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Abstract

The dynamical model for the L-H confinement mode transitions consisting of three ordinary differential equations (3-ODE model) for the essential state variables is proposed. The model is derived from the energy balance equations for the resistive pressure-gradient-driven turbulence and describes temporal evolutions of three characteristic variables (u, k, f) the potential energy contained in the pressure gradient, the turbulent kinetic energy and the shear flow energy. The energy input to the peripheral plasma region is included as an external control parameter in the model. The model equations have stationary solutions corresponding to the L and H modes. We obtain the L to H and H to L transition with the energy input parameter changed. The type of the L-H transition, whether a first-order or second-order transition, is shown to be determined by the shear flow damping coefficient. We also find the parameter region in which the H mode stationary solution becomes unstable and bifurcates to the limit cycle which shows periodic oscillations characteristic of the ELM confinement state.

I Introduction

In order to explain the mechanism of the L-H transitions observed in many tokamaks and in some stellarators, various theoretical models have been proposed in recent years.¹⁻⁷ The key points that such models attempt to describe are how the radial electric field or the poloidal shear flow suppresses the turbulence and anomalous transport and how the electric field or the shear flow is produced. In our previous paper,⁷ the spontaneous shear flow generation and the resultant transport reduction similar to the L-H transition are shown in the single-helicity nonlinear simulation of the resistive g modes⁸⁻¹¹ and it is suggested that the shear flow generation is triggered by the decrease of the collisional shear flow damping due to the increase of the temperature. Shaing *et al.*³ and Hassam *et al.*⁴ also argues in their L-H transition models that the bifurcation of the poloidal flow is caused by the decrease of the poloidal flow damping. However, the poloidal rotation is driven by the torque due to the ion orbit loss in the theory of Shaing *et al.* and due to the poloidal asymmetric radial diffusion in the model of Hassam *et al.*, while our simulation showed that the Reynolds stress directly generates the shear flow.

The plasma mass flow generation by the divergence of the Reynolds stress arises from the nonlinear convective derivative in the mass acceleration. This mechanism for flow generation is based on the self-consistent hydrodynamics for the interaction between the plasma turbulence and the background mass flow and has been supported by several other theoretical works.¹³⁻¹⁶ In the L-H transition model presented in the present work, the Reynolds stress is considered to be the cause of the shear flow generation. Diamond *et al.* presented the L-H transition model which describes the temporal behavior of the turbulence fluctuation and the shear flow⁶ although the pressure gradient is fixed as an external parameter. In actual experiments, the pressure gradient also changes at the L-H transition according to the

change in the transport. Therefore, our model contains the pressure gradient as a basic state variable as well as the fluctuation energy density and the mass flow kinetic energy density. Thus, the state of the system is specified by these three dynamical variables which interact through the hydrodynamic nonlinearities. Our model consists of the three ordinary differential equations (3-ODE model) which govern the temporal evolution of the three system state variables. The model reduces to the 2-ODE model of Diamond *et al.* if the pressure gradient variable is fixed. In the Diamond *et al.* model, the H mode stable solution appears when some criterion is satisfied by the external parameter. There, the L-H transition is considered as a second-order phase transition: the magnitudes of the fluctuation and the flow are continuously changed at the critical point of the transition. In contrast, most of other L-H transition models have claimed that the L-H transition should be a first-order phase transition where the fluctuation and the flow are changed discontinuously at the transition and the hysteresis property appears.

In our 3-ODE model, the energy input to the peripheral plasma region is regarded as an external control parameter. It will be shown that the type of the L-H transition is either a first-order or a second-order phase transition depending on the form of the collisional shear flow damping. In addition, we show that under certain conditions the present model predicts the transition from the H mode to a time dependent state with relaxation oscillations characteristic of the edge localized modes (ELM) observed in sufficiently high-beta H mode plasmas. In order to explain the edge localized modes (ELM), Leboeuf *et al.* proposed another 2-ODE model which has the same form as the Volterra-Lotka equations.¹⁷ Our 3-ODE model has the ELM mode behavior occurring as a subset of the parameter region in which the H mode stationary solution becomes unstable and bifurcates to a limit cycle which shows periodic relaxation oscillations like those observed in the ELM state.

This work is organized as follows. In Sec. II, our three-state variables are defined, and the 3-ODE model is derived from the energy balance equations of the resistive plasma.

gradient-driven turbulence by modeling the energy transfer and dissipation terms. There we also explain the L and H mode stationary solutions and the condition for the existence of the H mode. In Sec. III, the linear stabilities of the L and H mode stationary solutions are examined and the criterion for the unstable H mode (or occurrence of the ELM) is derived. In Sec. IV, numerical solutions of our model equations are given. We show the temporal behavior of the pressure gradient, turbulence and shear flow energy for various values of the external thermal power input parameter giving examples of the L and H modes and the ELM-like limit cycle. In Sec. IV numerical solutions are shown for the case where the power input is temporally varied to show the difference between the first-order and second-order phase transitions. Finally, the conclusions and discussion are given in Sec. V.

