Contribution from The Kellex Corporation

and Carbide and Carbon Chemicals Corporation

Research Paper A-3 June 24, 1946

UNSTEADY-STATE SEPARATION PERFORMANCE OF CASCADES

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1. INTRODUCTION

Multi-stage cascades for close fractionation of nearly identical components are becoming of increasing industrial importance. Distillation columns are being designed for isomers and other closely related components whose boiling points are only a few degrees apart, and whose relative volatilities are 1.05 or less. A number of isotope separation plants making use of fractional distillation, thermal diffusion, chemical exchange, [redacted] have been built. Such a cascade or column is generally characterized by a long start-up time and a slow response to changes in operating conditions. This is the case when the holdup of product component in the cascade is many times the daily rate of production. A significant fraction of the operating life of such a cascade is spent in starting up, in transition from one set of operating conditions to another, and in recovery from transient disturbances.

[redacted]

Procedures for setting optimum start-up and transition procedures and (for minimizing) transient losses in such a cascade are of comparable practical importance to procedures for setting optimum steady-state operating conditions.

The purpose of this paper is to develop the theory of the unsteady-state behavior of these cascades and to present simple ways of thinking about this behavior. Unfortunately, exact calculating procedures for a cascade of complicated structure are generally very difficult and cannot be made a matter of routine.

In this paper the following mode of attack has been adopted: A cascade with an idealized structure resembling cascades of practical interest is specified. The specification is chosen so that the unsteady state behavior of the idealized cascade can be calculated exactly. Comparison of the unsteady state behavior of the idealized cascade with its steady state behavior has suggested approximate methods of estimating one from the

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Throughout this paper, we regard the light component as the desirable one, and so we speak of the material withdrawn at the top of the cascade as "product," and that at the bottom as "waste." Our main interest is in the case in which the rectifier's inventory of light component is many times the daily take-off of light component, and the stripper's inventory is the same order of magnitude as the daily waste rate. (This is generally true if the light component is appreciably enriched, and if the recovery of light component from feed is not extremely high.) In this case, the principal feature of interest during unsteady-state operation is the change in the rectifier's light component inventory (either its magnitude or distribution), and the results of this paper largely involve ways of thinking about this change. Our results can readily be modified if the heavy component is desired, appreciably enriched and not at extremely high recovery (the rectifier is now below the feed point, and again its product component inventory is an important cascade variable). The case in which both appreciable enrichment and extremely high recovery are desired (both rectifier and stripper inventories are important) is more complicated, but many of our results can be generalized to provide useful information about this case as well.

other. Since exact steady state calculations for complicated cascades are simple [redacted] good estimates of the unsteady state behavior of cascades of practical interest can readily be obtained. It is felt that this mode of attack may often have value when solutions are sought for important time-dependent problems (heat flow, diffusion, etc.) in systems too complicated for direct calculation of time-dependent phenomena, but amenable to calculation of steady states.

bution times are discussed briefly, with special reference to the case of a uniform concentration in a large number of stages at the top of the plant.

Section 3 gives an exact procedure for evaluating the loss in output resulting from a transient disturbance which upsets the steady-state distribution of light component in the cascade. This procedure is valid for a cascade of any design.

Section 4 discusses the problems connected with the search for an optimum start-up procedure for a cascade, such that the desired steady state is reached with a minimum loss of production.

Section 5 gives the exact solution of the time-dependent problem for a special continuously tapered cascade (interstage flows monotonically decreasing with stage number counting away from the feed point), operating in the range of light component mol fraction small relative to one. The taper is chosen to resemble cascades of practical interest, and is specified by functions selected such that the solution of the time-dependent problem can be written as a generalized Fourier Series of products of two elementary functions, one of time alone and the other

of the stage number. The degree of taper of the special idealized cascade can be varied between wide limits by suitable choice of a parameter in the equation specifying the cascade's interstage flows. The square cascade (interstage flows constant throughout cascade) is included as a special case.

Section 6 gives an exact solution of the time dependent problem for square cascades operating in concentration ranges where mol fraction is not small relative tq one.

Sections 2, 3, and 4 can be read by the non-mathematical reader without reference to Sections 5 and 6.

2. APPROXIMATE EQUATION FOR VARIATION OF TOP STAGE CONCENTRATION WITH TIME.

This section gives an approximate method of calculating the concentration at the top stage of a cascade as a function of time during unsteady-state operation. Results obtained by this method, which relies only on knowledge of the steady-state behavior of the cascade [redacted] are compared with exact results for several special cascades.

Unsteady-state problems will be divided into three classes:

1. Approach to steady-state of a cascade initially filled with inventory at feed concentration.
2. Approach to steady-state of a cascade initially operating at another steady-state.
3. Approach to steady-state of a cascade whose initial concentration distribution is arbitrary.

In this section, we shall treat problems of classes 1 and 2 in the most detail.

It is found that a cascade may be said to approach steady-state through a succession of intermediate steady-states; that is, at any instant during the approach to the final steady-state the concentration distribution in the cascade is very nearly the same as the steady-state distribution appropriate to the top stage concentration at that instant. This remark holds if no product is withdrawn during the approach to steady-state (in which case the steady-state corresponds to zero production), and also if product is withdrawn continuously during the approach to the steady-state corresponding to production at the rate of interest. Use is made of this observation to derive the approximate equation (2.6) for the approach to steady-state in terms of cascade steady-state properties.

Problems of class 3 (redistribution times) will be considered briefly at the close of this section, and in greater detail in the next section.

8.1 Examples of Exact Curves Describing Approac.. to Steady-State - The exact methods of Sections 5 and 6 have been used to determine the behavior of three cascades during the approach to the steady-state corresponding to no product withdrawal (total reflux), when the cascades are operated from the outset without product withdrawal and are initially filled with feed.

(i) Tapered Cascade in Low Concentration Range - The special tapered cascade, developed in Section 5 for the case of constant ψ and $N \ll 1$, and described by equation (5.26), has been studied. Numbers have been obtained for the case $t\psi = 6.96$, $\propto t = 3.84$, and $\pi = 0$ (zero production). This cascade has a taper closely resembling an ideal cascade with the same value of $t\psi$. The total pumping required by the cascade is 0.3% less than that of the ideal cascade, and the output at a product purity $N_p = N_0 e^{t\psi/2}$, the design purity of the ideal cascade, is about 2% less. Figure 1 compares the interstage flows of the two cascades under discussion. The tapered cascade considered in this paper closely resembles cascades of practical interest. Practical cascades are apt to be

very nearly ideal in the lower stages, where pumping and materials requirements are high, and sound design seeks to minimize flow and equipment size. Practical cascades are apt to be blunt-topped (i.e. flows at the top many times ideal). The principal cost here is the unit cost per stage, which is nearly independent of equipment size. The ideal cascade's property of minimizing flow and equipment size is no longer an advantage, since these requirements are small in any event, and there is great practical advantage of squaring off the cascade, with the accompanying reduction in number of stages and reduction in number of equipment types.

Most practical cascades will probably lie between the tapered cascade here described and the square cascade. There is risk in designing a cascade with a taper sharper than ideal (even though costs may be apparently reduced by such a design) because failure of equipment to perform according to specification has a worse effect on overall cascade performance in stages smaller than ideal that in stages larger than ideal. It is felt

that an understanding of the time-dependent behavior of our tapered cascade and of square cascades will cover a wide range of interest for cascades of practical design.

Figure 2 gives curves of light component concentration versus stage number at several times during the tapered cascade's approach to the steady state corresponding to no product withdrawal. Figure 4 gives a curve of top stage concentration versus time.

(ii) Square Cascade In Low Concentration Range - Curves have been developed for a square cascade having the same $t\psi = 6.96$ as the tapered cascade. Figure 3 gives curves of concentration versus stage number, and Figure 5 gives a curve of top stage concentration.

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2.2 Approach to Find Steady-State Through a Succession of Intermediate Steady-States - Inspection of Figures 2 and 3 suggests that one of the time-dependent curves of concentration versus stage number may be approximated by the steady-state curve for production of material at the same top stage concentration as the time dependent curve. The steady state curves ploted in Figures 2 and 3 (see dashed curves) demonstrate that this is the case. The cascade may be said to approach final steady-state through a succession of intermediate steady-states.

Use is made of this observation to derive an approximate expression for the approach to steady state. At a given instant during the approach, the cascade's inventory of light component is $X(N_p)$, where N_p is the top stage concentration at this instant. We use the steady-state inventory corresponding to N_p as an estimate of the instantaneous inventory during the approach. Figure 7 compares the instantaneous inventory values, given by the curve, with representative steady-state values, given by the open circles. The time required for top stage concentration to change from $N_p(T)$ to $N_p(T) - dN_p$ is equal to

the time required to achieve the corresponding increment in inventory $d X(N_p)$. The latter time is readily seen to be the quotient of $d X(N_f)$ divided by the net transport of light component into the cascade at that instant.

We can obtain an estimate of the instantaneous net transport in the following way: The net transport of light component Φ at a stage is given by:

$$\Phi = F\left\{\psi N(1-N) - \frac{dN}{d\alpha}\right\} + PN \qquad (2.1)$$

where P is the net transport of total material at the stage. If P, N, and $dN/d\alpha$ are changing with time, the equation holds for the instantaneous values, and Φ in general will have a different value at each stage. During steady-state operation at purity N_p, $\Phi = PN_p$ at every stage. In particular, at the feed stage during steady-state operation,

$$PN_p = F_o\left\{\psi N_o(1-N_o) - \left(dN/d\alpha\right)_o\right\} + PN_o \qquad (2.2)$$

During the approach to steady-state (with no product withdrawal), at the feed stage

$$\Phi_o = F_o\{\psi N_o(1-N_o) - (dN/da)_o\} . \qquad (2.3)$$

If we assume that the instantaneous concentration gradient $(dN/da)_o$ at the feed stage is the same as the steady-state gradient corresponding to the same N_p, we find

$$\Phi_o = P(N_p - N_o) , \qquad (2.4)$$

for the instantaneous net transport. Figure 8 compares the instantaneous net transport for the tapered and square cascades with the values obtained from steady-state values of P and N_p through the use of equation 2.4. (Notice the sharp reduction in net transport at very low purities in the case of the tapered cascade, and the relative constancy of the net transport up to fairly high purities in the case of the square cascade; these effects reflect the rapid increase in $(dN/da)_o$ shown by Figure 2 for the tapered cascade, and the long persistance of a value near zero shown by Figure 3 for this derivative in the square cascade.)

Thus, we find the following approximate expression

$$\left(\frac{dT}{dN_p}\right) = \frac{dX/dN_p}{P(N_p)[N_p(T) - N_o]} , \qquad (2.5)$$

where $N_p(T)$ stop stage concentration at time T during a transition, and $P(N_p)$ is the steady-state product removal rate corresponding to the product

purity $N_p(T)$. Integration of (2.6) yields the following implicit relation for top stage concentration as a function of time:

$$T(N_p^*) = \int_{X_o}^{X(N_p^*)} \frac{dX(N_p)}{P(N_p)[N_p(T) - N_o]} \qquad (2.6)$$

where X_o = initial light component inventory;

$X(N_p^*)$ = steady-state light component inventory corresponding to product purity N_p^*;

N_p^* = top stage concentration at time $T(N_p^*)$, which may be evaluated from the equation.

For a given cascade (Ψ, H, F fixed) at a steady-state, specification of any one of the three variables N_p, P, and X determines the other two. Integration of (2.6) in closed form is usually difficult if not impossible because of the complicated relationship between P, N_p, and X. Numerical values of the integral may be conveniently obtained by plotting $\frac{1}{[P(N_p)\quad][N_p(T) - N_o]}$ versus X and estimating the area under the curve between X_o and $X(N_p^*)$.

The open circles of Figures 4, 5, [illegible] were obtained using equation (2.6). The agreement between the exact curves and the approx- 16m363

imate equation is excellent for the three cascades studied. The product purities of greatest interest are those in the neighborhood of $N_p/N_o = 50$ to 100 in Figures 4 and 5, [redacted] (Figure 8 shows the sharp decrease in steady-state transport of light material for values of N_p/N_o greater than about 100.) It is seen that equation (2.6) overestimates by about 5% the time for the tapered cascade to reach purities of interest. This is a consequence of the fact that for the tapered cascade instantaneous transports and inventories are somewhat higher and lower respectively than the steady-state values.

2.3 Range of Applicability of Approximate Equation - The approximate equation (2.6) should apply to any blunt-top cascade in the low concentration range with taper between a square cascade that of the tapered cascade studied here. It does not apply to an ideal cascade. The small size of the top stages of this cascade causes the concentration gradient at the top of the cascade to be greatly different at steady state from the instantaneous gradient during approach to steady state. For any cascade the latter gradient is always given by

$$\left(\frac{dN}{dz}\right)_t = \Psi N_p(1-N_p). \qquad (2.7)$$

For an ideal cascade the top stage steady-state gradient at the design product purity is just one-half this value, and for production at lower purities the steady-state gradient is flatter still, with negative values below a certain purity (and a corresponding maximum purity at an intermediate stage; this is of course an absurd mode of operation, since one will abandon the top stages if one requires material of an appreciably lower purity than design). It is for this reason that the light component inventory of an ideal cascade is about 30% short of the steady-state de-

sign inventory when the top stage concentration reaches design purity during the approach to steady state [redacted] while the inventory of the tapered cascade studied here is practically complete when this occurs.

A blunt-top cascade has the same top stage concentration gradient at steady-state as during transition (given by 2.7). However, even in the case of our tapered cascade, at low product purities the gradient at stages near the top is much flatter in the steady state than it is during transition. This gives rise to the discrepancy in Figure 4 between the exact curve and approximate equation (2.6) at early times.

2.4 Generalization of Approximate Equation - Equation (2.6) may be generalized to include any approach to a new steady-state when the cascade is initially operating at a steady-state. Suppose the schedule of actual product withdrawal set by the operator is $P(T)$ during the transition, and suppose $P(T) < P(N_f)$ at all times. The rate of increase of inventory of light component is now equal to the net transport of light component at the feed stage minus the rate of withdrawal at the top, $P(T)N_p(T)$. The instantaneous net transport at the feed is now given by:

$$\Phi_0 = F_0\{\psi N_0(1-N_0) - (dN/d\Delta)_0\} + P(T)N_0 . \qquad (2.8)$$

Again assuming the instantaneous gradient to be identical with the steady state gradient corresponding to the same N_p, we find:

$$\Phi_0 = [P(N_p) - P(T)][N_p(T) - N_0] , \qquad (2.9)$$

And thus

$$\frac{dT}{dN_p} = \frac{dX/dN_p}{[P(N_p) - P(T)][N_p(T) - N_0]} ; \qquad (2.10)$$

Integration of (2.10) from the initial steady-state inventory yields:

$$T(N_p^*) = \int_{X_0}^{X(N_p^*)} \frac{dX(N_p)}{[P(N_p) - P(T)][N_p(T) - N_0]} \qquad (2.11)$$

A number of exact calculations of this type have been made for both square cascades and the tapered cascade described by equation (B.26) Cascades initially in steady-state production at a given rate were allowed to approach new steady-states corresponding either to no production or to a new lower product rate. In all cases the approximate equation (2.11) is an excellent estimate of the

cascade's behavior. (The cumulative loss of output in the latter case, where the product rate is suddenly reduced to a lower value greater than zero, may be calculated exactly for any cascade from equation (3.1) of the next Section.)

A consequence of the generalization of (2.6) given by (2.11) may be noted: Once a curve for the approach to steady-state at total reflux has been prepared for the case of a cascade initially filled with feed, the curve for the approach to steady state at total reflux for the case of a cascade initially operating at some steady state is the same curve with the time scale displaced so that time reads zero at the initial steady state.

2.5 Simpler Approximate Equation for Square Cascades - Numerous calculations for square cascades at low concentration

ranging in size from $\ell = 1$ to ∞ show that the following simple expression, suggested by K. Cohen, is a fair estimate of the top stage concentration at early times:

$$T(N_P^*) = \frac{X(N_P^*) - X_0}{P(N_P^*)\left[N_P^* - N_0\right]} \qquad (2.12)$$

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This equation uses the steady-state net transport at the feed stage corresponding to the final purity N_p^* as an estimate of the time-average net transport between zero time and time $T(N_p^*)$. It has been found that this is a good estimate for top stage concentrations N_p^* such that

$$\frac{N_p^*}{N_0} < e^{\frac{r-1}{r+1}} \qquad (2.13)$$

The plausibility of this result may be seen from Figure 8 which indicates the comparative independence of the net transport up to $N_p/N_0 \cong 100$.

Use can be made of this result to obtain an approximate equation for the time dependence of the concentration at the top stage of a square cascade. Equation (2.12) is inadequate at long times; this inadequacy is corrected in an alternative expression which has the same functional form as the exact solution in the limit of long times. The exact solution is a linear combination of exponential functions. This suggests the functional form which attempts to approximate $N_p(T)$ by a single exponential:

$$N_p(T) = N_{\infty p} - (N_{\infty p} - N_0)e^{-T/\Theta} \qquad (2.14)$$

where $N_{\infty p}$ steady-state value of N_p at total reflux. Clearly, (2.14) is exact at times $T=0$ and $T=\infty$. The single time constant (H) can be determined by using (2.12) to determine the time $T(N_p^*)$ corresponding to N_p^*, where N_p^* is carefully chosen. Experience has shown that (2.14) is a fair approximation of the approach to steady state if the N_p^* chosen for evaluation of (H) lies in the range defined by:

$$\frac{1}{4} < \ln \frac{N_{\infty p} - N_0}{N_{\infty p} - N_p^*} < \frac{1}{2} \quad . \tag{2.15}$$

Equation (2.15) is not as good an approximation as equation (2.6) but has the advantage of requiring knowledge of the cascade's steady-state performance at only one product purity, rather than knowledge over the entire range.

Equation (2.12) gives a serious over-estimate of the time required for the tapered cascade developed in Section 5 to reach top stage concentrations in the range of interest ($N_p/N_0 = 50$ to 100).

2.6 <u>Arbitrary Initial Distribution</u> - Before discussing the general problem of treating approach to steady state from an initial ar-

bitrary distribution, we will describe a typical example.

In general one will not bring all of a cascade into operation simultaneously. Suppose a cascade is brought into operation in two parts. Suppose the first part reaches a steady state for a given product rate and produces a stock pile of enriched material which is used to fill the second part when it is ready. The curve of Figure 9 labelled " θ " illustrates the situation immediately after the second part is put in operation, and other solid curves give concentration versus stage number at several later times. The solid curve of Figure 10 shows the top stage concentration versus time. These curves are exact, and were obtained by the methods given in Section 5.

