \psi_P} (JB^2) - \frac{\partial}{\partial \chi} [c(\psi_P, Q, \chi) + a(\psi_P, Q, \chi) \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi)] - J \frac{\partial p}{\partial \psi_P} \right\} + \frac{1}{J} \nabla_Q \left\{ \frac{\partial}{\partial Q} (JB^2) - \frac{\partial}{\partial \chi} a(\psi_P, Q, \chi) \right\} = 0 \quad (51)$$

yielding, besides Eq.(36),

$$\frac{\partial}{\partial \psi_P} (JB^2) - \frac{\partial}{\partial \chi} [c + a \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi)] = J \frac{\partial p}{\partial \psi_P}, \quad (52)$$

which requires the solubility constraint on rational magnetic surfaces $[\bar{s}(\psi_P) = m/n, m \in \mathbb{N}, n \in \mathbb{N}_0]$:

$$\frac{\partial}{\partial \psi_P} \int_0^{2\pi m} d\chi JB^2 - a \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi') \Big|_{\chi'=\chi}^{\chi'=\chi+2\pi m} = \frac{\partial p}{\partial \psi_P} \int_0^{2\pi m} d\chi J. \quad (53)$$

This implies, in addition, assuming that $J(\psi_P, Q, \chi)$ is at least of class C^2 ,

$$\frac{\partial^2}{\partial \psi_P \partial Q} \int_0^{2\pi m} d\chi J B^2 - \frac{\partial}{\partial Q} \left\{ a \chi' \frac{\partial}{\partial \psi_P} s(\psi_P, \chi') \right\} \Big|_{\chi'=\chi}^{\chi'=\chi+2\pi m} = \frac{\partial p}{\partial \psi_P} \frac{\partial}{\partial Q} \int_0^{2\pi m} d\chi J. \quad (54)$$

Hence, the constraint equations Eqs.(38) and (44) imply that

$$\frac{\partial}{\partial Q} \left\{ a \chi' \frac{\partial}{\partial \psi_P} s(\psi_P, \chi') \right\} \Big|_{\chi'=\chi}^{\chi'=\chi+2\pi m} = 0, \quad (55)$$

From Eqs.(36),(52) and (55) it follows the general solution:

$$a(\psi_P, Q, \chi) = a_0(\psi_P, Q) + \frac{\partial}{\partial Q} \int_0^\chi d\chi J B^2 \quad (56)$$

where generally $\frac{\partial}{\partial Q} J B^2 \neq 0$. We notice that, in the particular case in which the canonical coordinates $(\psi_P, \vartheta, \chi)$ are defined so that $J B^2 = f_1(\psi_P, \chi)$, with $f_1(\psi_P, \chi) > 0$ an appropriate smooth function, there results

$$a = a(\psi_P) \equiv a_0(\psi_P). \quad (57)$$

The proof of existence of a non-unique coordinate system of this type will be given below. Hence, thanks to the previous periodicity constraint, the solution of Eq.(51) reads:

$$c(\psi_P, Q, \chi) = c_0(\psi_P) - \int_0^\chi d\chi' \left[\frac{\partial}{\partial \psi_P} (J B^2) + a \frac{\partial}{\partial \psi_P} \psi_T(\psi_P, \chi') + \frac{\partial p}{\partial \psi_P} J \right] \quad (58)$$

with c satisfying the condition of periodicity on rational magnetic surfaces [i.e., with $\bar{s}(\psi_P) = m/n$ and $m \in \mathbb{N}, n \in \mathbb{N}_0$]

$$c(\psi_P, Q, \chi + 2\pi m) = c(\psi_P, Q, \chi). \quad (59)$$

On the other hand, c , being an observable, must be periodic both in ϑ and χ with period 2π [see Eqs.(A.2)-A.4) in the Appendix], i.e., $\forall k \in \mathbb{N}$: $c(\psi_P, Q(\psi_P, \vartheta, \chi), \chi) = c(\psi_P, Q(\psi_P, \vartheta, \chi + 2k\pi), \chi + 2k\pi)$, where $Q(\psi_P, \vartheta, \chi + 2k\pi) = Q(\psi_P, \vartheta, \chi) - 2\pi k \bar{s}(\psi_P)$. It follows:

$$\int_0^{2\pi} d\chi' \left[\frac{\partial}{\partial \psi_P} (J B^2) + a \frac{\partial}{\partial \psi_P} \psi_T(\psi_P, \chi') + \frac{\partial p}{\partial \psi_P} J \right] = 0. \quad (60)$$

To recover the solutions for the coefficients a and c , corresponding to the general case of canonical coordinates with $J B^2$ of the type defined by Eq.(47), it is then sufficient to introduce a transformation of the form given by Eqs.(19)-21).