II Model Equations

Basic variables for our dynamical model of L-H transitions are the turbulent kinetic energy K , the background shear flow kinetic energy F , and the potential energy related to the pressure profile U , which are defined by

$$K \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \frac{1}{2} \langle \tilde{v}^2 \rangle, \quad F \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \frac{1}{2} v_0^2, \quad U \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \frac{(-x)}{L_c} \frac{P_0}{n_0 m_i} \quad (1)$$

respectively, where we define the peripheral region by $-\delta \leq x \leq 0$. Here, the angle bracket $\langle \cdot \rangle$ denotes the average over the (y, z) -plane (or the magnetic surface). The velocity and the pressure are divided into the x -dependent average parts and the fluctuating parts as $\mathbf{v} = \mathbf{v}_0(x) + \tilde{\mathbf{v}}$ and $p = p_0(x) + \tilde{p}$, respectively. The average mass density is denoted by $n_0 m_i$. The unfavorable average magnetic curvature is represented by $1/L_c$ and is assumed to be constant. From the reduced resistive MHD equations (7,10,18) in the electrostatic approximation, we obtain the following energy balance equations

$$\frac{dU}{dt} = P_U - P_K$$

$$\frac{dK}{dt} = P_K - P_F - \epsilon_K \quad (3)$$

$$\frac{dF}{dt} = P_F - \epsilon_F \quad (4)$$

where the production and dissipation terms in the right-hand sides are given by

$$P_U \equiv \frac{\langle \bar{p}\bar{v}_x \rangle|_{x=-\delta}}{L_C n_0 m_i} \quad (5)$$

$$P_K \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \frac{\langle \bar{p}\bar{v}_x \rangle}{L_C n_0 m_i} \quad (6)$$

$$P_F \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \langle \bar{v}_x \bar{v}_y \rangle \frac{dv_0}{dx} \quad (7)$$

$$\epsilon_K \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \left\{ \mu \left\langle \left(\frac{\partial \bar{v}_i}{\partial x_j} \right)^2 \right\rangle + \frac{\eta}{n_0 m_i} \langle \bar{J}_{\parallel}^2 \rangle \right\} \quad (8)$$

$$\epsilon_F \equiv \frac{1}{\delta} \int_{-\delta}^0 dx \mu \left(\frac{dv_0}{dx} \right)^2. \quad (9)$$

Here μ denotes the (kinematic) viscosity, η the resistivity and $\bar{J}_{\parallel} = \eta^{-1} \nabla_{\parallel} \phi$ the parallel current. The electrostatic potential ϕ gives the velocity as $\mathbf{v} = -(c/B_0) \nabla \phi \times \hat{\mathbf{z}}$ in the reduced MHD model. In the temporal evolution equation for U , we have neglected the collisional dissipation term by assuming that the turbulent thermal transport is much larger than the collisional one. In order to solve Eqs. (2)-(4), the production and dissipation terms (5)-(9) need to be expressed in terms of U , K and F . Before proceeding to this closure problem, some remark on these terms should be given here. The potential energy production P_U is given by the energy input to the peripheral region through the inner boundary at $x = -\delta$, the turbulent energy production P_K is expressed in terms of the pressure transport multiplied by the unfavorable curvature, which is originally from the potential energy, and the background flow production is represented by the product of the Reynolds stress and the flow shear, which comes from the turbulent kinetic energy. As for the dissipation terms, ϵ_K stands for the viscous and Joule dissipations of the fluctuation, and ϵ_F the viscous dissipation of the

average flow. In equilibria, we obtain $P_U = P_K = \epsilon_K + \epsilon_F$ and $P_F = \epsilon_F$. From the definitions, we also find that $P_F = \epsilon_F = 0$ in the case of no shear flow (since $dv_0/dx = 0$) as in the L mode phase. When P_F is positive, the stationary value of the turbulent energy dissipation $\epsilon_K = P_U - P_F$ becomes smaller than in the no shear flow case, which implies the reduction of the turbulence level as observed in the H mode phase.

Now let us consider the closure problem. The time scale τ_c in the g mode turbulence is given by $\tau_c \sim [t] \equiv (L_c n_0 m_i / |dP_0/dx|)^{1/2}$. Approximating the background pressure profile as $P_0(x) \simeq P_0(x=0) + x dP_0/dx$, we have

$$U \simeq \frac{1}{3} \frac{\delta^2}{L_c n_0 m_i} \left| \frac{dP_0}{dx} \right| + \frac{1}{2} \frac{\delta P_0(x=0)}{L_c n_0 m_i} \quad (10)$$

where we assumed that $dP_0/dx < 0$ and $n_0 \simeq \text{const.}$ Hereafter U is redefined by the pressure gradient term in Eq. (10) as $U \equiv \delta^2 |dP_0/dx| / (3L_c n_0 m_i)$ assuming that the pressure at the outer boundary $P_0(x=0) = \text{const.}$ Then we obtain

$$\tau_c \sim \delta U^{-1/2} \quad (11)$$

Estimating the anomalous pressure diffusivity as $D \sim \tau_c K$, we have

$$P_K \sim \frac{D |dP_0/dx|}{L_c n_0 m_i} \sim \tau_c^{-1} K \sim \delta^{-1} U^{1/2} K. \quad (12)$$