The dashed curve of Figure 9 is the steady-state concentration distribution corresponding to exactly the same light component inventory as is contained in the cascade at zero time. We notice that the concentration distribution rapidly approaches this steady state distribution; this redistribution of inventory occurs during the initial sharp rising portion of the curve of Figure 10.

In fig. 10
The dashed curve labelled A gives exactly the top stage concentration versus time if the initial cascade concentration distribution is the dashed steady-state curve of Figure 9. It is seen that after the short period of redistribution, the solid curve closely parallels the dashed curve. The small time lag between the two curves results from the fact that during the redistribution time the concentration gradient at the feed point is steeper than in the steady state, and the net transport of light material is correspondingly less.

We also notice an interesting property of the stage at which the dashed steady-state curve of Figure 9 crosses the initial concentration curve. The concentration at this point remains nearly constant during the redistribution, which may be thought of as a movement of light component past this stage from the lower region of excess to the upper region of deficiency. This stage may be termed the "pivot point" of the redistribution. The stages above the pivot point may be regarded as a "partial cascade", and the redistribution described as the approach to steady state of this partial cascade fed by a reservoir at the pivot point concentration. In Figure 10, the dashed curve labelled C gives

the exact time variation of top stage concentration of the partial cascade, regarding it as isolated from the remainder of the cascade and fed by material at the pivot point concentration. It is seen that the solid curve may be estimated fairly well by a composite of dashed curves A and C.

Generally speaking, if the initial concentration distribution is arbitrary, the cascade quickly redistributes its light component inventory to approach the steady-state distribution corresponding to the same inventory. If one is interested in predicting top stage concentrations after a long time, a fair estimate may be obtained from equation 2.6, setting X_0 equal to the initial light component inventory, even though the inventory distribution is not a steady-state distribution. For predicting concentrations after short times, the concept of a partial cascade described above may frequently be of use, if the arbitrary distribution crosses the corresponding steady-state distribution only once, and if the distribution above the pivot point defined by this crossing is not too irregular.

3. LOSS OF OUTPUT DUE TO TRANSIENT DISTURBANCES

During steady state operation of a cascade, disturbances frequently occur which cause the distribution of light component in the cascade to depart from the normal steady state distribution. Power failures, removal of equipment for repairs, temporary changes in equipment characteristics, temporary interruption of feed, waste, or product streams, temporary interruption of interstage flow at some point in the cascade --- many such disturbances cause the shifting and mixing of inventory from one part of the cascade to another. After such a disturbance, a plot of concentration versus stage number differs from the steady-state curve and the cascade's output of light component differs from the steady-state rate, but as time goes by the concentration curve and output return to normal. One wishes to be able to calculate the overall loss in light component output caused by the disturbance. In practice it is impossible to observe the loss in output caused by a particular disturbance because of the many factors causing day to day fluctuations in the product rate and purity, and because in most instances the loss in output due to a particular disturbance is felt

over a long period of time and has a hardly noticeable effect on a given day's production. One would therefore like to be able to compute the loss in output from the only remaining observable at one's disposal: The curve of light component concentration versus stage number immediately after the disturbance.

It will be shown in this section that the loss in light component output, if the rate of product removal remains constant, is given by the following expression:

$$\text{Loss in light component output} = \int_0^{\tau} H(s)\left[N_\infty(s) - N^\circ(s)\right] J(s)\, ds \qquad (3.1)$$

where

$H(s)$ = holdup of s^{th} stage.

$N_\infty(s)$ = steady-state light component concentration at s^{th} stage. (This is the concentration a long time after the disturbance.)

$N^\circ(s)$ = light component concentration at s^{th} stage immediately after disturbance.

t = number of stages in cascade.

$J(\Delta)$ = a positive weighting function (see equation 3.25) whose value is small compared with unity near the bottom of the cascade, and increases to a value near unity at the top of the cascade.

This equation is valid for all cascades, whatever the dependence of ψ and F on stage number. It holds whether the total light component inventory immediately after the disturbance is the same as the steady state light component inventory or not. If a graph of $J(\Delta)$ versus stage number has been prepared, the integral of equation (3.1) can easily be evaluated by numerical methods.

First, the expression for $J(\Delta)$ will be derived. Second, the effect of a particular disturbance (the uniform mixing of inventory in a region of the cascade) will be considered, and illustrated by examples calculated for a typical cascade. Last, the duration of a disturbance's effect on product purity will be discussed briefly.

3.1 Derivation of Expression for $J(\Delta)$ - Suppose a cascade of t stages (u stages in the stripper and $t-u$ stages in the rectifier) is operating with a constant product withdrawal rate P and waste rate W (feed rate $I = P+W$). The following argument applies to a cascade whatever the dependence of overall stage enrichment factor $\psi(\Delta)$, stage holdup $H(\Delta)$, and interstage flow $F(\Delta)$ on stage number Δ. Suppose that at $T=0$ a disturbance occurs which upsets the concentration distribution in the cascade. Let $N^0(\Delta)$ be the light component mol fraction at the Δ^{th} stage at zero time. In general $N^0(t)$ and $N^0(0)$, the initial concentration at the top and bottom of the cascade, are different from N_P and N_W, the steady state product and waste purities. Moreover, the initial light component inventory, $\int_0^t H(\Delta) N^0(\Delta) d\Delta$ may differ from the steady state inventory $\int_0^t H(\Delta) N_0(\Delta) d\Delta$ where $N_0(\Delta)$ is the steady state mol fraction at the Δ^{th} stage. Our problem is to find the overall loss of light component output caused by the disturbance; this is equal to the product rate times the time integral from zero to infinite time of the discrepancy between steady state

product purity and the transient top stage concentration.

Loss of light component output = $P\int_0^\infty [N_P - N(t,T)]\,dT$ (8.2)

where $N(t,T)$ is top stage concentration at time T. Since N and δ (defined in terms of N) are the only variables depending on T, independent variables will be indicated only when special emphasis is needed.

Let us write the material balance about the part of the rectifier between stages Δ and t. At any time T:

$$F(\Delta)N'(\Delta,T) - [F(\Delta)-P]N(\Delta-1,T) = PN(t,T) + \frac{\partial}{\partial T}\int_\Delta^t M(\Delta)N(\Delta)\,d\Delta \quad (8.3)$$

Using the overall stage enrichment factor $\psi(\Delta)$ relating $N'(\Delta,T)$ with $N(\Delta,T)$, we obtain:

$$N'(\Delta,T) = N(\Delta,T) + \psi(\Delta)N(\Delta,T)[1-N(\Delta,T)] \quad (8.4)$$

$$\frac{\partial N}{\partial \Delta} - \psi N(1-N) - N\frac{P}{F} = -N(t,T)\frac{P}{F} - \frac{1}{F}\frac{\partial}{\partial T}\int_\Delta^t HN\,d\Delta \quad (8.5)$$

The differential equation for the steady state concentration is as follows:

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$$\frac{dN_0}{d\Delta} - \psi N_0(1-N_0) - N_0 \frac{P}{F} = -N_p \frac{P}{F} \qquad (8.6)$$

Let us subtract equation (8.5) from equation (8.6), and let

$\delta(\Delta,T) = N_0(\Delta) - N(\Delta,T)$, $\delta_p(T) = N_p - N(t,T)$; approximate ($N_0^2 = N^2$)

by $2N_0\delta$.

$$\frac{\partial \delta}{\partial \Delta} - [\psi(1-2N_0) + P/F]\delta = -\delta_p P/F + \frac{1}{F}\frac{\partial}{\partial T}\int_\Delta^t HN\,d\sigma \qquad (8.7)$$

It is convenient to introduce the functions:

$$A(\Delta) = e^{-\int_u^\Delta [\psi(1-2N_0) + P/F]\,d\sigma} \qquad (8.8)$$

and

$$Q(\Delta) = \int_u^\Delta \frac{A(\sigma)}{F(\sigma)}\,d\sigma \qquad (8.9)$$

If we multiply equation (8.7) by A , we get

$$\frac{\partial (A\delta)}{\partial \Delta} = -PQ'\delta_p + Q'\frac{\partial}{\partial T}\int_\Delta^t HN\,d\sigma, \qquad (8.10)$$

where primes denote differentiation with respect to Δ .

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Integrating between u and t, we get

$$A(t)\delta_p - \delta_J = -P\delta_p Q(t) + \frac{\partial}{\partial T}\int_u^t Q' \int_s^t HN \, d\sigma \, ds \qquad (8.11)$$

where $\delta_J(T) = N_\infty(u) - N(u,T)$ is the concentration difference at the junction between stripper and rectifier. The integral on the right may be evaluated by parts:

$$\int_u^t Q' \int_s^t HN \, d\sigma \, ds = Q \int_s^t HN \, d\sigma \Big|_u^t + \int_u^t HNQ \, ds . \qquad (8.12)$$

Since the first term on the right vanishes at each endpoint, we have, finally, for the rectifier:

$$[Q(t) + A(t)/P]P\delta_p - \delta_J = \frac{\partial}{\partial T}\int_u^t HNQ \, ds . \qquad (8.13)$$

A similar argument, starting from a material balance around the part of the stripper between stages 0 and s, gives the following sequence of equations, where $\delta_W(T) = N_W - N(0,T)$:

$$\frac{\partial N}{\partial s} - \psi N(1-N) + N\frac{W}{F} = N(0,T)\frac{W}{F} - \frac{1}{F}\frac{\partial}{\partial T}\int_0^s HN \, d\sigma \qquad (8.14)$$

$$\frac{\partial N_\infty}{\partial s} - \psi N_\infty(1-N_\infty) + N_\infty\frac{W}{F} = N_W\frac{W}{F} \qquad (8.15)$$

$$\frac{\partial \delta}{\partial \Delta} - \left[\psi(1-2N_\infty) - \frac{W}{F}\right]\delta = \delta_W \frac{W}{F} + \frac{1}{F}\frac{\partial}{\partial T}\int_0^\Delta HN\, d\sigma. \tag{8.16}$$

It is convenient to introduce the functions:

$$B(\Delta) = e^{\int_\Delta^u [\psi(1-2N_\infty) - W/F]\, d\sigma} \tag{8.17}$$

and

$$R(\Delta) = \int_\Delta^u \frac{B(\sigma)}{F(\sigma)}\, d\sigma . \tag{8.18}$$

If we multiply equation (8.16) by B, we get

$$\frac{\partial (B\delta)}{\partial \Delta} = -WR'\delta_W - R'\frac{\partial}{\partial T}\int_0^\Delta HN\, d\sigma \tag{8.19}$$

Integrating between 0 and u, we get

$$\delta_J - B(0)\delta_W = WR(0)\delta_W - \frac{\partial}{\partial T}\int_0^u R'\int_0^\Delta HN\, d\sigma\, d\Delta \tag{8.20}$$

Evaluating the integral on the right by parts, we obtain:

$$\delta_J - [R(0) + B(0)/W]W\delta_W = \frac{\partial}{\partial T}\int_0^u HNR\, d\Delta . \tag{8.21}$$

Now add equations (8.13) and (8.21), integrate from $T=0$ to $T=\infty$ (equilibrium), and denote the initial state $N(\Delta,0)$ by N°.

$$[Q(t)+A(t)/P]P\int_0^\infty \delta_p\,dT - [R(0)+B(0)/W]W\int_0^\infty \delta_w\,dT$$

$$= \int_0^u H(N_0-N^\circ)R\,d\Delta + \int_u^t H(N_0-N^\circ)Q\,d\Delta . \quad (8.22)$$

Overall material balance supplies a relation permitting the elimination of $W\int_0^\infty \delta_w\,dT$:

$$P\int_0^\infty \delta_p\,dT + W\int_0^\infty \delta_w\,dT = \int_0^t H(N_0-N^\circ)\,d\Delta . \quad (8.23)$$

We arrive at the following expression for the loss of light component output:

$$P\int_0^\infty [N_p-N(t,T)]\,dT = \int_0^t H(N_0-N^\circ)\,J\,d\Delta \quad (8.24)$$

where

$$J(\Delta) = \frac{G(\Delta)+R(0)+B(0)/W}{Q(t)+A(t)/P+R(0)+B(0)/W}, \quad u \le \Delta \le t$$

$$J(\Delta) = \frac{R(\Delta)+R(0)+B(0)/W}{Q(t)+A(t)/P+R(0)+B(0)/W}, \quad 0 \le \Delta \le u . \quad (8.25)$$

The loss in light component output can be obtained for any initial concentration distribution $N^\circ(\Delta)$ by a simple numerical integration, if

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$J(\alpha)$ is known.

The function $J(\alpha)$ has the following physical interpretation: If at a given moment the inventory of stage α does not contain the normal amount of light component, but is deficient by an amount $H(N_\alpha - N^\circ)$, there will be a deficiency in the quantity of light component withdrawn in product until the steady state normal inventory and product rate has been restored. The quotient of the resulting total cumulative reduction in light component output divided by the initial light component inventory deficiency at stage α is given by $J(\alpha)$. An initial light component inventory deficiency at stage α may be caused by a cascade disturbance (which may produce a deficiency in some parts of the cascade and a gain in others).

One could also create an initial deficiency at stage α by instantaneously withdrawing some of the stage's inventory and replacing it by an equal amount of pure heavy component. It is interesting therefore to note the close relationship between the function $J(\alpha)$ and the steady-state loss in output if partially enriched material is continuously removed at a fixed rate from stage α and is replaced pound

for pound by pure heavy component. In this case, as well as in the unsteady-state case described above, $J(\Delta)$ is the quotient of the loss in light component output divided by the quantity of light component removed:

Clearly, equation (3.24) gives the cumulative loss of output during a transition from one steady state to another. If the product rate is suddenly decreased, the equation gives the integral of the discrepancy between actual top stage concentration and the steady state value corresponding to the new product rate, if N_{∞} and J are calculated for the new product rate, and N° for the old.

Also, equation (3.24), with N° set equal to N_0, gives an estimate of cumulative loss of output when a cascade is first put in service. This is discussed more fully in Section 4.

The evaluation of $J(\Delta)$ for a cascade in which V and F vary continuously with stage number generally requires numerical integration. Evaluation of $A(\Delta)$ and $B(\Delta)$ is straight-forward for a cascade constructed of square sections (a section is a group of stages with constant V and F; for such a cascade, a good estimate of $Q(\Delta)$ and $R(\Delta)$ is readily

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obtained by replacing $(N'/4)$ within a section by the section average $\overline{N'}$

8.2 Example of Loss Due to Transient Disturbance - In this sub-section the loss due to a particular transient disturbance will be calculated to illustrate the use of equation (8.24). Suppose the inventories of a number of stages are uniformly mixed. This would produce a flat region in the curve of light component concentration versus stage number, without altering the total inventory of light component in the cascade. (When stages are removed from the cascade for repairs, it is frequently necessary to mix their inventories before returning the stages to onstream service.) The initial uniform concentration of the flat region is given by:

$$\bar{N} = \frac{\int_{A_1}^{A_2} H(A) N_0(A)\, dA}{\int_{A_1}^{A_2} H(A)\, dA} \qquad (8.27)$$

where A_1 and A_2 are the stages at the ends of the flat region. The loss in light component output is given by:

$$\int_{A_1}^{A_2} H(A)(N_0 - \bar{N})\, J(A)\, dA \qquad (8.28)$$

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Equations (8.27) and (8.28) can be evaluated numerically, but a more convenient expression can be found for the limiting case as $(\Delta_2 - \Delta_1)$ approaches zero. If the integrands of (8.27) and (8.28) are expanded in Taylor's series about the value at Δ_1, an expansion of the loss in powers of $(\Delta_2 - \Delta_1)$ can be derived. The first non-vanishing term is:

$$\text{loss} \approx \frac{(\Delta_2-\Delta_1)^2}{12} \frac{H(\Delta_1)}{F(\Delta_1)} \frac{A(\Delta_1)\left(dN_c/d\Delta\right)_{\Delta_1}}{G(t)+A(t)/F+R(\infty)+B(\infty)/N} \quad (8.29)$$

the transient loss is closely related to the steady state loss caused by removing stages permanently from the cascade. Remembering that $H/2F = \tau$, the stage holdup time, we find that the transient loss due to mixing the inventories of stages between Δ_1 and Δ_2 (expressed as total cumulative loss of light component produced) is equal to $\frac{(\Delta_2-\Delta_1)^2}{6}\tau$ times the steady state loss due to permanent removal of the stages between Δ_1 and Δ_2 (expressed as the

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decrease in light component output per unit time). It is as if the mixing of inventory causes the stages between d_1 and d_2 to be removed from productive work in the cascade for a time equal to $\frac{(n_2 - n_1)^2}{6} \tau$. The square relationship is plausible since the amount of light component shifted down-cascade as a result of the mixing is proportional to the square of the number of stages involved, and the time to restore the normal cascade concentration gradient is roughly proportional to the shift in light component inventory.

Illustrative loss calculations are given for the special tapered cascade, developed in Section 5 for the case of constant ψ and $N \ll 1$. The taper is described by equation (5.24); numbers have been developed for the case, $\psi = 6 \cdot 8$, $\alpha t = 3.23$, and $\pi = \frac{tP}{F_0} = 0.075$, where F_0 is ~~feed flow~~ interstage upflow at bottom stage. The cascade consists of a rectifier only, operating over an infinite reservoir of feed. An expression for J/δ for this cascade is obtained by setting R and B equal to zero in equation (5.28).

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$$J(\Delta) = \frac{P N_p}{F_0 N_0} \frac{1 - e^{-(\psi - \alpha)\Delta}}{\psi - \alpha} \qquad (8.50)$$

Figure 11 is a plot of $J(\Delta)$ for this cascade.

Figure 12 gives the loss of light component output from this cascade due to flattening of the curve of light component concentration [ ] in a part of the cascade. The curve which gives the loss for the limiting case of a small flat region, was calculated from the expression:

$$\text{Loss} \Big/ \left(\frac{\Delta_2 - \Delta_1}{t}\right)^3 = t^2 \gamma P N_p \frac{e^{-(\psi-\alpha)\Delta}}{6} \left\{ t\psi \frac{N}{N_0} \frac{F}{F_0} - \frac{tP}{F_0}\left(\frac{N_2}{N_2} - \frac{N}{N_1}\right) \right\} \qquad (8.51)$$

where $\frac{N}{N_0}$ and $\frac{N_2}{N_0}$ are given by (8.40) and (8.41). The open circle gives the loss for the case of a flat region consisting of one-half of the cascade's stages, symmetrical about the middle stage of the cascade. The solid circle gives the loss when the entire cascade's inventory is uniformly mixed. The loss divided by the cube of the number of stages affected is only about 30% greater in the latter case than it is when the flat region consists of only a few stages at the middle of the cascade.