We conclude, therefore, that provided the canonical coordinate system $(\psi_P, \vartheta, \chi)$, is defined in such a way that Eqs.(38) and (45) are fulfilled the quasi-symmetric magnetic field admits the covariant representatzion

$$\underline{B} = g(\psi_P, \chi) \nabla \chi + a(\psi_P) \nabla Q + \{c(\psi_P, Q, \chi) + a(\psi_P) \chi \frac{\partial}{\partial \psi_P} s(\psi_P, \chi)\} \nabla \psi_P, \quad (61)$$

where $a(\psi_P)$ and $c(\psi_P, Q, \chi)$ are defined by Eqs. (29) and (30) and obey equations (56) and (58) [which can be viewed as constraint equations for the elements of the metric tensor g^{ij}]. Moreover, in this case, the magnetic field strength $B(\psi_P, Q, \chi)$ has the general form

$$\frac{1}{B^2} = f_1(\psi_P, \chi) \sum_{p, q=-\infty}^{\infty} e^{ip\vartheta + iq\chi} J_{pq}(\psi_P). \quad (62)$$

As an application, it is instructive to identify the coordinate system $(\psi_P, \vartheta, \chi)$ with customary coordinate systems. Thus, for example, in the case of Boozer coordinates²¹ Eq.(61) is analogous to the representation of the magnetic field given by Garren and Boozer^{19,20}, provided the function g is assumed of the form $g=g(\psi_P)$. Instead, in the case of the natural (Hamada) coordinates⁴ ($h^1 \equiv \psi_P, h^2 \equiv \vartheta, h^3 \equiv \chi$), it follows that

$$\underline{B} = B^2(\psi_P, h^3) \nabla h^3 + a(\psi_P) \nabla Q + \{c(\psi_P, h^3) + a(\psi_P) h^3 \frac{\partial}{\partial \psi_P} \bar{s}(\psi_P)\} \nabla \psi_P, \quad (63)$$

where $Q = h^3 \bar{s}(\psi_P) h^2$ and

$$a(\psi_P) = \{g^{13} \nabla Q \cdot \nabla \psi_P - g^{11} \nabla Q \cdot \nabla h^3\}, \quad (64)$$

$$c(\psi_P, h^3) = \nabla Q \cdot \nabla \psi_P \nabla Q \cdot \nabla h^3 - |\nabla Q|^2 g^{13} - a(\psi_P) h^3 \frac{\partial}{\partial \psi_P} \bar{s}(\psi_P). \quad (65)$$

Here it is interesting to point out that the constraint (45) is fulfilled identically both by the Boozer and Hamada coordinates.

A particular case is represented by quasi-orthogonal coordinate systems. Thus, denoting ($q^1 \equiv \psi_P, q^2 \equiv \vartheta, q^3 \equiv \chi$), an example is given by the requirement that $g^{12} \equiv \nabla \psi_P \cdot \nabla q^2 = 0$, $g^{13} \equiv \nabla \psi_P \cdot \nabla q^3 = 0$ and $g^{23} \equiv \nabla q^2 \cdot \nabla q^3 \neq 0$. Such coordinates, however, cannot exist in the case of general quasi-symmetric toroidal equilibria. In fact, the previous orthogonality conditions can be proven generally to hold only on isolated magnetic ψ_P -surfaces. Another example is obtained requiring $g^{12} \equiv \nabla \psi_P \cdot \nabla q^2 = 0$, $g^{13} \equiv \nabla \psi_P \cdot \nabla q^3 \neq 0$ and $g^{23} \equiv \nabla q^2 \cdot \nabla q^3 = 0$, which corresponds, in particular, to toroidal axisymmetric equilibria, if ϑ is an ignorable coordinate both for the magnetic field

components and the metric tensor. In the first case there results generally $s \equiv s(\psi_P, q^3)$ [i.e., is a non-constant local rotational transform factor], implying that

$$a(\psi_P) = -J(\psi_P, q^3) g^{11} \{ g^{23} - s(\psi_P, q^3) \nabla g^{33} \} \quad (66)$$

$$c(\psi_P, Q, q^3) = 0. \quad (67)$$

Therefore

$$\underline{B} = (JB^2)(\psi_P, q^3) \nabla q^3 + a(\psi_P) \nabla q^2 \quad (68)$$

with

$$B^2(\psi_P, Q, q^3) = g^{11} \{ g^{22} + s^2(\psi_P, q^3) g^{33} - 2s(\psi_P, q^3) g^{23} \}, \quad (69)$$

$$J^{-2}(\psi_P, Q, q^3) = g^{11} \{ g^{22} g^{33} - \{ g^{23} \}^2 \}. \quad (70)$$

Eq.(81) implies, in turn, that $\underline{B} \cdot \nabla q^2 = a(\psi_P) g^{22} + JB^2 g^{23} = s(\psi_P, q^3)/J$, and $\underline{B} \cdot \nabla q^3 = JB^2 g^{33} + a(\psi_P) g^{23} = 1/J$, i.e.

$$g^{23} = \frac{1}{aJ} \left\{ 1 - J^2 B^2 g^{33} \right\}, \quad (71)$$

$$g^{22} = \frac{s}{aJ} - \frac{B^2}{a} \left\{ 1 - J^2 B^2 g^{33} \right\}. \quad (72)$$

An analogous result holds for other quasi-orthogonal coordinates, such as in the case in which $g^{12} \equiv \nabla \psi_P \cdot \nabla q^2 = 0$, $g^{13} \equiv \nabla \psi_P \cdot \nabla q^3 \neq 0$ and $g^{23} \equiv \nabla q^2 \cdot \nabla q^3 = 0$. This suggests that only equilibria which are strictly symmetric can admit quasi-orthogonal coordinates.