In the same way as in the background pressure profile, we approximate the background flow profile as $v_0(x) \simeq v_0(x=0) + x dv_0/dx$. Then F is given by

$$F \simeq \frac{1}{2} (\bar{v}_0)^2 + \frac{1}{24} \delta^2 \left(\frac{dv_0}{dx} \right)^2 \quad (13)$$

where $\bar{v}_0 \equiv \delta^{-1} \int_{-\delta}^0 v_0(x) dx$ denotes the volume-averaged poloidal flow. Hereafter we also redefine F as $F \equiv \delta^2 (dv_0/dx)^2 / 24$ since the volume-averaged velocity $\bar{v}_0$ is not accelerated by the divergence of the Reynolds stress $-d \langle \tilde{v}_x \tilde{v}_y \rangle / dx$. The Reynolds stress is approximated by $\langle \tilde{v}_x \tilde{v}_y \rangle \sim \tau_c (dv_0/dx) K$. We obtain

$$P_F \sim \tau_c K (dv_0/dx)^2 \sim \delta^{-1} U^{-1/2} F K. \quad (14)$$

Assuming that the Joule dissipation is dominant in (8), the turbulent energy dissipation can be written¹⁹ as

$$\epsilon_K \sim D_L^{-1} K^2 \quad (15)$$

where $D_L = D_L(U)$ is, as seen later, the L mode anomalous diffusivity and a function of U through its dependence on the background temperature. This form of the turbulent energy dissipation $\epsilon_K \propto K^2$ is the same as in the Diamond's 2-ODE model. Finally the background flow energy dissipation is written as

$$\epsilon_F \sim \mu \delta^{-2} F \quad (16)$$

where the ion collisional viscosity $\mu = \mu(U)$ is also given as a function of U due to the temperature dependence of the viscosity. Thus a closed set of the equations for U , K and F are obtained as follows

$$\frac{dU}{dt} = P_U - C_K \delta^{-1} U^{1/2} K \quad (17)$$

$$\frac{dK}{dt} = C_K \delta^{-1} U^{1/2} K - C_F \delta^{-1} U^{-1/2} F K - C'_K D_L^{-1} K^2 \quad (18)$$

$$\frac{dF}{dt} = C_F \delta^{-1} U^{-1/2} F K - C'_F \delta^{-2} \mu F \quad (19)$$

where the potential energy U input power $P_U (> 0)$ is regarded as an external parameter and C 's are nondimensional numerical constants.

The constants C 's may be estimated either through comparisons of the solutions of Eqs. (17)–(19) with numerical simulations or empirically from modelling time series obtained for L-H and ELM dynamics in tokamaks.

Stationary solutions of Eqs. (17)–(19) can be easily obtained. The L mode solution in the L mode case is given by

$$F = 0, \quad K = \delta^{-1} C_K C'^{-1}_K D_L(U_L) U_L^{1/2}, \quad U = U_L \quad (20)$$

where $U_L = U_L(P_U)$ is a function of P_U given by the solution of

$$C_K C_K'^{-1} D_L(U_L) U_L = C_K^{-1} \delta^2 P_U . \quad (21)$$

In this case, the anomalous pressure diffusivity $D \sim \tau_c K$ is written as

$$D = C_P \delta U^{-1/2} K = C_P C_K C_K'^{-1} D_L(U_L) \quad (22)$$

which confirms the previous statement that D_L agrees with the L mode diffusivity except for the nondimensional numerical factor.

Another stationary solution, i.e., the H mode stationary solution is given by

$$\begin{aligned} F &= C_K' C_F^{-1} U_H D_L^{-1}(U_H) [C_K^{-1} D_L(U_H) - C_F^{-1} C_F' \mu(U_H)] \\ K &= \delta^{-1} C_F^{-1} C_F' \mu(U_H) U_H^{1/2} \\ U &= U_H . \end{aligned} \quad (23)$$

In this stationary state the fluctuation level is controlled by the viscous damping of the mean shear flow. The H mode solution exists only when

$$C_K C_K'^{-1} D_L(U_H) > C_F^{-1} C_F' \mu(U_H) . \quad (24)$$

From Eqs. (20) and (23) we see that when the H mode condition (24) is satisfied the turbulence level K is decreased. Here $U_H = U_H(P_U)$ is a function of P_U given by the solution of

$$C_F^{-1} C_F' \mu(U_H) U_H = C_K^{-1} \delta^2 P_U . \quad (25)$$

The anomalous pressure diffusivity in the H mode is written as

$$D = C_P \delta U^{-1/2} K = C_P C_F^{-1} C_F' \mu(U_H) . \quad (26)$$

From Eqs. (22) and (26), we obtain the ratio of the H mode to L mode pressure diffusivity for $P_U > P_c$, which is given by

$$\frac{H}{L} = \frac{C_F^{-1} C_F' \mu(U_H)}{C_K C_K'^{-1} D_L(U_L)} = \frac{U_L}{U_H} < 1$$

where Eqs. (21) and (25) are used.

Now let us define (P_{c1}, U_{c1}) by the solution of

$$C_K^{-1} \delta^2 P_{c1} = C_K C_K'^{-1} U_{c1} D_L(U_{c1}) = C_F^{-1} C_F' U_{c1} \mu(U_{c1}) . \quad (28)$$

We assume that Eq. (28) has only a unique solution (P_{c1}, U_{c1}) and consider the two cases where $U D_L(U)$ and $U \mu(U)$ are functions of U as shown schematically in Figs. 1(a) and (b). The condition for the existence of the H mode stationary solution is given by $P_U > P_{c1}$ in the case of Fig. 1(a). In the case of Fig. 1(b), the H mode solution exists for $P_U > P_{c2}$ where $C_K^{-1} \delta^2 P_{c2}$ denotes the minimum value of $C_F^{-1} C_F' U_{c1} \mu(U)$ at $U = U_{c2} (> U_{c1})$ as shown in Fig. 1(b). In the latter case, Eq. (25) has the two solutions for $P_{c2} < P_U < P_{c1}$, which are denoted by U_{H-} and $U_{H+} (> U_{H-})$. As seen in the next section, U_{H-} is unstable and U_{H+} corresponds to the real H mode.