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8.3 Duration of Disturbance's Effect on Product Purity - The method given in this section will not permit the calculation of top stage concentration versus time after the disturbance. There is some interest in knowing approximately the time during which the top stage concentration differs appreciably from normal. Generally speaking, unless the disturbance is confined to a small region near the top of the cascade, this time is very long (the order of the cutting time of product component in the cascade), and the effect of the disturbance on a single day's output is only a small fraction of the total cumulative effect.

This is illustrated by Figure 18, which shows the top stage concentration versus time in the special instance discussed above, after uniform dilution of the inventory in each of the cascade's stages, symmetrical about the middle stage. (The integrator loss in this case is indicated by the open circle of Figure 18.) Figure 18 was obtained by the method given in Section 8. The top stage concentration

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first falls 2% below the normal purity, and then rises to 0.1% above normal purity. This alternation of the arithmetic sign of the discrepancy between observed and normal purity may be explained as reflecting the arrival in succession at the top of the cascade of the region of concentration deficiency and the region of excess. These regions may be regarded as spreading and travelling up and out of the top of the cascade.

A cascade removes a discrepancy from the normal concentration curve by spreading it out over all the stages in the cascade (thus reducing its magnitude on a given stage), and by gradually moving the discrepancy in inventory out at the ends of the cascade. The first process is fairly rapid, the latter a slow process involving a slow movement of the discrepancy up the cascade. If the discrepancy is originally located near the top of the cascade, the deviation from normal product purity plays a greater role in erasing the discrepancy in cascade inventory than does the deviation from normal waste purity. Thus J is nearer to one the closer the discrepancy is to the top of the cascade. The duration of the effect of the disturbance on the concentration is longer the farther the disturbance is from the top.

In many respects a discrepancy from the normal concentration curve behaves like a packet of contaminant. The behavior of a packet of contaminant may be used to judge the duration of a disturbance effect on product purity. In an infinite square plant operation without product withdrawal the travel time for the center of gravity of a small packet of contaminant is $\psi/2$ stages per stage holdup time, where ψ is the stage enrichment factor for the contaminant versus the working substance. The rate of spread of the packet (defined as the rate of increase in the number of stages in which the contaminant concentration exceeds $1/e^2$ of the higher concentration in the packet at the same instant) is given by $\sqrt{2/\epsilon T}$ stages per holdup time, where T is the time measured from the moment the contaminant was introduced to the cascade. The rate of spread exceeds the travel time until $8/\psi^2$ holdup times have elapsed after introduction of the contaminant.

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4. OPERATION DURING START-UP OF A CASCADE

The problem of finding the best operating procedure during the start-up of a cascade is complicated and highly dependent upon the specific character of the cascade of interest and the goals set for its operator. One may wish to deliver a specific quantity of material by a given date at the highest possible purity; or to deliver as much material as possible by a given date at a specified purity; or one may wish to be able to start deliveries at specified purity and rate as soon as possible. These problems may be complicated by a staggered schedule of completion and first operation of various parts of the cascade. We are not in a position to give a general discussion of these problems, but some observations seem worth making.

4.1 Operating Procedure at Start-Up - A choice must be made of the steady state toward which one is working. This choice may be conditioned by a desire to reduce the cascade's inventory of light component, in order to be able to reduce the time required to reach the steady state, even though reduction of inventory may be accom-

panied by a reduction in output.

After one's choice of the final steady state, one will seek ways of minimizing the time required to obtain the corresponding light component inventory. This may well lead to operating conditions at early times different from those selected for ultimate use. Immediately after the start-up, high interstage flows are desirable to maximize the net transport of light component into the cascade. It may prove desirable to operate stages at the feed point in parallel or to set up two separate cascades, if these tricks are possible, during early times. A higher feed rate than that ultimately contemplated may be desirable.

In some cases it may prove desirable to produce material at an intermediate purity to be fed back to the cascade as soon as the amount of light component stockpiled, plus that in cascade inventory, is equal to the amount required for operation at the ultimate steady state desired. After the stockpile is fed back to the cascade, a short

time is required for its redistribution before production can begin at the final purity. This procedure enables the operator to take advantage of the fact that the net transport of light component is higher the lower the purity. Thus as the top stage concentration rises during initial operation without product withdrawal, a point is reached beyond which it does not pay to allow the concentration to rise, even if one's object is to produce ultimately at a higher purity. The inventory required for production at the higher purity is obtained more quickly by producing it at a lower purity and returning it to the cascade.

4.2 Loss of Production During Start-Up - An estimate of the loss of production during start-up is given by Equation (3.1), with N^* set equal to N_0. This equation is a nearly exact expression for the start-up loss if product is withdrawn at the design rate from the outset and if $(N_p^2 - N_0^2)$ can be approximated by $2N_p(N_p - N_0)$. Usually the schedule of product withdrawals during start-up will call for no product removal until the top stage concentration approaches the desired product purity. If the product rate as a function of time is $P(T)$,

and if the waste rate is held fixed at the design value, the loss during start-up is formally given by the following expression:

$$\int_0^{\infty}[PN_P - P(T)M(t,T)]\,dT = \int_0^{\tau}[M(t_0) - N^*]\,dt + \frac{\int_0^{\tau}[P - P(T)]\{N(t,T)A(t)/P + A\}\,dT}{Q(t) + A(t)/P + R(t) + B(t)/W} \qquad (4.1)$$

The additional term is always positive if $P > P(T)$, but cannot be readily evaluated. It is seen that equation (3.1) is a lower bound for the cumulative loss during start-up, provided the actual withdrawal rate during start-up at no time exceeds the rate of the final steady state. Generally the difference between the loss in light component output incurred during the start-up if no product is removed and the loss incurred if product is removed at design rate from the outset is worthwhile for two reasons: The value of a given quantity of light component removed as product is usually greater the higher the concentration of light component, and normal steady-state operation is reached sooner if no product is permanently removed from the cascade. (The last remark is not inconsistent with the stock piling operation mentioned above as desirable in some cases.)

Expressed as a time, the lower bound on the loss of output during start-up may be written:

$$\underline{T(N_p)} = \frac{1}{P N_p} \int_0^{\tau} H(N_\infty - N_0) J \, da \,. \qquad (4.2)$$

This time is also a lower bound on the actual physical time which must elapse (if the schedule of product withdrawals always calls for $P(T) < P$) before steady-state production at N_p can be put into effect. (The possibility of reducing this time by product withdrawals $P(T) > P$ which are later returned to the cascade was indicated in 4.2 above.) An upper bound $\overline{T(N_p)}$ for the time which must elapse before the steady-state concentration distribution is achieved when no product is withdrawn during this time may be obtained as follows: Set $N(t,T)$ and $N(T)$ in the additional term of (4.1) equal to their largest values, N_p and N_∞ respectively. Assume $P(T) = 0$ for all $T < \overline{T}$, and $P(T) = P$ for all $T > \overline{T}$. Equation (4.1) may then be solved for $\overline{T(N_p)}$.

The solid circles of Figures 4 and 5 give lower bound times $\underline{T}$ obtained from (4.2). At intermediate purities, these times coincide with those calculated from (2.6); at high purities, the times are shorter, reflecting a larger net transport when product is withdrawn than when it is

not and indicating the possibility of reducing the time required to reach these purities.

The half-open circles of Figures 4 and 5 give upper bound times obtained as described above. For the tapered cascade, $\overline{T}$ is 10% greater than $\underline{T}$ in the range of greatest interest ($N_p/N_o = 60$ to 100), and is about 40% greater at $N_p/N_o \cong 400$. It is interesting to note that the exact curve of Figure 4 reaches $N_p/N_o = 50$ more quickly than $\underline{T(N_p)}$, thus seeming to contradict our statement that $\underline{T(N_p)}$ is a lower bound. However, reference to Figures 2 and 7 shows that the cascade does not have the complete inventory required for production at $N_p/N_o = 50$ when that purity is reached, and so production at the full steady-state rate cannot begin immediately. For the square cascade, $\overline{T}$ is 5 to 10% greater than $\underline{T}$ in the range of greatest interest, and about 30% greater at $N_p/N_o \cong 400$.

If $2N_p(N_p - N_o)$ is not a good approximation for $(N_p^2 - N_o^2)$, equation (4.1) can be formally corrected by adding

$$\int_0^\infty \int_u^\tau \psi \delta^2 A(s)\, ds\, du\, dT + \int_0^\infty \int_0^u \psi \delta^2 B(s)\, ds\, du\, dT$$

to the numerator of the second member of the right-hand side. It is

seen that equation (4.2) is still a lower bound for the cumulative loss during start-up, and an upper bound may be obtained as before with the additional provision that $\delta = (N_\infty - N_0)$, its largest value.

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5. EXACT SOLUTION FOR TAPERED CASCADE AT LOW CONCENTRATIONS

In this section a method is derived for calculating the concentrations at all times at all stages of a tapered cascade fed by an infinitely large reservoir of constant feed concentration. The method is restricted to light component mol fractions small compared with one. The cascade initially has an arbitrary concentration distribution.

5.1 General Solution of Time-Dependent Equation for Tapered Cascade -

The time rate of change in concentration at a given stage of the cascade is described by the following differential equation:

$$-\frac{\partial}{\partial \Delta}\left[F\left\{\psi N(1-N) - \frac{\partial N}{\partial \Delta}\right\} + PN\right] = H\frac{\partial N}{\partial T} = 2\tau F\frac{\partial N}{\partial T} \qquad (5.1)$$

where Δ is the running stage number;

F is the flow in the light stream leaving a stage (F varies with Δ);

ψ is the overall enrichment of a stage $=\frac{N'-N}{N(1-N)}$, where N' is the concentration of a light component in light stream and N in heavy; ψ is assumed small relative to one;

P is the production rate;

H is the stage hold-up; (H varies with Δ)

τ is the hold-up time per stage = $H/2F$;

N is the concentration of light component at the stage at time T .

This equation will be solved subject to the following boundary conditions:

$$\text{(i)} \quad \text{At } \Delta = 0. \quad N = N_0 \text{ at all times after } T = 0.$$

$$\text{(ii)} \quad \text{At } \Delta = t. \quad \frac{\partial N}{\partial \Delta} = \psi N(1-N) \text{ at all times after } T = 0. \qquad (5.2)$$

$$\text{(iii)} \quad \text{At } T = 0. \quad N = N^0(\Delta) \text{ where } \Delta \text{ varies from } 0 \text{ to } t.$$

where N_0 is the concentration of the infinite reservoir of feed.

$N^0(\Delta)$ is the concentration distribution in the cascade at $T = 0$.

t is the total number of stages in the cascade.

The tapered cascade selected for study is specialized in the following respects:

(1) ψ and τ are constants, whereas in cascades of practical interest this may not be the case: (2) F is a continuously varying function of Δ, whereas in practical cascades F is apt to vary in a step-wise manner, being constant throughout a large number of stages:

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(3) the equation is linearized by setting $\psi N(1-N)$ equal to ψN.

New dimensionless variables are defined as follows:

$$\begin{aligned} \theta &= T/\tau t^2 \\ \rho &= \Delta/t \\ \pi &= tP/F_0 \\ \lambda &= F/F_0 \\ \xi &= N/N_0 \end{aligned} \qquad (5.3)$$

where F_0 is the flow in light stream from the zero-th stage. It will be recognized that θ and ρ are independent variables, F_0 and π are constants characteristic of the plant and the product rate chosen, λ is a function of ρ, and ξ a function of ρ and θ.

The linearized differential equation becomes

$$\frac{1}{\lambda}\frac{\partial}{\partial\rho}\left[\lambda\left(\frac{\partial\xi}{\partial\rho} - \psi t\xi\right) = \pi\xi\right] = 2\frac{\partial\xi}{\partial\theta}, \qquad (5.4)$$

with the boundary conditions

$$\begin{aligned} &(i) \quad \text{At } \rho = 0, \ \xi = 1 \text{ for all } \theta > 0. \\ &(ii) \quad \text{At } \rho = 1, \ \frac{\partial\xi}{\partial\theta} = \psi t\xi \text{ for all } \theta > 0. \\ &(iii) \quad \text{At } \theta = 0, \ \xi = \xi^0(\rho), \text{ where } \rho \text{ varies from } 0 \text{ to } 1. \end{aligned} \qquad (5.5)$$

Solution

For the present, let the function $\lambda(\rho)$ be left unspecified. A solution having the following form is sought:

$$\xi = R(\rho)\,Q(\theta) \quad (*) \qquad (5.6)$$

(*) Not to be confused with R and Q used in Section 5.

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where R is a function of ρ only, and Q of θ only.

Substitution in the differential equation and division by RQ give:

$$\frac{1}{\lambda R}\frac{d}{d\rho}\left[\lambda(k'-\psi t R)-\pi R\right] = 2\frac{\dot{Q}}{Q} \qquad (5.7)$$

where prime over the R and the dot over the Q denote differentiation with respect to ρ and θ respectively.

Since the left hand side of the equation is a function of ρ only and the right a function of θ only, each must be equal to a constant. Let this constant be denoted by the new symbol $(-2m)$.

Then

$$\dot{Q} + mQ = 0 \qquad (5.8)$$

from which

$$Q = e^{-m\theta} \qquad (5.9)$$

The equation for R is

$$\frac{d}{d\rho}\left[\lambda(R'-\psi t R)-\pi R\right] + 2m\lambda R = 0, \qquad (5.10a)$$

or

$$R''-(\psi t + \pi/\lambda - \lambda'/\lambda)R'+(2m-\psi t\,\lambda'/\lambda)R=0. \qquad (5.10b)$$

To remove the term in the first derivative,

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$$R = G(\rho)\, e^{\frac{1}{2}\int_0^\rho (\psi t + \pi/\lambda - \lambda'/\lambda)\, d\rho} = \frac{G(\rho)}{\sqrt{\lambda}}\, e^{\frac{1}{2}\left[\psi t \rho + \pi \int_0^\rho \frac{d\rho}{\lambda}\right]} \qquad (5.11)$$

Then, the differential equation for $G(\rho)$ is

$$G'' + \left[2m + \frac{1}{2}\frac{d}{d\rho}(\pi/\lambda - \lambda'/\lambda) - \left(\frac{\psi t + \pi/\lambda - \lambda'/\lambda}{2}\right)^2 - \psi t\, \lambda'/\lambda\right] G = 0. \qquad (5.12)$$

To proceed further with the solution, it is now desirable to specify λ as a function of ρ. Instead of specifying λ arbitrarily, functions are sought for $\lambda(\rho)$ which permit solutions of equation (5.12) in terms of elementary functions. The resulting forms of λ must then be investigated to see if they can be made to resemble cascades of practical interest, by appropriate specification of parameters.

Equation (5.12) will have simple solutions if the coefficient of G is equal to a constant. This will occur if the function $\lambda(\rho)$ is chosen so that it satisfies the following equation:

$$\frac{1}{2}\frac{d}{d\rho}\left(\frac{\pi}{\lambda} - \frac{\lambda'}{\lambda}\right) - \left(\frac{\psi t + \pi/\lambda - \lambda'/\lambda}{2}\right)^2 - \psi\pi\frac{\lambda'}{\lambda} = -\left(\frac{\psi t - \alpha t}{2}\right)^2 \qquad (5.13)$$

where α is a new constant. It will be shown in the next sub-section that it is possible to select functions for $\lambda(\rho)$ which satisfy this equation and resemble cascades of practical interest. The coefficient of G in equation (5.12) then becomes

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$$2m - \left(\frac{\psi t - \alpha t}{2}\right)^2$$

This constant quantity will be denoted by $\varphi^2(m)$. Thus;

$$\varphi^2(m) = 2m - \left(\frac{\psi t - \alpha t}{2}\right)^2. \tag{5.14}$$

Equation (5.12) then becomes

$$G'' + \varphi^2(m)\, G = 0 \tag{5.15}$$

which has as solutions

$$\left.\begin{aligned} G &= \sin[\varphi(m)\rho] \\ \text{and} \quad G &= \cos[\varphi(m)\rho] \end{aligned}\right\} \tag{5.16}$$

The solution to the boundary value problems (5.4 and 5.5) can now be written as the sum of all functions satisfying (5.4) including the final steady-state solution $\xi_\infty(\rho)$. (The function $\xi_\infty(\rho)$ satisfies equation (5.4) and boundary conditions (5.5) when $\frac{\partial \xi}{\partial \theta}$ is set equal to zero. The concentration distribution $\xi_\infty(\rho)$ is the limit of $\xi(\rho,\theta)$ as $\theta \to \infty$.

$$\xi(\rho) = \xi_\infty(\rho) + \frac{1}{\pi\lambda} e^{\cdot \frac{1}{2}\left[\nu\tau\rho + \pi\int_0^\rho \frac{d\rho}{\lambda}\right]} \cdot \sum_m \left[A(m)\cos\varphi(m)\rho + B(m)\sin\varphi(m)\rho\right] e^{-m\theta} \tag{5.17}$$

The series is summed over all permissable values of m. These

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are determined below, along with the values of the coefficients $A(m)$ and $B(m)$, from the boundary conditions.

To satisfy boundary condition (i), all of the $A(m)$'s must be zero. Permissable values of m follow from boundary condition (ii). Substitution of equation (5.16), evaluated at $\rho = 1$, into the boundary condition gives:

$$\tan \varphi(m) = \frac{2\varphi(m)}{\psi t - \pi/\lambda(1) + \lambda'(1)/\lambda(1)} \quad . \tag{5.18}$$

This equation can be true for only certain values of φ (known as the eigen-values) and hence only these values of φ can be used in the solution. If $\left[\psi t - \frac{\pi}{\lambda(1)} + \frac{\lambda'(1)}{\lambda(1)}\right] > 2$, then just one $\varphi(m)$, say $\varphi_1(m)_1$ will be imaginary. The remaining φ's are all real. Let all φ's be numbered in the order of increasing size, with j the running index. The value of m_j corresponding to each $\varphi_j(m_j)$ is given by equation (5.14). A more complete discussion of (5.18) is given in sub-section 5.4 below.

