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Magnetic Flux Reconstruction Methods for Shaped Tokamaks

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**MAGNETIC FLUX RECONSTRUCTION
METHODS FOR SHAPED TOKAMAKS**

by

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ABSTRACT

The use of a variational method permits the Grad-Shafranov (GS) equation to be solved by reducing the problem of solving the 2D non-linear partial differential equation to the problem of minimizing a function of several variables. This high speed algorithm approximately solves the GS equation given a parameterization of the plasma boundary and the current profile (p' and FF' functions). We treat the current profile parameters as unknowns. The goal is to reconstruct the internal magnetic flux surfaces of a tokamak plasma and the toroidal current density profile from the external magnetic measurements. This is a classic problem of inverse equilibrium determination. The current profile parameters can be evaluated by several different matching procedures. We found that the matching of magnetic flux and field at the probe locations using the Biot-Savart law and magnetic Green's function provides a robust method of magnetic reconstruction.

The matching of poloidal magnetic field on the plasma surface provides a unique method of identifying the plasma current profile. However, the power of this method is greatly compromised by the experimental errors of the magnetic signals. The Casing Principle [60] provides a very fast way to evaluate the plasma contribution to the magnetic signals. It has the potential of being a fast matching method. We found that the performance of this method is hindered by the accuracy of the poloidal magnetic field computed from the equilibrium solver.

A flux reconstruction package have been implemented which integrates a vacuum field solver using a filament model for the plasma, a multi-layer perceptron neural network as a interface, and the volume integration of plasma current density using Green's functions as a matching method for the current profile parameters. The flux reconstruction package is applied to compare with the ASEQ and EFIT data. The results are promising. Also, we found that some plasmas in the tokamak

Alcator C-Mod lie outside our operationally valid region, given the current set of the trial functions inside the variation method.

Thesis Supervisor: Ian H. Hutchinson, Jeff P. Freidberg
Title: Professors of Nuclear Engineering

Dedication

To my wife **Suk-yi**

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Chapter 1

Introduction

1.1 Motivation

The success of the Alcator C tokamak in fusion research at MIT has proven the merits of the high-field, high-density tokamak in obtaining hot, well confined plasma. As a next step in fusion research, MIT has built a new high-field compact tokamak named Alcator C-Mod. Alcator C-Mod has the same major radius as Alcator C, but it will have an elongated plasma, a divertor, and higher plasma current. It is designed to be heated by ion cyclotron range of frequencies (ICRF). Also, it will be the first tokamak [1] which can be operated with electron densities substantially above $10^{20}m^{-3}$ in a divertor configuration. Hence, we expect the experimental data from this tokamak will be very important to future ignition devices. The attainment of an ignited and controlled thermonuclear plasma has long been a dream shared by all fusion energy researchers.

In the operation of a tokamak, the measurements of the external magnetic field and flux provide the most basic information on the electromagnetic properties of the confined plasma. Knowing the plasma position, shape, pressure and some other global quantities such as internal inductance is vital to the real time feedback

control of the plasma, plasma tuning and optimization between shots, and long term MHD stability analysis, plasma confinement and transport studies. Thus the magnetic diagnostics are extremely productive and practical in routine operation of Alcator C-Mod. Moreover, these global quantities are essential when unfolding information from most other diagnostics. Plasma physicists who are directly involved with the magnetic diagnostics on a tokamak, those who are involved with the interpretation of other basic plasma diagnostics or with machine control, and researchers in computational MHD should be interested in this topic.

The simplest way to measure the magnetic field in the vicinity of a point in space is to use a tiny coil of conducting wire. The principle of this diagnostic is based on Faraday's law [2].

If the plasma temperature is low, a multicoil, internal magnetic probe [3, 4] can be inserted into the plasma without being melted. The probe signals can be actively integrated to measure the total poloidal field profile. The toroidal current density profile can be calculated from the differential form of the Ampere's law.

Since the plasma temperature in modern tokamaks is extremely high, the internal probe will be melted by the hot plasma. Moreover, the plasma equilibrium will be perturbed by the probe. Hence, all such direct magnetic measurements have to be done external to the plasma.

The design, test, calibration and installation of the magnetic diagnostics package on Alcator C-Mod was fully described by Granetz, et al[5]. Fig. 1.1 shows the locations of the magnetic coils and the flux loops.

The vacuum vessel is protected from the hot plasma by molybdenum tiles. The magnetic field coils and flux loops are installed between the tiles and the vessel wall. The locations of the magnetic probes are chosen under engineering constraints such as the thermal conduction and radiation, the electrical insulation, mechanical protection and ultra-high vacuum outgassing behavior of material. Hence, the number

Figure 1.1: The locations of the magnetic coils and the flux loops.

and locations of the probes are considered as fixed in the flux reconstruction.

1.2 Challenges of the Flux Reconstruction

Given the measurements from the magnetic coils and the flux loops, the total plasma current, and the PF coil currents, our objectives are listed below approximately in order of increasing difficulty:

- to find the centroid of the plasma current.
- to trace out the plasma boundary.
- to distinguish the limiter plasma and divertor plasma.
- to locate the position of the magnetic axis.
- to calculate the different forms of internal plasma inductance.
- to calculate the different forms of β_p .
- to determine the current density distribution.
- to reconstruct the internal magnetic flux.
- to find the current profile parameters.
- to calculate the safety factor profile.

The magnetic flux contours and the corresponding plasma parameters of a hypothetical plasma is shown in Fig. 1.2. Although it is beyond the capability of our machine, it is useful for theoretical studies [6].

Table 1.1 summarizes the relevant parameters which describe the plasma equilibrium. The current profile parameters b_p , α_f and α_p will be discussed in detail later.

Figure 1.2: The magnetic flux contour of a hypothetical plasma proposed for theoretical studies

R_0	major radius of the plasma
a	minor radius of the plasma
κ_s	plasma elongation on the boundary
δ_s	plasma triangularity
B_0	toroidal vacuum magnetic field at $R = R_0$
I_p	total plasma current
b_p	current profile parameter
α_f	current profile parameter
α_p	current profile parameter

Table 1.1: List of Notations

Since this method may be applied in the real time analysis of the tokamak experiment, both the speed and the accuracy of this flux reconstruction are important.

With the assumptions of toroidal axisymmetry and a scalar plasma pressure, the ideal magnetohydrodynamics equations can be reduced to an elliptical partial differential equation for the poloidal magnetic flux, the Grad-Shafranov equation[7].

In order to reconstruct the magnetic flux surface inside an isotropic plasma, this elliptical partial differential equation has to be solved.

$$\Delta^* \psi = -\mu_0 R J_\phi(R, \psi) \quad (1.1)$$

where

$$\Delta^* \psi(R, Z) \equiv R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} \quad (1.2)$$

and

$$\mu_0 J_\phi(R, \psi) = \mu_0 R p'(\psi) + \frac{1}{R} F F'(\psi) \quad (1.3)$$

The function ψ has two physical meanings, it serves as a stream function of the poloidal magnetic field, and is also related to the signals measured by the flux.

The poloidal magnetic field can be calculated from the derivatives of the function ψ by the following equation.

$$\mathbf{B}_p = \frac{\nabla \psi \times \mathbf{e}_\phi}{R} \quad (1.4)$$

The unit vector $\mathbf{e}_\phi$ points in the toroidal direction. This equation shows that the poloidal field lies on the flux surface. The different components of the poloidal field can be expressed as derivatives of the stream function.

$$B_R = -\frac{1}{R} \frac{\partial \psi}{\partial Z} \quad (1.5)$$

$$B_z = \frac{1}{R} \frac{\partial \psi}{\partial R} \quad (1.6)$$

We use Ψ to represent the poloidal flux and ψ to denote the poloidal stream function.

Then,

$$\Psi = \int_0^{2\pi} d\phi \int_0^R dR R B_z(R, Z=0) \quad (1.7)$$

$$\Psi = 2\pi\psi \quad (1.8)$$

Assume $\psi(R=0, Z=0)$ is zero for convenience. The signals from the flux loops after the integration are the value of Ψ .

Fig. 1.1 shows that the magnetic coils and the flux loops are implemented in discrete position on the vessel wall. Hence, in the mathematical sense, the signals from the flux loops and the magnetic coils are the discrete boundary values of ψ and $\frac{\partial \psi}{\partial n}$. Here, n is the unit normal vector of the contour which defines the boundary.

In order to understand the nature of this physical problem, let us consider a cylindrical symmetric plasma current flow. The cross section of the magnetic flux surfaces are concentric circles. These surfaces have constant pressure, density and temperature. A set of magnetic coils are placed on a concentric circle outside the current flow. Using Ampere's law, the total current can be determined. For the same set of magnetic signals, there is an infinite set of radial current distribution which satisfy the Maxwell's equations. Hence, in this case, even though we know the shape of all the flux surfaces and the total current, we still can not determine the current profile.

The difficulty of the magnetic flux reconstruction is that the radial distribution of toroidal current is unknown. The Grad-Shafranov equation does not provide direct information on the current profile, and the existence and uniqueness of solutions

are not guaranteed.

Christiansen and Taylor[8] have proposed that the current distribution in a toroidal configuration can be determined completely from purely geometric information about the shape of the magnetic surfaces obtained from X-ray tomography, cyclotron emission or laser scattering. Since these surfaces coincide with those of constant density and temperature it is possible that observations of plasma density and temperature could be sufficient to determine the current distribution. However, Braams[10] has pointed out that their argument is incorrect. He performs an analysis on this problem and concludes that, except in the circular case, the current profile in a toroidal configuration can be determined from knowledge of the geometry of the flux surfaces together with a measurement of the total plasma current. In the degenerate case, the external magnetic measurements need not define uniquely the current profile. There can be a family of equilibria which give the same flux surface structure and the same total current.

Christiansen, Callen, Ellis, and Granetz[9] had shown that the current distribution in JET can be determined from the geometry of the flux surfaces inferred from surfaces of constant X-ray emissivity.

Since we do not know the toroidal current profile, this is an inverse non-linear problem. A non-linear optimization will be performed to obtain the internal flux surfaces. The resulting equilibrium will have the plasma current profile which is an optimal fit to the external magnetic measurements.

Two principal macroscopic parameters which characterize the plasma equilibrium in a tokamak are the ratio of the plasma pressure to the pressure of the magnetic fields, β_I and the inductance per unit length of the plasma l_i . Knowing these parameters as a function of time can provide information on the energy containment time of the plasma[11].

Shafranov[12] has shown that the quantity $\beta_p + (l_i/2)$ can be obtained from line integrals of products of the poloidal magnetic field components outside the plasma, evaluated on any contour in a poloidal plane that encloses the plasma. No knowledge of the plasma current distribution is assumed. This method is convenient especially when the contour is the plasma boundary, because it encloses only internal magnetic energy, and the normal component of the poloidal field is absent. In principle, the poloidal field outside the plasma can be measured or calculated. However, this method works best when the aspect ratio is high, otherwise the difference between the geometric center and the mean radius has to be estimated first.

To achieve higher plasma betas and higher toroidal current densities, plasma cross sections with high vertical elongation and some degree of triangularity are desired. To have better impurity control, a divertor was installed in Alcator C-Mod. Hence, the plasma cross-section is not necessary circular nor symmetric. As the plasma cross-section becomes more and more complicated, it is a mathematical challenge to formulate the plasma flux contours in a general expression. For mathematical analysis, a family of analytical functions with several adjustable parameters will be chosen to describe the plasma equilibrium with different shapes.

1.3 Approach and Outline of This Work

The flux reconstruction problem can be divided into three parts, the vacuum region, the plasma region and the interface.

The vacuum field can be solved using a filament model[15]. We have applied a filament model to simulate the plasma, the PF coil currents, the eddy currents inside the vacuum vessel and the stainless steel supporting structures such as the cover and the cylinder. The magnetic signals can be perceived as a combination of contributions from the plasma, the PF coil currents, and eddy currents inside

vacuum vessel wall and supporting structure. In order to determine all the filament currents, the least square method is used to fit the magnetic signals.

Once the vacuum field is solved, the separatrix can be found by tracing flux values inside the mesh which matches the flux either on the limited point or at the X-point. Hence, the plasma boundary can be determined quickly and accurately from the magnetic measurements.

The fixed boundary variational method provides a potentially fast solution to the plasma equilibrium. In contrast to solving the Grad-Shafranov equation on a two-dimensional spatial grid, an algebraic Lagrangian is minimized. The internal magnetic flux coordinates are transformed to a flux and angle-like coordinate. Since there are many physical quantities such as poloidal flux and the plasma pressure which are constant on a flux surface, this transformation will result in simplifying of the formulation and saving of computation time.

The two free functions $p(\psi)$ and $FF'(\psi)$ are written in the form due to Strickler[13]. The Strickler's function takes only one parameter as input and it is a convenient choice. Since it models a wide range of current profiles, it provides a large base for the solution set.

We will apply the variational method to solve the fixed boundary equilibrium. The current profile parameters will be determined from the matching of magnetic field and flux at the probe locations.

Combining the vacuum solver and the fixed boundary plasma equilibrium solver, we create a full flux reconstruction package. However, the outputs of the vacuum solver are the vacuum flux and a set of points on the separatrix, while the input of the fixed boundary solver requires 4 shape parameters. Hence, the output of the vacuum solver can not be fed directly into the input of the fixed boundary equilibrium solver. Here we solve this non-linear mapping problem by applying a neural network.

In order to show the capability of the integrated flux reconstruction package, it is compared with ASEQ and EFIT.

Experimental data from Alcator C-mod will also presented.

Chapter 2

Equilibrium Calculation and Flux Reconstruction

Given the plasma current profile, $J_\phi(R, \psi)$, of a toroidal plasma at MHD equilibrium, the plasma equilibrium calculation is usually performed with a computer program which solves the Grad-Shafranov equation and determines the poloidal flux $\psi(R, Z)$ in and around the plasma.

The purposes of equilibrium calculation are:

- to program the plasma shape as a function of the external coil currents.
- to determine the poloidal flux function in space coordinates, $\psi(R, Z)$.
- to determine the toroidal current distribution as function of space coordinates, $J_\phi(R, Z)$.
- to interpret experimental results.
- to calculate of the global plasma quantities.
- to design of new confinement devices, coil placement, scaling.

- to analyze the stability of various machine configurations.

Given the external magnetic measurements and the external coil currents, assuming the toroidal current profile is unknown, the magnetic flux reconstruction is executed with a computer code which solves the Grad-Shafranov equation and determines the current profile by matching to the external magnetic measurements. The values of ψ and its derivatives are assumed to be known at discrete points of the magnetic coil locations and flux loop locations. Furthermore, the poloidal flux in and around the plasma is reconstructed.

The purposes of magnetic reconstruction are similar to those of the equilibrium calculation and are listed in Section 1.2. After the determination of the current profile parameters from the given set of magnetic measurement, the plasma equilibrium can be solved and the internal magnetic flux can be calculated. Once the plasma shape and position are known, they can be in principle input to the real time feedback control for the plasma operation.

In the equilibrium calculation, the current profile parameters are given as inputs, while in the flux reconstruction, they are among the desired outputs. Obviously, the equilibrium calculation and the flux reconstruction are twin problems. Moreover, the flux reconstruction problem can be treated as an inverse equilibrium determination.

The equilibrium calculation can be classified as a free-boundary solver or fixed-boundary solver depending on whether the plasma boundary is given. Hence the flux reconstruction can also be classified in the same way.

From the physical background of the problem, three regions of space (see Fig. 2.1) can be defined. The first region is a toroid containing all of the plasma current. The second region is the vacuum region which surrounds the first region and is bounded by the magnetic probes. The third region is the region outside the second region which contains the vessel wall and the PF coils. The mathematical

Figure 2.1: The three regions of space

formulations and boundary conditions can be different in different regions.

2.1 The problem of the vacuum field reconstruction

In a tokamak experiment, the plasma shape and position are important for equilibrium and stability study. This information can be obtained by solving the poloidal field in the vacuum region.

In the vacuum region, the plasma current is assumed to vanish and the Grad-Shafranov equation is homogeneous:

$$\Delta^*\psi = 0 \quad (2.1)$$

where

$$\Delta^*\psi(R, Z) \equiv R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} \quad (2.2)$$

The region under consideration lies between the contour of the sensor coil locations and the plasma separatrix. The flux loops will measure the ψ at discrete points on the inner surface of the vessel wall. The Mirnov coils will measure the $n \cdot \nabla \psi$ at another set of discrete points. These are the boundary conditions of the homogeneous second order partial differential equation.

The magnetic probes are external to the plasma and they are fairly insensitive to the details of the plasma current distribution. For a fixed set of magnetic measurements, the solution of the corresponding current distribution is not unique. For example, a pair of coaxial equal and opposite currents will not change the total current. It is impossible to detect this current distribution when the probes are external to the plasma.

The reconstruction of the magnetic flux and field in the vacuum region from the external magnetic measurements is an ill-posed problem if no further restrictions on the current profile are given. Since Δ^* is an elliptic operator, the mathematical problem of interest requires the solution of an elliptic partial differential equation, subject to Cauchy boundary-conditions. This is well known to be an ill-posed problem because small changes in the boundary data will lead to large changes in the solution a short distance away[14]. Swain and Neilson[15] have shown that the assumptions about currents in the poloidal field coils have a strong effect on the calculation of vacuum field. Thus, the mathematical problem can be better specified if the current and locations of the poloidal field coils are known.

2.2 The vacuum field solution

It is mathematically not possible, from any external magnetic measurements, to completely determine the plasma current distribution, but only to determine its multipole moments[16]. There exists an infinite number of current distributions that possess a truncated set of multipole moments. This suggests that the plasma can be represented by a set of discrete filaments at pre-specified locations.

For the calculation of the contribution to the magnetic field in the vacuum region from the plasma current, some researchers [17, 18] chose a multi-filament model to represent the plasma current.

The multi-filament model totally neglects the force balance of the magnetic and kinetic pressures. It solves the magnetostatics part of the problem conveniently because the influences from filaments to sensors can be pre-calculated since the locations of the filaments and the poloidal field coils are known, which leads to a simple matrix treatment of the problem.

Wootton [17] proposed a fast method of determining the plasma boundary from

external magnetic measurements. The plasma current distribution was represented as a number of toroidal filaments, whose positions are chosen to give the experimentally measured toroidal multipole moments. This technique was applied to a small tokamak, TOSCA. Flux surfaces generated by three-filament plasma model were compared to a plasma distorted by a hexapole magnetic field. The plasma boundary matched with the prediction from free-boundary equilibrium calculations. The multipole moments are expressed as an expansion in terms of the inverse aspect ratio. This method assumes large aspect ratio and small shift of the current center from the geometric center.