Before ending this section, let us introduce the following normalized variables, functions and parameters

$$\begin{aligned} u &= U/U_{c1}, \quad k = K/U_{c1}, \quad f = F/U_{c1}, \quad \tau = C_K \delta^{-1} U_{c1}^{1/2} t \\ d(u) &= C_K C_K'^{-1} \delta^{-1} U_{c1}^{-1/2} D_L(U), \quad m(u) = C_F^{-1} C_F' \delta^{-1} U_{c1}^{-1/2} \mu(U) \\ q &= C_K^{-1} \delta U_{c1}^{-2/3} P_U, \quad c = C_F/C_K \end{aligned} \quad (29)$$

in terms of which, Eqs. (17)–(19) are rewritten as

$$\frac{du}{d\tau} = q - u^{1/2} k \quad (30)$$

$$\frac{dk}{d\tau} = u^{1/2} k - c u^{-1/2} f k - d^{-1}(u) k^2 \quad (31)$$

$$\frac{df}{d\tau} = c u^{-1/2} f k - c m(u) f . \quad (32)$$

The L and H mode stationary points are give by the normalized variables as

$$(u_L, k_L, f_L), \quad (u_H, k_H, f_H)$$

respectively, where

$$\begin{aligned} k_L &= u_L^{1/2} d(u_L) , \quad f_L = 0 \\ k_H &= u_H^{1/2} m(u_H) , \quad f_H = c^{-1} u_H d^{-1}(u_H) (d(u_H) - m(u_H)) \end{aligned} \quad (34)$$

and u_L and u_H are functions of the external parameter q defined by

$$u_L d(u_L) = u_H m(u_H) = q . \quad (35)$$

The condition for the existence of the H mode stationary solution is written as

$$d(u_H) > m(u_H) \quad (36)$$

The critical value q_{c1} is give by solving

$$u_{c1} d(u_{c1}) = u_{c1} m(u_{c1}) = q_{c1} . \quad (37)$$

where $u_{c1} \equiv 1$ from the normalization in Eq. (29). In the case of Fig. 1(a), the condition (37) is equivalent to $q > q_{c1}$. In the case of Fig. 1(b), the H mode solution exists for $q > q_{c2}$ where q_{c2} is the minimum value of $um(u)$ at $u = u_{c2} (> u_{c1} \equiv 1)$. In this case, for $q_{c2} < q < q_{c1}$, the equation $um(u) = q$ has two solutions u_{H-} and $u_{H+} (> u_{H-})$ where u_{H-} is unstable and u_{H+} corresponds to the real H mode.

III Linear Stability of the L and H States

In this section, we examine linear stabilities of the L and H mode stationary solutions obtained in the previous section.

Putting $(u, k, f) = (u_L, k_L, f_L) + (\tilde{u}, \tilde{k}, \tilde{f})$ and linearizing Eqs. (30)–(32) around the L mode stationary solution, we obtain the following equations

$$\frac{d}{d\tau} \begin{pmatrix} \tilde{u} \\ \tilde{k} \\ \tilde{f} \end{pmatrix} = M_L \begin{pmatrix} \tilde{u} \\ \tilde{k} \\ \tilde{f} \end{pmatrix}$$

where M_L is a matrix defined by

$$M_L = \begin{pmatrix} -\frac{1}{2} d_L & -u_L^{1/2} & 0 \\ \frac{1}{2} d_L + u_L d'_L & -u_L^{1/2} & -c d_L \\ 0 & 0 & c(d_L - m_L) \end{pmatrix}. \quad (39)$$

Here $d_L \equiv d(u_L)$, $m_L \equiv m(u_L)$, $d'_L \equiv d'(u_L)$ and ' denotes the derivative with respect to u .

The eigenvalues of M_L are easily obtained as

$$\lambda_{L1} \equiv c(d_L - m_L), \quad \lambda_{L+}, \quad \lambda_{L-} \quad (40)$$

where λ_{L+} and λ_{L-} are the solutions of the quadratic equations

$$\lambda^2 + \left(\frac{1}{2} d_L + u_L^{1/2} \right) \lambda + u_L^{1/2} (u_L d_L)' = 0. \quad (41)$$

We find that $\text{Re } \lambda_{L+} < 0$ and $\text{Re } \lambda_{L-} < 0$ since $(ud)' > 0$ is assumed as seen in Figs. 1(a) and (b). The eigenvectors corresponding to λ_{L+} and λ_{L-} are both tangential to the (u, k) -plane defined by $f = 0$ and the L mode solution $(u_L, k_L, 0)$ is a sink for the orbits on this plane. The L mode solution is unstable if and only if $d_L > m_L$ which is equivalent to the condition $q > q_{cl}$.

Next, in the similar way as above, putting $(u, k, f) = (u_H, k_H, f_H) + (\tilde{u}, \tilde{k}, \tilde{f})$ and linearizing Eqs. (30)-(32) around the H mode solution yield the following equations

$$\frac{d}{d\tau} \begin{pmatrix} \tilde{u} \\ \tilde{k} \\ \tilde{f} \end{pmatrix} = M_H \begin{pmatrix} \tilde{u} \\ \tilde{k} \\ \tilde{f} \end{pmatrix} \quad (42)$$

with

$$M_H = \begin{pmatrix} -\frac{1}{2} m_H & -u_H^{1/2} & 0 \\ m_H - \frac{1}{2} m_H^2/d_H + u_H m_H^2 d'_H/d_H^2 & -u_H^{1/2} m_H/d_H & -c m_H \\ -(1 - m_H/d_H) \left(\frac{1}{2} m_H + u_H m'_H \right) u_H^{1/2} (1 - m_H/d_H) & 0 \end{pmatrix} \quad (43)$$