The solution can now be written:

$$\xi = \xi_\infty + \frac{1}{\sqrt{\lambda}} e^{\frac{1}{2}\left[\psi t \rho + \pi \int_0^\rho \frac{d\rho}{\lambda}\right]} \sum_{j=1}^{\infty} B_j \operatorname{sen}(\varphi_j \rho)\, e^{-m_j \theta} \quad . \tag{5.19}$$

The B_j's are yet to be determined. This is done by use of boundary condition (iii), the initial distribution $\xi^0(\rho)$ at zero time. Sub-

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stituting $\sigma = 0$ in equation (5.19), multiplying by $\sin \phi_k \rho$, where k is a particular value of j, and integrating with respect to ρ, with ρ going from 0 to 1, we find

$$B_k = - \frac{\int_0^1 (\xi_\infty - \xi^0) \sqrt{\lambda}\, e^{-\frac{i}{2}\left[\nu t \rho + \pi \int_0^\rho \frac{d\rho}{\lambda} \right]} \sin \phi_k \rho \, d\rho}{\int_0^1 \sin^2 \phi_k \rho \, d\rho} \quad (5.20)$$

This expression for B_k results from the fact that

$$\int_0^1 \sin \phi_j \rho \, \sin \phi_k \rho \, d\rho = 0 \quad \text{for } k \neq j \quad (5.21)$$

when the ϕ's satisfy equation (5.18).

The expression for the steady state, $\xi_\infty(\rho)$, is

$$\xi_\infty(\rho) = e^{\nu t \rho + \pi \int_0^\rho \frac{d\rho}{\lambda}} \left\{ 1 - \pi \xi_\infty(1) \int_0^\rho \frac{e^{-\nu t \rho - \pi \int_0^\rho d\rho/\lambda}}{\lambda} d\rho \right\} \quad (5.22)$$

where

$$\xi_\infty(1) = \frac{e^{\nu t + \pi \int_0^1 \frac{d\rho}{\lambda}}}{1 + \pi e^{\nu t + \pi \int_0^1 d\rho/\lambda} \int_0^1 \frac{e^{-\nu t \rho - \pi \int_0^\rho d\rho/\lambda}}{\lambda} d\rho} \quad (5.23)$$

5.2 Expression for Taper - Expressions for $\lambda(\rho)$ satisfying equation (5.13) will be derived. This equation can be written in the form

$$\left(\frac{\nu t + \pi/\lambda + \lambda'/\lambda}{2} \right)' = \left(\frac{\nu t - \alpha t}{2} \right)^2 - \left(\frac{\nu t + \pi/\lambda + \lambda'/\lambda}{2} \right)^2 \quad (5.24)$$

In each of the following solutions, one constant of integration has already been so chosen that $\lambda = 1$ when [redacted] while integration constants

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which have been left arbitrary are denoted by C, D, and E.

If $\psi = \infty$ in (5.24), the general solution is

$$\lambda = (\rho + C)^2 e^{-\psi t \rho} \left[\frac{1}{C^2} - \pi \int_0^\rho \frac{e^{\psi t \rho}}{(\rho + C)^2} d\rho \right] \tag{5.25}$$

If $\psi \neq \alpha$ in (5.24), a solution exists for which each member of (5.24) vanishes identically; this solution is:

$$\lambda = -\frac{\pi}{\alpha t} + \left(1 + \frac{\pi}{\alpha t}\right) e^{-\alpha t \rho} \tag{5.26}$$

If $\psi > \alpha$ in (5.24), and the right member of (5.24) is positive, the general solution is:

$$\lambda = e^{-\psi t \rho} \cosh^2 \left(\frac{\psi t - \alpha t}{2} \rho + D \right) \left[\operatorname{sech}^2 D - \pi \int_0^\rho e^{\psi t \rho} \operatorname{sech}^2 \left(\frac{\psi t - \alpha t}{2} \rho + D \right) d\rho \right] . \tag{5.27}$$

The solution (5.26) is the limit of this solution as D tends to infinity.

If $\psi < \alpha$ in (5.24), and the right member of (5.24) is negative, the general solution is:

$$\lambda = e^{-\psi t \rho} \sinh^2 \left(\frac{\psi t - \alpha t}{2} \rho + E \right) \left[\operatorname{csch}^2 E - \pi \int_0^\rho e^{\psi t \rho} \operatorname{csch}^2 \left(\frac{\psi t - \alpha t}{2} \rho + E \right) d\rho \right] . \tag{5.28}$$

The solution (5.26) is the limit of this solution as E tends to infinity.

Solutions (5.25), (5.26), (5.27) and (5.28) provide expressions by means of which a wide variety of cascade taper can be approximated.

~~Section 4 of~~

Section 3 of this paper gives numerical calculations making use of (5.26), which can be made to resemble cascades of practical interest by suitable choice of the single parameter α . The results of Section 3 are calculdted for a cascade which closely resembles the ideal cascade except in one respect: the inter-stage flow rates at the top of the plant are large compared with the largest production rate of practical interest, whereas the light stream flow rate at the top stage of an ideal cascade just equals the design production rate. The blunt, or squared-off, nature of the cascade studied in Section 3 is characteristic of practical cascades, since the process advantage of the very small stages at the tip of an ideal cascade is slight, and the practical advantage of replacing them by fewer large stages of uniform size is great.

If one selects a particular cascade for study at a number of different production rates, a different value of α must be chosen for each value of π . As a consequence slightly different approximation of the cascade's taper is used for each production rate.

It will be noted that (5.26) includes the square cascade as a special case (let $\alpha = -\pi$).

5.3 Solution of Time-Dependent Equation for Special Tapered Cascade

If the expression for λ given in Equation (5.26) is substituted in equation (5.14), the solution becomes:

$$\xi = \xi_\infty + \frac{e^{(\nu t - \alpha t)\rho/2}}{\lambda} \sum_{j=1}^{\infty} B_j \sin \varphi_j \rho \, e^{-m_j \theta} \tag{5.29}$$

where the φ_j's are determined from

$$\tan \varphi_j = \frac{\varphi_j}{(\nu t - \alpha t)/2 - \pi/\lambda(1)} \tag{5.30}$$

$$m_j = \pm \left\{ \varphi_j^2 + \left(\frac{\nu t - \alpha t}{2} \right)^2 \right\} \tag{5.31}$$

$$B_j = - \frac{\int_0^1 (\xi - \xi^\circ) \lambda \, e^{-(\nu t - \alpha t)\rho/2} \sin \varphi_j \rho \, d\rho}{\int_0^1 \sin^2 \varphi_j \rho \, d\rho} \tag{5.32}$$

$$\xi_\infty = \frac{1}{\lambda} \left\{ e^{(\nu t - \alpha t)\rho} - \pi \xi_\infty(1) \frac{e^{(\nu t - \alpha t)\rho} - 1}{\nu t - \alpha t} \right\} \tag{5.33}$$

$$\xi_\infty(1) = \frac{e^{(\nu t - \alpha t)} - 1}{\lambda(1) + \pi \dfrac{e^{(\nu t - \alpha t)} - 1}{\nu t - \alpha t}} \tag{5.34}$$

In terms of the original variables this last set of equations becomes:

$$F = F_0 \left\{ - \frac{P}{\alpha F_0} + \left(1 + \frac{P}{\alpha F_0} \right) e^{-\alpha \Delta} \right\} \tag{5.35}$$

$$N = N_\infty + N_0 \frac{e^{(\nu - \alpha)\Delta/2}}{F/F_0} \sum_{j=1}^{\infty} B_j \sin(\varphi_j \Delta/t) \, e^{-m_j T/\tau t^2} \tag{5.36}$$

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$$\tan \phi_j = \frac{}{(\psi t - \alpha t)/2 - 2P/F_t} \qquad (5.37)$$

$$m_j = \tfrac{1}{2}\left\{ \phi_j^2 + \left(\frac{\psi t - \alpha t}{2} \right)^2 \right\} \qquad (5.38)$$

$$B_j = - \frac{\int_0^t \left(\frac{N_0}{N_0} - \frac{N^*}{N_0} \right) \frac{F}{F_0} e^{-(\psi - \alpha) s/2} \sin(\phi_j s/t)\, ds}{\int_0^t \sin^2(\phi_j s/t)\, ds} \qquad (5.39)$$

$$N_n = \frac{F_0}{F} \left\{ N_0 e^{(\psi - \alpha)t} - \frac{1}{F_0} N_p \frac{e^{(\psi-\alpha)t} - 1}{(\psi - \alpha)} \right\} \qquad (5.40)$$

where

$$N_p = \frac{N_0 e^{(\psi - \alpha)t}}{\frac{F_t}{F_0} + \frac{1}{F_0} \frac{e^{(\psi-\alpha)t} - 1}{\psi - \alpha}} \qquad (5.41)$$

5.4 Roots of the Transcendental Equation $\mu = \tan \beta\mu$

The numerical applications of the results of this section (and of Section 6) require a knowledge of the characteristic roots of the transcendental equation

$$\mu = \tan \beta\mu \qquad (5.42)$$

This is equivalent to (5.18) if $\beta\mu$ is identified with ψ and $\beta = \frac{1}{2}[\pi - \pi/\lambda(1) + \lambda'(1)/\lambda(1)]$; and to (5.19) if $\beta = \frac{\pi}{2}$. Each of the roots of (5.42) correspond to an intersection of the curves,

$$f_1(\mu) = \mu$$
$$f_2(\mu) = \tan \beta\mu$$

As is evident from Fig. 15, when $\beta < 1$ there is one intersection in each of the intervals $\mu = (0, \pi/2\beta)$, $(\pi/2\beta, 3\pi/2\beta)$, $(3\pi/2\beta, 5\pi/2\beta)$, etc. However, when $\beta > 1$, there is no intersection in the interval $\mu = (0, \pi/2\beta)$. Defining the interval $((2j-1)\pi/2\beta, (2j+1)\pi/2\beta)$ as the j^{th} interval, and the root in the j^{th} interval as μ_j, one readily sees that as $j \to \infty$

$$\mu_j \longrightarrow (2j+1)\pi/2\beta. \qquad (5.43)$$

This is the result used in evaluating $S(\pm\psi)$ in equation (6.80).

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For the determination of μ_0. it is convenient to remember

(a) as $\beta \to 0$, $\mu_0 \to \pi/2\beta$

(b) as $\beta \to 1$, $\mu_0 \to 0$

In this case we can use the first terms of the power series as an approximation,

$$\tan \mu_0 \beta = \beta \mu_0 + \frac{1}{3}(\beta \mu_0)^3 + \dots$$

and obtain from (8.42)

$$\mu_0 \cong \frac{1}{\beta}\sqrt{3\left(\frac{1}{\beta}-1\right)} \qquad (8.44)$$

(c) When $\beta > 1$, there is no intersection corresponding to μ_0 Then, however, there exists a purely imaginary root to (8.42)

If we substitute $\mu = i\nu$ in (8.42) we get

$$\nu = \tanh \nu\beta = \frac{e^{\nu\beta} - e^{-\nu\beta}}{e^{\nu\beta} + e^{-\nu\beta}} = 1 - 2e^{-2\nu\beta} + 2e^{-4\nu\beta} - \dots \qquad (8.45)$$

The single real root of (8.45) is the intersection point of $y_1(\nu) = \nu$ with $y_2(\nu) = \tanh \nu\beta$. It can be shown that as $\beta \to \infty$, $\nu \to 1$.

As an approximation for large β. we find

$$\nu = i\mu_0 = 1 - 2e^{-2\beta} - (2\beta - 2)e^{-4\beta} + \dots \qquad (8.46)$$

6. METHOD FOR SOLVING SQUARE CASCADE ENRICHMENT EQUATION IN HIGH CONCENTRATION RANGE

In Section 5, a solution of the enrichment equation (5.1) was obtained for tapered cascade operating in the low concentration range. The approximation $N(1-N) \cong N$ there used is no longer valid at high concentrations. In this section an exact solution, not restricted to low concentrations, will be given for operation of a square cascade without product withdrawal. The following non-linear equation gives the concentration in a square cascade operating without product withdrawal:

$$-\frac{\partial}{\partial \lambda}\left\{\psi N(1-N) - \frac{\partial N}{\partial \lambda}\right\} = 2\tau \frac{\partial N}{\partial T} . \tag{6.1}$$

If the cascade is fed by an infinite reservoir at feed concentration, the boundary conditions are:

$$\begin{aligned} &\text{(i) at } \lambda = 0 . \quad N = N_0 \text{ for all } T > 0 . \\ &\text{(ii) at } \lambda = t . \quad \frac{\partial N}{\partial \lambda} = \psi N(1-N) \text{ for all } T > 0 . \\ &\text{(iii) at } T = 0 . \quad N = N_0 \text{ for } 0 \le \lambda \le t . \end{aligned} \tag{6.2}$$

A solution of (6.1) will be developed in this section which will involve a power series in the abundance ratio of light component in feed:

$$Y_0 = N_0/(1-N_0)$$

This quantity is small in many important natural isotope mixtures. For example: in natural carbon, Y_0 or C_{13} is 0.011; in uranium, Y_0 or U-235 is 0.0071; and in nitrogen Y_0 or N_{15} is 0.0036.

6.1 Steady-State Solutions - In the steady state $\frac{\partial N}{\partial t} = 0$, and the enrichment equation for the steady-state without product withdrawal reduces to

$$\frac{\partial N}{\partial z} = \psi N(1-N) \qquad (6.3)$$

Although this non-linear ordinary equation can readily be integrated by separation of the variables, we shall solve it first by linearizing it with the introduction of the abundance ratio of light component.

$$Y(z) = N(z)/(1-N(z)) \qquad (6.4)$$

As $N \to 1$, $Y \to \infty$ and as $N \to 0$, $Y \to 0$.

Substitution of (6.4) into (6.3) yields

$$\frac{\partial Y}{\partial z} = \psi Y, \qquad (6.5)$$

a linear ordinary equation whose solution is

$$Y(z) = N(z)/[1-N(z)] = Y_0 e^{z\psi}, \qquad (6.6)$$

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where $Y_0 = N_0/(1-N_0)$.

The well known expression for the steady-state light component concentration $N(\lambda)$ when no product is drawn follows immediately from (6.6):

$$N(\lambda) = N_0 e^{-\lambda\tau}/[1+N_0(e^{\tau\tau}-1)] \quad . \qquad (6.7)$$

6.2 Quasilinear Equation for Abundance Ratio - Let us now transform the complete enrichment equation (6.1) into an equation involving abundance ratio by substituting (6.4) into (6.1). This substitution yields the new non-linear equation:

$$\frac{\partial}{\partial\lambda}\left\{\frac{\partial Y}{\partial\lambda} - \psi Y\right\} - 2\tau\frac{\partial Y}{\partial T} = \frac{2}{1+Y}\frac{\partial Y}{\partial\lambda}\left\{\frac{\partial Y}{\partial\lambda} - \psi Y\right\} \qquad (6.8)$$

with the linear boundary conditions

$$(a)\quad Y(\lambda,T) = Y_0 = N_0/(1-N_0) \qquad (6.9a)$$

$$(b)\quad \frac{\partial Y}{\partial\lambda}\Big]_{\lambda,T} = \psi Y(\tau,T) \qquad (6.9b)$$

and the initial condition

$$(c)\quad Y(\lambda,0) = Y_0 = N_0/(1-N_0) \quad [\text{when } T=0]. \qquad (6.10)$$

As we shall see presently the new non-linear equation (6.8) has some advantages over (6.1), especially when the feed concentration is small.

To integrate (6.8) let us assume

$$Y(\xi,\tau) = Y_0 c^{(1)}(\xi,\tau) + Y_0^2 c^{(2)}(\xi,\tau) + \dots \qquad (6.11)$$

Substitution of this series into (6.8), multiplying $(1+Y)$ on both sides of (6.8), and equating coefficients of Y_0^k on both sides of (6.8), one obtains the following sequence of <u>linear</u> equations for $c^{(k)}(\xi,\tau)$:

$$\frac{\partial}{\partial\xi}\left[\frac{\partial c^{(1)}}{\partial\xi} - \psi c^{(1)}\right] - 2\tau\frac{\partial c^{(1)}}{\partial\tau} = 0 \qquad (6.12a)$$

$$\frac{\partial}{\partial\xi}\left[\frac{\partial c^{(2)}}{\partial\xi} - \psi c^{(2)}\right] - 2\tau\frac{\partial c^{(2)}}{\partial\tau} = 2\frac{\partial c^{(1)}}{\partial\xi}\left(\frac{\partial c^{(1)}}{\partial\xi} - \psi c^{(1)}\right) \qquad (6.12b)$$

$$\frac{\partial}{\partial\xi}\left[\frac{\partial c^{(3)}}{\partial\xi} - \psi c^{(3)}\right] - 2\tau\frac{\partial c^{(3)}}{\partial\tau} = 4\frac{\partial c^{(1)}}{\partial\xi}\frac{\partial c^{(2)}}{\partial\xi} - 2\psi\left(c^{(2)}\frac{\partial c^{(1)}}{\partial\xi} + c^{(1)}\frac{\partial c^{(2)}}{\partial\xi}\right) - 2c^{(1)}\frac{\partial c^{(1)}}{\partial\xi}\left(\frac{\partial c^{(1)}}{\partial\xi} - \psi c^{(1)}\right). \qquad (6.12c)$$

etc., and the boundary conditions:

$$\partial c^{(k)}/\partial\xi\,]_{\xi=\tau} = \psi c^{(k)}(\tau,\tau), \quad k=1,2, \qquad (6.13a)$$

$$c^{(1)}(0,\tau) = 1; \quad c^{(k)}(0,\tau) = 0, \quad k>1. \qquad (6.13b)$$

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The initial conditions become :

$$C^{(1)}(\Delta,0)=1, \quad C^{(k)}(\Delta,0)=0 \quad \text{for } k>1. \tag{6.13c}$$

From equations (6.12) and (6.13) one can determine the coefficients of γ_0, γ_0^2, etc., in the series for the abundance ratio $Y(\Delta,T)$. If N_0 is small, the first term gives the main contribution to the variation in abundance ratio with time.