Swain and Neilson [15] have shown that given measurements of the fields, the external winding currents and the total plasma current, the vacuum poloidal fields everywhere outside the plasma boundary can be reconstructed by assuming a multi-filament plasma model. The reconstructed fields will satisfy Maxwell's equations and also fit the magnetic measurement. This technique was used to provide a complete time history of ISX-B high-beta discharges within two to three minutes of each shot. The calculated boundary is relatively insensitive to the pattern chosen for the plasma current filaments provided the filaments are not too close to the boundary. The currents in the filaments are determined by a weighted least-square fit. The quantity $\beta_p + (l_i/2)$ can be determined by taking integrals along the plasma boundary[12] or by generalization of the Shafranov formula[19] for the vertical field. Knowing the geometric form of the safety factor, the plasma internal inductance can then be estimated. β_p can of course be determined by subtraction.

Besides the filament model, there are alternative approaches to rapid plasma shape diagnostics.

Lee and Peng [20] have expanded the poloidal flux function in terms of Legendre functions. The coefficients are determined by fitting with the magnetic measurement. The plasma current density is decomposed into multipole moments. Since the solution is a infinite series, it has to be truncated for computation. It should

provide reasonable good fitting under different noise levels. However, the optimum value of the number of terms kept is not known *a priori*.

Hakkarainen [21] developed a fast algorithm to reconstruct the vacuum flux surfaces from magnetic diagnostics using the Vector Green's theorem and Fourier transform. The method requires no knowledge of the plasma current distribution. However, the high currents in the PF coils require special caution. The PF coils are designed to provide plasma shaping and positioning control in addition to significant Ohmic drive. The currents inside these PF coils are in the order of MA-turns and the generated magnetic field will be overwhelming. Before the discrete Fourier transformation is applied to the magnetic signals, caution should be taken on the high harmonic content in the measurement from those probes which are very close to the PF coils. To eliminate these high harmonic content, the contribution of the PF coils to the magnetic measurement are subtracted. All computations related to the Green's functions and the experimental geometry are pre-calculated. This method has potential for a practical algorithms in machines where the magnetic diagnostics are arranged on a wall conforming to the plasma shape. For C-Mod, this is not the case, the high spatial harmonics required to describe the diagnostic surface introduce errors which are difficult to deal with in a consistent way.

2.3 Analytic Solution of the Equilibrium Calculation

Accurate solution of the Grad-Shafranov equation is crucial to both the equilibrium calculation and the flux reconstruction. If there exist some analytic solutions that can be applied in experimental plasma, it will allow for a quantitative description of the interaction of the plasma current with the external maintaining fields. Instead of taking this task as a numerical analysis, some researchers tried to solve it analytically.

ically, even though the mathematical functions involved are extremely complicated.

In the case of equilibrium calculation, the current profile, $J_\phi(\psi)$, is known. As boundary conditions for this differential equation, the poloidal flux function is prescribed to vanish at infinity and along the toroidal axis of the tokamak. These are Dirichlet boundary conditions, which, for an elliptic partial differential equation, lead to a well-posed problem. Several researchers have investigated different forms of analytic solutions.

Zakharov and Shafranov [22] have studied the equilibrium of a straight plasma column with elliptical cross-section and a flat current distribution. They also investigated the exact solution for a toroidal plasma with circular cross-section and a quasiuniform current density.

Nevertheless, very little can be done on this problem analytically. The systematic analysis of the properties of an axisymmetric system with noncircular cross-sections is still very incomplete. Only simple plasma shapes, like the ellipse, and/or quasi-uniform current density distributions have been studied[23].

Mazzucato[24] has derived a class of analytic solutions of the ideal MHD equations of equilibrium of an axisymmetric configuration using the generalized hypergeometric series solutions. He assumed:

$$p(\psi) \equiv c_0 + c_1\psi \quad (2.3)$$

$$F^2(\psi) \equiv c_2 + c_3\psi + c_4\psi^2 \quad (2.4)$$

where c_i 's are constants and $c_4 \neq 0$.

Scheffel [25] has solved analytically the Grad-Shafranov equation for a toroidal, axisymmetric plasma. Exact solution are expressed in terms of confluent hypergeometric functions. He had assumed a parabolic pressure profile.

$$p(\psi) \equiv a + b\psi + c\psi^2 \quad (2.5)$$

$$F^2(\psi) \equiv d + e\psi + f\psi^2 \quad (2.6)$$

where a , b , c , d , e and f are arbitrary constants.

The Grad-Shafranov equation can be solved analytically in some limiting cases, and given some assumption. While these solutions tend to be unphysical, they are useful to develop intuition and to test numerical calculations.

Analytical solutions give little information for real systems where the external coil currents must be specified. Also, they are only of limited value in detailed tokamak design. Thus, this kind of problem has to be solved by numerical calculations.

2.4 Free Boundary Equilibrium Solver

In the past, there were different tokamak designs aiming at different physics issues. Also, there were different numerical methods in solving the plasma equilibrium problem. Free boundary equilibrium solvers are useful in tokamak design and plasma control. By setting different PF currents, the computer generated plasma will provide a lot of important information about the equilibrium and stability of the confinement. The control of a plasma equilibrium is extremely important in tokamak experiment.

However, for an arbitrary set of PF coil currents, there is no guarantee of plasma equilibrium. On the other hand, for a given plasma equilibrium, the solution of PF coil currents may not be unique.

Furthermore, the vertical axisymmetrical instabilities which are intrinsically be associated with any vertically elongated plasma can enter the numerical model and present a challenge to the free boundary approach.

McClain and Brown [26] developed a computer code called GAQ to find and analyze axisymmetric MHD plasma equilibria of the General Atomic experimental devices - Doublet IIA and Doublet III. The magnetic flux on or near the external conductors and the plasma current profile are known. To solve the linear problem of inverting Δ^* , a combination of the Green's function method and the Buneman's method [27] is adopted. Owing to the nonlinear nature of the Grad-Shafranov equation, an iterative scheme is used to determine the plasma current distribution. Numerous examples of nonconvergence and multiple solutions have been found. Four different types of initial guess of current density profile are available and they may converge to different equilibria. The bifurcation properties were studied by Helton and Wang[28].

The Tokamak Simulation Code(TSC) [29] is a computer program that assumes a time-dependent model combining MHD and transport. It is a powerful tool to analyse a plasma discharge and was used to simulate the formation of a bean-shaped plasma in the Princeton Beta Experiment(PBX). This method evolves a plasma through the entire shot, including full resistive effects of the vessel wall and and PF coils, while running self-consistent transport calculations. However, due to the complexity and long execution time, it is not suitable for routine magnetic flux reconstruction.

PEST [30] which stands for Princeton Equilibrium, Stability, and Transport package was developed in Princeton Plasma Physics Laboratory. It was designed to study the MHD spectrum of the perturbations from plasma equilibrium. To reduce the number of mathematical terms, it adopts a natural coordinate system. In order to solve the equilibrium for asymmetric plasma, the code PEST was modified into another code called ASEQ[31] to handle up-down asymmetric plasma. Since both the plasma boundary and the current distribution $J_\phi(R, Z)$ are unknowns, iterative schemes and finite different methods are used to solved the elliptic nonlinear partial differential equation for the poloidal flux. The toroidal current distribution is given

by specifying the pressure and either the poloidal current or the safety factor profiles as functions of the poloidal flux. The matrices of the Green's functions are pre-calculated and stored. Although the position of the magnetic axis is prescribed on the mid-plane, it drifts as the iterations proceed. Feedback currents in some additional external coils solve this problem.

PEST 2 [32] is an upgrade version of PEST. A scalar form of the ideal MHD energy principle is found to be more accurate than the vector form for determining the stability of an axisymmetric toroidal equilibrium. Special attention has paid to the inverse coordinate transformation system.

2.5 Fixed Boundary Equilibrium Solver

In practice, the plasma is shaped by external fields produced by currents flowing in the PF coils. Sometimes, in tokamak design, one prefers to prescribe the shape of the plasma boundary rather than the specific coil configuration. Hence, it is useful to investigate the properties of the plasma when the plasma boundary is fixed.

ASEQ[31] can be applied to cases in which the plasma boundary is specified. It has been used for the conceptual design of tokamak with non-circular cross section plasmas. It determines the PF currents which are necessary to support an equilibrium with a given shape.

PEST(J-solver)[33] is another version of PEST. It considers both fixed boundary and free boundary problem. Instead of modelling the toroidal current density, the average along magnetic field lines of the the component of plasma current density which is parallel to the magnetic field is written in a functional form with two profile parameters. The successive over-relaxation method is applied to solve the finite difference equations. The method is useful for obtaining equilibria to use in tokamak stability and transport calculations.

Lao, et al [34, 35] has developed a semi-analytical approach to solve a few equations of the moments of the Grad-Shafranov equation using variational methods. The internal flux surfaces are obtained by solving a few ordinary differential equations, which are moments of the Grad-Shafranov equation. Assuming up-down symmetry of the plasma, the flux surface coordinates are expanded in Fourier series. A few terms (about 3) in the Fourier series are generally sufficient to describe many plasma equilibria, including those for high-beta, strongly D-shaped plasmas. This method has been applied to the Impurity Study Experiment (ISX-B) and the Engineering Test Facility (ETF)/International Tokamak Reactor (INTOR) geometries. The results agree well with another numerical code RSTEQ[36]. The main advantage of the variational moment method is that it reduces the computational time without sacrificing accuracy.

However, this moment method requires the driving functions such as $p(\rho)$ and $I(\rho)$ as well as the outermost flux surface expanded in Fourier series to be specified in particular forms.

2.6 Free Boundary Flux Reconstruction

Blum, et al[37, 38] have formulated the free boundary problem in variational form and applied the finite-element method to solve the Grad-Shafranov equation. The non-linearities of the plasma current density and the free boundary are handled by Picard iterations. The plasma diffusion is coupled into the code and provides the Grad-Shafranov equation with the plasma pressure and toroidal field profiles. This code is applied to analyze the magnetic measurements of non-circular limited plasma in JET and TFR equilibrium configurations.

Luxon and Brown [39] determined the plasma shape, the inductance and the pressure for non-circular cross-section plasma from the external magnetic measure-

ments of the Doublet-III tokamak. The current profile was represented by choosing a particular function characterized by three free parameters. Their calculation starts with a crude estimate of the plasma shape, using iterative numerical techniques with a code called GAQ[26], the free-boundary Grad-Shafranov equation was solved self-consistently to search for the best fitted current density profile. Least squares minimization techniques were introduced to compare the experimental data with the calculated equilibrium. This method treats the magnetostatic and force balance aspects of the problem in a self-consistent manner. However, the computer time spent on the iterative process was very significant. For sufficiently elongated plasma, it was shown statistically that the average poloidal beta can be determined independent of the plasma internal inductance.

Lao et al [18] have derived the integral relations for the average poloidal beta and the plasma internal inductance when the diamagnetic flux measurement is known[40]. From external magnetic field measurements, for elongated plasma $\bar{\beta}_p$ and l_i can be separately determined and for nearly circular plasma only the sum $\bar{\beta}_p + l_i/2$ can be determined. The volume-dependent parameters involved depend only weakly on the actual current density distribution. The plasma current distribution is approximated by using a few filament currents. The determination of the plasma shape and the boundary magnetic field from the external magnetic measurement is similar to those developed by Swain and Neilson[15]. However, the external coil currents are taken as unknown and placed into the minimization procedure. This method was shown to be able to accurately determine most of the limited and diverted plasma shapes produced in Doublet III. However, the method becomes inaccurate for large-size plasmas which simultaneously touch the top, the inner limiter and the outer limiter, and have flat current profile.

Another method called EFIT[41] was shown to be efficient to reconstruct the current profile parameters, the plasma shape, and a current profile. It based on fast Buneman's method[27] and a Picard iteration approach[42] which approximately

conserve the external magnetic measurements. The two free functions $P'(\psi)$ and $FF'(\psi)$ are modelled as polynomials in ψ , with unknown linear coefficients. The total plasma current, the average poloidal beta, the plasma internal inductance and the axial safety factor are the imposed constraints. This method was applied to reconstruct the current profiles and plasma shapes in ohmically and auxilarily heated Doublet III plasmas. The $\bar{\beta}_p$ and l_i can be determined separately for non-circular plasma from external magnetic measurements alone. For circular plasma, an additional diamagnetic measurement is required. The internal reconstructed magnetic surfaces depend more strongly on the value of q_0 imposed than on the particular form of the current parametrization used. Neither the fine structure of the current distribution nor the derivatives of the internal magnetic surfaces can be determined. Also, the fast Buneman's method has set certain restrictions on the choice of grid size.

Braams et al [43] have demonstrated the method of function parametrization in the context of controlled fusion research. This method was originally developed in high energy physics by H. Wind[44]. It relies on a statistical analysis of a large database of simulated experiments to obtain a functional representation for intrinsic physical parameters of a system in terms of the values of the measurements. This method was employed on the ASDEX experiment. It rapidly determines the characteristic equilibrium parameters. The simulated measurements were generated by using the Garching equilibrium code[45]. The relevant measurements consist of poloidal field and flux measurements, the external coil current and the total plasma current.

Lister and Schnurrenberger[46] have determined some selected equilibrium parameters from the magnetic signals from the tangential field probes and flux-loops. The non-linear mapping is performed by a particular configuration of Neural Network known as the multi-layer perceptron. This technique was applied to the DIII-D tokamak single-null diverted plasmas. The selected equilibrium parameters include

plasma shape parameters, current centroid and other parameters that are important to the dynamical plasma control. No information can be deduced for the internal magnetic flux shape.

2.7 Fixed Boundary Flux Reconstruction

If the plasma boundary, the total plasma current and the toroidal vacuum field are given, the internal magnetic flux geometry can be reconstructed by matching the external magnetic measurements. The toroidal current profile parameters can be determined by running iteratively a fixed boundary equilibrium solver. A fast equilibrium solver can be used in a non-linear optimization code to solve the inverse problem.

Haney[47] has developed a fixed boundary equilibrium solver. The Grad-Shafranov equation is solved by minimizing a Lagrangian. The use of a variational method permits the Grad-Shafranov equation to be solved by reducing the problem of solving the 2D non-linear partial differential equation to the problem of minimizing a function of a few variables. This high speed algorithm approximately solves the plasma equilibrium given a parameterization of the plasma boundary and the current profile (p' and FF' functions). This method will be illustrated in next chapter.

2.8 Summary and Conclusions of this chapter

As mentioned before, the solution of the vacuum field can be obtained by various fast algorithms[15, 21]. Hence, the flux reconstruction can be separated into two parts, one for vacuum region, the other for plasma region. Combining a vacuum flux solver and a fast fixed boundary code with a non-linear optimization algorithm

SOLVER	INPUT	OUTPUT
Vacuum flux solver	magnetic measurements	separatrix
Interface	separatrix	shape parameters
Fixed boundary solver	shape parameters	equilibrium

Table 2.1: The inputs and outputs of different solvers.

yields a flux reconstruction package.

In the vacuum region, the magnetic flux can be solved by treating the plasma as a few current filaments[15]. This is basically a magnetostatic problem. Once the magnetic field in the vacuum region is evaluated, the plasma boundary can be determined by tracing the flux surface that touches the limiter or the x-point. The remaining part of the magnetic flux reconstruction is in the plasma region. That is, the force balance of the magnetic and kinetic pressures and the optimization of the current profile. This is a magnetohydrodynamic equilibrium problem.

In order to solve the Grad-Shafranov equation, the free functions $p'(\psi)$ and $FF'(\psi)$ are written as functions of some profile parameters, so that the shape of these functions can be controlled.

Here, we treat the current profile parameters as unknowns. The goal is to reconstruct the internal magnetic flux surfaces of a tokamak plasma and the toroidal current density profile from the external magnetic measurements. This is a classic problem of inverse equilibrium determination.

Fig. 2.2 shows that different flux solvers are applied to different regions. Different mathematical methods are involved since the physical conditions are different in each region. Table 2.1 summarizes the inputs and outputs of those solvers. We will discuss the formulation and application of all the solvers in the coming chapters.

Figure 2.2: The flux solvers in different regions.

Chapter 3

The Variational Method and the Current Profile

3.1 The Variational Method

For a fixed plasma boundary, the internal magnetic flux can be solved without taking the external PF coils into account. The physical situation can be thought of as the plasma being enclosed by a superconducting shell. There is no transport of energy or particles. The plasma is in MHD equilibrium and the system under consideration is non-dissipative. With a known current profile, the purpose of the fixed boundary equilibrium solver is to determine the internal flux through solving the Grad-Shafranov equation. This problem can be treated by applying the finite-difference method, but usually it takes very considerable computer execution time.

The variational method [48, 49, 50] has played an important role in the development of both classical and quantum theoretical physics. Moreover, the energy concept in the variational method helps physicists to understand more about the system under consideration. In Ideal MHD stability studies, the variational formulation is crucial for finding the normal-mode eigenvalue[51] of instabilities.

In order to formulate this non-dissipative system for the variational method, the Grad-Shafranov equation is written as a self-adjoint operator [52] on the flux function.

$$\mathbf{H}\psi = 0 \quad (3.1)$$

From the self-adjointness of the operator,

$$\int \phi \mathbf{H}\psi dv = \int \psi \mathbf{H}\phi dv \quad (3.2)$$

where ψ and ϕ are real functions. The integral is taken over the plasma volume.

A Lagrangian function and a set of trial functions have to be constructed. The trial functions represent different possible physical forms of the plasma with a given boundary. Obviously, the more degree of freedom these functions have, the closer to reality the calculated flux surfaces will be. The Lagrangian is a function of these trial functions and it can be perturbed by varying those variables in the trial function.

$$\mathbf{L} = \int \psi \mathbf{H}\psi dv \quad (3.3)$$

When the Lagrangian $\mathbf{L}$ is perturbed by those virtual displacements and it becomes stationary at a certain point in the multi-variable co-ordinate, it will generate a set of "Euler equations". Here the displacement of the trial function is deemed to first order.

$$\delta \mathbf{L} = \int \delta \psi \mathbf{H}\psi dv + \int \psi \mathbf{H}\delta \psi dv \quad (3.4)$$

$$= 2 \int \delta \psi \mathbf{H}\psi dv \quad (3.5)$$

When $\mathbf{L}$ is stationary for arbitrary $\delta \psi$, it may mean a minimum, a maximum or a saddle point. The non-linearity of the operator $\mathbf{H}$ causes the unpredictability.