where $d_H \equiv d(u_H)$, $m_H \equiv m(u_H)$ and $m'_H \equiv m'(u_H)$. The eigenvalues of M_H are given by the solutions of

$$\lambda^3 + A \lambda^2 + B \lambda + C = 0 \quad (44)$$

where

$$\begin{aligned}
A &= m_H \left(\frac{1}{2} + \frac{u_H^{1/2}}{d_H} \right) \\
B &= u_H^{1/2} m_H ((1+c)(1 - m_H/d_H) + m_H(u_H d_H)' / d_H^2) \\
C &= c u_H^{1/2} m_H (u_H m_H)' (1 - m_H/d_H) .
\end{aligned} \tag{45}$$

Here we consider the case in which the condition for the existence of the H mode solution $d_H > m_H$ is satisfied. It is found that both A and B are positive. In the case of $u_H = u_{H-}$ in Fig. 1(b) where $q_{c2} < q < q_{c1}$, we obtain $C < 0$ since $(um)' < 0$ for $u_{c1} < u < u_{c2}$. Then the stationary point (u_{H-}, k_{H-}, f_{H-}) is unstable since Eq. (46) has one real and positive solution for $C < 0$. For the H mode stationary point in Fig. 1(a) and for that at $u > u_{c2}$ in Fig. 1(b), we have $C > 0$ and therefore at least one real and negative solution of Eq. (46). In these cases where $C > 0$, other two solutions of Eq. (46) are both real and negative or a pair of complex conjugate values since both A and B are positive. Thus the marginal stability of the H mode solution except for (u_{H-}, k_{H-}, f_{H-}) occurs when $C = AB$ which implies that Eq. (46) has a pair of pure imaginary solutions. Then these H mode solutions become unstable when $C > AB$ which is rewritten as

$$\frac{(u_H m_H)'}{m_H} > \left(1 + c^{-1} + c^{-1} \frac{m_H/d_H}{1 - m_H/d_H} \frac{(u_H d_H)'}{d_H} \right) \left(\frac{1}{2} + \frac{u_H^{1/2}}{d_H} \right) . \tag{46}$$

This criterion for the instability of the H mode depends on the values of q (through $u_H = u_H(q)$), c as well as the functional forms of $d(u)$ and $m(u)$. As shown by the numerical examples in the next section, there appears a Hopf-bifurcation from the stable H mode solution to the limit cycle around the unstable H mode solution just when the criterion (46) is satisfied.

IV Numerical Results

Here our model equations (30)–(32) are solved numerically. Considering the anomalous diffusivity to be proportional to the background pressure gradient as in the case of the resistive g turbulence, we use the following form of $d(u)$ in all the numerical calculations here,

$$d(u) = u. \quad (47)$$

For the functional form of $m(u)$, we examine two cases as follows

$$\text{Case A : } m(u) = u^{-1}(0.97 + 0.03u^2)$$

$$\text{Case B : } m(u) = u^{-1.5}(0.95 + 0.05u^{2.5}). \quad (48)$$

Here it is assumed in both cases that, as u increases, from small u (low temperature), the drag $m(u)$ decreases due to the weaker collisional viscosity caused by the temperature increase. For sufficiently large u the frictional drag $um(u)$ increases due to the increase of the anomalous or turbulent contribution to the viscosity. Figures 2(a) and (b) show $ud(u)$ and $um(u)$ in Case A and Case B. We can see that Case A and Case B correspond to the cases in Fig. 1(a) and (b), respectively. It should be noted that the qualitative properties shown in the numerical results depend not on the detailed expressions of $d(u)$ and $m(u)$ written in Eqs. (49) and (50) but on the qualitative behaviors of them shown in Fig. 2(a) and (b). We put $c = 5$ for which we can obtain the parameter region where the H mode instability criterion (46) is satisfied.

Case A.

By the normalization in Eq. (29), we have $u_{c1} \equiv 1$. From Eqs. (37) and (46), we obtain $q_{c1} = 1$, $u_{c3} = 6.28$ and $q_{c3} = 2.15$, which are shown in Fig. 2(a). Here (q_{c3}, u_{c3}) denotes

critical point for the marginal stability of the H mode solution. The solid and dashed curves in Fig. 2 correspond to the stable and unstable stationary solutions, respectively.

For $q < q_{c1}$, there exist only one stationary solution, i.e., the stable L mode solution. Figures 3(a), (b) and (c) show the numerical solution of Eqs. (30)–(32) as the trajectory in the (u, k, f) phase space, its projection onto the (k, f) -plane and the temporal dependence of (u, k, f) , respectively, where the external parameter $q = 0.8$ and the initial condition $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$ are used. Here we see that the trajectory is approaching to the stable L mode solution $(u_L, k_L, f_L) = (0.89, 0.85, 0)$. For $q_{c1} < q < q_{c3}$, the L mode stationary solution is unstable and the stable H mode solution appears. Figures 4(a)–(c) show the numerical solution for $q = 1.05$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$ in the same way as in Figs. 3(a)–(c). Here the trajectory starting from $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$ first approaches to the L mode solution $(u_L, k_L, f_L) = (1.02, 1.04, 0)$ tangentially to the (k, f) -plane at $f = 0$ and next leave it at angle to the (k, f) -plane and finally goes to the H mode solution $(u_H, k_H, f_H) = (1.63, 0.82, 0.20)$. For $q > q_{c3}$, the L mode solution remains unstable and the H mode solution also becomes unstable. Figures 5(a)–(d) show the numerical solution for $q = 2.3$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. Here we find that the trajectory finally approaches to a limit cycle existing around the H mode solution $(u_H, k_H, f_H) = (6.66, 0.89, 1.2)$. Thus we obtain the Hopf-bifurcation at $q = q_{c3} = 2.15$.