6.3 First Order Approximation of $Y(\Delta,T)$. - The first order approximation $\gamma_0 C^{(1)}(\Delta,T)$ to $Y(\Delta,T)$ is the product of γ_0 and the solution of equation (6.13a):

$$\frac{\partial}{\partial\Delta}\left(\frac{\partial C^{(1)}}{\partial\Delta}-\psi C^{(1)}\right)-2\tau\frac{\partial C^{(1)}}{\partial T}=0 \tag{6.12a}$$

with the boundary conditions (6.13a), (6.13b), and (6.13c). This equation is a special case of those discussed in Section 2. Using the methods presented there one can show that:

$$C^{(1)}(\Delta,T)=e^{\psi\Delta}-\psi e^{\left(\frac{\psi(\Delta-\ell)}{2}-\frac{\psi^2 T}{8\tau}\right)}\cdot\sum_{j=1}^{\infty}\frac{\sin\left(\frac{1}{2}\mu_j\ell\psi\right)\sin\left(\frac{1}{2}\mu_j\Delta\psi\right)e^{-T\psi^2\mu_j^2/8\tau}}{(1+\mu_j^2)\left[\ell\psi-2\cos^2\left(\frac{1}{2}\mu_j\ell\psi\right)\right]}, \tag{6.14}$$

where the numbers μ_j are roots of the transcendental equation

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$$\mu = \tan\left(\tfrac{1}{2}\mu \cdot \psi t\right). \qquad (6.15)$$

The first order approximation to the abundance ratio is therefore

$$Y(\Delta,T) = N(\Delta,T)/[1-N(\Delta,T)] = C^{(1)}(\Delta,T)\,N_0/1-N_0 \,. \qquad (6.15a)$$

6.4 Second Order Approximation To The Abundance Ratio $Y(\Delta,T)$.

The second order approximation to $Y(\Delta,T)$ is the solution of the differential equation (6.12b);

$$\frac{\partial}{\partial\Delta}\left\{\frac{\partial C^{(2)}}{\partial\Delta} - \psi C^{(2)}\right\} - 2\tau\frac{\partial C^{(2)}}{\partial T} = 2\frac{\partial C^{(1)}}{\partial\Delta}\left(\frac{\partial C^{(1)}}{\partial\Delta} - \psi C^{(1)}\right)$$

under the following conditions

$$\left.\frac{\partial C^{(2)}}{\partial\Delta}\right]_{\Delta=t} = \psi C^{(1)}(t,T)$$

$$C^{(2)}(0,T) = 0$$

$$C^{(2)}(\Delta,0) = 0 \,.$$

To solve (6.12b), the non-homogeneous term

$$= \frac{\partial C^{(1)}}{\partial \Delta}\left(\frac{\partial C^{(1)}}{\partial \Delta} - \psi C^{(1)}\right)$$

must first be determined explicitly from (6.14). If it is assumed that the characteristic roots $\{\mu_j\}$ of (6.15) are ordered so that

$$|\mu_0| < |\mu_1| < |\mu_2| < \cdots$$

the term in (6.14) involving μ_0 gives practically the entire time-dependent contribution to $C^{(1)}(\Delta, T)$, except at very early times. Therefore to estimate the contribution of the second order term in (6.11) to $Y(\Delta, T)$, we shall employ the μ_0-term in the non-homogeneous part of (6.12b). On this basis:

$$\frac{\partial C^{(1)}}{\partial \Delta}\left(\frac{\partial C^{(1)}}{\partial \Delta} - \psi C^{(1)}\right) \simeq \frac{4\psi^2 \sin\left(\tfrac{1}{2}\mu_0 T\psi\right)\left\{\sin\left(\tfrac{1}{2}\mu_0 \Delta\psi\right) - \mu_0 \cos\left(\tfrac{1}{2}\mu_0 \Delta\psi\right)\right\} e^{\psi(\Delta - T)/2 - \psi T(1+\mu_0^2)/8\tau}}{(1+\mu_0^2)\left\{T\psi - 2\sin^2\left(\tfrac{1}{2}\mu_0 T\psi\right)\right\}}$$

$$- \frac{8\psi^2 \sin^2\left(\tfrac{1}{2}\mu_0 T\psi\right)\left[(1-\mu_0^2) - (1+\mu_0^2)\cos\mu_0 \Delta\psi\right] e^{\psi(\Delta - T) - 2T\psi^2(1+\mu_0^2)/8\tau}}{(1+\mu_0^2)\left[T\psi - 2\sin^2\left(\tfrac{1}{2}\mu_0 T\psi\right)\right]^2}$$

Substituting this term into (6.12b), we have

$$\frac{\partial}{\partial \Delta}\left\{\frac{\partial C^{(2)}}{\partial \Delta} - \psi C^{(2)}\right\} - 2I\frac{\partial C^{(2)}}{\partial T} = \frac{8\psi^2 \sin^2\tfrac{1}{2}\mu_0 T\psi\left[\sin\left(\tfrac{1}{2}\mu_0 \Delta\psi\right) - \mu_0 \cos\left(\tfrac{1}{2}\mu_0 \Delta\psi\right)\right] e^{\psi(\Delta - T)/2 - \psi T(1+\mu_0^2)/8\tau}}{(1+\mu_0^2)\left\{T\psi - 2\sin^2\left(\tfrac{1}{2}\mu_0 T\psi\right)\right\}}$$

$$- \frac{16\psi^2 \sin^2\left(\tfrac{1}{2}\mu_0 T\psi\right)\left[(1-\mu_0^2) - (1+\mu_0^2)\cos\mu_0 \Delta\psi\right] e^{\psi(\Delta - T) - \psi^2 T(1+\mu_0^2)/4\tau}}{(1+\mu_0^2)^2\left\{T\psi - 2\sin^2\left(\tfrac{1}{2}\mu_0 T\psi\right)\right\}^2} \qquad (6.16)$$

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To solve this equation we transform $C_A^{(2)}$ to a new variable $g(s,T)$ which satisfies the "diffusion" or "heat conduction" equation. The boundary conditions are made time dependent by this transformation. This type of equation can be solved by the method of Stokes (c.f. Carslaw, "Conduction of Heat in Solids," Dover Press (1945) p. 68.).

Let us define a new function $g(s,T)$ by the equation

$$C_A^{(2)} = \frac{\gamma e^{3\psi s/2 - \psi^2 T(1+\mu_0^2)/8\tau}}{\psi^2(1+\mu_0^2)} \left\{(1-\mu_0^2)\sin(\tfrac{1}{2}\mu_0\psi s) - 2\mu_0\cos(\tfrac{1}{2}\mu_0\psi s)\right\}$$

$$- \frac{\gamma^2 e^{\psi s - 2\psi^2 T(1+\mu_0^2)/8\tau}}{2\psi^4(1+\mu_0^2)} \left\{(1-\mu_0^2) - [2\mu_0\sin s\mu_0\psi + (1-\mu_0^2)\cos s\mu_0\psi]\right\} \qquad (6.17)$$

$$+ \frac{\gamma g(s,T)}{\psi^2(1+\mu_0^2)} e^{(\frac{1}{2}\psi s - \psi^2 T/8\tau)}$$

where

$$\gamma = \frac{8\psi^2 e^{-\tau\psi/2}\sin(\tfrac{1}{2}\mu_0\psi t)}{(1+\mu_0^2)[t\psi - 2\cos^2(\tfrac{1}{2}\mu_0 t\psi)]} . \qquad (6.17a)$$

Since

$$\mu_j\cos(\tfrac{1}{2}\mu_j\psi t) = \sin(\tfrac{1}{2}\mu_j\psi t) \qquad (6.17b)$$

we have

$$\sin^2(\tfrac{1}{2}\mu_0 t\psi) = \mu_0^2/(1+\mu_0^2) \quad \text{and} \quad \cos^2(\tfrac{1}{2}\mu_0 t\psi) = 1/(1+\mu_0^2) \qquad (6.17c)$$

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so that

$$\gamma = 8\psi^2 e^{-t\psi/2}\mu_0 \Big/ \left[\sqrt{1+\mu_0^2}\left\{t\psi(1+\mu_0^2)-2\right\}\right]. \qquad (6.18)$$

Substituting (6.17) into (6.16) we have

$$\frac{\partial^2 g}{\partial s^2} = 2\tau \frac{\partial g}{\partial T} \qquad (6.19a)$$

with

$$\left.\frac{\partial g}{\partial s}\right]_{s=\tau} = \frac{\psi}{2} g(t,T) \qquad (6.19b)$$

$$g(0,T) = 2\mu_0 e^{-\psi^2 T \mu_0^2/4\tau}$$

and

$$g(s,0) = e^{\frac{1}{2}\psi}\left[\cos\left(\tfrac{1}{2}\mu_0 s\psi\right) - (1-\mu_0^2)\sin\left(\tfrac{1}{2}\mu_0 s\psi\right)\right] \qquad (6.19c)$$
$$+ \frac{\psi e^{\psi s/2}}{2\psi^2}\left[1-\mu_0^2-(1-\mu_0^2)\cos(\mu_0\psi t) - 2\mu_0\sin(\mu_0\psi t)\right].$$

Equation (6.19) can be solved by the methpd of Stokes. Let us assume

$$g(s,T) = \sum_{j=0}^{\infty} a_j(T)\sin\left(\tfrac{1}{2}\mu_j s\psi\right) \qquad (6.20)$$

where μ_j is the $j^{\underline{th}}$ root of the characteristic equation (6.15).

By integration by parts one can, with the aid of the boundary conditions (6.19b) establish the identity

$$\int_0^\tau \frac{\partial^2 g}{\partial \alpha^2} \sin\left(\tfrac{1}{2}\mu_j \psi t\right) d\alpha - \psi \mu_j \mu_0 e^{-\psi^2 \mu_0^2 T/s\tau} - \left(\tfrac{1}{2}\mu_j \psi\right)^2 \int_0^\tau g(\alpha,T) \sin\left(\tfrac{1}{2}\mu_j \psi \alpha\right) d\alpha \qquad (6.21)$$

Incorporating the differential equation (6.19) into this identity we have

$$2\tau \frac{\partial}{\partial T} \int_0^\tau g(\alpha,T) \sin\left(\tfrac{1}{2}\mu_j \psi \alpha\right) d\alpha = \psi \mu_j \mu_0 e^{-\psi^2 \mu_0^2 T/s\tau} - \left(\tfrac{1}{2}\mu_j \psi\right)^2 \int_0^t g(\alpha,T) \sin\left(\tfrac{1}{2}\mu_j \psi \alpha\right) d\alpha \qquad (6.22)$$

The Fourier coefficients $a_j(T)$ of $g(\alpha,T)$ follow from (6.22) and the orthogonality of the functions $\sin\left(\tfrac{1}{2}\mu_j \alpha \psi\right)$:

$$a_j(T) = \frac{\int_0^\tau g(\alpha,T) \sin\left(\tfrac{1}{2}\mu_j \psi \alpha\right) d\alpha}{\frac{1}{\psi}\left[\frac{\tau\psi}{2} - \omega^2\left(\tfrac{1}{2}\mu_j \psi t\right)\right]} \qquad (6.23)$$

Thus, (6.22) becomes:

$$\frac{da_j}{dT} + \frac{\mu_j^2 \psi^2 a_j}{s\tau} = \frac{\psi^2 \mu_0 \mu_j e^{-\mu_0^2 \psi^2 T/s\tau}}{2\tau\left[\frac{\tau\psi}{2} - \omega^2\left(\tfrac{1}{2}\mu_j t \psi\right)\right]} \qquad (6.24)$$

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which has the solution:

(a) When $j = 0$

$$a_0(T) = e^{-\nu^2\mu_0^2 T/\tau}\left[a_0(0) + \frac{\mu_0^2\psi\, T}{\tau[\tau\psi - 2\cos^2(\frac{1}{2}\mu_0\tau\nu)]}\right] \qquad (6.25a)$$

(b) When $j > 0$

$$a_j(T) = a_j(0)e^{-\nu^2\mu_j^2 T/\tau} + \frac{4\mu_0\mu_j[e^{-\nu^2\mu_j^2 T/\tau} - e^{-\nu^2\mu_0^2 T/\tau}]}{(\mu_0^2-\mu_j^2)[\frac{\tau\psi}{2} - \cos^2(\frac{1}{2}\mu_j\tau\nu)]} \qquad (6.25b)$$

One employs the initial condition (6.19c) and (6.23) to obtain $a_j(0)$:

$$a_j(0) = \frac{\psi\int_0^\tau g(x,0)\sin(\frac{1}{2}\mu_j x\nu)\,dx}{[\frac{\tau\psi}{2} - \cos^2(\frac{1}{2}\mu_j\tau\nu)]}$$

$$= \frac{\left\{\int_0^\tau e^{x\psi}[2\mu_0\cos(\frac{1}{2}\mu_0\nu x) - (1-\mu_0^2)\sin(\frac{1}{2}\mu_0\nu x)]\sin(\frac{1}{2}\mu_j\nu x)\,dx + (3/2-\psi)\int_0^\tau e^{x\psi/2}[(1-\mu_0^2)\sin\tfrac{1}{2}\mu_0\nu x + 2\mu_0\cos\tfrac{1}{2}\mu_0\nu x]\sin(\tfrac{1}{2}\mu_j\nu x)\,dx\right\}}{[\frac{\tau\psi}{2} - \cos^2(\frac{1}{2}\mu_j\tau\nu)]}$$

Direct integration and introduction of the definition of γ yields

$$a_0(0) = \frac{1}{[\tau\psi - 2\cos^2(\frac{1}{2}\mu_0\tau\nu)]}\left\{\mu_0^2\left(e^{\tau\psi} - 1 + \frac{\psi}{\gamma\mu_0^2}\right) - \frac{64\mu_0^4[2\sqrt{1-\mu_0^2} + (3\mu_0^2-5)e^{-\gamma+1/2}]}{[\sqrt{1-\mu_0^2}(\tau\psi(1-\mu_0^2)-2)](1-\mu_0^2)(1+\gamma\mu_0^2))}\right\} \qquad (6.26)$$

Since we shall in most of our calculations neglect the higher harmonics

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in (6.20) we do not record the rather lengthy general expression for

$a_j(\tau)$ when $j > 2$.

Combining (6.25a) and (6.26) we have

$$a_1(T) = \frac{\mu_1^2 e^{-\tau\mu_1^2 T/\tau}}{\tau - 2\tau^2(1+\mu_1^2)} \left\{ \left(e^{\tau\tau} - 1 + \frac{\tau}{1+\mu_1^2}\right) - \frac{(4\mu_1^2[2\sqrt{1+\mu_1^2} + (3\mu_1^2 - 5)e^{-\tau\sqrt{2}}]}{(1+\mu_1^2)^{3/2}(1+3\mu_1^2)(\tau + (1+\mu_1^2)-2)} + \frac{\mu_1^2 T}{2} \right\} \quad (6.27)$$

and (neglecting higher harmonics)

$$c_2^{(1)} = \frac{\tau e^{-\tau(1+\mu_1^2)/\tau} e^{\tau\sqrt{2}}}{\tau^2(1+\mu_1^2)} \left\{ [(1+\mu_1^2)\sin(1\cdot\mu_1 + \Delta) - 2\mu_1\cos(1\cdot\mu_1 + \Delta)]e^{-\tau} \right.$$

$$- 2(\mu_1^2+\tau^2)e^{-\tau - \tau(1+\mu_1^2)/\tau}\left((1-\mu_1^2)\cdot - [2\mu_1\sin\tau\mu_1 + (1+\mu_1^2)\cos\tau\mu_1]\right)$$

$$+ \frac{\mu_1^2\sin(1\cdot\mu_1+\Delta)}{\tau - 2\tau^2(1+\mu_1^2)}\left[e^{\tau\tau} - 1 + \frac{\tau}{1+\mu_1^2} + \frac{\mu_1^2 T}{\tau} - \frac{(4\mu_1^2[2\sqrt{1+\mu_1^2} + (3\mu_1^2-5)e^{-\tau\sqrt{2}}}{(1+\mu_1^2)^{3/2}(1+3\mu_1^2)[\tau + (1+\mu_1^2)-2]}\right] \quad (6.28)$$

$$\left. - 5\mu_1 \sum_{j=2}^{\infty} \frac{\mu_j \sin(1\cdot\mu_j + \Delta)}{(\mu_j^2 - \mu_1^2)[\tau - 2\tau^2(1+\mu_j^2)]} \right\}$$

In particular, at the top stage of the plant

$$c_2^{(1)}(1,T) = \frac{5\mu_1^2 e^{-\tau(1+\mu_1^2)/\tau}}{(1+\mu_1^2)[\tau + (1+\mu_1^2)-2]^2}\left\{ -e^{-\tau\tau}[\tau + (1+\mu_1^2)-2] + \frac{5\mu_1^2}{1+\mu_1^2}\left[e^{\tau\tau - \tau(1+3\mu_1^2)/\tau} - 1 + \frac{\tau[(1+\mu_1^2)T}{5\tau}\right]\right.$$

$$+ \mu_1^2\left[e^{\tau\tau} - 1 + \frac{\tau\tau}{1+\mu_1^2} - \frac{(4\mu_1^2(2\sqrt{1+\mu_1^2} + 3\mu_1^2 - 5)e^{-\tau\sqrt{2}}}{(1+\mu_1^2)^{3/2}(1+3\mu_1^2)(\tau + (1+\mu_1^2)-2)}\right]$$

$$\left. - \frac{\tau[\tau + (1+\mu_1^2)-2]}{\tau(1+\mu_1^2)} \sum_{j=2}^{\infty} \frac{(1+\mu_j^2)\mu_j^2\sqrt{1+\mu_j^2}}{(\mu_j^2 - \mu_1^2)[\tau + (1+\mu_j^2)-2]} \right\} \quad (6.29)$$

where we have used

$$\sin(1\cdot\mu_j + \Delta) = (-1)^{j+1}\mu_j/\sqrt{1+\mu_j^2}.$$

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The sum

$$S(t\psi) = \sum_{j=1}^{\infty} \frac{(-1)^j \mu_j^2 \sqrt{1+\mu_j^2}}{(\mu_0^2 - \mu_j^2)[t\psi(1+\mu_j^2) - 2]} \qquad (6.30)$$

can be readily evaluated by employing the asymptotic expression for μ_j: $\mu_j \sim (2j+1)\pi/t\psi$. The j^{th} term in $S(t\psi)$ therefore asymptotically approaches $-(-1)^j/\pi(2j+1)$.