We conclude, therefore, that Eq.(61) yields a covariant representation of the magnetic field which, for the canonical coordinate system (ψ_P, Q, χ) , identically fulfills the periodicity constraints required for the solubility of the magnetostatic equilibrium equation [i.e., Eqs.(38) and (45)]. The implications are obvious. In fact, assume that for some suitable realization of the canonical coordinate system (ψ_P, Q, χ) the magnetic field is quasi-symmetric. Then, the corresponding representation in *any* other canonical coordinate system, such as for example the Hamada coordinates, is uniquely determined [see related discussion in Sec.II]. It follows that, in general, in the transformed coordinates [such as the Hamada coordinates] the magnetic field is *not* quasi-symmetric; however, solubility conditions analogous to (38) and (45), evidently will still be fulfilled.

Thus, due to the weakness of the constraint placed on the coordinate system (ψ_P, Q, χ) , the generalness of the present results appears established.

V - GENERALIZED EQUILIBRIUM EQUATION FOR $\psi_P(\mathbf{r})$

To complete the evaluation of the magnetostatic equilibrium, an equation for the flux function $\psi_P(\mathbf{r})$ is here obtained. In this section we prove, in fact, that as a consequence of the assumed existence of a "good" magnetostatic equilibrium, exhibiting nested magnetic surfaces, an equilibrium equation for $\psi_P(\mathbf{r})$ can actually be obtained. The striking feature is that, as before, the canonical coordinate system used, (ψ_P, Q, χ) is still essentially arbitrary, which requires an interpretation of the subsequent equilibrium equation. This permits to determine for $\psi_P(\mathbf{r})$ a *generalized equilibrium equation* extending the well known Grad-Shafranov equation, valid for toroidal axisymmetric equilibria, to the case of an *arbitrary quasi-symmetric magnetostatic equilibrium*, expressed in terms for the canonical coordinates $(\psi_P, \vartheta, \chi)$ [or, equivalent, (ψ_P, Q, χ)]. On the other hand, in view of the discussion introduced in the previous section, once $(\psi_P, \vartheta, \chi)$ are determined it is then immediate to express such an equation in terms of an arbitrary canonical coordinate system, for which the magnetic field is not necessarily quasi-symmetric.

The approach closely parallels the usual method required to construct the Grad-Shafranov equation in the case of axisymmetric equilibria. For this purpose, one evaluates the ∇y^2 -component of Ampere's law in the coordinate system $(\psi_P, y^2 \equiv \vartheta, y^3 \equiv \chi)$, obtaining

$$\nabla y^2 \cdot \nabla \times \mathbf{B} = -\nabla \cdot (\nabla y^2 \times \mathbf{B}), \quad (73)$$

$$\nabla y^2 \cdot \mathbf{J} = -\nabla y^2 \cdot \frac{\nabla \psi_P \times \mathbf{B}}{B} + \nabla y^2 \cdot \mathbf{b} J_{\parallel}. \quad (74)$$

It follows, in particular, that

$$\nabla y^2 \times \mathbf{B} = \nabla y^2 \times (\nabla \psi_P \times \nabla y^2) - s(\psi_P, y^3) \nabla y^2 \times (\nabla \psi_P \times \nabla y^3); \quad (75)$$

namely

$$\nabla y^2 \times \mathbf{B} = \nabla \psi_P \{ g^{22} - s(\psi_P, y^3) g^{23} \} - \nabla y^2 g^{12} + s(\psi_P, y^3) \nabla y^3 g^{12}. \quad (76)$$

Similarly, projecting the perpendicular and parallel components of the current density, one finds

$$\nabla y^2 \cdot \mathbf{J}_{\perp} = c \frac{\partial p}{\partial \psi_P} \{ g^{11} g^{22} - (g^{12})^2 + s(\psi_P, y^3) [g^{12} g^{13} - g^{11} g^{23}] \} / B^2, \quad (77)$$

$$\nabla y^2 \cdot \mathbf{b} J_{\parallel} = J_{\parallel} s(\psi_P, y^3) / JB.$$