The Lagrangian is constructed in such a way that the "Euler equation" generated from the variation of the Lagrangian will be equivalent to the Grad-Shafranov equation.

When δL vanishes for arbitrary $\delta\psi$, the Euler equation is the same as Eq. 3.1 to the first order.

Hence, the use of a variational method permits the Grad-Shafranov equation to be solved by reducing the problem of solving the 2D non-linear partial differential equation to the problem of minimizing a function of several variables.

3.2 The Variational Method: General Remarks

The might of the variation method lies on its extraordinary speed. However, the accuracy of its solution is a mystery that needs to be explored.

Generally speaking, the variational method is much faster in terms of cpu time than the finite differencing technique. In our case, the problem of finding 2D equilibria is reduced from solving a non-linear partial differential equation to minimizing a function of a few variables. The tradeoff is that solutions obtained are approximate and depend on the choice of the trial functions.

Although the variational method is an approximation, it is good for developing a physical model for complex problems and it often provides satisfactory approximate solution. It can be a reasonable accurate method for the estimation of the global quantities.

The trial function of variational method is only valid to the first order and the functional is not unique, but the functional will yield exactly the Grad-Shafranov equation as its Euler equation.

A non-linear problem, in general, may not have any solution, or may have more

than one solution. However, the solution depends continuously on the adjustable parameters. Theoretically, infinite sets of trial functions and infinite number of adjustable parameters will provide the exact solution. Given a finite set of trial functions, the Lagrangian is perturbed by varying the adjustable parameters. Then the role of the trial function is to represent or approximate the solution of the partial differential equation.

Obviously, if finite number of trial functions and finite number of adjustable parameters are employed, some of the detailed information of the physical system may be lost. For example, the choice of a simple inverse coordinate system may neglect the higher harmonics of the flux surfaces. Hence, the designer of the trial functions has to decide what the major physical nature of the problem is.

In the preliminary survey or in the absence of any reliable information, One has to guess the form of the trial functions. Good guesses come from good insight. Good insight is based on intuition, experience, and understanding of the complex problem. In other words, the accuracy of the variational method is limited by the descriptive power of the trial functions which rely on the designer's knowledge of the solution. In general, the convergence of the Lagrangian minimization is not guaranteed, however, the accuracy of the variational method will be improved progressively as the understanding of the problem deepened.

3.3 The formulation of the variational equilibrium solver

The variational technique has been shown to be a fast method to construct an approximate solution to the Grad-Shafranov equation [47].

The derivation of this method starts from the rewriting of the Grad-Shafranov equation (Eq. 1.1) in terms of a normalized flux:

$$\psi_0 \Delta^* \tilde{\psi} = -\mu_0 R J_\phi(R, \tilde{\psi}) \quad (3.6)$$

and

$$\tilde{\psi} \equiv \frac{\psi}{\psi_0} \quad (3.7)$$

where ψ_0 is the flux at the magnetic axis.

For this fixed boundary problem, the plasma shape is given as a curvilinear contour described by several shape parameters. Since some of the physical quantities are constants of flux surfaces, it is wise to choose a coordinate system which coincides with the plasma boundary as well as the internal flux surfaces.

From magnetohydrodynamics, there are physical quantities such as plasma pressure and safety factor that are functions of the magnetic flux. These quantities are constant and have distinct values on each of the nested magnetic flux surfaces. In Fig. 3.1, an inverse coordinate is plotted on the cross-section of the plasma. The variable ρ serves as a radial parameter and a label to mark each flux surface, while the variable μ acts like an angular parameter. The surface elements of both coordinate systems can be related through the Jacobian $\mathcal{J}$.

$$dRdZ = \mathcal{J} d\rho d\mu \quad (3.8)$$

The Jacobian is defined as:

$$\mathcal{J} = \begin{vmatrix} R_\rho & R_\mu \\ Z_\rho & Z_\mu \end{vmatrix} = R_\rho Z_\mu - R_\mu Z_\rho \quad (3.9)$$

where the subscription means partial differentiation.

The relations between the derivatives in cylindrical coordinates and inverse coordinates can be simplified by the chain rule:

$$\frac{\partial}{\partial R} = \frac{Z_\mu}{\mathcal{J}} \frac{\partial}{\partial \rho} - \frac{Z_\rho}{\mathcal{J}} \frac{\partial}{\partial \mu} \quad (3.10)$$

Figure 3.1: The contours of constant ρ and μ for a typical plasma

$$\frac{\partial}{\partial Z} = -\frac{R_\mu}{\mathcal{J}} \frac{\partial}{\partial \rho} + \frac{R_\rho}{\mathcal{J}} \frac{\partial}{\partial \mu} \quad (3.11)$$

As the range of variation of the ψ function is not known before the minimization of the Lagrangian, it is reasonable to assume that ψ_0 is an unknown in the process of numerical calculation. $\tilde{\psi}$ varies from 1 to 0.

The Grad-Shafranov equation can be rewritten in terms of the inverse coordinate:

$$\psi_0 \left[\frac{Z_\mu}{R} \frac{\partial \tilde{\psi}}{\partial R} - \frac{R_\mu}{R} \frac{\partial \tilde{\psi}}{\partial Z} \right]_\rho - \psi_0 \left[\frac{Z_\rho}{R} \frac{\partial \tilde{\psi}}{\partial R} - \frac{R_\rho}{R} \frac{\partial \tilde{\psi}}{\partial Z} \right]_\mu = -\mu_0 \mathcal{J} J_\phi(\rho, \mu, \tilde{\psi}) \quad (3.12)$$

The free functions can be written in a form which depends only on the shape of the flux rather than its magnitude.

$$J_\phi(R, \tilde{\psi}) = -\frac{C}{\mu_0} \left[b_p \frac{R}{R_0} h'_p(\tilde{\psi}) + (1 - b_p) \frac{R_0}{R} h'_f(\tilde{\psi}) \right], \quad (3.13)$$

where

$$h'_{p,f} = \frac{e^{-\alpha_{p,f}(1-\tilde{\psi})} - e^{-\alpha_{p,f}}}{1 - e^{-\alpha_{p,f}}} \quad (3.14)$$

Eq. 3.14 is known as Strickler's function and is only one example of representing the current profiles [13].

The quantities β_p and l_i are important parameters consisting physical and engineering implications, representing ratio of kinetic to magnetic pressure and the dimensionless internal inductance, respectively. Their definitions are:

$$l_i = \frac{\int B_p^2 dV}{\bar{B}_p^2 V_p} \quad (3.15)$$

and

$$\beta_p = \frac{2\mu_0 \langle p \rangle}{\bar{B}_p^2} \quad (3.16)$$

Here $\langle p \rangle$ is the volume averaged plasma pressure and $\bar{B}_p$ is the poloidal line averaged B_p on the plasma surface,

$$\langle p \rangle = \frac{\int p dV}{V_p} \quad (3.17)$$

$$\bar{B}_p = \frac{\oint B_p dl}{\oint dl}. \quad (3.18)$$

Hence, the ratio of l_i and β_p is:

$$\frac{l_i}{\beta_p} = \frac{\int B_p^2 dV}{2\mu_0 \int p dV} \quad (3.19)$$

Although different definitions for l_i and β_p can be used, however, the ratio is usually the same, since the differences are generally in the definition of $\bar{B}_p$.

Given the plasma shape and the current profile parameters (p' and FF' function), the Grad-Shafranov equation can be solved by minimizing the Lagrangian

density $\tilde{\mathcal{L}}$ with respect to these variational parameters. This gives an approximate solution to the Grad-Shafranov equation.

Note that the flux function $\tilde{\psi}$ is independent of μ . The calculation of Lagrangian is faster if it is performed on an inverse coordinate such as shown in Fig. 3.1.

$$\tilde{\mathcal{L}} = \int_0^{2\pi} \int_0^1 \tilde{L} d\rho d\mu \quad (3.20)$$

where the integration is over the plasma cross-section and $\tilde{L}$ is the Lagrangian density.

$$\tilde{L} = \psi_0 \frac{R_\mu^2 + Z_\mu^2}{R\mathcal{J}} [\tilde{\psi}'(\rho)]^2 + 2C\mathcal{J} \left[b_p \frac{R}{R_0} h_p(\tilde{\psi}) + (1 - b_p) \frac{R_0}{R} h_f(\tilde{\psi}) \right] \quad (3.21)$$

The Lagrangian can be varied with respect to $\tilde{\psi}$ by letting $\tilde{\psi} \rightarrow \tilde{\psi} + \delta\tilde{\psi}$. This variation in $\tilde{\psi}$ produces a corresponding variation in $\tilde{\mathcal{L}}$ of the form

$$\delta\tilde{\mathcal{L}} = -2 \int_0^{2\pi} \int_0^1 \tilde{I} \delta\tilde{\psi} d\rho d\mu \quad (3.22)$$

where

$$\tilde{I} = \psi_0 \left[\frac{Z_\mu}{R} \frac{\partial \tilde{\psi}}{\partial R} - \frac{R_\mu}{R} \frac{\partial \tilde{\psi}}{\partial Z} \right]_\rho - \psi_0 \left[\frac{Z_\rho}{R} \frac{\partial \tilde{\psi}}{\partial R} - \frac{R_\rho}{R} \frac{\partial \tilde{\psi}}{\partial Z} \right]_\mu - C\mathcal{J} \left[b_p \frac{R}{R_0} h'_p(\tilde{\psi}) + (1 - b_p) \frac{R_0}{R} h'_f(\tilde{\psi}) \right] \quad (3.23)$$

When the Lagrangian is stationary, $\tilde{I}$ vanishes. As Eq. 3.23 is set to zero, it gives exactly the the Grad-Shafranov equation in the inverse coordinate same as Eq. 3.12.

Although Eq. 3.12 and Eq. 3.23 are different from those in the original literature[47], one can easily verify that the above equations are correct. After consulting the author, it is found that the differences arise from typographical errors.

In the minimization of the Lagrangian, there are two unknown constants, ψ_0 and C , that must be evaluated by some additional conditions. The selection of these conditions is based on the physical nature of the problem and is independent of the variational formulation. These conditions can be interpreted as constraints imposed on the variational method. C can be related to the total plasma current by taking the surface integral of Eq. 3.13, and ψ_0 can be evaluated by an argument based on the magnetic energy.

The plasma current can be perceived as a bunch of filamentary current loops with current flowing parallel to each other. The energy associated with the interaction of the current of one loop and the magnetic field generated by the other can be expressed [54] as a volume integral over the plasma volume as follows:

$$W = \frac{1}{2} \int_{\tau} \mathbf{A} \cdot \mathbf{J}_{\phi} dV \quad (3.24)$$

where

$$\mathbf{A} = \frac{\psi}{R} \hat{\mathbf{e}}_{\phi}$$

and

$$\mathbf{B}_p = \nabla \times \mathbf{A}$$

Since

$$\mathbf{J}_{\phi} = \frac{1}{\mu_0} \nabla \times \mathbf{B}_p$$

we can rewrite the energy as follows:

$$W = \frac{1}{2\mu_0} \int_{\tau} \mathbf{A} \cdot (\nabla \times \mathbf{B}_p) dV \quad (3.25)$$

$$= \frac{1}{2\mu_0} \int_{\tau} \mathbf{B}_p \cdot (\nabla \times \mathbf{A}) dV - \frac{1}{2\mu_0} \int_{\tau} \nabla \cdot (\mathbf{A} \times \mathbf{B}_p) dV \quad (3.26)$$

$$= \frac{1}{2\mu_0} \int_{\tau} \mathbf{B}_p \cdot (\nabla \times \mathbf{A}) dV - \frac{1}{2\mu_0} \oint (\mathbf{A} \times \mathbf{B}_p) \cdot d\mathbf{S} \quad (3.27)$$

The second term on the right hand side of the above equation will vanish if the surface of the integral taken is expanded to infinity, since the field of a current loop drops as a magnetic dipole field.

Hence,

$$W = \int_{\tau} \frac{B_p^2}{2\mu_0} dV \quad (3.28)$$

The above equation shows that the energy is stored as magnetic energy in the form of magnetic field. Thus, the magnetic energy can be written in two equivalent forms as follows:

$$\int_{\tau} \frac{B_p^2}{2\mu_0} dV = \frac{1}{2} \int_{\tau} \mathbf{A} \cdot \mathbf{J}_{\phi} dV \quad (3.29)$$

Since

$$B_p^2 = \frac{\psi_0^2}{R^2} (\nabla \tilde{\psi})^2 \quad (3.30)$$

and the magnitude of $\mathbf{J}_{\phi}$ is defined by Eq. 3.13, a relation of ψ_0 can be found from Eq. 3.29.

Hence, ψ_0 and C can be evaluated by the following simultaneous equations.

$$\psi_0 = -C \frac{\int_0^1 \int_0^{2\pi} [b_p \frac{R}{R_0} h'_p(\tilde{\psi}) + (1 - b_p) \frac{R_0}{R} h'_f(\tilde{\psi})] \tilde{\psi} \mathcal{J} d\mu d\rho}{\int_0^1 \int_0^{2\pi} \frac{1}{R} (\nabla \tilde{\psi})^2 \mathcal{J} d\mu d\rho}, \quad (3.31)$$

$$\frac{1}{C} = \frac{1}{\mu_0 I_p} \int_0^1 \int_0^{2\pi} \left[b_p \frac{R}{R_0} h'_p(\tilde{\psi}) + (1 - b_p) \frac{R_0}{R} h'_f(\tilde{\psi}) \right] \mathcal{J} d\mu d\rho \quad (3.32)$$

Actually, Eq. 3.31 can be found by an alternative method. If Eq. 3.6 is multiplied by $\frac{\tilde{\psi}}{R}$ on both sides, and integrated over the plasma cross-section. Hence, Eq. 3.31 is the poloidal field energy, and it is also a first moment of the Grad-

Shafranov equation. Eq. 3.32 is the normalization for the toroidal plasma current, since I_p is an input.

3.4 The Implementation of the variational equilibrium solver

In order to develop the equilibrium solver, the current density profile and the trial function have to be selected. To evaluate the variational parameters and the values of ψ_0 and C , special numerical algorithms and procedures are required for the minimization of the Lagrangian. After the equilibrium solver is developed, its speed is tested on different computational platforms. The accuracy will be tested by comparing with MHD theory and other equilibrium solver.

As mentioned before, Eq. 3.14 is known as Strickler's function and is chosen to describe the current density profile as shown in Eq. 3.13.

The Strickler function is a convenient choice because the function has only one parameter but scans a wide range. Fig. 3.2 shows that it scans a wide range of current profiles as the parameters α_p or α_f is varied.

Nevertheless there are some limitations of the Strickler's function. For example, the function is not defined when the profile parameter is zero. When the profile parameter drops to about -80.0, the computed value is not reliable any more as the exponential terms grow beyond the numerical limit of the computer system.

A set of trial functions [53] which represent the poloidal flux profile and positions can be written in terms of several variational parameters.

$$\psi = \psi_0[1 - \nu\rho^2 - (1 - \nu)\rho^4] \quad (3.33)$$

$$R(\rho, \mu) = R_0 + aS(\rho) + aM(\rho, \mu) \quad (3.34)$$

Figure 3.2: The current profile parameter.

ν	control the radial shape of ψ
σ	control the shift of the flux surfaces
κ_0	control the central elongation
η	control the triangularity

Table 3.1: Variational parameters

$$Z(\rho, \mu) = aK(\rho, \mu) \quad (3.35)$$

$$S(\rho) = \sigma(1 - \rho^2) \quad (3.36)$$

$$M(\rho, \mu) = \rho \cos \left\{ \mu + D_a \rho^2 [\eta + (1 - \eta) \rho^2] \sin \mu \right\} \quad (3.37)$$

$$D_a = \sin^{-1}(\delta_a) \quad (3.38)$$

$$K(\rho, \mu) = \rho [\kappa_0 + (\kappa_s - \kappa_0) \rho^4] \sin \mu \quad (3.39)$$

The variational parameters are ν, σ, κ_0 and η . Table 3.1 summarizes their physical meanings. They are all dimensionless variables and of order 1 or 0.

The trial functions mentioned before describe limited plasma with up-down symmetry. If this method is viable for limited plasma, it may be extended to diverted plasma or up-down asymmetric plasma by rewriting the trial functions.

The evaluation of the Lagrangian involves a lot of surface integrals. Since the functional form of the integrand is known, it is possible to determine the points where the integrand is to be evaluated such that the accuracy of the numerical integration is increased and the computation time is minimized.

The function of Gaussian quadrature[56] is to determine these optimal points. The Gaussian quadrature allows us to choose the weighting coefficients and the location of the abscissas freely. They will be no longer equal space. Gaussian integration is implemented in the radial direction and a simple finite difference

scheme is used in the angular direction. In practice, eight points in the Gaussian integration results in accurate answers.

The Strickler's function we choose has certain limitation under numerical consideration. Although the Strickler function scans a wide range of current profile, it is not defined when the profile parameter is zero. Mathematically, the profile will give another form when it trends to zero.

$$\lim_{\alpha \rightarrow 0} h'_{p,f} = \tilde{\psi}$$

$$\lim_{\alpha \rightarrow 0} h_{p,f} = \frac{\tilde{\psi}^2}{2}$$

When the profile parameter drops below -80.0, the exponential terms in the Strickler function will grow beyond the numerical limit of the computer system.

On the other hand, the eight points selected by the Gauss-Legendre integration will be meaningless if the profile parameter increases beyond about 10.0. Judging from the physical situation of the plasma current profile, if the profile parameter lies between -60.0 to 10.0, it has already scanned the extrema of plausible profiles accessible in this family.

Moreover, the execution time can be further reduced by pre-calculating certain frequently used quantities and integrals, as well as choosing a fast minimization algorithm.

The minimization of the Lagrangian is accomplished by repeated one-dimensional minimization of the Lagrangian with respect to each of the variational parameters. In each iteration, the Lagrangian is minimized with to all the variational parameters but one at a time. During the one-dimensional minimization, the other variational parameters are kept fixed. The minimization of the Lagrangian with respect to the variational parameters will end if the relative difference of the Lagrangian between consecutive iteration is less than a certain tolerance.