Let us regard a value of u for the stable stationary solution as an ‘order parameter’ in our system and consider it as a function $u = u(q)$ of the external parameter q . In the critical point $q = q_{c1}$, the order parameter u is continuous while its first derivative du/dq is not. Thus the L-H transition in Case A is a second-order phase transition. In Fig. 6, we find the results where the external parameter is temporally varied. Figure 6(a) shows the external parameter $q = q(\tau)$ as a function of time. The temporal dependence of (u, k, f) trajectories in the (u, k, f) -space and in the (k, f) -plane are shown in Figs. 6(b), (c) and (d), respectively. Even though we mentioned earlier that this case corresponds to the case

order transition, we find clear L to H and H to L transitions with a certain time lag required to reach the bifurcated stable solution caused by the finite difference in q and accordingly sudden changes in (u, k, f) . While $q > q_{c3} = 2.15$, the ELM-like instability grows approaching to the periodic oscillation represented by the limit cycle.

Case B.

Here, we have $u_{c1} \equiv 1$, $q_{c1} = 1$, $u_{c1}^* = 3.01$, $u_{c2} = 1.86$, $u_{c2}^* = 0.93$, $q_{c2} = 0.87$, $u_{c3} = 4.40$ and $q_{c3} = 1.42$, which are shown in Fig. 2(b).

For $q < q_{c2}$, there exist only one stationary solution, i.e., the stable L mode solution. Figures 7(a), (b) and (c) show the numerical solution of Eqs. (30)–(32) for $q = 0.8$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$ in the same way as in Figs. 3(a)–(c). Here the stable L mode solution $(u_L, k_L, f_L) = (0.89, 0.85, 0)$ attracts the trajectory which is almost the same as in Fig. 3.

For $q_{c2} < q < q_{c1}$, the L mode stationary solution remains unstable while there appear one stable H mode solution and another unstable stationary solution. Figures 8(a)–(d) show the numerical solutions for $q = 0.9$ with two different initial conditions $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.08)$ and $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.1)$. Here, both the solutions firstly are close to each other along the stable manifold shrinking to the unstable stationary point $(u_{H-}, k_{H-}, f_{H-}) = (1.41, 0.76, 0.15)$ before reaching the unstable manifolds expanding from the point. (The linearized matrix around this unstable point has one positive real eigenvalue and a pair of complex-conjugate eigenvalues with negative real parts.) Then the two solutions deviate from each other: the one starting from $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.08)$ goes to the L mode stable point $(u_L, k_L, f_L) = (0.95, 0.92, 0)$ and the other starting from $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.1)$ to the H mode stable point $(u_{H+}, k_{H+}, f_{H+}) = (2.39, 0.58, 0.40)$. Due to this parameter region where the two stable stationary solutions exist, the hysteresis curve as shown by arrows in Fig. 2(b) is predicted with respect to the dependence of the order parameter u on the

external parameter q .

For $q_{c1} < q < q_{c3}$ and $q > q_{c3}$, the stabilities of the L and H stationary solutions and the behavior of the trajectories are similar to those in the corresponding parameter regions in Case A as shown in Fig. 9 for $q = 1.05$ and in Fig. 10 for $q = 1.6$. We find the Hopf-bifurcation at $q = q_{c3} = 1.42$, where the stable H mode solution is replaced by the attracting limit cycle.

As mentioned earlier, in Case B, we obtain the hysteresis curve of the order parameter $u = u(q)$ as a function of the external parameter q . At the critical points $q = q_{c1}$ and $q = q_{c2}$ of the L to H and H to L transitions, the order parameter u changes discontinuously with respect to q as $u_{c1} \rightarrow u_{c1}^*$ and $u_{c2} \rightarrow u_{c2}^*$. They represent the characteristics of the first-order phase transition. In Figs. 11(a)–(d), the results where the external parameter is temporally varied are shown in the same way as in Figs. 6(a)–(d). The hysteresis nature can be clearly seen in that the L to H transition occurs when $q > q_{c1} = 1$ (occurring at $t \simeq 120$ and giving $u \simeq 4.1$) while the H to L transition occurs from a lower H mode with ($u \simeq 2.8$ at $t = 850$), when $q < q_{c2} = 0.87$ small changes in q can yield the transitions with larger changes in the pressure. Thus, in Case B the sensitivity of the confinement to small changes in the input power is much greater than in Case A.

The L to H transition at time $t \simeq 140$ in Fig. 11(b) may also be described as a hard onset of the H mode using the term from stability theory that describes the fast transition of u from $u \simeq 1$ to $u \simeq 4$ in a subcritical bifurcation.