Here $J_{\parallel} \equiv J_{\parallel}(\psi_P, y^2, y^3)$ can be evaluated by integrating Eq.(43) [and using the solubility condition

given by Eq.(45)], yielding

$$\{J_{\parallel}/B + J\nabla y^3 \cdot \underline{J}_{\perp}\}_{(\psi_P, Q, y_o^3)} = F(\psi_P, Q, y_o^3) + c \frac{\partial p}{\partial \psi_P} \frac{\partial}{\partial Q} \int_{y_o^3}^{y^3} d\chi J \quad (78)$$

with $F(\psi_P, Q, y_o^3)$ a suitable constant of integration and $\nabla y^3 \cdot \underline{J}_{\perp}$ given by

$$\nabla y^3 \cdot \underline{J}_{\perp} = c \frac{\partial p}{\partial \psi_P} \left\{ g^{11} g^{23} - g^{12} g^{13} + s(\psi_P, y^3) [(g^{13})^2 - g^{11} g^{33}] \right\} / B^2. \quad (79)$$

Thus, for example, assuming that y_o^3 exists such that $\nabla y^3 \cdot \underline{J}_{\perp} \big|_{(\psi_P, Q, y_o^3)} = 0$, there results

$F(\psi_P, Q, y_o^3) = \{J_{\parallel}/B\}_{(\psi_P, Q, y_o^3)}$. Requiring that $F(\psi_P, Q, y_o^3)$ be an observable then implies that $F = F(\psi_P, y_o^3)$. Hence, thanks to Eqs.(75)-(79) it follows that a *generalized equilibrium equation*

$$\begin{aligned} \nabla \cdot \left\{ \nabla \psi_P \{ g^{22} - s(\psi_P, y^3) g^{23} \} - \nabla y^2 g^{12} + s(\psi_P, y^3) \nabla y^3 g^{12} \right\} = \\ = - 4\pi \frac{\partial p}{\partial \psi_P} - \frac{4\pi}{c} s(\psi_P, y^3) B(\psi_P, Q, y^3) F(\psi_P, y_o^3) / (JB)(\psi_P, y^3) - \\ - \frac{4\pi}{c} \frac{\partial p}{\partial \psi_P} \frac{s(\psi_P, y^3)}{J} \frac{\partial}{\partial Q} \int_{y_o^3}^{y^3} d\chi J. \end{aligned} \quad (80)$$

exists for ψ_P , extending the well known Tokamak Grad-Shafranov equation to fully three-dimensional equilibria. As before, in this equation the matrix elements g^{ij} , here defined in terms of the coordinate system $(\psi_P, \vartheta, \chi)$, should generally be considered unknowns of the problem. This is evidently analogous to the case of Boozer [or Hamada] coordinates [see, for example, the related discussion in Refs.[19,20]], for which the curvilinear coordinates are part of the unknowns to be determined. A similar Grad-Shafranov equation was obtained by Shafranov and his colleagues²³ who used it to obtain 3-D equilibria numerically.

Notice, however, that, according to another viewpoint, the angle coordinates ϑ and χ can be considered prescribed. In particular, if they are indentified with the corresponding angle-coordinates defined on a suitable reference axisymmetric toroidal surface, and are such that $g^{23} \equiv 0$ [*quasi-orthogonal coordinate system*], then Eq.(80) reduces to the customary elliptic form.

It is useful to point out that the equation has properties that are remarkably similar to those of the Grad-Shafranov equation. In particular, it is apparent that in order to obtain a well-posed problem for the generalized equilibrium equation it is necessary to prescribe the functions $p(\psi_P)$ and $F(\psi_P, y_O^3)$. We stress, furthermore, that the solution of Eq.(80) must be an observable, which is evidently a prerequisite for well-posedness, thanks to the periodicity constraint Eq.(45).

As before, it is instructive to identify the canonical coordinates $(\psi_P, \vartheta, \chi)$ with customary coordinate systems. Thus in the case of Hamada coordinates (h^1, h^2, h^3) [$J=1$], the previous constraint is identically fulfilled, which simplifies the previous equation to the form

$$\begin{aligned} \nabla \cdot \left\{ \nabla \psi_P \{ g^{22} \bar{s}(\psi_P) g^{23} \} - \nabla h^2 g^{12} + \bar{s}(\psi_P) \nabla h^3 g^{12} \right\} = \\ = - 4\pi \frac{\partial p}{\partial \psi_P} - \frac{4\pi}{c} \bar{s}(\psi_P) F(\psi_P, h_O^3). \end{aligned} \quad (81)$$

Notice that the generalized equilibrium equation [either Eq.(80) or (81)] defines in general a non-elliptic equation because, in particular, the quantity $\nabla \psi_P \{ g^{22} \bar{s}(\psi_P) g^{23} \}$ need not be positive definite [i.e., *unless* the coordinates ϑ and χ are orthogonal, as in the case indicated above]. An analogous conclusion holds for magnetostatic equilibria admitting the quasi-orthogonal coordinate system for which $g^{12} \equiv 0$. In this case, Eq.(80) reduces to the form:

$$\nabla \cdot \left\{ \nabla \psi_P \{ g^{22} - s(\psi_P, q^3) g^{23} \} \right\} = - 4\pi \frac{\partial p}{\partial \psi_P} - \frac{4\pi}{c} s(\psi_P, q^3) F(\psi_P, q_O^3). \quad (82)$$