The one-dimensional minimization is performed by applying the Brent's method[58]. This method needs only evaluations of the function to be minimized and hence calculation of the derivatives are not required. However, this algorithm has assumed a rigorous initial bracketing which is achieved by another algorithm. The initial bracketing[57] of the minimization can be attained by the combination of the parabolic fit and the golden section search.

Eq. 3.31 and Eq. 3.32 are the constraints in the minimization process. Initially, C is set to unity and ψ_0 is evaluated. The minimization of $\tilde{L}$ is performed. Afterward C is evaluated from Eq. 3.32 and ψ_0 is updated from Eq. 3.31. The minimization of $\tilde{L}$ is repeated according to the current values of C and ψ_0 until ψ_0 converges to sufficient accuracy.

The equilibrium solver are written in ANSI/C and tested on different platforms. Table 3.2 summarizes the CPU times for the minimization of a Lagrangian on different computation platforms. The plasma under consideration is describe in Fig. 1.2.

If we use another fixed boundary equilibrium solver called PEST to solve a similar plasma, the CPU time is about 70 seconds in CRAY/C machine.

This comparison shows that the strength of the variational equilibrium method lies in its great speed. This fixed boundary equilibrium solver has the potential of being a powerful tool for analyzing the magnetic measurements between plasma shots in modern tokamak experiments. It could be considered as a real-time equilibrium solver and to be included in the flux reconstruction routine or the plasma feedback control system. In principle, information from other diagnostics can be incorporated into the reconstruction.

Node Name	Computer Model	Compiler	CPU TIME (sec)
CRAY/A	Cray Y-MP C90/16256	cc *	0.051987
CRAY/C	Cray-2S/8128	cc *	0.115505
CRAY/F	Cray-2S/4128	cc *	0.116258
RADIO	VAXstation.4000.90	VAXC	0.24
ILM	SG IRIX System V.3	cc	0.27
ATHENA	DECstation.5000.25	gcc	0.648396
LEADER	VAXstation.4000.60	VAXC	0.78
REX	Sun/SPARCstation IPC	acc	1.31661
SCORCH	VAXstation 3100 M38	VAXC	2.45
ALCVAX	VAX8350	VAXC	8.47
NERUS	VAX8300	VAXC	10.48
CONAN	VAXstation 2000	VAXC	12.42

* For comparison, the automatic vectorization in CRAY/UNICOS is turned off by choosing the "cc -h vector0" option.

Table 3.2: The CPU time for the minimization of the Lagrangian

Figure 3.3: A high aspect ratio circular plasma.

3.5 Benchmark of the variational method

In order to study the convergence and the accuracy of this variational method, the Shafranov shift of a high aspect ratio circular plasma (Fig. 3.3) was computed and compared with the analytical results.

The Shafranov shift is the displacement of the magnetic axis with respect to the geometric center. It can be estimated by assuming ohmically heated plasma in the tokamak and high aspect ratio. The zero and first orders of the Grad-Shafranov equation are expanded. Freidberg[55] has shown that the Shafranov shift of the plasma is given by:

$$\Delta(0) = -\frac{\psi_1(0)}{\psi'_0(0)} \quad (3.40)$$

where

$$\psi_1(0) = B_{\theta 1}(0) \int_0^a \frac{dx}{x B_{\theta 1}^2(x)} \int_0^x \left[2\mu_0 y^2 \frac{dp_2(y)}{dy} - y B_{\theta 1}^2(y) \right] dy \quad (3.41)$$

$$\psi_0'(0) = R_0 B_{\theta}(0) \quad (3.42)$$

$$B_{\theta 1}(r) = \frac{1}{R_0} \frac{d\psi_0}{dr} \quad (3.43)$$

Suppose the plasma has high aspect ratio, constant current profile and parabolic pressure profile:

$$B_{\theta}(r) = \frac{\mu_0 I_p r}{2\pi a^2} \quad (3.44)$$

$$p(r) = p_0 \left[1 - \frac{r^2}{a^2} \right] \quad (3.45)$$

$$h'_{p,f} \simeq 1 \quad (3.46)$$

It can be shown that:

$$p_0 = \frac{C b_p I_p}{4\pi} \quad (3.47)$$

where C is the normalization constant of the plasma current.

The formula of the Shafranov shift $\Delta(0)$ in high aspect ratio plasma is then as follows:

$$\Delta(0) = \frac{a^2}{8R_0} \left[1 + \left(\frac{4\pi a^2}{\mu_0 I_p} \right) C b_p \right] \quad (3.48)$$

From Eq. 3.39, the computed Shafranov shift is the difference between $R(\rho = 0, \mu)$ and R_0 .

$$R(\rho = 0, \mu) = R_0 + a\sigma \quad (3.49)$$

b_p	Theoretical value	Numerical result	Error(%)
0.1	1.0871e-4	1.0795e-4	0.7
0.2	1.3987e-4	1.39110e-4	0.5
0.3	1.71024e-4	1.70263e-4	0.4
0.4	2.02184e-4	2.01414e-4	0.4
0.5	2.33345e-4	2.3256e-4	0.3
0.6	2.64509e-4	2.63696e-4	0.3
0.7	2.95672e-4	2.9482e-4	0.3
0.8	3.26837e-4	3.25932e-4	0.3
0.9	3.58000e-4	3.57034e-4	0.3

Table 3.3: Shafranov Shift (in meters)

Table 3.3 shows the theoretical values and the numerical values from the variational equilibrium of the Shafranov shifts of high aspect ratio circular plasma. The errors are found to be less than 1%. This shows the accuracy of the variational method for high aspect ratio circular plasma. Although the high aspect ratio circular plasma is convenient for analytical calculation, it is far from the highly shaped plasma in the modern tokamak experiments.

In order to test the convergence and accuracy of the variational equilibrium code when the plasma is highly shaped, it is compared to an exact equilibrium code called J-solver[33] which is a fixed boundary version of the PEST code developed in Princeton.

The comparison of the two codes is complicated by the fact that they use different inputs and profiles.

The pressure profile is described by:

$$p = p_0(1 - \hat{\psi}^{BETA})^{ALPHA} \quad (3.50)$$

where

p_0 is the pressure on axis in units of $\mu_0 B^2$, and

$$\hat{\psi} = \frac{\Psi - \Psi_{axis}}{\Psi_{lim} - \Psi_{axis}}$$

The shape of the pressure profile can be adjusted by two parameters, *ALPHA* and *BETA*.

The current profile is adjusted indirectly by the ohmic current j_{oh} defined as follows:

$$j_{oh} = \frac{\langle \mathbf{J} \cdot \mathbf{B} \rangle}{R_{axis} \langle \nabla \varphi \cdot \mathbf{B} \rangle} \quad (3.51)$$

$$j_{oh} = j_0(1 - \hat{\psi}^{BETAQ})^{ALPHAQ} \quad (3.52)$$

where

j_0 is the toroidal current on the magnetic axis.

The shape of the ohmic current profile can be adjusted by two parameters, *ALPHAQ* and *BETAQ*.

Some of the inputs of the J-solver code are different from the variational code. For example, the I_p is an input in the variational method while it is an output in J-solver code. In order to have a rigid comparison, the J-solver has to be run until it gives the same I_p .

By setting some parameters, the J-solver can produce up-down symmetric plasma with the same elongation and triangularity as that of the variational code. The comparison is done on the proposed highly shaped plasma (Fig. 1.2).

By meticulous and painful fine tuning, the two pressure profiles and the two j_{oh} profiles can be adjusted to produce the same profiles of the h'_p and h'_f . After

Parameter	PEST	Variational
$R_{axis}(m)$	0.64862	0.64873
$a\sigma(m)$	0.00392	0.00401
κ_0	1.5180	1.5180
l_i	0.8368	0.8038
β_p	0.10	0.09
$I_p(MA)$	3.7004	3.7*

Table 3.4: A comparison between the variational method and PEST.

we choose the current profile parameters, the maximum plasma pressure and safety factor on axis can be computed from the variational code. In order to get the values of $ALPHA$ and $BETA$ for the pressure profile and the values of $ALPHAQ$ and $BETAQ$ for the j_{oh} profile, we have matched them with the pressure profile and the toroidal current profile. The matching can be done through minimization of a multivariable function. Since there are so many differences between these two fixed boundary solvers, they are brought closer together to each other by adjusting the desired safety factor until the J-solver produces the accurate plasma current.

Table 3.4 provides a comparison on a particular plasma equilibrium between the variational method and the J-solver. The plasma current, I_p , is an input to the variational code, but an output from PEST. For these important parameters, we found that the variational code produced results quite close to the J-solver.

This proves that the variational equilibrium code is reliable for the high aspect ratio plasma as well as the proposed plasma. A detailed comparison of the variational code and the J-solver code will be done in the next chapter.

3.6 The accuracy of the poloidal field

The poloidal field calculated from the equilibrium solver is a very important physical quantity for both the study of the plasma equilibrium and engineering problem. It has been pointed out that the poloidal field calculated from the derivative of the flux function in this variational method was not accurate [47]. In this section, this issue is explored in further detail to see whether any improvement can be obtained.

The poloidal field can be calculated from the derivatives of the flux function as follows:

$$\mathbf{B}_p = \frac{\psi_0 \nabla \tilde{\psi} \times \mathbf{e}_\phi}{R} \quad (3.53)$$

The flux function is written in terms of variational parameters in an analytical form. Thus, we can deduce analytically the derivative of the flux function. Hence, this is a fast way to compute the poloidal field. Explicitly, the magnitude of the poloidal field can be written as following:

$$B_p = \frac{\sqrt{R_\mu^2 + Z_\mu^2}}{R\mathcal{J}} \psi_0 \tilde{\psi}_\rho \quad (3.54)$$

where $\tilde{\psi}_\rho$ is the partial derivative of $\tilde{\psi}$ with respect to ρ and $\mathcal{J}$ is the Jacobian.

According to Ampere's law, a line integral of poloidal field around the plasma boundary divided by the permeability of vacuum will give the total plasma current.

Here we apply Ampere's law to the plasma in Fig. 1.2. Since the total plasma current is an input of the variational equilibrium solver, we can check the accuracy of poloidal field calculated from Eq. 3.54 by taking its line integral around the plasma boundary.

We find that the calculated value changes with the number of angular grid points on the line integral. If we choose only 4 grid points on the line integral, the

Figure 3.4: The percentage error in the total plasma current

Figure 3.5: The length of the integration contour.

result of integration is obviously smaller since the plasma boundary is replaced by 4 straight lines. However, as the number of grid points increased, the calculated value converges to about 6%. In fact, the calculated value of the total plasma current is larger than the input one. Fig. 3.4 shows the percentage errors of the absolute values of the difference of the calculated value and the input value of the total plasma current of the plasma in Fig. 1.2.

There are two approaches in calculating the length of the integral contour:

$$\oint dl = \oint \sqrt{R_\mu^2 + Z_\mu^2} d\mu \quad (3.55)$$

where R_μ and Z_μ are the partial derivatives of R and Z with respect to μ . They are calculated from analytical equations.

$$\oint dl = \sum_i \sqrt{[R(i+1) - R(i)]^2 + [Z(i+1) - Z(i)]^2} \quad (3.56)$$

where $R(i)$ and $Z(i)$ are the R and Z coordinates of the points around the integration contour.

Moreover, the error of the calculated value may come from the poloidal field or the line integral. From the Fig. 3.5, the dotted line represents the length around the plasma calculated from Eq. 3.55. The solid line represents the contour length calculated from Eq. 3.56. Both methods approach the same values as the number of grid points around the plasma edge increases.

In order to check whether the error is in the line integral, a circular plasma is chosen (Fig. 3.6). Fig. 3.7 shows the difference of the calculated value and the input value of the line integral of a circular plasma boundary. The error in the contour length is basically zero as the number of the grid points rises above 100. Hence, it can be concluded that the poloidal field on the plasma edge is not consistent with the toroidal current density inside the plasma.

Figure 3.6: A circular plasma and the error of the plasma current

Figure 3.7: The error of contour length in the line integral

If we chose another option for the trial function in Eq. 3.37 [59] as follows:

$$M(\rho, \mu) = \rho \cos \{ \mu + D_a \rho [\eta + (1 - \eta) \rho^2] \sin \mu \} \quad (3.57)$$

It is found that, the error for the total plasma is slightly increased. That means, taking another trial function like the above equation does not decrease the error in the calculated poloidal field.

It seems that there are substantial errors in the calculated poloidal field from the derivative of the flux function.

The variational method provides a solution for the flux function that is, by construction, first order accurate in the trial function. However, this technique has no guarantee for the derivative of the flux function.

The accuracy problem arises from the fact that the poloidal field is calculated from the derivatives of the flux function. The variational method will not give an exact solution since only finite set of trial functions have been used. The inaccuracy of the flux function will be propagated or even amplified to the poloidal field.

To obtain an accurate calculation of the poloidal field, one has to apply the Biot-Savart law or more precisely, to integrate the current density with the Green's function over the plasma volume. Obviously, this integration method will take much longer cpu time.

3.7 Determination of current profile from poloidal field

In this section, we explore the possibility of matching the current profile parameters by comparing the poloidal field on the plasma boundary. Although we have shown that the poloidal field calculated from the derivative of the flux function has certain

error, we do not know how seriously this error will affect the matching of current profile parameters without further investigation.

The poloidal magnetic field on the plasma surface changes when the current profile parameters are changed. This provides a possible method to determine the current profile parameters and hence it could be used as a scheme for the magnetic flux reconstruction.

In order to test this idea, a parameter ϵ is defined as following:

$$\epsilon(b_p, \alpha) = \frac{\int_{S_p} [B_p(b_p, \alpha) - \hat{B}_p]^2 dS}{\int_{S_p} \hat{B}_p^2 dS}, \quad (3.58)$$

where $\hat{B}_p$ is the poloidal field on the plasma boundary of a reference equilibrium. This parameter measures the square of the differences between calculated B_p from the equilibrium solver and $\hat{B}_p$.

The reference equilibrium has the same input as shown in Fig. 1.2. $\hat{B}_p$ is the poloidal magnetic field calculated from the variational equilibrium solver for the particular current profile.

By changing the current profile parameters b_p and α , we can generate different plasma equilibria from the variational code. For each of these equilibria, ϵ can be calculated from the above equation. Fig. 3.9 shows a contour plot of the ϵ versus the current profile parameters.

The output ϵ surface is smooth. The true value is surrounded by a nested set of closed contours. This plot shows the existence of a minimum region.

If we interpret $\hat{B}_p$ as poloidal field at the plasma edge calculated from the vacuum solver, with this smooth surface and well defined minimum region, almost any minimization algorithm will help for the determination of the current profile.

At first glance, the matching of poloidal magnetic field on the plasma surface provides a neat method of identifying the plasma current profile. However, there

Figure 3.8: The smooth surface of the epsilon values

Figure 3.9: The contour of the ϵ versus the current profile parameters

Figure 3.10: The contour of the ϵ versus l_i and β_p

are at least two questions attached to this idea.

Firstly, is the poloidal field calculated from the vacuum solver accurate enough to provide a reliable reference? Secondly, is the poloidal field calculated from the variational code accurate enough for this purpose? If the answer of either of these questions is negative, then it will be hard to make a good match for the purpose of flux reconstruction.

In the vacuum region, there is no plasma current. In terms of mathematics, we are solving an elliptic partial differential equation, subject to Cauchy boundary-conditions. The error will increase as it move towards the plasma surface. The error of poloidal field on the plasma surface will be greater than those at the probe locations. Assume the experimental error of the magnetic measurements is about 1% at the probe locations. The error of the poloidal field near the plasma surface will be greater than 1%. On top of that, the accuracy of the poloidal field computed

from the variational equilibrium solver has been investigated in the previous section. It will be much worse than 1%. This makes the situation even worse. Since ϵ is related to the square of $\hat{B}_p$, the reference point will be anywhere inside a contour which is even bigger than the 10^{-4} contour of Fig. 3.9. Then the error of b_p , α , l_i and β_p will be very big.

In short, the power of this method is greatly limited by the experimental errors of the magnetic signals, the numerical error of solving the elliptic partial differential equation, subject to Cauchy boundary-conditions, and the accuracy of the poloidal field computed from the variational equilibrium solver.

3.8 The Application of the Virtual-Casing Principle

If we consider the contribution from the plasma current only, the magnetic flux and field at probe locations can be related to the integral of the plasma current density and Green's functions across the plasma cross-sectional area:

$$\Psi(R, Z) = \mu_0 \int \int dR' dZ' J_\phi(R', Z') \frac{(RR')^{1/2}}{2\pi} \left[\frac{(2 - k^2)K(k) - E(k)}{k} \right] \quad (3.59)$$

$$B_R(R, Z) = \mu_0 \int \int dR' dZ' J_\phi(R', Z') \frac{1}{2\pi R} \frac{(Z - Z')}{[(R + R')^2 + (Z - Z')^2]^{1/2}} \left\{ -K(k) + \left[\frac{R^2 + R'^2 + (Z - Z')^2}{(R - R')^2 + (Z - Z')^2} \right] E(k) \right\} \quad (3.60)$$

$$B_Z(R, Z) = \mu_0 \int \int dR' dZ' J_\phi(R', Z') \frac{1}{2\pi} \frac{1}{[(R + R')^2 + (Z - Z')^2]^{1/2}} \left\{ K(k) - \left[\frac{R^2 - R'^2 + (Z - Z')^2}{(R - R')^2 + (Z - Z')^2} \right] E(k) \right\} \quad (3.61)$$

where

$$k^2 = \frac{4RR'}{(R + R')^2 + (Z - Z')^2}$$

$K(k)$ and $E(k)$ are the complete elliptical integrals of the first and second kind respectively.

The above equations come from the Biot-Savart Law and involve integration of Green's functions over the plasma volume. The Green's functions are related to the probe locations and the flux function. The probe locations are fixed and the flux function is the output of the variational method.

Many of the numerical results in this thesis depend on the accuracy of the elliptic and Green's function for field and flux. The results of some tests of the algorithms for calculating these functions are collected in Appendix A.

Eqs. (3.59, 3.60, 3.61) provide a direct way to calculate the magnetic signals contributed from the plasma current. The drawback is that they involve the R and Z coordinates of all the internal flux surfaces and a large amount of Green's functions evaluation.