Thus, we can transform the $S(t\psi)$ series into

$$S(t\psi) = -\frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^j/(2j+1) + \sum_{j=1}^{\infty}(-1)^j \left\{ \frac{\mu_j^2\sqrt{1+\mu_j^2}}{(\mu_0^2-\mu_j^2)[t\psi(1+\mu_j^2)-2]} + \frac{1}{\pi(2j+1)} \right\}$$

Since $\pi/4 = 1 - 1/3 + 1/5 - \ldots$

$$S(t\psi) = \left(\frac{1}{\pi} - \frac{1}{4}\right) + \sum_{j=1}^{\infty}(-1)^j \left\{ \frac{\mu_j^2\sqrt{1+\mu_j^2}}{(\mu_0^2-\mu_j^2)[t\psi(1+\mu_j^2)-2]} + \frac{1}{(2j+1)\pi} \right\}. \qquad (6.31)$$

The sum on the right converges very rapidly. Since

$$Y(\Delta,T) = Y_0 C^{(1)}(\Delta,T) + Y_0^2 C^{(2)}(\Delta,T) + \ldots,$$

$$\frac{N(\Delta,T)}{1-N(\Delta,T)} = Y(\Delta,T) = Y_0\left\{ e^{\frac{t\psi}{2}} S \sum_{j=0}^{\infty} \frac{\mu_j^2 e^{-T+t\psi(1+\mu_j^2)/8\tau}}{(1+\mu_j^2)[t\psi(1+\mu_j^2)-2]} \right\}$$

$$+ Y_0^2 \frac{S\mu_0^2 e^{-\psi^2 T(1+\mu_0^2)/8\tau}}{(1+\mu_0^2)[t\psi(1+\mu_0^2)-2]^2}\left\{ -e^{\frac{t\psi}{2}}[t\psi(1+\mu_0^2)-2] + \frac{S\mu_0^2}{1+\mu_0^2}\left[e^{\psi^2 T(1-\mu_0^2)/8\tau} - 1 + \frac{\psi^2(1+\mu_0^2)T}{8\tau} \right] \right.$$

$$\left. + \mu_0^2\left[e^{\frac{t\psi}{2}} - 1 + \frac{12}{1+\mu_0^2} - \frac{64\mu_0^2(2\sqrt{1+\mu_0^2} + (3\mu_0^2-5)e^{-t\psi/2})}{(1+\mu_0^2)^{3/2}(1+9\mu_0^2)[t\psi(1+\mu_0^2)-2]} \right] \right\}.$$

$$- \frac{S[t\psi(1+\mu_0^2)-2]}{\sqrt{1+\mu_0^2}} S\{t\psi\} + O(Y_0^3)$$

Higher order terms could be determined from (6.12). When Y_0 is the order of 10^{-2}, the two term equation given above seems to be valid for concentrations as high as 50% light component.

7. TERMINOLOGY

A, B, Q, R = special functions defined by equations (3.8) and (3.17).

$C^{(0)}, C^{(1)}$, etc. = coefficients of Y_0 in equation (6.11).

F = interstage upflow rate, i.e. flow in light stream leaving stage.

H = holdup per stage of both light and heavy component.

I = feed rate.

$J(\Delta)$ = a weighting function, used in equation (5.1), and given by equation (5.25).

$N(\Delta, T)$ = mol fraction light component in heavy stream leaving stage Δ at time T.

$N'(\Delta, T)$ = mol fraction light component in light stream leaving stage Δ at time T.

N_0 = mol fraction light component in feed.

$N^\circ(\Delta)$ = light component concentration at stage Δ immediately after a disturbance.

N_p = steady-state product purity, mol fraction light component.

N_w = waste concentration, mol fraction light component.

P = steady-state product rate.

$P(T)$ = schedule of product withdrawals during a transition; $P(T) \leq P$ and $P(T) \to P$ as $T \to \infty$.

Δ = stage number, running from 0 to t.

81 M 26

t = total number of stages in cascade, including both rectifier and stripper.

$\underline{T(N_p)}$ = lower bound on the time which must elapse before steady-state production of material of ~~production of specified~~ N_p concentration can be put into effect (see equation 4.2), provided $P(T) < P$ at all T.

$\overline{T(N_p)}$ = upper bound (see Section 4.2).

u = number of stages in stripper.

W = waste rate.

$X(N_p)$ = steady-state light component inventory corresponding to product purity N_p.

X_0 = initial light component inventory.

Y = abundance ratio of light component $N/(1-N)$.

α = special parameter used in defining taper of cascade described by equation (5.26).

$\delta_J = N_\infty(u) - N(u,T)$ = concentration difference at junction of stripper and rectifier.

$\delta_P = N_p - N(t,T)$ = where N_p is steady-state product purity.

$\delta(\ell) = N_\infty(\ell) - N(\ell,T)$ = difference between steady-state concentration at stage ℓ (approached as $T \to \infty$), and concentration at time T.

$\delta_W = N_W - N(0,T)$ = where N_W is steady-state waste purity.

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Φ = net transport of light component.

$\tau = H/2F$ = holdup time of a stage.

Ψ = overall stage enrichment factor $(N'-N)/N(1-N)$.

Ⓗ = time constant introduced in Section 2.5 (see equation 2.16).

$\theta, \rho, \pi, \lambda, \xi$ = special dimensionless variables used in Section 5 (defined by equation 5.3).

φ = defined by 5.14.

FIGURE 1. Comparison of Interstage Flows of Tapered Cascade Studied in This Paper with Flows of an Ideal Cascade

Both cascades have the same number of stages, given by $t = \frac{6.96}{\Psi}$. Flows of the tapered cascade are given by $F = F_o e^{-3.84\, s/t}$, a special case of Equation (5.26) with zero production; flows of the ideal cascade are given by $F = F_o \left(e^{3.48(1-s/t)} - 1\right)/\left(e^{3.48} - 1\right)$. Both cascades operate at mol fractions small relative to one, and are fed by infinite reservoirs at feed concentration.

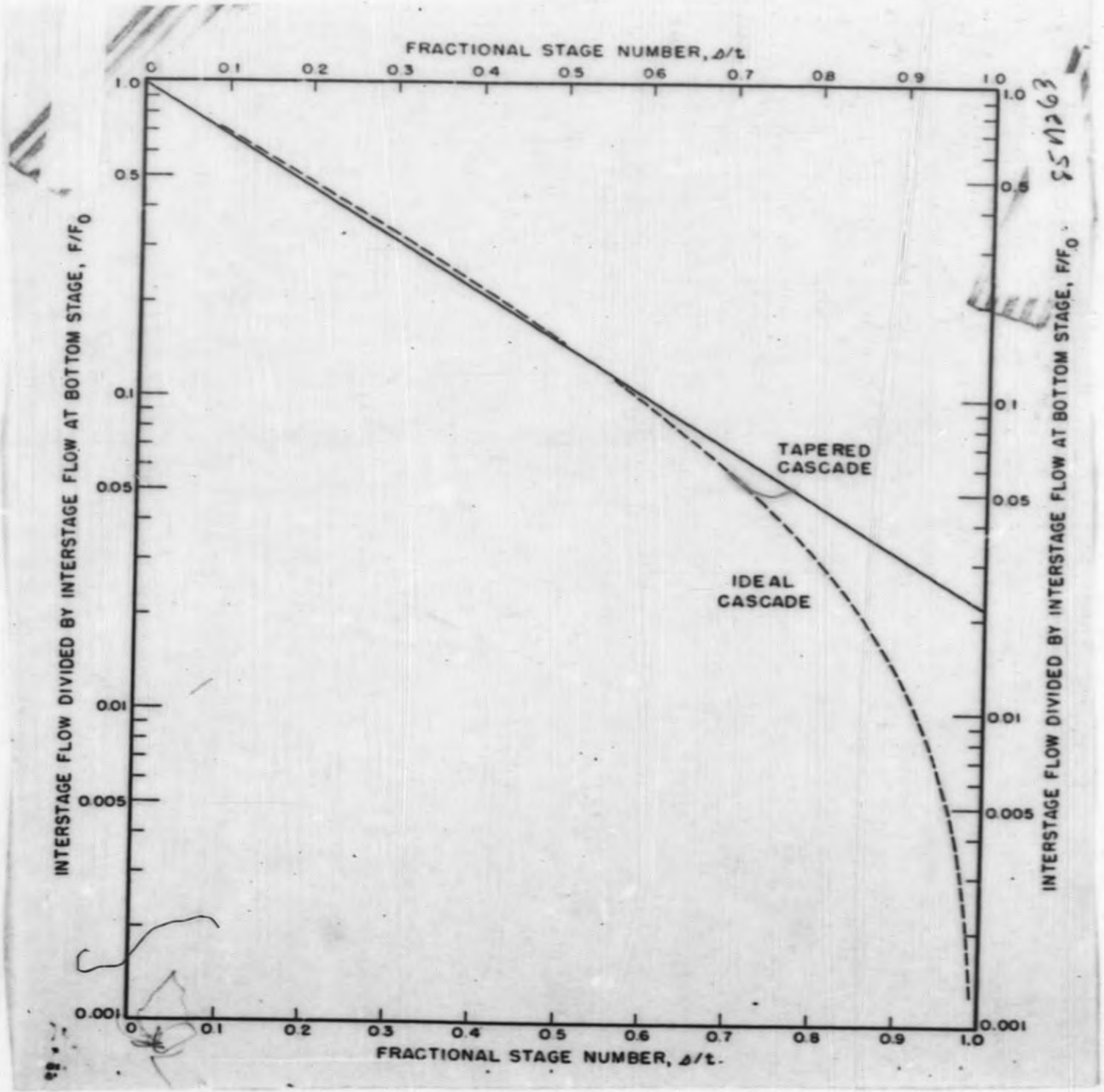

FRACTIONAL STAGE NUMBER, Δ/t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
INTERSTAGE FLOW DIVIDED BY INTERSTAGE FLOW AT BOTTOM STAGE, F/F0
1.0
0.5
0.1
0.05
0.01
0.005
0.001
TAPERED CASCADE
IDEAL CASCADE
FRACTIONAL STAGE NUMBER, Δ/t.
55 1263

FIGURE 2. Concentration in Tapered Cascade Versus Stage Number during Approach to Steady State

The cascade is filled with feed concentration at zero time. The numbers on the curves indicate elapsed time after start-up of the cascade in $t^2\gamma$ units (note: $t\Psi = 6.96$). The dashed curves are concentration distributions during steady-state production, and illustrate the conclusion that the cascade may be said to approach the steady state at infinite time through a succession of intermediate steady states.

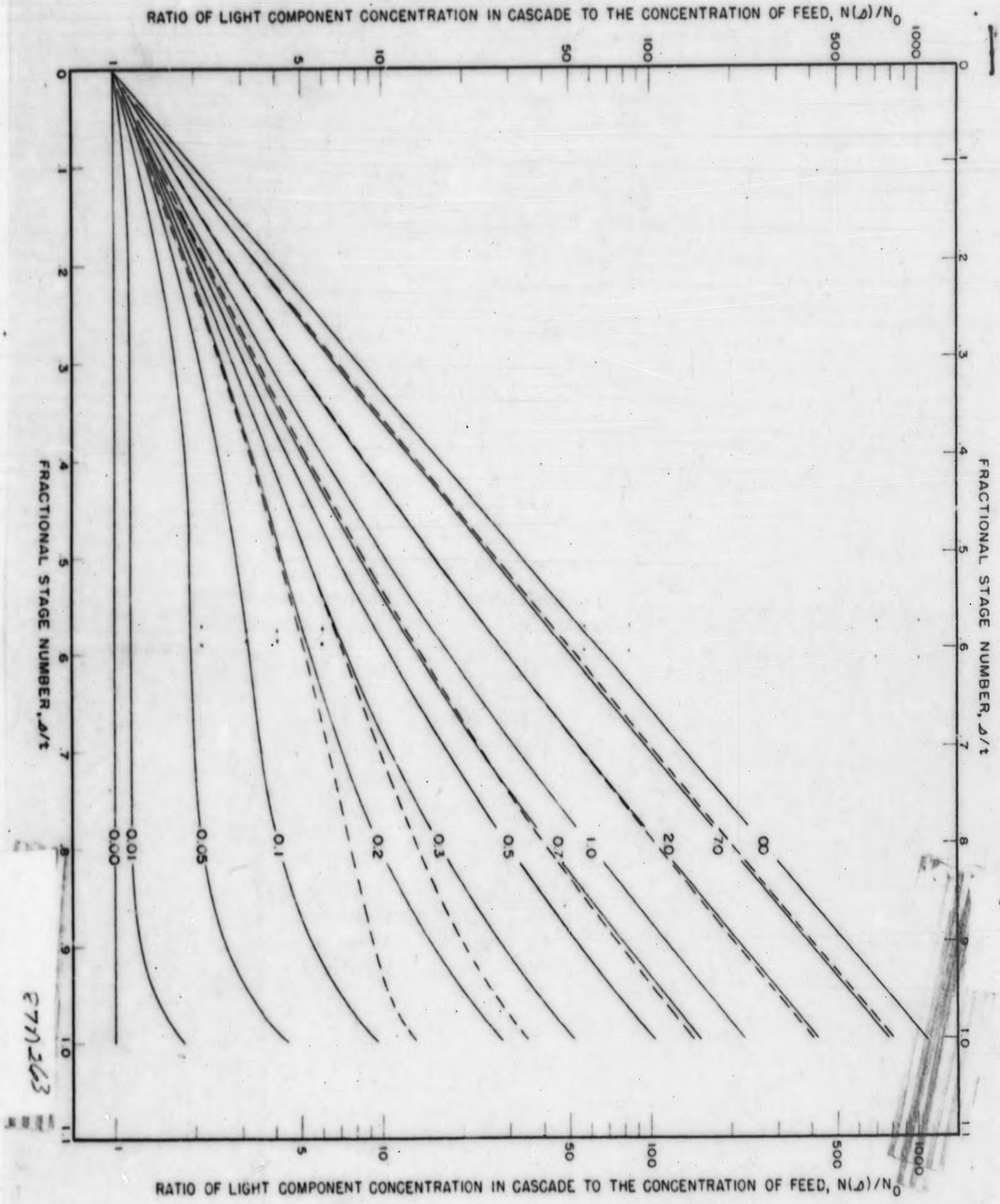

RATIO OF LIGHT COMPONENT CONCENTRATION IN CASCADE TO THE CONCENTRATION OF FEED, N(δ)/N0
1
5
10
50
100
500
1000
FRACTIONAL STAGE NUMBER, δ/t
0.00
0.01
0.05
0.1
0.2
0.3
0.5
0.7
1.0
2.0
7.0
∞

FIGURE 3. Concentration in Square Cascade versus Stage Numer During Approach to Steady State Corresponding to Zero Production

The cascade is filled with feed concentration at zero time. The numbers on the curves indicate elapsed times after start-up of the cascade in $t^2\mathcal{C}$ units (note: $t\Psi = 6.96$). The dashed curves are concentration distributions during steady-state production.

8PM3G3

FRACTIONAL STAGE NUMBER, δ/t

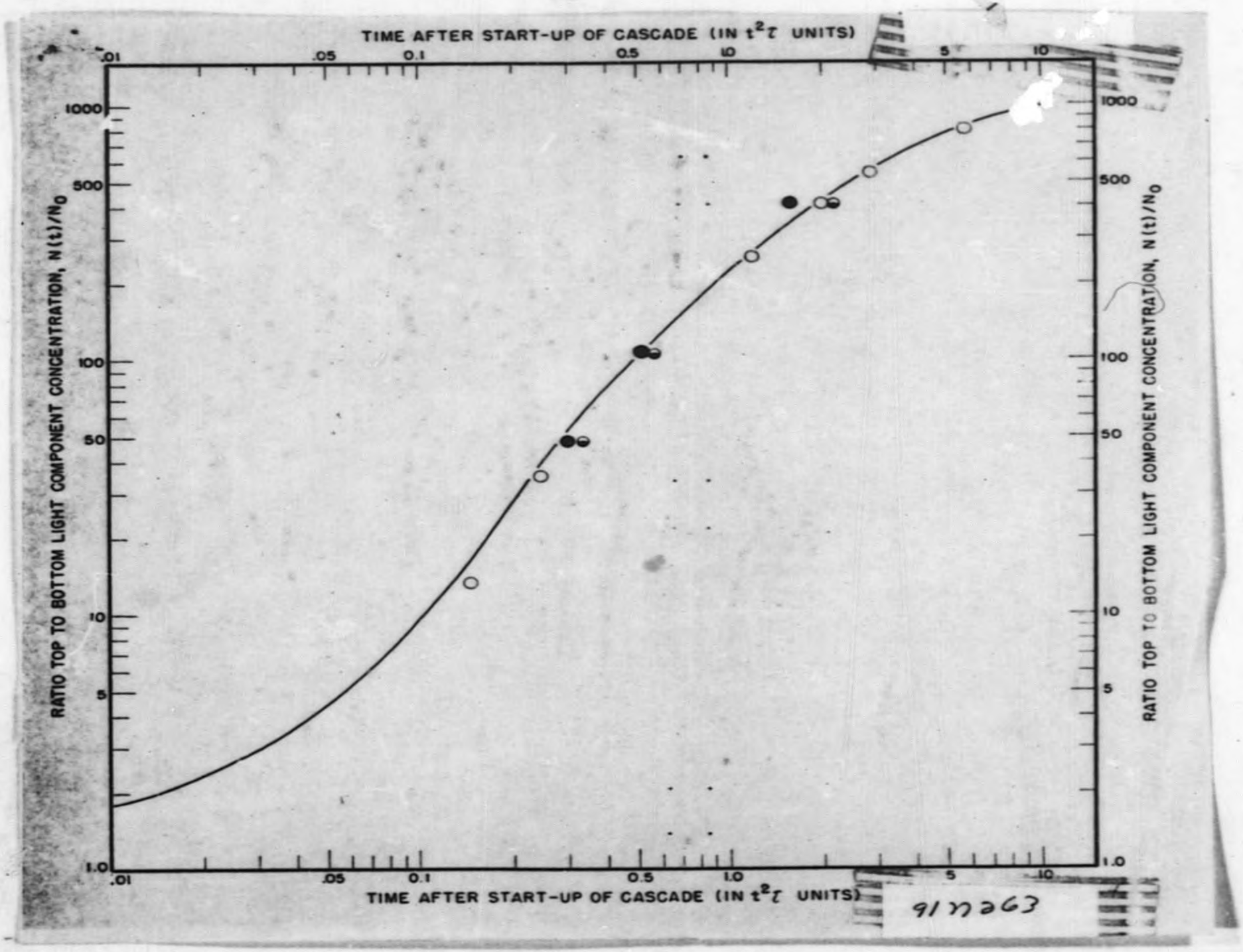
RATIO OF LIGHT COMPONENT CONCENTRATION IN CASCADE TO THE CONCENTRATION OF FEED, $N(\delta)/N_0$

∞
50
15
7.0
2.0
1.0
0.2
0.1
0.0

89 1263

FIGURE 4. Concentration at the Top Stage of Tapered Cascade versus Time during Approach to Steady State Corresponding to Zero Production