V Conclusions and Discussion

In this work, we have derived a 3-dimensional state space dynamical model of the L-H dynamics and transitions. The model predicts the temporal evolution of the potential energy $U(t)$ contained in the pressure gradient, the turbulent kinetic energy $K(t)$ and the background shear flow energy $F(t)$. The model is derived from the energy balance equations by

modeling the energy transfer and dissipation terms in terms of the three energy variables (U, K, F) . The thermal power input P_U into the peripheral region is the external control parameter in our model. Kukharkin *et al.*²⁰ also proposed an L-H transition model consisting of three ODEs although that model has terms of a different form from those presented here, and especially it does not contain the external parameter corresponding to P_U . If we freeze U , the equations for K and F reduce to the 2-ODE model of the same form as that of Diamond *et al.*

Our dynamical model has the stationary solutions corresponding to the L and H confinement modes. Depending on the value of the external parameter P_U , their stabilities change. The pressure gradient energy U of the stable stationary solution is considered as the order parameter of our system which then yields a function $U = U(P_U)$. We have examined two cases: one is Case A (second-order phase transition) in which not $U(P_U)$ but dU/dP_U is discontinuous at the transition. This may also be called the soft onset of the H mode. In the second case, and another Case B, (first-order phase transition) in which $U(P_U)$ is discontinuous at the transition and the system exhibits the hysteresis property. Case B may also be called the hard onset of the H mode.

The model also predicts the criterion for the loss of stability of the H mode, and the observed Hopf bifurcation from the stable to the unstable H mode gives the attracting limit cycle which behaves like the ELM plasma state. The limit cycle is a relaxation oscillation with intermittent bursts (Fig. 10(d)) of high turbulence levels and associated transport that allow the system to pass through the higher input power P_U values that trigger the ELM state. In the 2-ODE model by Diamond *et al.* where U is treated as an external parameter, we have neither a first-order phase transition nor the unstable H mode. In the L-H transition model by Hinton,⁵ the relation between the thermal flux and the temperature gradient similar to the (P_U, U) -curve in Case B is presented. In Hinton's model, the poloidal flow is given from the temperature gradient by the neoclassical theory while the poloidal shear

flow effect on the anomalous transport is separately modeled using the argument by Biglari *et al.*² and the transport coefficient of drift wave turbulence. However, in our model, the interrelation between the pressure gradient as a turbulence source, the turbulence energy as a cause of anomalous transport and the shear flow as a stabilizer of the turbulence are self-consistently treated by taking into account energy transfer nonlinearities due to the product of the thermal transport flux and the pressure gradient and due to that of the momentum transport from the Reynolds stress and the shear flow gradient. Because of the abrupt change and the hysteresis property, the first-order phase transition seems to be supported by the experimental results although we found clear transition from the temporal evolution of the numerical solution in Case A due to the time lag for reaching to the bifurcated stable point. Figure 4(c) shows an example of the time lag for a case just above the critical power required in Case A.

There remain some issues for the investigation in our model, such as the functional forms of the diffusivity and viscosity as well as the values of the dimensionless numerical constants. Here we have explained the general behavior of the L-H transition dynamics within the context of a self-consistent turbulent transport model. However, further investigation on the model is necessary especially on the viscosity function since its form affects the critical point and the type of the transition as well as the occurrence of the ELM-like instability. As an extension of our model, the macroscopic spatial structures of the three variables should be described by the partial differential equations (PDE model) taking into account their transport. This spatial extension is required to address the issue of the nonlocal versus local nature of the L-H transition. In our previous work,⁷ the shear flow generation by the Reynolds stress of the resistive g mode is confirmed by the single-helicity nonlinear simulation although in that simulation the energy input is modeled by fixed pressure gradient (which is a conventional technique). In such simulations, we observed that the reduced 2-ODE model describes well the qualitative behavior of the solutions. Comparison between our 3-ODE

model with numerical turbulence simulation, which includes the energy input as an external parameter and allowing the self-consistent variation of the pressure gradient, is an interesting research problem for future studies.

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Figure Captions

1. The functional forms of $UD_L(U)$ and $U\mu(U)$. $UD_L(U)$ is assumed to be a monotonically increasing function of U . (a) $U\mu(U)$ is a monotonically increasing function of U . For $P_U > P_{c1}$, there exist the L and H mode solutions, which are denoted by U_L and U_H , respectively. (b) $U\mu(U)$ has a minimum value at $U = U_{c2}(> U_{c1})$. For $P_{c2} < P_U < P_{c1}$, there are three stationary solutions, i.e., U_L , U_{H-} and U_{H+} .
2. The stationary solutions plotted in the (u, q) -plane. The solid and dashed curves denote the stable and unstable stationary solutions, respectively. The expressions for $d(u)$ and $m(u)$ are given in Eqs. (49) and (50). (a) and (b) correspond to Case A and Case B, respectively. Arrows in (b) represent the second-order phase transitions showing the hysteresis property.
3. Case A: numerical solution of Eqs. (30)–(32) for $q = 0.8(< q_{c1})$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. The stable L-mode solution is given by $(u_L, k_L, f_L) = (0.89, 0.85, 0)$. (a) The trajectory in the (u, k, f) phase space. (b) The projection of the trajectory onto the (k, f) -plane. (c) The temporal dependence of (u, k, f) .
4. Case A: numerical solution of Eqs. (30)–(32) for $q = 1.05(q_{c1} < q < q_{c3})$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. The unstable L-mode solution and the stable H mode solution are given by $(u_L, k_L, f_L) = (1.02, 1.04, 0)$ and $(u_H, k_H, f_H) = (1.63, 0.82, 0.20)$, respectively. (a) The trajectory in the (u, k, f) phase space. (b) The projection of the trajectory onto the (k, f) -plane. (c) The temporal dependence of (u, k, f) .
5. Case A: numerical solution of Eqs. (30)–(32) for $q = 2.3(> q_{c3})$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. The unstable L-mode solution and the stable H mode solution are given by $(u_L, k_L, f_L) = (1.52, 1.87, 0)$ and $(u_H, k_H, f_H) = (6.66, 0.89, 1.26)$, respectively.