The variational method described before is a fixed boundary solver, the plasma boundary is defined by a set of shape parameters. For a given set of current profile parameters, the minimization of the Lagrangian will provide an approximate solution of the Grad-Shafranov equation. The current profile parameters are fixed in the process of the variational method. However, in flux reconstruction the current profile parameters are unknown. If we keep the total plasma current fixed and change only the current profile parameters, the variational method will produce another set of internal flux surfaces with the same plasma boundary. In other words, the R and Z coordinates of all the internal flux surfaces change whenever the current profile parameters changes. Hence, integrals involving only the points on the plasma boundary can be done by pre-calculating the Green's functions involving those points on the fixed plasma boundary. While integrals involving the

internal points have to re-calculate their Green's functions whenever the internal flux surfaces changes.

This simple fact, if wisely applied, may save a great deal of computation time.

In this section, a potentially fast flux reconstruction method is attempted by taking advantage of the above fact. The Virtual-Casing Principle [60] provides a very fast way to evaluate the plasma contribution to the magnetic signals. It involves magnetic Green's functions which refer to the coordinates of the plasma boundary only. If we know the shape of the plasma boundary, these Green's functions can be pre-calculated and stored.

The magnetic field of an equilibrium toroidal plasma is a superposition of the vacuum field produced by external sources and the self field produced by the plasma. The plasma is being held in equilibrium by the vacuum field. Instead of the vacuum field, let us assume the plasma is held by a closed superconducting sheath. If the superconducting sheath surrounds the equilibrium configuration such that it coincides with the outermost plasma flux surface, the plasma is separated from outside and is balanced by the induced surface current in the superconductor. Obviously, the field outside this sheath is zero. The only magnetic field sources are the plasma current and the surface current induced in the superconductor. Assuming that we know the poloidal field on the plasma surface B_s , the relation between the induced current and B_s can be found by Ampere's law. The induced current in the casing can be determined directly from the B_s . Outside the casing, the field generated by the induced current is the same as the self-field produced by the plasma current but with opposite sign. Inside the sheath, the field generated by the induced current is exactly the same as the vacuum field.

Using the Biot-Savart law, we have the following equations for the magnetic flux and field contributed by the plasma.

$$\Psi(R, Z) = \oint B_s \frac{(RR')^{\frac{1}{2}}}{k} [(2 - k^2)K(k) - 2E(k)] dl \quad (3.62)$$

$$B_R(R, Z) = \oint B_s \frac{1}{R} \frac{(Z - Z')}{[(R + R')^2 + (Z - Z')^2]^{1/2}} \left\{ -K(k) + \left[\frac{R^2 + R'^2 + (Z - Z')^2}{(R - R')^2 + (Z - Z')^2} \right] E(k) \right\} dl \quad (3.63)$$

$$B_Z(R, Z) = \oint B_s \frac{1}{[(R + R')^2 + (Z - Z')^2]^{1/2}} \left\{ K(k) - \left[\frac{R^2 - R'^2 + (Z - Z')^2}{(R - R')^2 + (Z - Z')^2} \right] E(k) \right\} dl \quad (3.64)$$

where

$$k^2 = \frac{4RR'}{(R + R')^2 + (Z - Z')^2}$$

B_s stands for the poloidal field on the plasma boundary. The line integrals are taken around the plasma edge. Here, R' and Z' are coordinates of the contour of integration while R and Z are coordinates outside the plasma.

Fig. 3.11 and Fig. 3.12 shows the values of the plasma contribution to the magnetic signals at different probe locations calculated from the Virtual-Casing Principle. B_s are calculated from Eq. 3.54. The configuration is the proposed highly shaped plasma (Fig. 1.2). Obviously, the signal calculated from the Virtual-Casing Principle (dotted lines) is different from the signal calculated from the volume integral of the toroidal current density (solid lines).

Judging from the small disagreement between these two methods, there may be a remedy to this discrepancy. If we expand the error of B_s in different orders, the higher order terms will drop quickly as the point under consideration moves outward to the probe locations. Hence, if we multiply B_s by a scaling factor so that it gives the correct total plasma current through Ampere's Law, we may have a chance to produce the correct signal at the probe location.

In Fig. 3.13 and Fig. 3.14, the solid lines are the differences between the

Figure 3.11: The signals at different B_p coils locations calculated from the Virtual-Casing Principle.

Figure 3.12: The signals at different flux loops locations calculated from the Virtual-Casing Principle.

Figure 3.13: The differences in B_p signals between the Casing method and the volume integration method

Figure 3.14: The differences in flux loops signals between the Casing method and the integration method

magnetic signals at different probe locations calculated from the Virtual-Casing Principle and the signals calculated from volume integral method (Eq. 3.61). The dotted lines are the differences between the corrected magnetic signals and the signals calculated from volume integral method. The corrected magnetic signals are the signals calculated from the Virtual-Casing Principle multiplied by a scaling factor.

In order to test whether the Virtual-Casing Principle can be applied to reconstruct the magnetic flux, a least square method is employed as following:

$$\chi^2(b_p, \alpha) = \sum_{m=1}^{Nb} \frac{[B_p(b_p, \alpha) - \hat{B}_p]^2}{\sigma_b^2} + \sum_{m=1}^{Nf} \frac{[\Psi(b_p, \alpha) - \hat{\Psi}]^2}{\sigma_f^2} \quad (3.65)$$

$$\sigma_b = 0.03 \quad (3.66)$$

$$\sigma_f = 0.03 \quad (3.67)$$

where Nb and Nf are the number of magnetic coils and flux loops respectively.

As before we set $\alpha = \alpha_f = \alpha_p$ for convenience. We have assumed 1% error of the average 3 Tesla signal from magnetic coil and 1% error of the average 3 Webber signal from flux loop. Hence, we set the values of σ_b and σ_f equal to 0.03.

Hoping that we can find a minimum region near the reference point, we varied the current profile parameters and plot the χ^2 values. We have tried plasmas with different shapes and sizes, we also moved the plasma slightly outward in the major radius direction. The results are not satisfactory. Unfortunately, even with rescaling to match the plasma current, no minimum region can be found in the χ^2 surface. For example, Fig. 3.15 shows a smooth χ^2 surface but there is no minimum region near the reference equilibrium (Fig. 1.2).

Since this method depends on the accuracy of the computed poloidal magnetic field, we conclude that the field is not computed accurate enough for the application of the Virtual-Casing Principle to the flux reconstruction.

Figure 3.15: χ^2 surface using the Casing Principle

3.9 Summary and Conclusions of this chapter

From the previous discussion, we know that the variational method is a very fast fixed boundary equilibrium solver. The poloidal field calculated from the derivative of the flux function has about 6% of error. If this error were smaller, the matching of the poloidal field on the plasma edge and the method based on the Virtual-Casing Principle may provide a very fast and accurate flux reconstruction tool.

Since B_s is calculated from the derivative of the flux function, its accuracy is obvious affected by the error in the flux function.

If we want to find a more accurate B_s , we may have to either use another equation to calculate the field or change the constraints of the variational method.

Eq. 3.13 is a formula for the toroidal current, it depends directly on the profile functions and the trial functions. However, Eq. 3.53 shows that the poloidal magnetic field depends on the derivatives of the trial function. Since the variation method is an approximate method, the trial function is by no means exact. The derivative of the trial function will have bigger error. The volume integration method described by Eq. 3.61 does not involve derivative of the flux function. It may be a better way to evaluate B_s .

On the other hand, we can change the constraints of the variational method and hopefully get a better B_s . One of the constraints we have imposed on the variational method is the normalization of the current density, a normalization constant C is adjusted so that the volume integral of the current density is exactly equal to the total plasma current.

$$I_p = \frac{C}{\mu_0} \int_{A_p} \left[b_p \frac{R}{R_0} h'_p(\tilde{\psi}) + (1 - b_p) \frac{R_0}{R} h'_f(\tilde{\psi}) \right] dR dZ \quad (3.68)$$

We may choose B_s calculate from Eq. 3.53 and apply Ampere's Law as a constraint to evaluate C . Knowing C , the value of ψ_0 can then be evaluated from

the energy moment equation(Eq. 3.31). We can set C equal to unity and start the iteration as before. This method of calculating B_s guarantees that we will get the correct value of I_p from the line integral of B_s , however, since it still involve the derivative of the flux function, it is less convincing as the previous one.

In the next chapter, we will investigate other methods of profile matching. We will avoid using poloidal field calculated from the derivative of flux function and prove that the volume integration method give a more accurate poloidal field.

Chapter 4

Current Integration Methods

4.1 The Idea of a Unified Lagrangian

In the magnetic flux reconstruction, the magnetic signals are considered as given. The goal is to find the plasma current profile parameters which fit the magnetic signals.

To match the current profile, we have to relate the magnetic measurements directly or indirectly to the some parameters which describe the shape of the current profile. One example can be the half width of the toroidal current density profile on the middle plane, this parameter is well defined, but it is too crude as information about the current profile.

Here, we use b_p , α_f , α_p as mentioned in the previous chapter to describe the current profile. Mathematically, the matching of current profile parameter is an optimization of a function:

$$\mathcal{L}_m = \mathcal{L}_m(\nu, \sigma, \kappa_0, \eta, b_p, \alpha_f, \alpha_p; B_{pm}, \Psi_m)$$

where B_{pm} and Ψ_m are the given set of magnetic measurements.

Inside this optimization, the plasma equilibrium has to be solved, which is another optimization of a function:

$$\mathcal{L}_e = \mathcal{L}_e(\nu, \sigma, \kappa_0, \eta; b_p, \alpha_f, \alpha_p)$$

Within the second optimization, the current profile parameters b_p, α_f, α_p are considered as fixed. The goal is to find those parameters which describe the plasma equilibrium such as ν, σ, κ_0 , and η .

This problem can be perceived as two different optimizations nested together. Firstly, the fixed boundary plasma equilibrium is optimized for any given set of current profile parameters. Secondly, the current profile parameters are optimized by some reliable matching methods to fit the magnetic measurements.

Philosophically, both optimizations can be merged into one optimization. The idea can be expressed abstractly:

$$\mathcal{L}_{new} = \mathcal{L}_e \oplus \mathcal{L}_m$$

where the sign $\oplus$ means that the two optimizations are merged together by combining them using one minimization algorithm or using Lagrangian multipliers.

If we can find the current profile parameters and solve the MHD equilibrium at the same time, it will save us the execution time substantially.

Actually, we have tried to rewrite the algorithm of the variational code to test this idea. Unfortunately, the result is a complete failure. Hence, before we have further evidence, we take the flux reconstruction problem as a combination of two different optimization.

4.2 Matching of Current Moments

The moments of the toroidal current density can be related to the line integrals of the linear combinations of the poloidal magnetic field measured outside the plasma[61]. Also, the current moments can be expressed in terms of the multipolar moments[62]. Hence, the comparison of current moments has the potential of determining the current profile parameters.

The following equations are the moments of toroidal plasma current density: The integrals are taken over the cross-section of the plasma.

$$M_0 = I_p = \int J_\phi dS \quad (4.1)$$

$$M_1 = Z_{cm} I_p = \int Z J_\phi dS \quad (4.2)$$

$$M_2 = R_{cm}^2 I_p = \int R^2 J_\phi dS \quad (4.3)$$

$$M_3 = \int R^2 Z J_\phi dS \quad (4.4)$$

$$M_4 = \int (R^2 Z^2 - \frac{1}{4} R^4) J_\phi dS \quad (4.5)$$

$$M_5 = \int (R^2 Z^3 - \frac{3}{4} R^4 Z) J_\phi dS \quad (4.6)$$

$$M_6 = \int (R^2 Z^4 - \frac{3}{2} R^4 Z^2 + \frac{1}{8} R^6) J_\phi dS \quad (4.7)$$

Each of them can be related to some physical plasma properties which are listed in Table 4.1. They are macroscopic parameters which are related to the plasma shape and position. They are important for plasma feedback control. From these definitions, all the odd terms will vanish if the plasma is up-down symmetric.

If we change the current density by changing the current profile parameters b_p , α_f and α_p , the current moments will be changed as a result. Hence, there may be a matching method for the determination of the current profile parameters for a given set of magnetic measurements. Therefore, it might be possible to use these

M_0	Total current
M_1	Upward displacement of current center
M_2	Outward displacement
M_3	Skew ellipticity
M_4	Vertical ellipticity
M_5	Upward triangularity
M_6	Outward triangularity

Table 4.1: Physical meaning of the current moments

moments to determine the profile parameters.

Here, we defined a parameter χ^2 which measures the sum of the squares of the difference of the calculated current moments and the given current moments. We try to use a least square fitting method to match the current profile parameters. Hence, the focus of this section is to test feasibility of this matching method.

To illustrate this idea, we take the proposed plasma (Fig 1.2) as an example. The current moments calculated for this plasma with its particular current profile parameters act as a reference. Then, we generate other plasmas by varying the current profile parameters. To see the differences in current moments with respect to the reference, we define the χ^2 as follows:

$$\chi^2(b_p, \alpha) = \sum_{k=0}^6 \frac{[M_k(b_p, \alpha) - \hat{M}_k]^2}{\sigma_k^2} \quad (4.8)$$

where $\hat{M}_k$ is the k th moment of the target equilibrium, and σ_k is defined as following:

$$\sigma_k = 0.01 \times \hat{M}_k$$

Figure 4.1: χ^2 versus l_i and β_p .

The definition is good for testing purpose and it assumes the current moments have about 1% error. For the sake of computation accuracy, the odd terms are taken out from the χ^2 , since the plasma under investigation are up-down symmetric.

Also, we have set the profile parameters α_f and α_p equal to each other and simply name them as α .

Fig. 4.1 is the result of this matching method. The χ^2 values are calculated by varying the current profile parameters, b_p and α . The χ^2 surface is smooth. This implies that the output from the variational method is continuous with respect to the current profile parameters, even though this is a non-linear problem.

We want to apply this matching method to resolve the l_i and β_p , since they have more physical and engineering implications than the current profile parameters. In general, l_i increases with α , since both are related to the peakedness of the current profile. Even though b_p has almost the same value as β_p for small b_p , we treat them

Figure 4.2: χ^2 versus l_i and β_p .

as two different parameters.

Fig. 4.2 is a contour plot of the χ^2 values with respect to the l_i and β_p . To our surprise, there is a long trough instead of a minimum region under the valley. The target values of the l_i and β_p are 0.8038 and 0.095 respectively.

Using the same method, we have investigated plasmas with different shape. All the χ^2 surfaces are smooth and have a long trough as a minimum. The orientation of the trough changes with the elongation and triangularity.

Other researchers[41] found that it is possible to resolve l_i and β_p for non-circular case. In order to check whether this trough may be a by-product of the variational method, we have to perform the same task using another fixed boundary solver called the J-solver version of PEST.

Since the input and output of the two solvers are different, as mentioned pre-

Figure 4.3: χ^2 versus b_p and α .

vious, we have to fine tune several parameters in order to get a close match for comparison. These fine tuning will certainly take a lot of CPU time. In order to have a valid comparison without over-spending in the CPU time in CRAY, we have to plan well and allocate points to cover the crucial part of the χ^2 surface. In Fig. 4.3, the 24 diamonds which cover the long trough are the selected points for testing.

Fig. 4.4, Fig. 4.5 and Fig. 4.6 show the result of the comparison. The solid lines are the χ^2 values from the variation code while the crosses are the result from PEST. They have almost the same values on the bottom parts. The most striking fact is that the trough is reproducible in Fig. 4.7.

Hence, we can concluded that the presence of the trough is not a by-product of the variational method. Also, this test can serve as a benchmark of the variational code.

Figure 4.4: Comparison of χ^2 when $\alpha = -1.5$

Figure 4.5: Comparison of χ^2 when $\alpha = -2.0$

2 of 2

Figure 4.6: Comparison of χ^2 when $\alpha = -2.5$

Figure 4.7: The trough is reproducible from PEST.

Since we are testing up-down symmetric plasma, we have taken out all the odd terms of the current moments from the χ^2 for better numerical computation. Also, since the zero moment is the plasma current which is an input and is constant, it is also taken out. Probably the information required to separate l_i and β_p is washed out as we have taken out four moments.

4.3 The Green's Function Method

If the plasma current density is known, the plasma contribution to the the magnetic measurement by the m th probe at location (R^m, Z^m) can be calculated from the Biot-Savart law and Green's function [63].

$$\Psi(R^m, Z^m) = \mu_0 \int \int dR' dZ' J_\phi(R', Z') \frac{(R^m R')^{1/2}}{2\pi} \left[\frac{(2 - k^2)K(k) - E(k)}{k} \right] \quad (4.9)$$

$$B_R(R^m, Z^m) = \mu_0 \int \int dR' dZ' J_\phi(R', Z') \frac{1}{2\pi R^m} \frac{(Z^m - Z')}{[(R^m + R')^2 + (Z^m - Z')^2]^{1/2}} \left\{ -K(k) + \left[\frac{(R^m)^2 + R'^2 + (Z^m - Z')^2}{(R^m - R')^2 + (Z^m - Z')^2} \right] E(k) \right\} \quad (4.10)$$

$$B_Z(R^m, Z^m) = \mu_0 \int \int dR' dZ' J_\phi(R', Z') \frac{1}{2\pi} \frac{1}{[(R^m + R')^2 + (Z^m - Z')^2]^{1/2}} \left\{ K(k) - \left[\frac{(R^m)^2 - R'^2 + (Z^m - Z')^2}{(R^m - R')^2 + (Z^m - Z')^2} \right] E(k) \right\} \quad (4.11)$$

where

$$k^2 = \frac{4R^m R'}{(R^m + R')^2 + (Z^m - Z')^2}$$

As the current profile parameters changes, the plasma contribution to the magnetic measurements at the probe locations changes. The above formula provides a way to determine the current profile parameters by matching the magnetic flux and field at the probe locations. Here, a least square method is employed as following:

$$\chi^2(b_p, \alpha) = \sum_{m=1}^{N_b} \frac{[B_p(b_p, \alpha) - \hat{B}_p]^2}{\sigma_b^2} + \sum_{m=1}^{N_f} \frac{[\Psi(b_p, \alpha) - \hat{\Psi}]^2}{\sigma_f^2} \quad (4.12)$$

$$\sigma_b = 0.03 \quad (4.13)$$

$$\sigma_f = 0.03 \quad (4.14)$$

N_b and N_f are the numbers of magnetic coils and flux loops respectively.