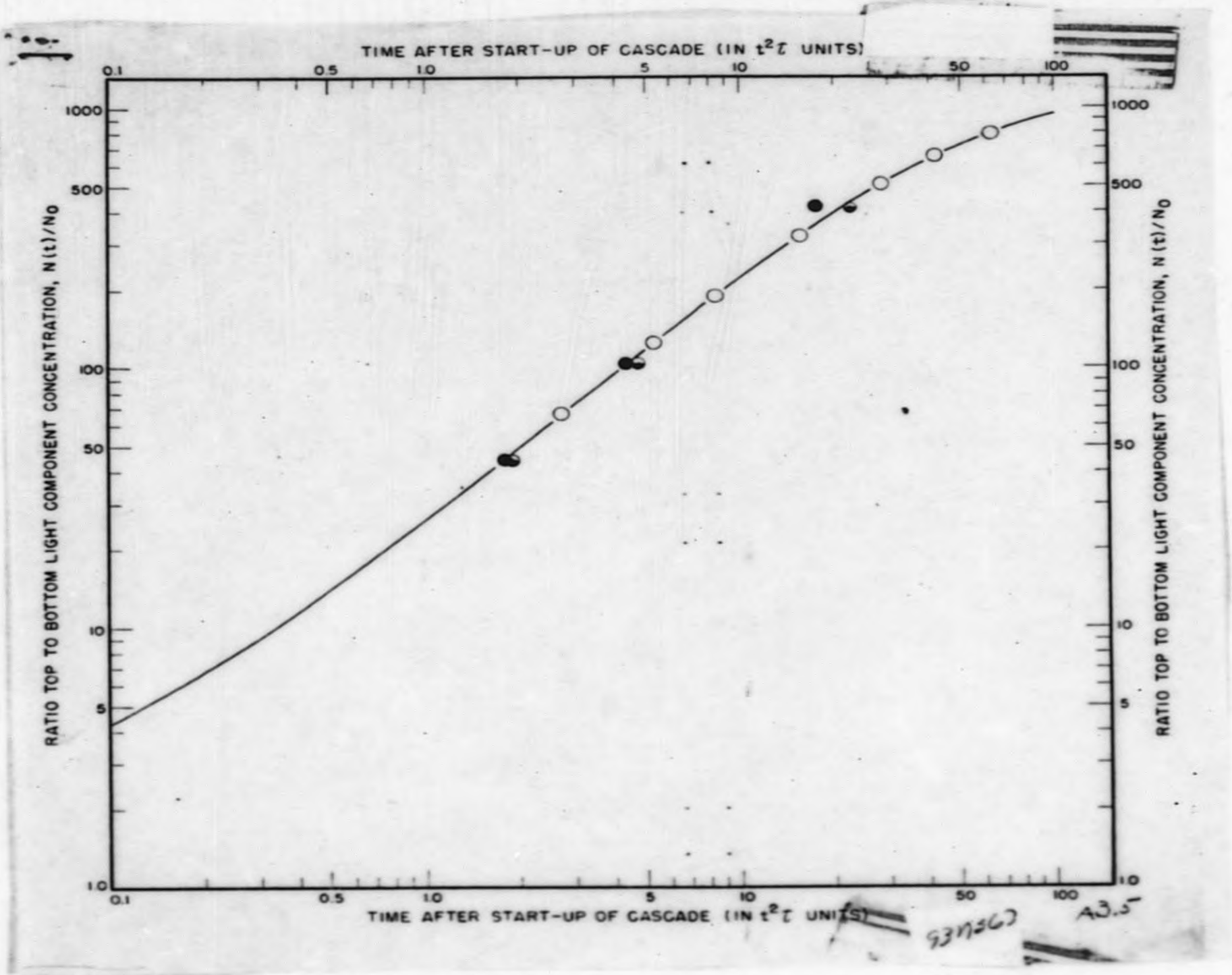
The curve, obtained by method given in Section 5, is exact. Open circles were obtained from approximate equation (2.6). Solid circles, calculated by equation (4.2), are lower bounds for the time which must elapse before the cascade can be in steady-state production at the corresponding product concentrations. Half-open circles are upper bound times (see Section 4.2).


TIME AFTER START-UP OF CASCADE (IN $t^2\tau$ UNITS)
.01
.05
0.1
0.5
1.0
5
10
1000
500
100
50
10
5
1.0
RATIO TOP TO BOTTOM LIGHT COMPONENT CONCENTRATION, $N(t)/N_0$
TIME AFTER START-UP OF CASCADE (IN $t^2\tau$ UNITS)
RATIO TOP TO BOTTOM LIGHT COMPONENT CONCENTRATION, $N(t)/N_0$

91n263

FIGURE 5. Concentration at the Top Stage of Square Cascade versus Time during Approach to Steady State Corresponding to Zero Production

The curve, obtained by method given in Section 5, is exact. Open circles were obtained from approximate equation (2.6). Solid circles, calculated by equation (4.2), are lower bounds for the time which must elapse before the cascade can be in steady-state production at the corresponding product concentration. Half-open circles are upper bound times (see Section 4.2).

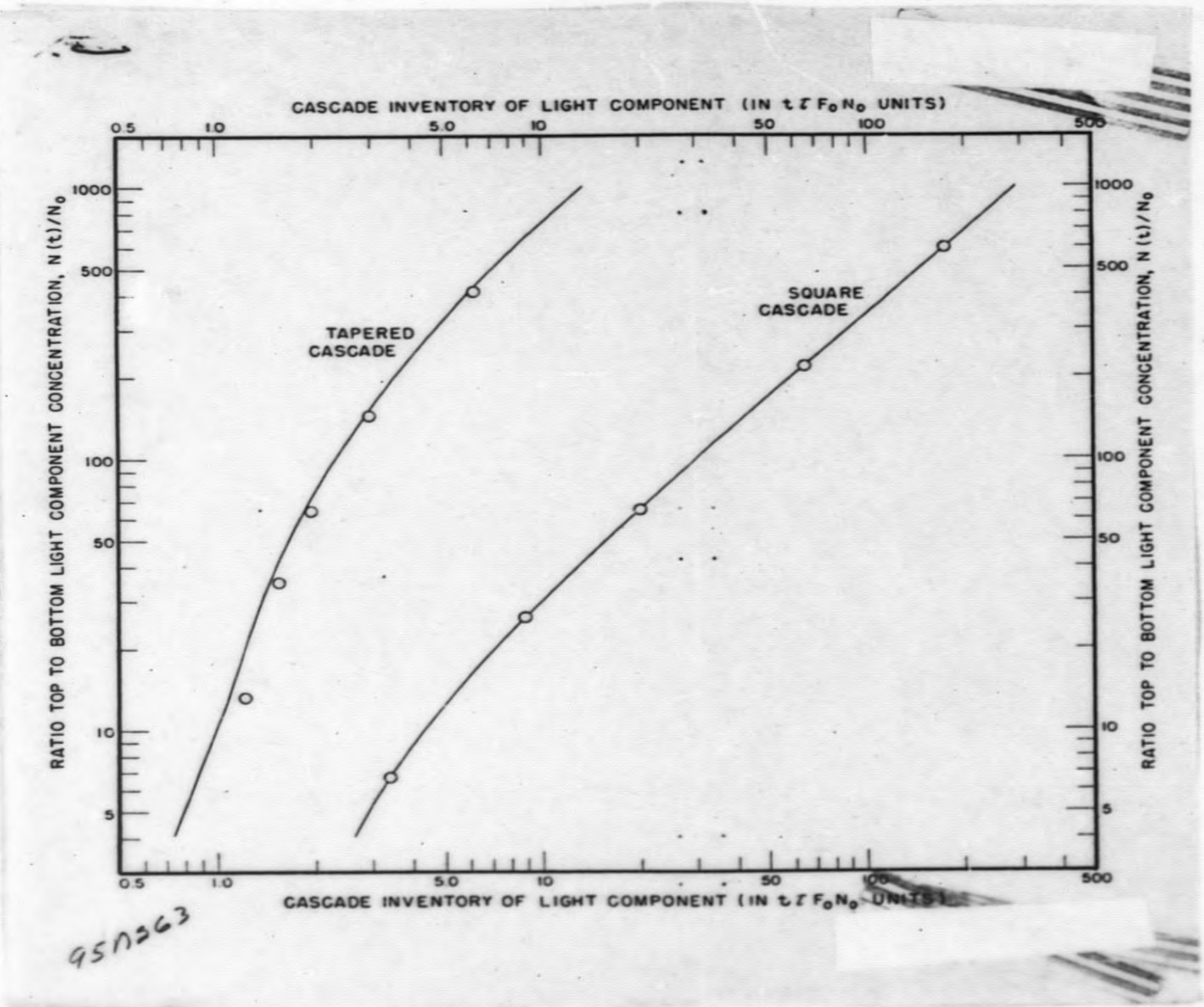

TIME AFTER START-UP OF CASCADE (IN $t^2\tau$ UNITS)
0.1
0.5
1.0
5
10
50
100
1000
500
100
50
10
5
1.0
RATIO TOP TO BOTTOM LIGHT COMPONENT CONCENTRATION, N(t)/N0
RATIO TOP TO BOTTOM LIGHT COMPONENT CONCENTRATION, N (t)/N0
TIME AFTER START-UP OF CASCADE (IN $t^2\tau$ UNITS)

93n56 A3.5

FIGURE 7. Comparison of Instantaneous Cascade Inventory of Light Component during Approach to Steady State with Inventory during Steady State Production

Curves give instantaneous inventory at moment top stage reaches a given concentration, and each open circle gives the steady-state inventory when cascade is producing material of the corresponding product concentration.

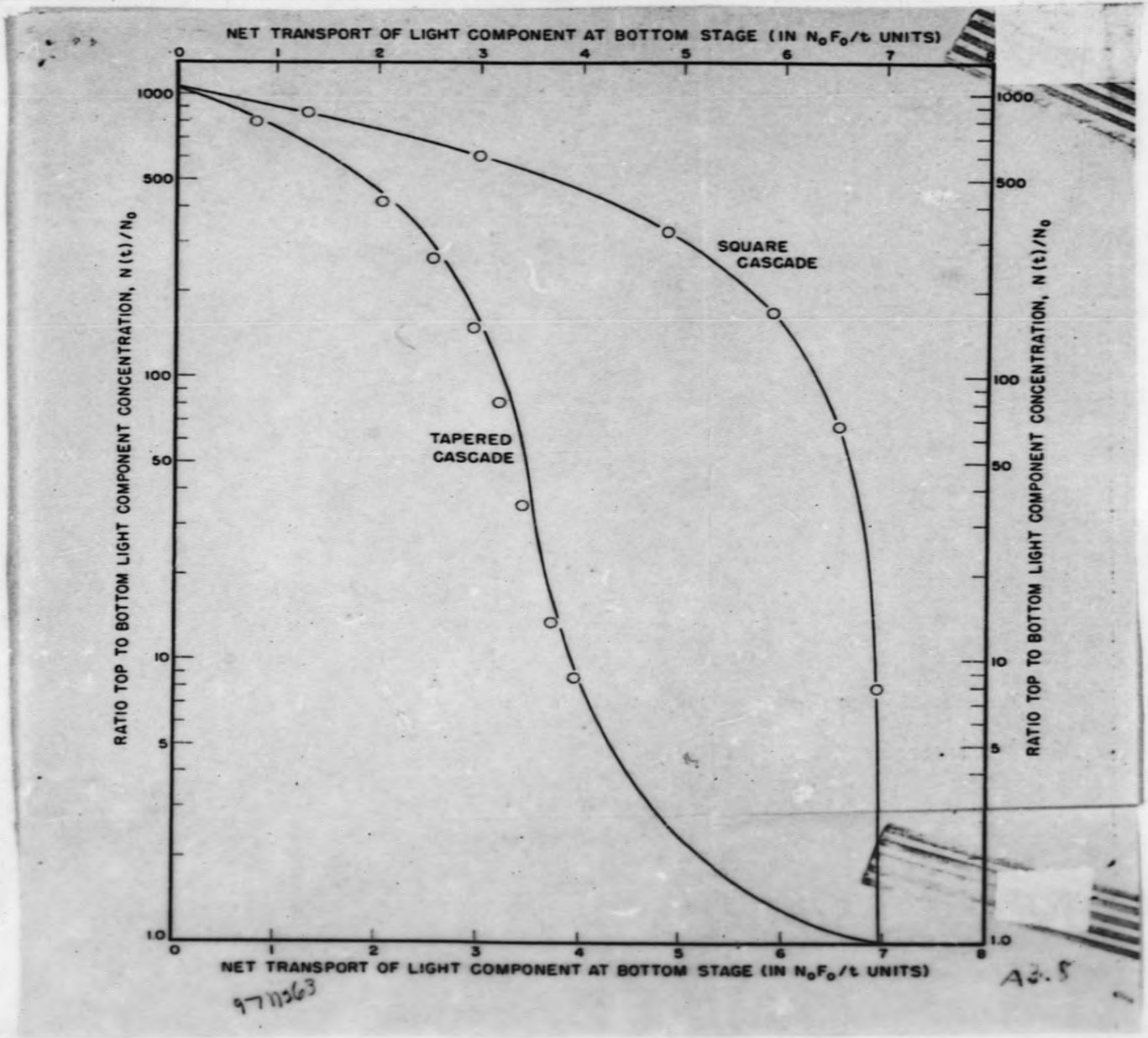
94N363


CASCADE INVENTORY OF LIGHT COMPONENT (IN $t\bar{\gamma}F_0N_0$ UNITS)
0.5
1.0
5.0
10
50
100
500
1000
500
100
50
10
5
RATIO TOP TO BOTTOM LIGHT COMPONENT CONCENTRATION, $N(t)/N_0$
TAPERED CASCADE
SQUARE CASCADE
CASCADE INVENTORY OF LIGHT COMPONENT (IN $t\bar{\gamma}F_0N_0$ UNITS)

95n263

FIGURE 8. Comparison of Instantaneous Net Transport of Light Component at Bottom Stage to Steady State with Net Transport during Steady-State Production

Curves give instantaneous net transport at moment top stage reaches a given concentration, and each open circle gives the steady-state net transport (PN_p) when cascade is producing material of the corresponding product concentration.


NET TRANSPORT OF LIGHT COMPONENT AT BOTTOM STAGE (IN N_0F_0/t UNITS)
0
1
2
3
4
5
6
7
8
1000
500
100
50
10
5
1.0
RATIO TOP TO BOTTOM LIGHT COMPONENT CONCENTRATION, $N(t)/N_0$
SQUARE CASCADE
TAPERED CASCADE
NET TRANSPORT OF LIGHT COMPONENT AT BOTTOM STAGE (IN N_0F_0/t UNITS)

9711263

A3.5

FIGURE 9. Concentrations in a Square Cascade during Redistribution of Light Component Inventory from an Arbitrary Initial Distribution

Originally, the lower half of the cascade was operating alone and was in steady-state production of material with concentration $N_p/N_o = 4.3$. The upper half is filled with material stock-piled from the product of the lower half. At zero time the two halves are placed in operation together with no product withdrawal; the numbers on the curves indicate elapsed time in $t^2\gamma$ units. (Note: $t\Psi = 6.0$). The dashed curve is the particular steady-state concentration distribution which leads to the same cascade light component inventory as was present at zero time.

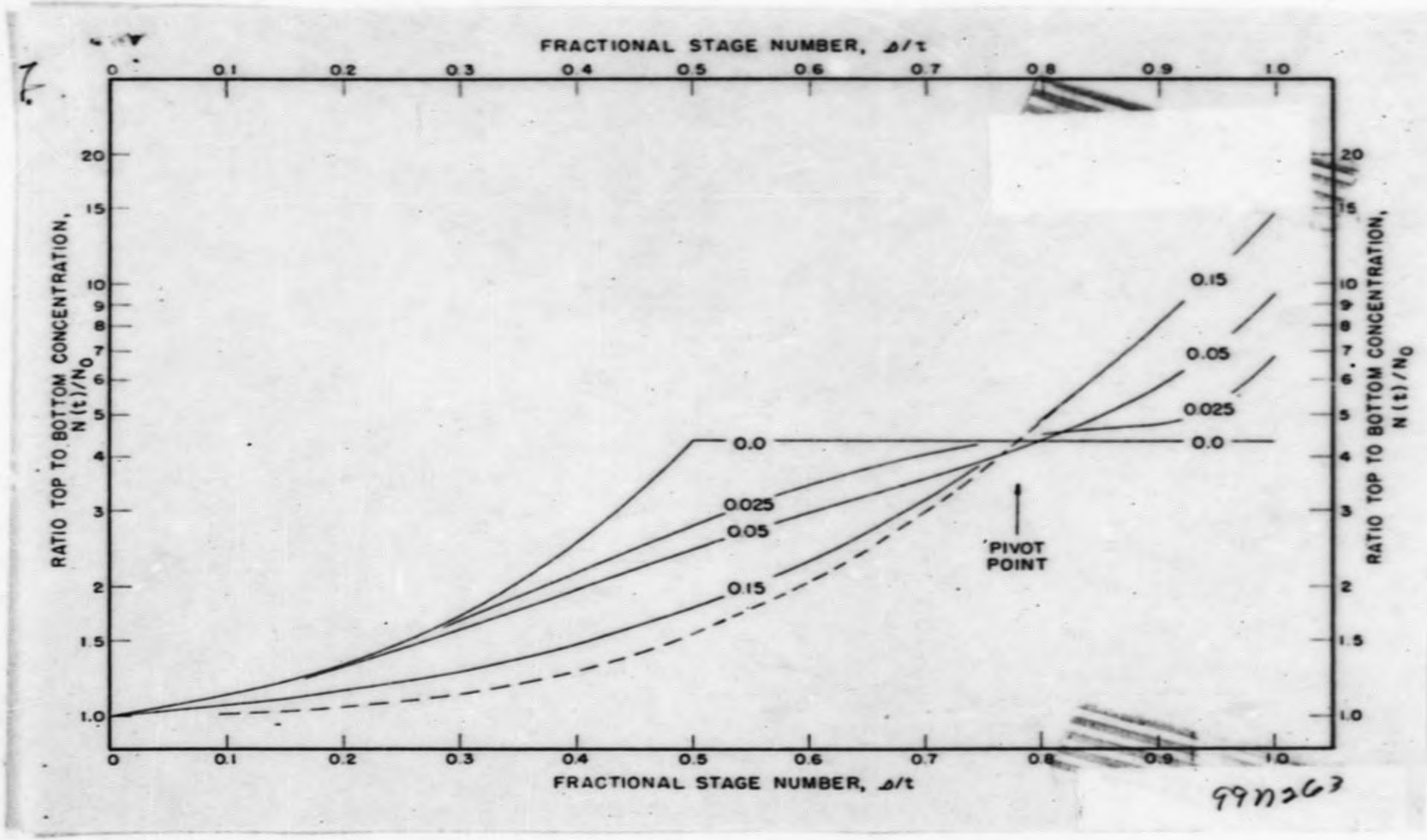
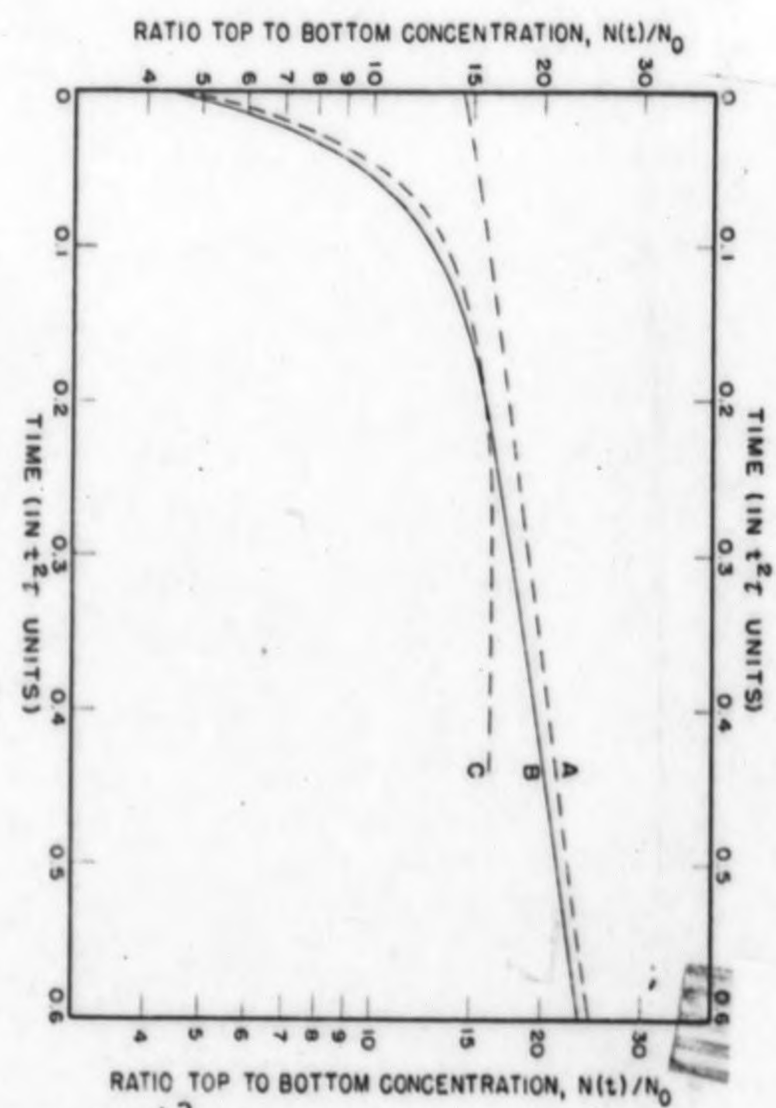