However, for axisymmetric equilibria, where $g^{23} \equiv 0$, the equation reduces to the customary [elliptic differential] Grad-Shafranov equation {see for example, Ref.[24]}. This results since if $\underline{B} = I(\psi_P) \nabla \vartheta + \nabla \psi_P \times \nabla \vartheta$, we can identify $s(\psi_P, q^3) F(\psi_P, q_O^3)$ with $\frac{1}{4\pi} \frac{1}{R^2} I(\psi_P) \frac{\partial}{\partial \psi_P} I(\psi_P)$ in Eq.(80),

where R denotes the distance from the symmetry axis. In particular, it follows that for sufficiently small perturbations from a toroidal axisymmetric equilibrium, such that

$$|\bar{s}(\psi_P) g^{23}| \ll |g^{22}|$$

the previous differential equation is elliptic, a property fundamental in order to establish the existence proof for symmetric equilibria¹⁰.

Ampere's law must be used to specify $F(\psi_P, y_O^3)$. In fact, the ∇y^3 -component of Ampere's law yields

$$\begin{aligned} & \nabla \cdot \left\{ \nabla \psi_P \{ g^{23} - s(\psi_P, y^3) g^{33} \} - \nabla y^2 g^{13} + s(\psi_P, y^3) \nabla y^3 g^{13} \right\} \\ & = - \frac{4\pi}{c} s(\psi_P, y^3) F(\psi_P, y_0^3) / J(\psi_P, y^2, y^3) - \end{aligned} \quad (83)$$

$$- \frac{4\pi}{c} \frac{\partial p}{\partial \psi_P} \frac{1}{JB} \frac{\partial}{\partial Q} \int_{y_0^3}^{y^3} d\chi J.$$

Another convenient form of the equation can be obtained using the covariant representation Eq.(61). For example, using the Hamada coordinates, one obtains

$$\begin{aligned} \frac{\partial a}{\partial \psi_P} &= - \frac{4\pi}{c} \tilde{s}(\psi_P) F(\psi_P, h_0^3) \\ c &= c(\psi_P) \end{aligned} \quad (84)$$

In particular, in the case of axisymmetric configurations, this equation relates $I(\psi_P)$ to the parallel current density²⁴,

$$\frac{\partial}{\partial \psi_P} I(\psi_P) = 4\pi \frac{1}{cB} J_{\parallel}. \quad (85)$$

To summarize, in this section we have proven that a generalized equilibrium equation for the flux function $\psi_P(\mathbf{r})$ can be formally constructed provided the magnetic field can be cast in quasi-symmetric form in a suitable canonical coordinate system. Evidently, this by no means constitutes, by itself, an existence proof of such a coordinate system nor of solutions of the boundary-value problem associated with the corresponding generalized equilibrium equation, i.e., Eq.(80).

The basic feature of the equilibrium equation is that, in difference with the customary Grad-Shafranov equation, it contains as unknowns the elements of the metric tensor g^{ij} ($i, j=1, 3$) and that generally it has the form of a non-elliptic second-order partial differential equation.

VI - APPLICATION: QUASI-HELICAL EQUILIBRIA

Numerical experiments performed by Nührenberg and Zille^{16,17} have pointed out that *quasi-helical* equilibrium solutions, i.e., in which the magnetic field strength exhibits the angle dependence defined by Eq.(1), could be viable magnetic confinement configurations.

It is therefore, important to investigate whether quasi-symmetric equilibria which are quasi-helical are admissible. In their original formulation¹⁶ Boozer coordinates were used. However, in principle the quasi-helical form of the magnetic field strength, could be obtained in *any* canonical coordinate system $(\psi_P, \vartheta, \chi)$. To recognize this it is sufficient, for example, to inspect Eq.(61), i.e., representing the quasi-symmetric magnetic field in the canonical coordinates $(\psi_P, \vartheta, \chi)$, assuming that the Jacobian be of the form of Eq.(48). Then, in the case in which $g \cong g(\psi_P)$ and $J(\psi_P, \vartheta, \chi) \cong J_n(\psi_P, \chi - n\vartheta)$, the magnetic field strength reduces evidently to the approximate quasi-helical form with respect to these coordinates, i.e.,

$$\frac{1}{B^2} \cong g(\psi_P) J_n(\psi_P, \chi - n\vartheta). \quad (86)$$

the general form being represented by Eq.(46). This representation holds evidently also in the case of Boozer coordinates²¹.

Notice, however, that this does not work for Hamada coordinates, since letting $J=1$, the previous representation for the magnetic field, given by Eq.(61) implies that $B=B(\psi_P, \chi)$, which is not of the required quasi-helical form. However, also in this case the construction of a representation for quasi-helical equilibrium solutions is elementary.