As before we set $\alpha = \alpha_f = \alpha_p$ for simplicity. We have assumed 1% error of the average 3 Tesla signal from magnetic coil and 1% error of the average 3 Weber signal from flux loop. Hence, we set the values of σ_b and σ_f equal to 0.03.

Fig. 4.8 shows the contour of χ^2 versus the current profile parameters. The surface is smooth and a minimum region appears around the true value.

In an experimental situation, instead of plotting out the whole χ^2 surface to locate the minimum, the minimum can be found by an iterative scheme. A multidimensional minimization algorithm called the downhill simplex method [64] was applied to test the convergence of the least square method. The downhill simplex method requires only function evaluations, not derivatives. For a two dimensional minimization such as this case, the simplex is a three dimensional geometrical figure. Given the initial three points, the simplex will move towards the minimum region by replacing the highest point with a lower point. Eventually, the simplex will enclose the minimum. Since the size and shape of the simplex keep changing, it is called "amoeba".

The disadvantage of this method is that it may require more function evaluations than other minimization method and hence it may not be the most efficient method. However, since it does not require derivatives, it is a robust minimization method.

It is well known that the result of the non-linear least square fit method has to be positive and it shows no further information if the reduced χ^2 is less than

Figure 4.8: The contour of χ^2 versus the current profile parameters

unity[65]. Hence, our search stops whenever the points of the simplex move inside the region where χ^2 is less than unity.

In general, the minimum value is an unknown and can be scaled by the values of σ_b and σ_f . We have developed several termination criteria for the case when the minimum value is greater than unity. Basically, we set up some parameters so that the final simplex is as small as we want and the differences between the values of the three points and the centroid of the triangle are insignificant.

The plasma boundary is defined and fixed by the shape parameters which are assumed known. The R and Z coordinates are related to the ρ and μ coordinates through the flux functions. Every time when the variational parameters change, the flux functions will change and so do the R and Z coordinates. The main difficulty here is that the R and Z coordinates of the internal plasma flux surfaces vary with the current profile parameters. The magnetic Green's functions for each grid point of the flux surface and for each probe location have to be recalculated each time the current profile parameters change. This is extremely time consuming.

Obviously, the simplex method can be replaced by other minimization scheme such as Powell's method[66]. However, a detailed analysis of the CPU time spent shows that about two thirds of the time is spent on the integrals in the evaluation of the magnetic Green's functions.

In Fig. 4.8, the triangles represent the starting and ending points of the simplex. As the algorithm starts working the simplex moves towards the minimum region. For simplicity, only the starting and ending simplex are shown. All the final points land in the region where χ^2 is less than unity. This proves a robust method of magnetic reconstruction.

4.4 The accuracy of the poloidal field

The poloidal field can be divided into the contribution from the plasma current and contribution from the PF coil currents. If we take a close integral of the poloidal field around the plasma boundary, the total plasma current will be obtained by Ampere's law.

$$\oint \vec{B}_p \cdot d\vec{l} = \oint \vec{B}_p(plasma) \cdot d\vec{l} + \oint \vec{B}_p(PFcoil) \cdot d\vec{l} \quad (4.15)$$

The second integral of the right hand side of the above equation vanishes since all the PF coils are excluded outside the integration path.

Hence, the close integral of the reduced poloidal field yields in the total plasma current times the permeability of vacuum.

$$\oint \vec{B}_p(plasma) \cdot d\vec{l} = \mu_0 I_p \quad (4.16)$$

The above equation provides a basis to check the accuracy of the calculated reduced poloidal field. The reduced poloidal field can be evaluated from the Biot-Savart law or more precisely, to integrate the current density with the Green's function over the plasma volume. The angles between the reduced poloidal field and the plasma surface have to be evaluated while taking the integral.

Fig. 4.9 shows the percentage errors of the absolute values of the difference of the calculated value and the input value of the total plasma current of the plasma in Fig. 1.2. We find that the calculated value changes with the number of output angular grid points on the line integral. In general, the calculated values of the total plasma current are smaller than the input one. However, the calculated value converges to the correct value.

Hence, this integration method of calculating poloidal field is far more accurate

Figure 4.9: The percentage error in the total plasma current from reduced field than the differentiation method even though it takes much longer cpu time.

4.5 Summary and Conclusions of this chapter

Comparing the performance of the moments method and the Green's function method, the later is robust because a minimum region is clearly shown.

However, the Green's function method takes a large amount of CPU time which is not suitable for the fast analysis of the plasma equilibrium.

Chapter 5

A magnetic flux reconstruction package

5.1 Introduction

We propose a magnetic flux reconstruction package which takes magnetic signals from the Mirnov coils and the flux loops inside a highly shaped tokamak and solves the flux function both inside and outside the plasma.

Fig. 5.1 shows all the components of the magnetic flux reconstruction package. This package consists of a vacuum solver called MFIL, a neural network as the interface, and a fixed boundary variational solver called MAGNEQ. The vacuum field is solved using a filament model. The output of the vacuum solver is a set of points on the plasma boundary, while the input of the fixed boundary solver requires 4 shape parameters. Thus, the output of the vacuum solver can not be fed directly into the input of the fixed boundary equilibrium solver. Here we solve this non-linear mapping problem by applying a neural network. The fixed boundary solver is an application of the variational method to solve the MHD equilibrium. The exact locations and orientations of Mirnov coils and the flux loops are implemented inside

Figure 5.1: The components of the magnetic flux reconstruction package.

the vacuum solver as well as the equilibrium solver. The current profile parameters are found by least square fitting of the calculated and measured magnetic signals contributed by the plasma current only. Volume integral of the current density and Green's functions are used.

This flux reconstruction has been tested with computer generated plasma and the results are compared with other equilibrium solvers such as ASEQ and EFIT. We find that this package can reproduce the main feature of the plasma equilibrium. Also, the I_i and β_p values have strong linear relation with the input values.

Furthermore, we attempted to implement this package to analyse the magnetic signal from the experimental data of Alcator C-Mod. We conclude that the variational trial functions are too crude for the evaluation of the current profile parameters in the experimental plasma produced in Alcator C-Mod.

5.2 The Computational Platform

In order to actualize our idea and test the feasibility of the magnetic flux reconstruction package, a particular computer platform is chosen to integrate the vacuum solver, the neural network and the equilibrium solver and to link them to the real time magnetic diagnostic database.

Generally speaking, the principle of physics will not change no matter what kind of software nor hardware we choose. The ability of the flux reconstruction package is based on the understanding of the physics aspect of the problem. The speed of the code comes from the genius and creative application of the mathematical principle and numerical method.

However, in the actual practice of computational physics and software development, the choice of hardware and the software often require clear vision and flexibility. In this information age, the capability of computer is upgraded every

few months. The understanding of advanced computational products can substantially promote the overall accuracy and performance. On the other hand, one may benefit from the abundant resources and the tremendous technical support of the well established but less advanced products. Considering also the operational cost, we have chosen different platforms in different phases of the software development.

To increase the portability of the software, the variational code was written in ANSI/C language. To lessen the burden of the software development process, most of the extensive testing were done on the CRAY/UNICOS supercomputer system. At the same time, the 64 bit floating point calculation in the supercomputer can offer a more precise answer. Hence, different matching methods can be tested and compared efficiently.

The raw magnetic diagnostics are integrated using differential input op-amp integrators, which have a sample-and-hold circuit to set and reset the baseline so to minimize the effect of drift. The integrated signals are digitized by CAMAC data acquisition and timing system and linked to the MDSplus data trees in the VAX/VMS system. Hence, the VAX/VMS system is chosen as a main base for all these. A software system called IDL(Interactive Data Language) is chosen as the main driver of the whole computation system. It is an interactive programming language designed for data analysis and image display. The variational code MAGNEQ which was developed on the CRAY/UNICOS system was then modified and compiled as a sharable image in VAX/VMS. To make the flux reconstruction package accessible to any member of the research team, all our VAXstations are linked with Ethernet to form a cluster. Any member can run the package on IDL to determine the plasma equilibrium and study the MHD instabilities.

Figure 5.2: The current filament model

5.3 The Vacuum Solver

MFIL is a code written by Prof. Ian H. Hutchinson, the head of the Alcator C-Mod. This code uses filaments to simulate the plasma currents and calculates the magnetic flux outside the plasma. The code retrieves the magnetic field and flux signals from the MDSplus data trees. A rectangular mesh of 65 by 65 grid points is arranged to cover most of the vessel cross-section. A filament model is applied to simulate the plasma, the PF coil currents, the eddy currents inside the vacuum vessel and the stainless steel supporting structures such as the cover and the cylinder. Fig. 5.2 shows 76 current filaments. They include all the PF coils and the plasma filaments as well as a simulation of all the supporting structures such as the dome and the TF coils. Note that the plasma is represented by 16 current filaments.

The magnetic signals can be expressed as a superposition of contribution from

the plasma and contribution from the PF coil currents, eddy currents inside vacuum vessel wall and supporting structure. The flux reconstruction method we describe in the previous chapter requires the signals purely contributed from the plasma. Here, we have a method of calculating this reduced signals for the given total magnetic signals and the currents of all the filaments except those which represent the plasma.

$$\mathbf{m}_i = \mathcal{L}_{ij} \mathbf{I}_j \quad (5.1)$$

Eq. 5.1 is the relationship between the axisymmetric currents and the magnetic signals measured at the probe locations. $\mathbf{I}_j$ is a vector represents the currents in the 76 current filaments, while $\mathbf{m}_i$ is a vector represents the magnetic measurements from both the mirnov coils and the flux loops.

The matrix $\mathcal{L}_{ij}$ represents the influence values between the particular set of coil currents and the magnetic measurements. The locations of the magnetic probes and the plasma filaments are fixed. This is a linear problem and the matrix elements can be pre-calculated.

$$\mathbf{I}_j = \mathcal{L}_{ji}^{-1} \mathbf{m}_i \quad (5.2)$$

Eq. 5.2 is the inverse of the linear system described by Eq. 5.1. Since the matrix $\mathcal{L}_{ji}$ may be singular or numerically ill-conditioned, the pseudo inverse is treated by the singular value decomposition method [67, 68]. The vector $\mathbf{I}_j$ contains the known values of the PF coil currents and the unknown values of the filament currents of the plasma.

$$\Psi_{rz} = \mathcal{M}_{rzi} \mathbf{m}_i \quad (5.3)$$

Eq. 5.3 is a combination of the above two equations, where the matrix Ψ_{rz} represents the flux function values on a 65X65 mesh, the three dimensional matrix

$\mathcal{M}_{rzi}$ contains two operations, one is to find out all the currents in the current filaments from the magnetic signals, the other is to find the flux functions values on the mesh from the currents in the current filaments through Green's function.

With the similar idea, a filter matrix can be designed so that when it operates on the total magnetic signals, it produces the reduced signals which are contributed from the plasma current only.

$$\mathbf{n}_k = \mathcal{P}_{ki} \mathbf{m}_i \quad (5.4)$$

Eq. 5.4 stands for the relationship between the reduced signals and the total magnetic signals. The filter matrix $\mathcal{P}_{ki}$ is independent of the plasma current distribution.

Fig. 5.3 shows the outputs of MFIL and EFIT, the dash line is the separatrix generated from MFIL for the same magnetic signals.

In order to determine all the filament currents, the least square method is used to fit the magnetic signals. Employing the singular value decomposition method, we can design a matrix operator which will operate on the magnetic measurements and estimates the signals purely contributed from the plasma.

Once the plasma boundary is solved, the separatrix is found by tracing flux values inside the mesh which matches the flux either on the limited point or at the X-point.

5.4 The mapping of the separatrix and the shape parameters

The separatrix found from the vacuum solver is a set of points at which the plasma boundary cuts the rectangular mesh frame. These points can not be input directly

* EFITD 33x33 08/11/93 *

date ran =29-SEP-93

shot # = 930929024

time(ms) = 200.

chi**2 = 13.0

rout(cm) = 66.326

zout(cm) = -2.860

a(cm) = 22.305

elong = 1.294

utriang = 0.298

ltriang = 0.396

indent = 0.000

vol(cm3) = 7.769e+05

energy(J) = 1.427e+04

betat(%) = 0.247

betap = 0.282

li = 1.370

error = 2.990e-03

q* = 5.519

qout = 5.966

qpsib = 4.731

dsep(cm) = -0.756

rm//rc = 67.5// 66.4

zm//zc = -1.7// -1.8

betapd = 1.049

betatd(%) = 0.918

wdia(J) = 5.313e+04

data used:

ip(kA) = 420.486

bt0(T) = 3.524

Figure 5.3: The outputs from MFIL and EFIT.

into the variational code. They have to be translated into the corresponding shape parameters.

The equations which describe the separatrix can be derived from the equations of the trial function (Eq. 3.39) by setting ρ equal to unity.

$$R(\rho = 1, \mu) = R_0 + a \cos(\mu + D_a \sin \mu) \quad (5.5)$$

$$Z(\rho = 1, \mu) = a \kappa_s \sin \mu \quad (5.6)$$

$$D_a = \sin^{-1}(\delta_a) \quad (5.7)$$

The angle-like parameter μ can be treated as a labelling parameter for different points on the separatrix. Given R_0 , a , D_a and κ_s , it is easy to generate a set of points to represent the separatrix from the above equations. However, the inverse of this calculation is not straight-forward.

If it is guaranteed that we can have the points corresponding to μ equals to 0 , $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$, the values of R_0 , a , D_a and κ_s can be calculated by substitution. However, in our case, those points are not guaranteed.

Starting from those points representing the separatrix, to find the corresponding set of R_0 , a , D_a and κ_s is a non-linear mapping. Here we need a tool to solve this non-linear multi-dimensional mapping. A neural network, which is by no means the only solution, has the advantage of great speed once the network is trained. However, its concept is not widely understood by researchers outside the field of artificial intelligence.

5.5 The concept of MLP neural network

Historically, the invention of the artificial neural network was inspired by the knowledge of the activities of neurons in the human brain. The artificial neural network

we use is a mathematical model which is much simpler than the biological neural network. The connection between different nodes can be viewed as similar to the axon-synapse-dendrite connections in the biological neuron.

The neural network we choose is called a multi-layer perceptron (Fig. 5.4). Typically, it consists of three groups of units — one input layer, at least one hidden layer and one output layer. The raw information is fed into the units of the input layer. The signals are sent from one layer to another through unidirectional communication channels. A weight will be multiplied to the signal in the middle of transmission. Different channels can have different weights. The weights serve as local memories and different channels can have different weights. By changing the values of the weights, the output signals can be changed.

Mathematically, an activation function is implanted into the network so that the simulated neuron remains on as soon as the sum of its weighted inputs and the bias term is above threshold. Originally, the activation function is a step function. However, for the sake of mathematical presentation, a differentiable function called sigmoidal function is applied. A sigmoidal function has upper and lower bounds and is differentiable. The smooth sigmoidal function allows the output to be a smooth function of the input.

$$S(x) = \tanh(x/2) \quad (5.8)$$

To train the neural network, we applied the back-propagation method [69]. This method requires supervised learning, which means it needs a “teacher” who knows the correct outputs (targets) to train the network. In the training phase, a gradient descent algorithm is applied to minimize the difference between the outputs and the targets. The squared errors are passed backward to adjust the weights. Although it takes a long time for all the weights to converge in the training phase, the application phase is quite fast since it is a pure feedforward algorithm.

Figure 5.4: A multi-layer perceptron.

Figure 5.5: The target curve and the output from the trained neural network

The purpose of the training is to determine the values of all the weights and bias terms. The back-propagation method is an effective learning algorithm. This method is similar to 'steepest descent' but it has the gradient calculated at each step. This algorithm has been applied to recognize handwritten digits, predict currency exchange rates and maximize the yields of chemical processes. The multi-layer perceptron provides a powerful and general technique for performing an arbitrary continuous non-linear multi-dimensional mapping.

Here is a simple one-input-one-output example for the illustration of the concept of a multi-layer perceptron.

Consider a non-linear equation as follows:

$$y = u^3 + 0.3u^2 - 0.4u \quad (5.9)$$

Figure 5.6: The square of the difference between the target value and the output value.

Given this equation, it is easy to find y for any u . However, if the equation is unknown and what we have is only a finite set of valid input-output examples, it is not straightforward to find the output for an input which does not belong to the example set. One may think of numerical methods of spline fitting or interpolation. Note that this example has only one input, one may find some numerical method to solve this non-linear problem. In case where the number of input is big, say 100, the complexity will be very significant.

To solve this problem, we can apply a one input one output multi-layer perceptron. Narendra and Parthasarathy [70] have demonstrated that the above identification problem can be solved using a neural network with two hidden layers. According to the paper, a neural network with 20 nodes in the first hidden layer and 10 nodes in the second hidden layer will converge after 50,000 iterations.

Here, we apply his idea but using a simpler structure since the computer is much faster nowadays and we can afford more iterations in the training phase. The neural network we built has a simpler structure of only 10 nodes in the first hidden layer and 5 nodes in the second layer. Since the structure is simpler, we expect the computation time will be much shorter once the network is trained.

The neural network is written in ANSI/C and compiled in CRAY/A supercomputer. The supervised training takes about 200,000 iterations and the cpu time is about 31 seconds. In each iteration, a random input whose amplitude was uniformly distributed in the interval $[-1, 1]$ was generated and fed to the network. Fig. 5.5 shows the target curve and the output from the trained neural network. The target curve is generated from Eq. 5.9. In order to see the different between the target values and the output values of the neural network, the square of the difference is plotted in Fig. 5.6. The squares of the difference are less than 10^{-3} except those points near the end.

5.6 The limitations of MLP neural network

Generally speaking, the computation time is proportional to the size of the architecture of the neural network. If the hidden layer has too many nodes, the neural net will exhibit overfitting. It will try to match the fluctuations instead of the correct contour. On the other hand, if there are too few nodes, the internal representation of knowledge will be inadequate. In case the number of weights and nodes are huge, it requires the high-speed processing through massively parallel VLSI implementations.

In general, the number of presentations of training data required for convergence is very large. Hence, the supervised training usually takes a long time. Moreover, the bigger networks will require more training examples. Sometimes this

method may fall into a local minimum of the error function instead of the global minimum.

Mathematically, it has been proven [71] that the approximate realization of any continuous mapping is guaranteed by multi-layer neural network with at least one hidden layer whose output functions are sigmoid functions. However, in practice, even though one understand the principle, the searching for the best architecture remains an art[72].