The trajectory in the (u, k, f) phase space for $0 \leq \tau \leq 20$ and for the limit cycle. (b) The projection of the trajectory onto the (k, f) -plane for $0 \leq \tau \leq 20$ and for the limit cycle. (c) The temporal dependence of (u, k, f) for $0 \leq \tau \leq 20$. (d) The temporal dependence of (u, k, f) for the limit cycle.

6. Case A: numerical solution of Eqs. (30)–(32) where the external parameter q is temporally varied. The initial condition is given by $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. (a) The external parameter $q = q(\tau)$ as a function of time. The two horizontal lines correspond to $q = q_{c1}(= 1)$ and $q = q_{c3}(= 2.15)$. (b) The temporal dependence of (u, k, f) . (c) The trajectory in the (u, k, f) phase space. (d) The projection of the trajectory onto the (k, f) -plane.
7. Case B: numerical solution of Eqs. (30)–(32) for $q = 0.8(< q_{c1})$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. The stable L-mode solution is given by $(u_L, k_L, f_L) = (0.89, 0.85, 0)$. (a) The trajectory in the (u, k, f) phase space. (b) The projection of the trajectory onto the (k, f) -plane. (c) The temporal dependence of (u, k, f) .
8. Case B: numerical solutions of Eqs. (30)–(32) for $q = 0.9(q_{c2} < q < q_{c1})$. Two different initial conditions $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.08)$ and $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.1)$ are used. There are two stable stationary solutions given by $(u_L, k_L, f_L) = (0.95, 0.92, 0)$ and $(u_{H+}, k_{H+}, f_{H+}) = (2.39, 0.58, 0.40)$, and one unstable stationary solution given by $(u_{H-}, k_{H-}, f_{H-}) = (1.41, 0.76, 0.15)$. (a) The two trajectories in the (u, k, f) phase space starting from $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.08)$ and from $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.1)$. (b) The projection of the two trajectories onto the (k, f) -plane. (c) The temporal dependence of (u, k, f) for $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.08)$. (d) The temporal dependence of (u, k, f) for $(u, k, f)_{\tau=0} = (1.5, 0.2, 0.1)$.

9. Case B: numerical solutions of Eqs. (30)–(32) for $q = 1.05(q_{c1} < q < q_{c3})$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. The unstable L-mode solution and the stable H mode solution are given by $(u_L, k_L, f_L) = (1.02, 1.04, 0)$ and $(u_H, k_H, f_H) = (3.23, 0.58, 0.58)$, respectively. (a) The trajectory in the (u, k, f) phase space. (b) The projection of the trajectory onto the (k, f) -plane. (c) The temporal dependence of (u, k, f) .
10. Case B: numerical solutions of Eqs. (30)–(32) for $q = 1.6(> q_{c3})$ and $(u, k, f)_{\tau=0} = (0.3, 0.2, 0.1)$. The unstable L-mode solution and the stable H mode solution are given by $(u_L, k_L, f_L) = (1.26, 1.42, 0)$ and $(u_H, k_H, f_H) = (4.83, 0.73, 0.90)$, respectively. (a) The trajectory in the (u, k, f) phase space for $0 \leq \tau/20$ and for the limit cycle. (b) The projection of the trajectory onto the (k, f) -plane for $0 \leq \tau \leq 20$ and for the limit cycle. (c) The temporal dependence of (u, k, f) for $0 \leq \tau \leq 20$. (d) The temporal dependence of (u, k, f) for the limit cycle.
11. Case B: numerical solutions of Eqs. (30)–(32) where the external parameter q is temporally varied. The initial condition is given by $(u, k, f)_{\tau=0} = (0.4, 0.4, 0.4)$. (a) The external parameter $q = q(\tau)$ as a function of time. The three horizontal lines correspond to $q = q_{c1}(= 1)$, $q = q_{c2}(= 0.87)$ and $q = q_{c3}(= 1.42)$. (b) The temporal dependence of (u, k, f) . (c) The trajectory in the (u, k, f) phase space. (d) The projection of the trajectory onto the (k, f) -plane.

Figs. 1(a)-(b)

Figs. 2(a)-(b)

Case A

$$g=0.8 < g_{c1}$$

L state

Fig.3(a)

Fig.3(b)

Fig.3(c)

Case A

$$g_{c1} < g = 1.05 < g_{c3}$$

Fig.4(a)

Fig.4(b)

Fig.4(c)

L → H transition

Case A

ELMy State

Fig.5(a)

Fig.5(b)

Case A

Fig. 6(a)

Case A

u, k, f

Fig. 6(b)

Figs. 6(a)-(b)

Case A

programmed $g(\tau)$

Fig.6(c)

limit cycle

Fig.6(d)

Case B

$$q = 0.8 < q_{c2} = 0.87$$

L state

Case B

Fig.8(a)

sensitivity to initial data

Fig.8(b)

Case B

Figs. 8(c)-(d)

Case B

$$q = 1.05 > q_{c1}$$

$$q_{c1} < q < q_{c3}$$

Fig.9(a)

L $\rightarrow$ H transition

Fig.9(b)

Fig.9(c)

Case B

$$g = 1.6 > g_{c3} = 1.4$$

Fig.10(a)

Fig.10(b)

Case B

$$q = 1.6 > q_{c3}$$

Figs. 11(a)-(b)

Case B

Programmed $g(\tau)$

Fig. 11(c)

limit cycle

Fig. 11(d)

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