Thus, identifying $(\psi_P, \vartheta, \chi)$ with the Hamada coordinates [however the following procedure works, evidently, also for general canonical coordinates], we introduce the coordinate transformation $(\psi_P, \vartheta, \chi) \rightarrow (\psi_P, \vartheta, \alpha)$, where $\alpha = \chi - N\vartheta$ and N is an arbitrary integer and consider the Hamiltonian equations given by Eqs.(4) and (5) by defining the coordinates y^2 and y^3 , respectively, as $y^2 = \vartheta$ and $y^3 = \alpha \equiv \chi - N\vartheta$. It follows that $\underline{B}$ can be represented in terms of the coordinate system $(\psi_P, \vartheta, \alpha)$ and, respectively, (ψ_P, Q, α) where the Clebsh potential $Q \equiv Q(\psi_P, \vartheta, \alpha)$ now reads

$$Q(\psi_P, \vartheta, \alpha) \equiv \vartheta - \alpha \bar{s}(\psi_P) \quad (87)$$

with $\bar{s}(\psi_P) = \frac{\partial \psi_T}{\partial \psi_P}(\psi_P)$. Therefore, the magnetostatic equilibrium equation, when expressed in terms of the coordinates (ψ_P, Q, α) now becomes

$$\frac{1}{J} \nabla \psi_P \left\{ \frac{\partial}{\partial \psi_P} (JB^2) - \frac{\partial}{\partial \alpha} [c + a \alpha \frac{\partial}{\partial \psi_P} \bar{s}(\psi_P)] - J \frac{\partial p}{\partial \psi_P} \right\} + \frac{1}{J} \nabla Q \left\{ \frac{\partial}{\partial Q} (JB^2) - \frac{\partial}{\partial \alpha} a \right\} = 0. \quad (88)$$

with $a(\psi_P, Q, \vartheta)$ and $c(\psi_P, Q, \vartheta)$ analogous to those of Eqs.(29) and (30) [see also below, i.e., Eqs.(104) and (105)]. On all rational magnetic surfaces such that $\bar{s}(\psi_P) = m/k$ this implies that its solubility conditions [i.e., periodicity constraints] have again the form

$$\int_0^{2\pi k} d\alpha \frac{\partial}{\partial Q} JB^2 = \frac{\partial}{\partial Q} \int_0^{2\pi k} d\alpha JB^2 = 0, \quad (89)$$

$$\frac{\partial p}{\partial \psi_P} \int_0^{2\pi k} d\alpha \frac{\partial}{\partial Q} J = \frac{\partial p}{\partial \psi_P} \frac{\partial}{\partial Q} \int_0^{2\pi k} d\alpha J = 0. \quad (90)$$

Hence, following the approach indicated in Section IV, one immediately obtains a covariant representation of the equilibrium magnetic field satisfying the previous periodicity constraints. In fact, requiring that the Jacobian $J=1$, i.e., transforming to the Hamada coordinates, one obtains the sought result, displaying the quasi-helical representation for the magnetic field, i.e.

$$\underline{B} = B^2(\psi_P, \alpha) \nabla \alpha + a(\psi_P) \nabla Q + \{c(\psi_P, Q, \alpha) + a(\psi_P) \alpha \frac{\partial}{\partial \psi_P} \bar{s}(\psi_P)\} \nabla \psi_P \quad (91)$$

where

$$a(\psi_P) = g^{13} \nabla Q \cdot \nabla \psi_P - g^{11} \nabla Q \cdot \nabla \alpha, \quad (92)$$

$$c(\psi_P, Q, \alpha) = \{\nabla Q \cdot \nabla \psi_P \nabla Q \cdot \nabla \alpha - |\nabla Q|^2 g^{13}\} - a(\psi_P) \alpha \frac{\partial}{\partial \psi_P} \bar{s}(\psi_P). \quad (93)$$

An analogous representation can be obtained by identifying $(\psi_P, \vartheta, \chi)$ with Boozer coordinates. In this case one infers that generally the more general condition:

$$J(\psi_P, Q, \alpha) B^2(\psi_P, Q, \alpha) = g(\psi_P, \alpha) \quad (94)$$

results.

In particular, it is evident that ψ_P fulfills, one again, the generalized equilibrium equation (80). It is sufficient to identify $y^2 = \vartheta, y^3 = \chi - N\vartheta$ in the previous equation to obtain it.

One notices that in the limit of infinite aspect-ratio (i.e., $R_0/a \rightarrow \infty$, with R_0 and a , respectively the major and minor semiaxes of the torus) Eq.(91) yields a helically symmetric equilibrium. Analogously, it is immediate to prove that Eq.(80) recovers the well known Grad-Shafranov equation for helical equilibria²⁵. This limit can be reached either by letting $R_0 \rightarrow \infty$ or $a \rightarrow 0$. In the former case, a helical solution exists for arbitrary $r: 0 \leq r \leq a$, r being the distance for the axis of symmetry. One, therefore, would expect that, if $a/R_0 \ll 1$, at least in the domain in which $r/a \gg a/R_0$, an approximate quasi-helical solution could be constructed.