5.7 The recognition of the plasma separatrix

We have applied a neural network with one hidden layer to solve the interface problem of the separatrix. The function of this network is to recognize quantitatively the plasma separatrix of a specific shape and express it in a parametric form.

The inputs are the R and Z coordinates of those points on the separatrix, the outputs are the four shape parameters, R_0 , a , D_a and κ_s .

Since the size of the architecture of our neural network is large, we choose the General Adaptive Recipe (GAR) which was developed by Lister et al[73] as our training algorithm. The GAR, based on the traditional back-propagation method, uses an adaptive step length instead of a constant step length. The automatic adjustment of step length accelerates the convergence in the training stage. All these inputs and outputs are normalized to $[-1, 1]$. The training code was written in IDL and compiled on a VAXstation.4000.90.

The best architecture of this network is the one which has the smallest number of nodes but carries most of the information of the plasma separatrix. The number of nodes in the output layer is fixed because there are four shape parameters. We studied 30, 60, 90, 180 nodes as input with different number of nodes in the hidden layer. It was found that the overall performance was improved slightly as the

Figure 5.7: The mean square residual versus the iteration during the training stage.

number of input nodes increased. However, for the case of 180 input nodes and 60 nodes in the hidden layer, the training was set to take 4500 iterations. In each iteration, we generate 11,200 samples of separatrix with different shape parameters. The computer crashed in the middle of the training process. It was found that the matrix size and accumulating memory allocation of the current IDL code are outside the range of the hardware setting. Hence, based on the limited set of cases we studied, it is hard to conclude a clear cut winner between different network architectures. We chose the best from our limited set of cases.

Here, we present the case of 90 input nodes and 30 nodes in the single hidden layer. The input consists of R and Z coordinates of 45 points on the separatrix. It takes about 13 hours cpu time for the supervised training in a VAXstation.4000.90.

Fig. 5.7 shows the mean square residual drops as the number of iterations increases. The training takes 4500 iterations. In each iteration, we generate 3195

Figure 5.8: The test results versus the true values of R_0

samples of separatrix with different shape parameters. From this figure, the mean square residual drops sharply in the beginning, but gradually in the later stage to below 10^{-4} , we know that the training of neural network is successful. The bumpiness implies change of the direction in the searching of the minimum. The slow asymptotic decrease means that the limit of the computation is near and further training may not give significant gain.

To examine how well the network behaves when some noise is added, we randomly generate 500 sets of R_0 , a , D_a and κ_s as inputs. From these shape parameters and Eq. 5.7, we then generate 45 point separatrices with $\pm 1\%$ random noise. These sets of 45 point separatrix are input to the trained network. The results are plotted versus the true values (Fig. 5.8 to Fig. 5.11). In general, the data show good agreement. There are a few points scattered off the lines of D_a and κ_s . This implies that the network finds it harder to resolve D_a and κ_s .

Figure 5.9: The test results versus the true values of a

Figure 5.10: The test results versus the true values of Da

Figure 5.11: The test results versus the true values of κ ,

5.8 The Comparison with ASEQ

To test the feasibility of the package, we choose one free boundary solver and one fixed boundary solver to generate the plasma and compare the output.

ASEQ is a variant of the PEST(Princeton Equilibrium, Stability and Transport) code[31]. The version we use here is a free boundary equilibrium solver. In general, the code takes values of PF currents, plasma current and Strickler function parameters like α_f , α_p as inputs.

Dr. S. Wolfe has input the Alcator C-Mod geometry and PF coil configuration to the ASEQ code, so that the computer generated plasma will be close to a real shot with up-down symmetry.

We have tested plasma with different beta values and elongations. Here, we present an elongated high beta plasma as an example. It has an elongation of

Figure 5.12: The output of ASEQ

1.2254 on the plasma boundary. The plasma current is 1.001 MA and the toroidal magnetic field is 8.0 Tesla. Fig. 5.12 shows the internal magnetic flux output from ASEQ for the computer generated shot number 3858. The limited plasma is up-down symmetric. Some flux surfaces appear around the PF1 coils because they are inside the grid of calculation and happen to have the same value of the magnetic flux values as those inside the plasma.

The vacuum flux and the separatrix calculated by the vacuum field solver MFIL, which simulates plasma as filaments, are shown in Fig. 5.13. The dotted line is the separatrix of the limited plasma. This particular separatrix is constructed from 116 points calculated by an algorithm which searches for X-points (or limiter point) and draws the separatrix. The flux calculated from this model is only valid outside the separatrix where the plasma current vanishes.

We select 45 points evenly from the 116 points and feed them into the trained

Figure 5.13: The vacuum field generated from the vacuum solver

Figure 5.14: The fitting of the separatrix from the trained neural network

neural network. Fig. 5.14 shows the 116 points(cross), a line joining all the 45 points as well as a contour. It is challenging to human eyes because these three entities are extremely close to each other. The contour is the separatrix generated from the 4 output shape parameters calculated from the trained neural network and Eq. 5.7. There are small discrepancies on the top and the bottom parts of the separatrix. These small discrepancies can be understood by considering the slight dispersion of data in Fig. 5.10 and Fig. 5.11. This proves that the neural network can handle the separatrix of a up-down symmetric plasma.

As mentioned before, in order to solve the MHD equilibrium, we apply the current integration method. We match the magnetic flux and field at the probe locations using the Biot-Savart law and Green's function [63].

To use the least square fitting for the searching of the current profile parameters, we define χ^2 as follows:

$$\chi^2(b_p, \alpha) = \sum_{k=1}^{N_b} \frac{[B_p(b_p, \alpha) - \hat{B}_p]^2}{\sigma_b^2} + \sum_{k=1}^{N_f} \frac{[\Psi(b_p, \alpha) - \hat{\Psi}]^2}{\sigma_f^2} \quad (5.10)$$

$$\sigma_b = 0.03 \quad (5.11)$$

$$\sigma_f = 0.03 \quad (5.12)$$

where $\alpha = \alpha_f = \alpha_p$.

Fig. 5.15 shows the surface plot of the χ^2 value as the b_p and α values are changed. At the bottom of the narrow valley there is a minimum region. Fig. 5.16 shows the contour plot of the minimum region. The contour plot shows a distinctive minimum region. Further analysis reveals that the minimum is about 2.55 and is greater than unity. Note that this is not a reduced χ^2 . This proves that there is a possibility of finding the current profile parameters using the method of volume integration of current density and Green's functions.

Given this smooth surface and well defined minimum region, we can apply a

Figure 5.15: The surface of the minimum region

Figure 5.16: The contour of the minimum region

	ASEQ input	MAGNEQ(P45) result
l_i	1.0113	0.93437321
β_{pV}	1.312	1.3800738
$\psi_e - \psi_a$	0.6966	0.636758
q_s	4.116	3.95585
q_o	0.8982	1.12384

Table 5.1: Comparison between ASEQ and the equilibrium solver

multi-dimensional minimization algorithm to find the current profile parameters.

The minimum searcher which looks for the current profile parameters is multi-dimensional minimization algorithm called AMOEBA[64]. The number of function evaluations is 29. In each of these function evaluations, the code will take a new set of b_p and α values and input them to the variational equilibrium solver.

In the minimization process, we have set some special criteria of convergence. First of all, the searching will be done once the χ^2 value of the highest point of the amoeba is less than unity. Right now, we have assumed 1% error of 3 Tesla poloidal field and 1% error of 3 Webers flux loop signal.

In case this criterion is too severe to fulfill, the searching will stop if the relative difference between the highest value and the lowest value of the amoeba is less than 1% and the average normalized distance between vertices is less than 0.1%.

The minimum value of χ^2 obtained is 2.54731. The package is compiled and run in VAXstation.4000.90. The cpu time for the searching is 19.79 seconds.

Fig. 5.17 shows the internal magnetic flux reconstructed from the variational equilibrium solver. Qualitatively, it has captured the important features of the ASEQ input. Quantitatively, we have to compare them term by term.

Figure 5.17: The flux reconstructed from the equilibrium solver

Table 5.1 compares the ASEQ input and the results from the equilibrium solver. In order to compare the values of the l_i and β_{pV} with ASEQ, we follow their definition in this particular section.

The definition of the l_i and β_{pV} in ASEQ:

$$l_i = \frac{2 \int B_p^2 dV}{R_m (\mu_0 I_p)^2} \quad (5.13)$$

and

$$\beta_{pV} = \frac{\int p dV}{\int \frac{B_p^2}{2\mu_0} dV} \quad (5.14)$$

where R_m is the magnetic axis of the plasma.

The calculated values of l_i and β_{pV} are quite close to the input. However, the values of q_0 is not as close. It is because the magnetic field solver starts from measurements outside the plasma and calculates towards inside, and the error grows

as the point of interest moves inward.

In order to design a robust equilibrium solver, different extreme conditions in the minimization process have been studied. Considering the Strickler profile function and the numerical limitation of the 8-point Gaussian Integration method, we set the largest acceptable α value to be 10.0. From Fig. 3.2, we know that the computer will not give correct value as the α drop below -80.0. Hence, we set the lowest acceptable α value to be -60.0. Judging from experience, these values of α are enough to describe the plasma current. Mathematically the computer has no problem with a negative bp value. However, from the physical point of view, we can only accept a positive β_p value. In other word, negative bp is forbidden.

The simplex algorithm consists of different movements— expansion, reflection and contraction[74]. Occasionally, the simplex may stretch outside the boundary mention above. We have modified the algorithm so that the simplex will bounce back and contract itself whenever it moves across the boundary. This will also prevent an endless cycle in which the simplex keeps hitting the boundary. Although, this modification may increase the cpu time a bit, the algorithm is robust for flux reconstruction.

5.9 The Comparison with EFIT

The version of EFIT we use is EFITD 33x33 03/16/93 and it is a fixed boundary solver. It is convenient for design and analysis. We use it to generate several plasmas with different plasma boundary. Then, we keep the plasma boundary fixed and vary some input parameters to generate plasma with different expected β_p . The MHD equilibrium is solved and the magnetic signals are calculated. We store this series of computer generated plasma equilibrium with corresponding magnetic measurements in the MDSplus database. We have investigated almost circular shaped plasmas and

EFIT			MAGNEQ		
l_i	β_p	$\beta_p + l_i/2$	l_i	β_p	$\beta_p + l_i/2$
0.852	0.206	0.632	0.911	0.081	0.537
0.850	0.308	0.733	0.887	0.188	0.632
0.841	0.410	0.830	0.898	0.279	0.728
0.840	0.515	0.935	0.897	0.377	0.826
0.830	0.620	1.035	0.953	0.449	0.926
0.820	0.727	1.137	1.986	0.003	0.996
0.810	0.835	1.240	2.012	0.092	1.098
0.805	0.944	1.347	1.846	0.291	1.214
0.794	1.055	1.452	1.689	0.484	1.329
0.783	1.167	1.559	1.560	0.662	1.442
0.771	1.281	1.667	1.421	0.845	1.556
0.759	1.396	1.776	1.309	1.014	1.669

Table 5.2: The output of EFIT and MAGNEQ for nearly circular plasma

elongated shaped plasmas.

We retrieve the computer generated equilibrium and their corresponding magnetic measurements. We test the magnetic flux reconstruction package and compare the result.

Table 5.2 shows a comparison between the output of the EFIT and the variational equilibrium solver MAGNEQ. The plasma has elongation of about 1.005. The plasma surface is almost circular.

Fig. 5.18 shows the separatrix of an almost circular plasma (shot number 4832). While Fig. 5.19 shows the separatrix of elongated plasma (shot number 6833). The separatrices are generated from the 4 output shape parameters calculated from the trained neural network. These plots prove that the neural network can manage plasma with different shapes.

Figure 5.18: The fitting of the interface

Figure 5.19: The fitting of the interface

Figure 5.20: The output l_i of MAGNEQ versus the output l_i of EFIT

We apply the flux reconstruction package to solve the plasma equilibrium and calculate the corresponding values of l_i and β_p .

Fig. 5.20 and Fig. 5.21 show that the l_i and β_p of the two solvers do not match each other. The straight lines represent the ideal case when both solvers completely agree with one other. There is no linear relationship found in both cases. However, Fig. 5.22 show a linear relationship in $\frac{l_i}{2} + \beta_p$ between the output of these two solvers. Although the points do not fall on the straight line, the values are quite close to each other.

To investigate the l_i and β_p resolution, we perform similar task to elongated plasma.

Table 5.3 shows a comparison between the output of the EFIT and MAGNEQ. The plasma has elongation of about 1.325. Fig. 5.23 and Fig. 5.24 show that the l_i and β_p have a linear relationship. Both the high and low beta values are shown.

Figure 5.21: The output β_p of MAGNEQ versus the output β_p of EFIT

EFIT			MAGNEQ		
l_i	β_p	$\beta_p + l_i/2$	l_i	β_p	$\beta_p + l_i/2$
1.053	0.111	0.637	1.039	0.066	0.586
1.047	0.213	0.737	1.027	0.171	0.684
1.041	0.316	0.837	1.017	0.275	0.784
1.034	0.421	0.938	1.004	0.379	0.881
1.028	0.527	1.041	0.995	0.483	0.981
0.997	0.957	1.456	0.935	0.932	1.400
0.989	1.066	1.561	0.921	1.040	1.500
0.981	1.175	1.665	0.907	1.149	1.603
0.973	1.284	1.771	0.895	1.257	1.704
0.965	1.393	1.875	0.880	1.366	1.806

Table 5.3: The output of EFIT and MAGNEQ for elongated plasma

Figure 5.22: The output $\frac{l_i}{2} + \beta_p$ of MAGNEQ versus the output $\frac{l_i}{2} + \beta_p$ of EFIT

Moreover, the values are quite close to each other.

5.10 Conclusion

The magnetic flux reconstruction package is successfully implemented in IDL. The vacuum solver, the trained neural network and the variational equilibrium solver are linked together on VAX/VMS. The comparison with ASEQ shows the potential of this package as a working tool for magnetic diagnostics. The comparison with EFIT shows that l_i and β_p can be resolved when the plasma is elongated.

Figure 5.23: The output l_i of MAGNEQ versus the output l_i of EFIT

Figure 5.24: The output β_p of MAGNEQ versus the output β_p of EFIT

Chapter 6

The Experimental Data from Alcator C-Mod

6.1 Introduction

Since our flux reconstruction package is designed for limited plasmas at present, we will test the package with those limited plasmas in the experiment. Here we present a plasma shot number 930623014. The plasma is more than 300kA, the plasma pulse duration is more than 0.3 second.

The plasma current is measured by a Rogowski coil while the toroidal vacuum magnetic field is diagnosed from the compensation coil of the diamagnetic coil. Fig. 6.1 shows the plasma current as a function of time. The maximum plasma current is above 0.3 MA for this plasma shot. Fig. 6.2 shows the toroidal vacuum magnetic field, it shows the ramp up and ramp down period of the TF coils. The maximum is almost 5 Tesla.

Fig. 6.3 shows the signals from the 26 Mirnov coils, while Fig. 6.4 shows the signals the 26 flux loops. They are stored in the MDSplus data tree.

Shot 930623014

Figure 6.1: The plasma current as a function of time

Shot 930623014

Figure 6.2: The toroidal vacuum magnetic field

Figure 6.3: The signals from the 26 Mirnov coils

Figure 6.4: The signals from the 26 flux loops

Figure 6.5: The B_p signals from MDS

Now, let us choose a particular time slice and examine the plasma equilibrium.

shot = 930623014

nt = 1270

time = 0.250 sec

For this particular time slice, we can retrieve the magnetic signals from the MDSplus data tree. Fig. 6.5 shows the B_p signals, while Fig. 6.6 shows the flux loop signals. They are retrieved from the MDSplus data tree. The signals shown are contributed from current of the plasma, the PF coils and other eddy currents.

Fig. 6.7 shows the vacuum field calculated from the MFIL for this particular time slice. Since the plasma is modeled by 16 filaments in the calculation of the vacuum field, the reconstructed vacuum field is valid only if all the filaments are located inside the plasma and the point of consideration is not too close to the

Figure 6.6: The flux loop signals from MDS

Figure 6.7: The vacuum field from MFIL

filaments. Thus, the plasma cross-section has to be large enough to include all these 16 filaments.

The plasma is shifted slightly vertically downward to match the divertor hardware, hence the center of a symmetric plasma is not necessary on the mid-plane of the tokamak.

We have therefore upgraded both the variational solver and the neural network so that they can handle the vertical offset of the magnetic axis of the plasma.

6.2 The upgraded package

Here we investigate a plasma with up-down symmetry but the magnetic axis is shifted vertically toward the divertor.

In order to handle this experimental configuration, the variational code and the neural network have to be upgraded. Mathematically, only two new equations have to be modified from Eq. 3.39 and Eq. 5.7:

$$Z(\rho, \mu) = Z_0 + aK(\rho, \mu) \quad (6.1)$$

$$Z(\rho = 1, \mu) = Z_0 + a\kappa_s \sin \mu \quad (6.2)$$

Inside the variational code, all the equations for Z have a Z_0 added. It is not surprising that the output values of the variational parameters do not change with the addition of Z_0 . For a fixed boundary equilibrium solver, the value of Z_0 is not as important as that of R_0 . For the same plasma equilibrium, if we increase or decrease R_0 , the internal flux surfaces must be changed a great deal for the equilibrium. However, if we increase or decrease Z_0 , the whole pattern of flux surfaces of the plasma will shift upward or downward. Hence, changing the value of Z_0 will not affect the outcome of the minimization of the Lagrangian, but since

Figure 6.8: The upgraded neural network

the relative distances of the plasma and the detectors are changed, the magnetic signals will change.

To upgrade the neural network, we can not simply add one more node in the output layer, because the old structure was not trained to accept this fact. Hence, after a series of training and testing of the MLP neural network, a completely new network evolves (Fig. 6.8).

The input consists of R and Z coordinates of 45 points on the separatrix. Thus, there are 90 nodes in the input layer. The hidden layer consists of 30 nodes while the output layer has 5 nodes. The output values are compared to the target values of R_0 , a , D_a , κ_s and Z_0 during the supervised training.

The supervised training stage was set to take 6000 iterations. In each iteration, we generate 3195 samples of separatrices with different shape parameters. The training takes more than 40 hours of cpu time in a VAXstation.4000.60.

Figure 6.9: The mean square residual versus the iteration during the training stage.