FRACTIONAL STAGE NUMBER, δ/t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
RATIO TOP TO BOTTOM CONCENTRATION, N(t)/N0
20
15
10
9
8
7
6
5
4
3
2
1.5
1.0
0.15
0.05
0.025
0.0
0.0
0.025
0.05
0.15
PIVOT POINT
FRACTIONAL STAGE NUMBER, δ/t
99n363

FIGURE 10. Top Stage Concentration versus Time during Redistribution of Light Component Inventory from an Arbitrary Initial Distribution

Solid curve B gives exactly the variation in top stage concentration with no product withdrawal, when the initial distribution is the curve of Figure 9 labelled O.O. Dashed curve A gives exactly the concentration versus time if the initial distribution is the steady-state distribution indicated by the dashed curve of Figure 9. (Note that B parallels A after the redistribution is complete.) Dashed curve C is an exact curve for the approach to steady state at zero production of the partial cascade above the pivot point designated on Figure 9, when the partial cascade operates over an infinite reservoir of feed at concentration 4.3 N_0. (Note that solid curve B may be said to be a composite of A and C.)

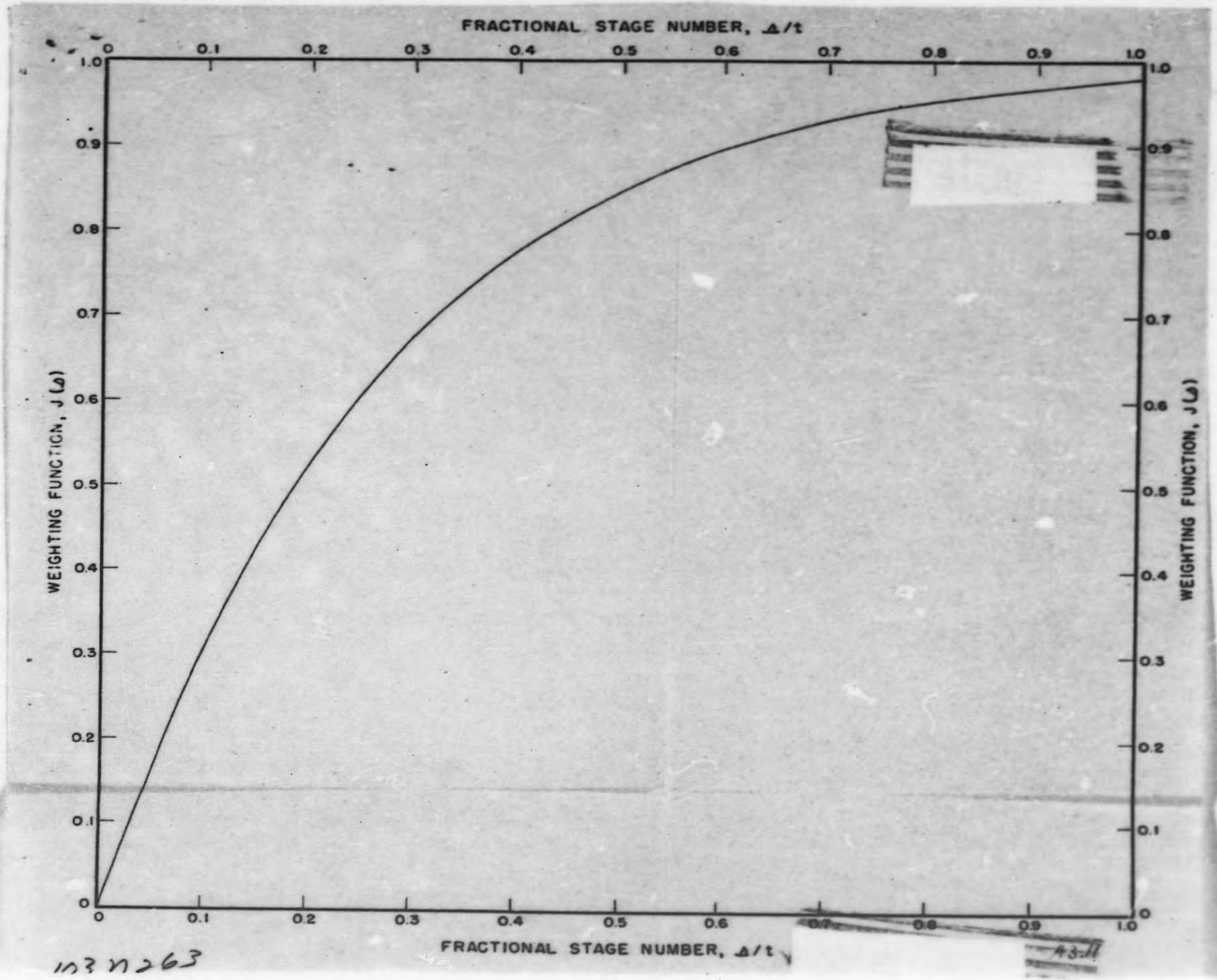
100n363


RATIO TOP TO BOTTOM CONCENTRATION, N(t)/N0
4
5
6
7
8
9
10
15
20
30
0
0.1
0.2
0.3
0.4
0.5
0.6
TIME (IN t²f UNITS)
C
B
A
RATIO TOP TO BOTTOM CONCENTRATION, N(t)/N0
TIME (IN t²f UNITS)

10/1267

FIGURE 11. Weighting Function J for Special Tapered Cascade

Weighting function is used in equation (3.1) for calculating losses due to transient disturbances. The flows in the tapered cascade are given by equation (8.26), with $\pi = \frac{tP}{r_o} = 0.075$, $\alpha t = 3.23$; $t\Psi = 6.96$. This cascade closely resembles the tapered cascade graphed in Figure 1, with slightly larger flows in the central stages.


FRACTIONAL STAGE NUMBER, Δ/t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
WEIGHTING FUNCTION, J(δ)
FRACTIONAL STAGE NUMBER, Δ/t

113263

43-11

FIGURE 12. Loss of Light Component Output Resulting from Flattering Concentration Gradient in Special Tapered Plant

Suppose a disturbance mixes uniformly the inventory of all stages between s_1 and s_2 of the special tapered plant described below Figure 11. The curve above gives the resulting loss of output in the limiting case $s_1 \longrightarrow s_2$. The loss is expressed in $t^2\tau$ units of time, i. e. the time during which the output lost would normally have been produced during steady-state operation. The open circle gives the loss for the case $s_1/t = 0.25$ and $s_2/t = 0.75$; and the solid circle gives the loss if the entire cascade's inventory is mixed uniformly.

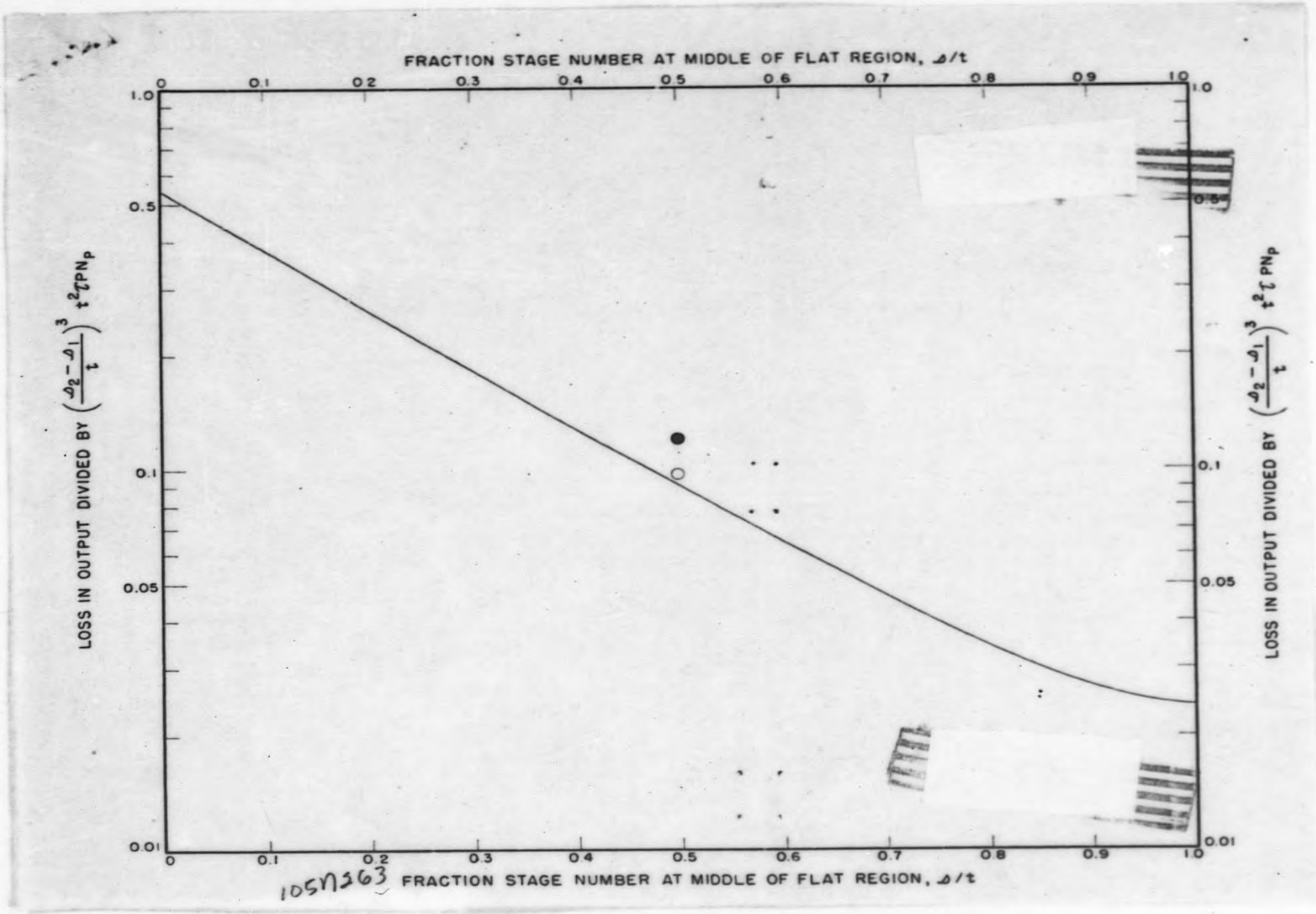

FRACTION STAGE NUMBER AT MIDDLE OF FLAT REGION, s/t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
LOSS IN OUTPUT DIVIDED BY $\left(\frac{s_2-s_1}{t}\right)^3 t^2\tau PN_p$
1.0
0.5
0.1
0.05
0.01
105N163
FRACTION STAGE NUMBER AT MIDDLE OF FLAT REGION, s/t

FIGURE 13. Product Purity versus Time after a Disturbance which Uniformly Mixes Inventory between Stages $s_1/t = 0.25$ and $s_2/t = 0.75$

The curve applies to the special tapered plant described below Figure 11. The dashed line indicates the normal steady-state product purity from this cascade at its product rate $P = 0.075\ F_o/t$. The time integral of P multiplied by the discrepancy between the dashed line and the curve is the loss in output given by the open circle of Figure 12.

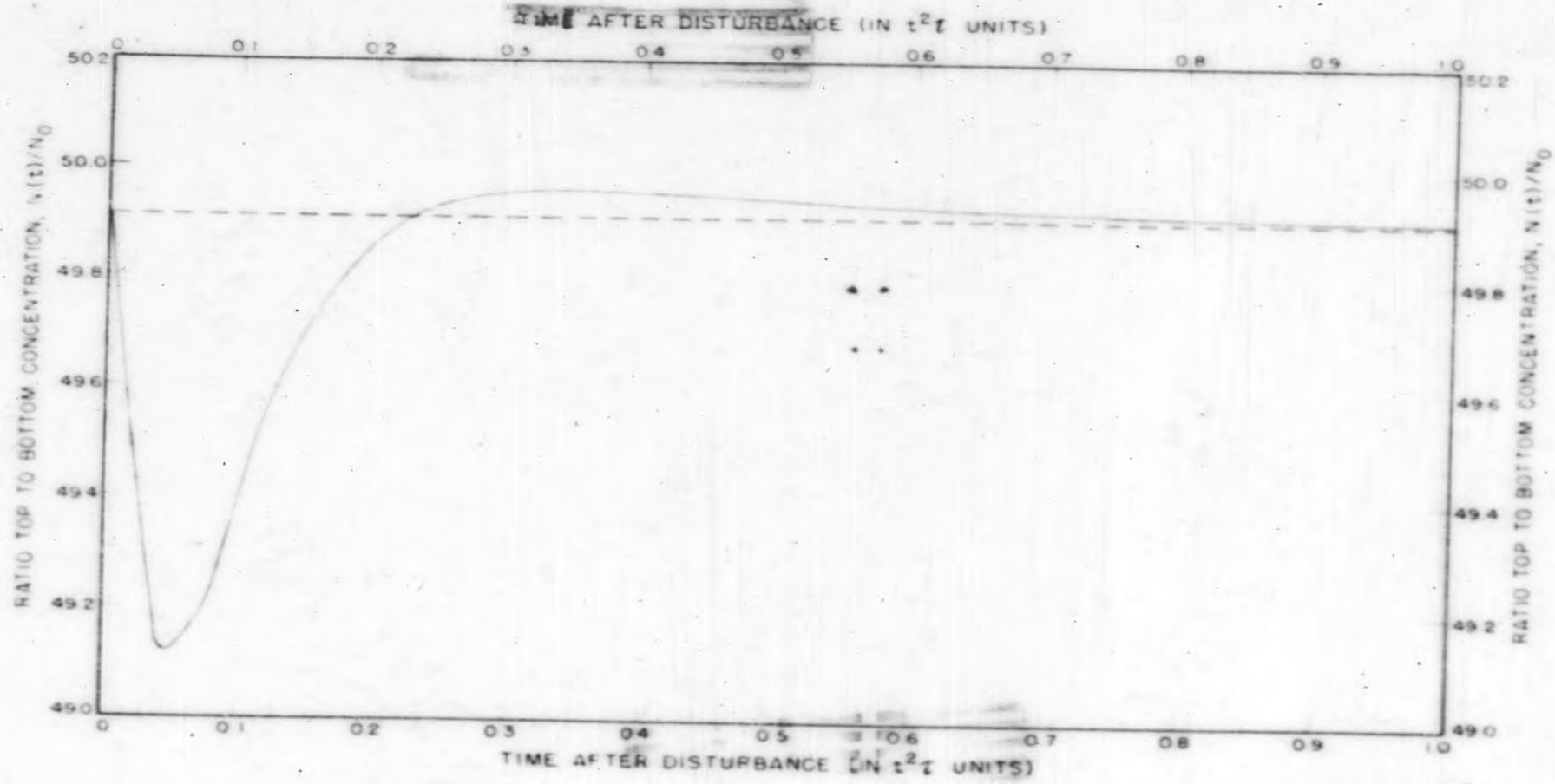

TIME AFTER DISTURBANCE (IN $t^2\tau$ UNITS)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
50.2
50.0
49.8
49.6
49.4
49.2
49.0
RATIO TOP TO BOTTOM CONCENTRATION, $N(t)/N_0$
TIME AFTER DISTURBANCE (IN $t^2\tau$ UNITS)
RATIO TOP TO BOTTOM CONCENTRATION, $N(t)/N_0$

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FIGURE 15. Illustration of solution of Equation (1.42) for Characteristic Roots

Dashed lines graph $f_1(\mu) = \mu$

Curved graphs $f_2(\mu) = \tan \mu$