In summary, we conclude that at least formally quasi-symmetric equilibria in which the magnetic field strength exhibits a quasi-helical dependence can be formally constructed in any coordinate system, and in particular in Boozer coordinates.

For the sake of completeness, here we mention the result obtained by Garren and Boozer²⁰, who have shown that exact quasi-helical solutions of this type cannot exist sufficiently close to the magnetic axis. Their approach is based on a suitable asymptotic ordering, i.e., it is an expansion sufficiently near the magnetic axis [so that strict helical symmetry cannot be recovered] and relies on the assumption that magnetic surfaces exist near the magnetic axis with non-negligible pressure gradient. Additionally, their expansion about the magnetic axis requires that the magnetic field be infinitely differentiable, while $\mathbf{B}$ generally needs to be at most $C^2(\Gamma)$.

Thus, their proof does not rule out the possibility that approximate, or even exact, quasi-helical equilibrium solutions could still exist sufficiently far from the magnetic axis, either admitting a possible ergodic region about the magnetic axis or requiring that the magnetic field be only $C^2(\Gamma)$.

VII - CONCLUSIONS

In this work we have developed a general Hamiltonian formalism for investigating the effect of imposing periodicity conditions on the solutions of the scalar-pressure magnetostatic equilibrium equation,

$$(\nabla \times \underline{B}(\underline{r})) \times \underline{B}(\underline{r}) = 4\pi \nabla p(\underline{r}).$$

A number of interesting results have been obtained. In particular, the periodicity constraints on the equilibrium magnetic field, arising from the solubility conditions of the magnetostatic equilibrium equation, have been pointed out and their consequences analyzed. The periodicity constraint imposed by the condition that the current be divergence-free, which requires

$$\oint dl \frac{\underline{B} \times \nabla p \cdot \nabla B^2}{B^4} = 0$$

on every closed field line belonging to a rational magnetic surface was used by Grad^{1,5,6} to state that solutions of the scalar-pressure magnetostatic equilibrium problem should only exist for symmetric configurations, the non-axisymmetric ones being "unlikely".

In this paper the periodicity constraints have been expressed in general coordinate systems, resulting in two conditions, Eq.(38) and (45). In particular, we have proven that these constraints can be fulfilled identically only by a suitable choice of the coordinate system in which the magnetic field is quasi-symmetric or that $\frac{\partial p}{\partial \psi_p} = 0$ on an isolated magnetic surface.

On the other hand, it has been demonstrated experimentally that three-dimensional stellarator equilibrium [i.e., non-symmetric] configurations indeed produce plasma confinement. This implies that magnetic surfaces must exist in most of the plasma [Γ], in the sense that the volume of the subdomain in which the magnetic field is ergodic must be very small, or even vanish, in comparison with the domain covered by nested magnetic surfaces. In particular, this includes the pathological case in which such a subdomain might be dense in Γ .

As for the theoretical models, there are various possible approaches. One is based on the observation that, as previously pointed out, the scalar-pressure magnetostatic equilibrium equation cannot describe correctly the actual physical magnetic field for sufficiently small spatial scale lengths [i.e., comparable to the ion Larmor radius]. In particular it follows that the periodicity constraints previously considered [i.e., Eqs.(38) and (45)], lose their meaning on those rational magnetic surfaces in which each magnetic flux line covers sufficiently densely each surface.

A simple approach, avoiding the need to model such complex small-scale physics, is to

assume that weak solutions, characterized by infinite currents (current sheets) on rational magnetic surfaces, are permitted. Indeed, based on this concept, several fully three-dimensional numerical codes including VMEC²⁶ and BETAS²⁷ have been developed and are currently in use for design and interpretation of many experimental devices and experiments.

A second one, which can be viewed as similar in some sense, is to adopt asymptotic expansions¹²⁻¹⁵, based on the assumption of small distortion from symmetry, as in the case of the "stellarator expansion"¹². Such expansions, which effectively replace the fully three-dimensional magnetostatic equilibrium problem with a truncated series of two-dimensional equations, can be formally constructed in such a way that an approximate asymptotic solution can be determined to all orders.

Another possible approach is to restrict the consideration to configurations for which, at least locally, classical solutions of the scalar-pressure magnetic field are permitted. This requires that the magnetic field satisfies exactly the periodicity constraints, i.e., it is quasi-symmetric, and in particular quasi-helical¹⁶⁻¹⁸. For this purpose, in this paper an Hamiltonian formulation has been utilized to describe the magnetic field lines and the magnetic field. This permits to determine the fundamental periodicity constraints related to the scalar-pressure magnetostatic equilibrium problem. As a consequence, assuming that rational magnetic surfaces are dense [i.e., requiring that the magnetic shear never vanishes], a covariant representation for the magnetic field is obtained which exhibits the feature of satisfying identically the periodicity constraints on arbitrary rational magnetic surfaces and of being quasi-symmetric. To facilitate its application to equilibrium problems as well as its physical interpretation, the formulation has been given in general canonical coordinates.