Fig. 6.9 shows the mean square residual drops as the number of iterations increases. The training takes 4500 iterations. In each iteration, we generate 3195 samples of separatrices with different shape parameters. The data are presented in logarithm scale with the first few epochs of iterations omitted since their values are too big. The bumpiness implies changes in direction as the minimization process takes place in a multi-dimensional space.

In the experiment, the size of Z_0 is of order of centimeters. The upgraded neural network is trained to handle Z_0 up to 1 decimeter.

To examine how well the network behaves when some noise are added, we randomly generate 500 sets of R_0 , a , κ_s , D_a and Z_0 as inputs. From these shape parameters and Eq. 5.7 and Eq. 6.2, we then generate a 45 point separatrices with $\pm 1\%$ random noise. These sets of 45 point separatrix are input to the trained network. The results are plotted versus the true values (Fig. 6.10 to Fig. 6.14).

Figure 6.10: The test results versus the true values of R_0

Figure 6.11: The test results versus the true values of a

Figure 6.12: The test results versus the true values of D_a

Figure 6.13: The test results versus the true values of κ_s

Figure 6.14: The test results versus the true values of Z_0

6.3 The interpretation of the experimental data

After the variational code and the neural network are upgraded, we apply the reconstruction package to the plasma at time equal to 0.25 second. It corresponds to the flat top region of the plasma current(Fig. 6.1).

Fig. 6.15 shows the fitting of the separatrix. The vertical shift of the magnetic axis is found to be about two centimeters below the mid-plane of the tokamak. This plasma is almost up-down symmetrical.

The surface of the χ^2 is shown in Fig. 6.16. The surface is smooth even when it is extended to the negative b_p domain.

Fig. 6.17 shows the relation between l_i and α . There are 21 lines close together to form a thick line. Basically, l_i increases with α . Fig. 6.18 shows the relation between β_p and b_p . There are 21 lines with some lines fan out. In general, negative b_p

Figure 6.15: The fitting of the separatrix

Figure 6.16: The surface of the χ^2 .

Figure 6.17: The relation between l_i and α

Figure 6.18: The relation between β_p and b_p

Figure 6.19: The relation between β_p and α

implies negative β_p which means negative plasma pressure. We know that negative plasma pressure is not physically possible, so we will restrict the minimum searcher on the positive b_p .

Fig. 6.19 shows the relation between β_p and α , while Fig. 6.20 shows the relation between l_i and b_p . Basically, their correlation are weak.

Fig. 6.21 shows the contours of the χ^2 . Judging from this plot, the minimum region is located in the negative b_p domain. If we restrict the minimum search from the negative b_p domain, we can find a minimum region of the χ^2 . From the result of the minimization we will find the plasma equilibrium. Before we start the simplex method of minimization, we know the position of the minimum from Fig. 6.21. The minimum will be located at b_p about zero and α about 8.

In the 2-d simplex minimization, 76 function evaluations are taken and the cpu time for the searching is 39.77 seconds on a VAXstation.4000.90. The final χ^2

Figure 6.20: The relation between l_i and b_p

Figure 6.21: The contours of the χ^2 .

Figure 6.22: The flux surface inside the plasma

is about 16.18. Fig. 6.22 shows the flux surfaces inside the plasma. Note that the elongation at the plasma boundary is greater than unity while the elongation on the magnetic axis is less than unity.

Table 6.1 shows all the variational parameters after the minimization of the Lagrangian. Mathematically, nobody has ever investigated what the shape of the elongation profile should be. Experimentally, this type of elongation profile is very

Trial function	k_0	ν	σ	η
Eq. 3.39	0.76915	1.72414	0.119643	-1.18085

Table 6.1: The output of the variational parameters.

Figure 6.23: The output from EFIT. (Shot 930623014 Time 0.250s)

strange and has never been seen before. However, we find that it is very easy to have κ_0 less than unity from the variational equilibrium solver.

Fig. 6.23 shows the output from EFIT based on the same magnetic measurement. The version of EFIT is EFITD 33X33 10/11/93. The elongation profile looks quite different from the output of MAGNEQ.

Fig. 6.24 shows the filtered and reconstructed signals of B_p , while Fig. 6.25 shows the filtered and reconstructed signals of flux loops. The reconstructed signals are close to the filtered signals. Fig. 6.26 shows the relative magnitude of the magnetic field error while Fig. 6.27 shows the relative magnitude of the flux error. At some points the errors are bigger because the original signals are small.

Since the elongation profile looks unphysical, we decided to do a series of investigations of this particular plasma equilibrium. We take all the relevant variables and then put into a stand alone version of variational code. We find exactly the

Figure 6.24: The filtered and reconstructed signals of B_p

Figure 6.25: The filtered and reconstructed signals of flux loops

Figure 6.26: The relative magnitude of the magnetic field error

Figure 6.27: The relative magnitude of the flux error

Parameter	PEST	Variational
$R_{axis}(m)$	0.648	0.648
$a\sigma(m)$	0.032922	0.024729
κ_0	1.0116	0.7692
l_i	4.259	2.671
β_p	0.00	0.005
$I_p(MA)$	0.305	0.306522*

Table 6.2: A comparison between the variational method and PEST.

same elongation profile. Hence, we know that this strange profile is not a product of the vacuum solver nor the neural network. We will focus our attention to the variational code.

Again we run the J-solver version of PEST to compare the plasma equilibrium from the variational code. Table 6.2 provides a comparison on a plasma equilibrium. It is between the variational method and the J-solver.

The plasma current in the variational code is an input. The input of J-solver is fine tuned until it gives almost the same total current as the variational code. Although the current profiles in both case have been matched closely, it happens that the l_i value is higher from the J-solver. The elongation on the magnetic axis from the J-solver is 1.0116 which is greater than unity. Now, we suspect the strange elongation profile is a product of the variational method.

6.4 Investigation of elongation profile

It seems that the trial functions (Eq. 3.39) of the variational method is not good enough to provide a reliable elongation profile. Here, we choose three different sets

of trial functions to test their performance in the variational method.

$$K(\rho, \mu) = \rho[\kappa_0 + (\kappa_s - \kappa_0)\rho^2] \sin \mu \quad (6.3)$$

$$\kappa(\rho) = \kappa_0 + (\kappa_s - \kappa_0)\rho^2 \quad (6.4)$$

$$K(\rho, \mu) = \rho[\kappa_0 + (\kappa_s - \kappa_0)\rho] \sin \mu \quad (6.5)$$

$$\kappa(\rho) = \kappa_0 + (\kappa_s - \kappa_0)\rho \quad (6.6)$$

$$K(\rho, \mu) = \rho[\kappa_s + (\kappa_0 - \kappa_s)(1 - \rho^2)] \sin \mu \quad (6.7)$$

$$\kappa(\rho) = \kappa_s + (\kappa_0 - \kappa_s)(1 - \rho^2) \quad (6.8)$$

These sets of trial functions describe plasma elongation profile with different shapes, concave, linear, or convex. We set up three different variational solvers and test them with this particular input file. The result is that the minimization of the Lagrangian does not converge in all the three variational solvers.

Now we take a higher degree of polynomial for the elongation profile as Eq. 6.10 and create another variational solver. We find that the minimization of Lagrangian does converge. The output of the variational parameters are presented in Table 6.3.

$$K(\rho, \mu) = \rho[\kappa_0 + (\kappa_s - \kappa_0)\rho^6] \sin \mu \quad (6.9)$$

$$\kappa(\rho) = \kappa_0 + (\kappa_s - \kappa_0)\rho^6 \quad (6.10)$$

However, if we lower the input value of α from 7.715 to 2.715, which means we present a less peaked current profile, the minimization of the Lagrangian converges

Trial function	k_0	ν	σ	η
Eq. 6.10	0.858875	1.68581	0.112712	-1.38796

Table 6.3: The output of the variational parameters.

Trial function	k_0	ν	σ	η
Eq. 3.39	1.06971	1.52624	4.693266e-02	0.09061
Eq. 6.4	0.97134	1.56247	4.857219e-02	0.29865
Eq. 6.6	0.80087	1.59256	4.800712e-02	0.41734
Eq. 6.8	0.53442	1.68102	4.387990e-02	0.73577
Eq. 6.10	1.11817	1.51450	4.383719e-02	-0.0296

Table 6.4: A comparison between the different trial functions.

in all four variational solvers. The results are listed in Table 6.4. It can be concluded that the minimization of the Lagrangian can be affected by the peakedness of the current profile.

Furthermore, when we take the proposed plasma as shown in Fig. 1.2 as an input, the minimization of the Lagrangian does converge in these four variational solvers. The outputs are listed in Table 6.5. It can be concluded that the change in the trial function can significantly affect the minimization of the Lagrangian and change the resulting equilibrium.

By examining the above data, it seems that the degree of the polynomials are not high enough to describe the elongation profile. We have attempted to fit the elongation profile retrieved from the ASEQ database. We fit the elongation profile

Trial function	k_0	ν	σ	η
Eq. 3.39	1.51625	1.22259	1.962912e-02	0.89585
Eq. 6.4	1.41397	1.26129	2.272207e-02	1.07496
Eq. 6.6	1.21355	1.28796	2.322057e-02	1.17647
Eq. 6.8	1.07351	1.36211	1.568040e-02	1.48285
Eq. 6.10	1.56009	1.20254	1.534556e-02	0.77869

Table 6.5: A comparison between the different trial functions.

with the polynomial interpolation in least square sense.

In the variational solver, the relationship between $\tilde{\psi}$ and ρ is defined by Eq. 6.11. Fig. 6.28 shows that the plot of $\tilde{\psi}$ versus ρ . The values of variational parameter ν are from -0.4 to 3.2 with 0.4 increment. We can conclude that if the variational parameter ν varies between 0.0 to 2.0, the curve is monotonic decreasing.

$$\tilde{\psi} = 1 - \nu\rho^2 - (1 - \nu)\rho^4 \quad (6.11)$$

However, the above relationship does not exist in the ASEQ database. Thus, instead of choosing ρ as the independent variable, we choose x because it has the following relationship with $\tilde{\psi}$.

$$x = 1 - \tilde{\psi} \quad (6.12)$$

$$\kappa(x) = \sum_{n=0}^D C_n x^n \quad (6.13)$$

The above equations are different from what we have been using and they serve as a preliminary survey on this problem.

Fig. 6.29 shows the typical result of polynomial fit of the elongation profile from

Figure 6.28: The relationship between $\tilde{\psi}$ and ρ

Figure 6.29: The polynomial fit of the elongation profile from ASEQ database

ASEQ database. The 29 crosses in each plot represent the calculated elongation at different flux surfaces for the computer generated plasma shot with shot number equal to 3858. The solid line represent the curve generated from the polynomial fit. Each plot corresponds to different value of D in Eq. 6.13.

Judging purely from the appearance of the output of the fitting we find that the value of D in Eq. 6.13 is equal to 4 for the given set of data. Since the elongation on the plasma boundary is known, the number of unknown coefficients in Eq. 6.13 is actually four. Also, when x equals to zero, the elongation is by definition equal to κ_0 which is one of the original variational parameters. It seems that we may need three more variational parameters to describe those elongation profiles, if we choose the above equations as the trial function. All in all, a better trial function is required for the elongation profile.

6.5 Operational region of the variational solver

From the previous experience, we find that the minimization of the Lagrangian will converge only within a certain region in the space of the current profile parameters. Even when it converges, it is hard to tell whether the answer is physically correct. Hence, we have to perform a series of surveys to determine the computationally valid region for the variational solver. At the same time, we need to setup practical criteria to find a physically valid region.

The trial function of the plasma elongation is to the 4th power of ρ .(Eq. 3.39)

There are 4 different types of plasmas under investigation:

1. the unphysical plasma. (Fig. 6.22).
2. the plasma in the proposal. (Fig. 1.2).
3. the plasma with high aspect ratio (Fig. 3.3).

Figure 6.30: The circular plasma under investigation.

4. the circular plasma. (Fig. 6.30).

Each type of plasma are examined by independently varying the b_p and α values, the 2 parameters which govern the shape of current density profile.

In order to find the boundary of the operational region, first, these two parameters are adjusted until the computation crashes. Then, we will focus on the physically valid zone.

From past experience, we know that the computation will certainly crash if the value of α is less than -80.0 or greater than 10.0. Hence, we restrain the α value within that particular range. Moreover, it carries not much significant physical meaning if the value of α lies outside this range.

From the survey we find that, for different plasma, different boundary of operational regions are found. In general, the computation will crash if the values of

PARAMETER	LOWER LIMIT	UPPER LIMIT
b_p	0	1.5
α	-60.0	4.0

Table 6.6: The general computational valid region.

b_p and α are too high.

1. the unphysical plasma. (Fig. 6.22).
 - The computation crashes when b_p reaches 4.
2. the plasma in the proposal. (Fig. 1.2).
 - The computation crashes when b_p reaches 1.5025 and l_i reaches 10.0.
3. the plasma with high aspect ratio (Fig. 3.3).
 - The computation crashes when b_p reaches a high value of 40.
4. the circular plasma. (Fig. 6.30).
 - The computation crashes when b_p reaches 4.

The crashes are closely examined. Basically, there is no minimum region and the minimum searcher drifts towards infinity to search the minimum and then the computation crashes. Table 6.6 shows the general computational valid region. No matter what kind of minimization algorithm we use, we now restrict the searching within this region.

It is clear that the operational zone can be viewed as a physically valid zone lying inside a computationally valid zone. Outside the computationally valid zone, the computation of the variation code will crash.

Inside the computationally valid zone, different iterative method can be applied in searching the correct current profile parameters since the computation is robust here.

Anyway, the operational zone we want is the physically valid zone. The points inside this zone represent both the numerical solution of the variational code and the experimental plasma.

Since we believe that for these plasma it is unphysical to have κ_0 less than unity, we set the κ_0 greater than or equal to one as a necessary condition, but not sufficient, for the physically acceptable plasma. For convenience, we choose κ_0 equals to one as a boundary to define the physically valid zone. Any point outside this zone is considered as unphysical.

It seems that in all the cases κ_0 drops below one if the α is greater than 0. The dropping may even start earlier than that. The value of α corresponds to different value of l_i in different type of plasma. Thus, it is hard to set a general criteria for l_i .

For ohmically heated tokamak, β_p is of order unity. For high beta tokamak, β_p is of order R/a . So it will be meaningless to have β_p equal to 30.

From the survey, it is found that the value of β_p is almost the same as the value of b_p . Thus, they can share the same criterion. The variational code provides a reliable solution with β_p from 0 to 1.5. However, the upper limit of l_i varies from case to case.

Table 6.7 and table 6.8 show the physically valid regions for the four different types of plasmas.

Fig. 6.31 shows the Strickler function when α is -60.0,-10.0,-4.0,-2.0,2.0,4.0. This range of α value describe from very flat to very peaked current profile of the experimental plasma.

PARAMETER	LOWER LIMIT	UPPER LIMIT
l_i	0.5	case dependent
β_p	0	1.5

Table 6.7: The general physically valid region.

Plasma type	α range	l_i range
Fig. 6.22	-60.0 to 3.0	0 to 1.6
Fig. 1.2	-60.0 to 4.0	0 to 2.0
Fig. 3.3	-60.0 to 0.0	0 to 1.0
Fig. 6.30	-60.0 to 0.0	0 to 1.0

Table 6.8: The range of α and l_i .

Figure 6.31: The Strickler function when α is -60.0,-10.0,-4.0,-2.0,2.0,4.0

6.6 Final Conclusions and Suggestions for Future Works

A demonstration of the feasibility of the application of the variational method is shown. The flux reconstruction package has been tested on the computer generated data from ASEQ and EFIT. It shows the potential of the volume integration of current density method as a working tool for magnetic diagnostics. The preliminary application of the package to the experimental data from Alcator C-Mod are performed.

At the present time, the performance of the package is affected by:

- the cpu time spent on the large amount of integration of the current density and Green's function.

- the descriptive power of trial function especially the elongation profile.

In order to fully actualize the power and speed of the variational method, there are several suggestion for future works:

- Find a better method to calculate the poloidal field on the plasma boundary.
- Investigate the elongation profile of different plasma.
- Increasing the number of trial functions or adjustable parameters.
- Replace Strickler's function by some polynomials.
- Up-grade the trial functions and the neural network for up-down asymmetric, diverted plasma.
- Include the effect of eddy current on the flux reconstruction.
- To better evaluate the current profile parameter by comparing data with other diagnostics such as ECE, X-ray tomography, and Thomson scattering.

Appendix A

The testing of the elliptic and Green's functions

The advantage of applying Green's function to solve a magnetostatic problem is that the plasma current can be replaced by a set of ring currents.

For a rectangular mesh with fixed points, the geometric part of the kernel of the integral equations can be calculated only once and for ever. The calculated kernel can be stored in the operative memory of the computer in the form of a two-dimensional matrix. In general, if the size of the grid is not too big, the storage of the matrix will not be a problem. However, if the mesh keeps changing, then the geometric part of the kernel will need to be changed every time and this costs a significant amount of computation time.

In order to provide a sufficiently accurate and high speed integration, the elliptic functions has to be chosen wisely.

Here, the numerical calculation of the first and second kind of complete elliptic functions, $K(k)$ and $E(k)$ respectively, are translated from the pseudo code written by Bulirsch [75, 76] to C programming language [77]. The computed values of the $K(k)$ function are compared with mathematical handbook [78] and are found to be

Figure A.1: The accuracy of the Green's function

accurate to seven decimal places.

The Green function which consisted both of the first and second kind of complete elliptic functions will be singular as the reference point and the source point coincided with each other. Numerically, this singularity could produce errors as the two points are close to each other. For the magnetic probes on the inboard side of the vacuum vessel, they can be as close as centimeters from the plasma edge. Hence, it is wise to test how close can the reference point moves to the source point and still keeps the accuracy of the Green functions.

Eqn. (A.1) is an approximation to the Green's function as the reference point gets close to the source point.

$$\lim_{(R,Z) \rightarrow (R',Z')} \Psi = \mu_0 R' \left[\ln \left(\frac{8R'}{|R - R'|} \right) - 2 \right] \quad (\text{A.1})$$

Figure A.2: The accuracy of the Green's function

Figure A.3: The accuracy of the Green's function

Figure A.4: The accuracy of the Green's function

Fig. A.1 to Fig. A.4 show the calculated values of the Green's function. The coordinates of the source point, (R', Z') , equals to $(1.0050\text{m}, 0\text{m})$. The solid lines represent

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