A basic aspect of such a representation is that the magnetic field must be quasi-symmetric in some suitable coordinate system as a consequence of the periodicity constraints imposed for general nonsymmetric toroidal equilibria on rational magnetic surfaces. As a fundamental consequence, we have obtained a generalized equilibrium equation which extends the Grad-Shafranov equation to general quasi-symmetric equilibrium magnetic fields.

Finally, as an application of the formalism, the case of quasi-helical equilibria has been investigated. We have found that a representation can be obtained for quasi-symmetric magnetostatic equilibria which is consistent from the physical viewpoint, in the sense that the periodicity constraints, placed on the equilibrium magnetic field as a consequence of the solubility conditions of the equilibrium equation, are indeed fulfilled on arbitrary rational magnetic surfaces. Garren and Boozer²⁰ have shown that, in the domain where an expansion about the magnetic axis is applicable, only solutions on isolated surfaces can be found for

quasi-helical configurations. Nevertheless, the present formulation may have relevance for the design of future magnetic confinement devices.

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APPENDIX: COORDINATE SYSTEM (ψ_P, Q, χ)

We first notice that the mapping $\mathbf{r} \rightarrow (\psi_P, \vartheta, \chi)$ defines a coordinate system if ϑ, χ belong to the semi-open interval $[0, 2\pi[$. On the other hand, the mapping

$$(\psi_P, \vartheta, \chi) \rightarrow (\psi_P, Q, \chi) \quad (\text{A.1})$$

is one-to-one provided the local rotational transform factor, $s(\psi_P, \chi)$, is strictly positive or negative on a given ψ_P -surface. Therefore, (ψ_P, Q, χ) is a coordinate system if Q, χ belong to a suitable restriction of $\mathbb{R}^2$. Notice that, without loss of generality, we can require, in particular, that the local rotational transform factor be constant, i.e., $s(\psi_P, \chi) \equiv \bar{s}(\psi_P)$. Hence, if $\bar{s}(\psi_P)$ is strictly positive, or negative, $(\vartheta, \chi) \rightarrow (Q, \chi)$ is one-to-one in the whole domain $\mathbb{R}^2$. The restriction to the domain $[0, 2\pi[\times [0, 2\pi[$ yields its corresponding image, where (ψ_P, Q, χ) is a coordinate system. Obvious periodicity constraints must be fulfilled by the observables if the variables (ϑ, χ) or (Q, χ) are allowed to belong to the whole domain $\mathbb{R}^2$. Thus if $\bar{a}(\psi_P, \vartheta, \chi)$ is an arbitrary observable, the periodicity constraint:

$$\bar{a}(\psi_P, \vartheta + 2\pi m, \chi + 2\pi n) = \bar{a}(\psi_P, \vartheta, \chi) \quad (\text{A.2})$$

must be fulfilled for arbitrary integers m and n . Denoting $a(\psi_P, Q, \chi) \equiv \bar{a}(\psi_P, Q - \chi \bar{s}(\psi_P), \chi)$ the same observable represented in term of the variables (ψ_P, Q, χ) , this implies that:

$$a(\psi_P, Q + 2\pi n - 2\pi n \bar{s}(\psi_P), \chi + 2\pi n) = a(\psi_P, Q, \chi). \quad (\text{A.3})$$

In particular, on an arbitrary rational magnetic surface for which $\bar{s}(\psi_P) = m/n$, the periodicity condition becomes:

$$a(\psi_P, Q, \chi + 2\pi n) = a(\psi_P, Q, \chi). \quad (\text{A.4})$$

It follows that the differential operators can be uniquely defined in terms of the variables (ψ_P, Q, χ) . Thus, if $\bar{a}(\psi_P, \vartheta, \chi)$ is differentiable, there results:

$$\begin{aligned}\frac{\partial}{\partial Q}a(\psi_P, Q, \chi) &= \frac{\partial}{\partial \vartheta}\bar{a}(\psi_P, \vartheta, \chi) \\ \frac{\partial}{\partial \chi}a(\psi_P, Q, \chi) &= \frac{\partial}{\partial \chi}\bar{a}(\psi_P, \vartheta, \chi) - \bar{s}(\psi_P)\frac{\partial}{\partial \vartheta}\bar{a}(\psi_P, \vartheta, \chi)\end{aligned}\tag{A.5}$$

Hence, $\frac{\partial}{\partial Q}a(\psi_P, Q, \chi)$ and $\frac{\partial}{\partial \chi}a(\psi_P, Q, \chi)$ are observables too.

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