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A DIFFERENTIAL EQUATION FROM THE THEORY OF NEUTRON THERMALISATION

by

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ABSTRACT

The "heavy gas model" using a synthetic kernel leads to the following coupled equations for the determination of the neutron flux $\phi = \phi(E)$ in a bare reactor:

$$\begin{cases} -(\Sigma_a + B^2 D)\,\phi + \dfrac{\partial q}{\partial E} + S = 0 \\ q = \xi \Sigma_s \left[(\beta E - T)\,\phi + ET\,\dfrac{\partial \phi}{\partial E}\right] - \epsilon \xi E S + \epsilon \xi E B^2 D \phi \ , \end{cases}$$

where

$$\beta = 1 + \epsilon \frac{\Sigma_a}{\Sigma_s} \quad .$$

This paper deals with the very special problem of solving analytically the defining equation for the energy dependent neutron flux in a region for which there are no source terms and wherein D, Σ_a, and Σ_s are constant. Thus, various forms of the solutions of the equation:

$$x\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + \lambda y = 0 \ ; \qquad 0 < \lambda < 1 \quad ,$$

may be derived for the two-point boundary value problem $(a < b)$:

$$\begin{cases} y(a) = A \\ y(b) = B \ , \end{cases}$$

ABSTRACT (continued)

where we have written

$$x = M E,$$

$$y = \phi,$$

$$\lambda = \frac{N}{M},$$

$$N = \frac{1}{T}\left[1 + \frac{\epsilon}{\Sigma_s}(\Sigma_a + B^2 D) - \frac{\Sigma_a + B^2 D}{\xi \Sigma_s}\right].$$

$$M = \frac{1}{T}\left[1 + \frac{\epsilon}{\Sigma_s}(\Sigma_a + B^2 D)\right].$$

Some of these forms are derived, and the problem of accurately evaluating them numerically is discussed.

The author is well aware of the rather artificial nature of the physical model embodied in this equation. However, the extreme simplicity of the functional forms of the coefficients poses the question whether we may use the results here obtained to interpret the nature of the flux values for a more realistic model –– one in which Σ_a, D, Σ_s, etc. are variable functions of the independent variable (energy) and not merely constants. A subsequent paper will include a discussion of this question, and an assessment of the accuracy attainable by substituting the 'constant–parameter' model successively and collectively over sufficiently small energy–intervals.

CONTENTS

1. INTRODUCTION

Calculations of the energy-dependence of the neutron flux in a bare reactor are in progress to aid in the determination of the performances of some thermal-intermediate power reactors (this work is being done by the Engineering Research Section of this establishment.) These involve the "numerical" solution, using a digital computer, of the following equations (for Equation 2 see Rubbra and Pollard 1962):

$$\begin{cases} -(\Sigma_a + B^2 D)\,\phi + \dfrac{dq}{dE} + S = 0 & (1) \\ q = \xi \Sigma_s \left[(\beta E - T)\,\phi + ET \dfrac{d\phi}{dE} \right] - \epsilon \xi E S + \epsilon \xi E B^2 D \phi & (2) \end{cases}$$

This system is equivalent to a linear second order inhomogeneous defining equation for ϕ or for q, the neutron slowing down density. During an investigation into the properties of the solutions of these equations for two-point boundary and initial-value problems, it was discovered that under certain simplifying conditions it is possible to solve this equation explicitly in a variety of ways. Further, it can be shown that all these forms can be numerically evaluated to a high degree of accuracy.

Reported here, then, are different forms of some analytical solutions of (1) and (2) for the special case of D, Σ_a, and Σ_s all independent of energy, no resonances present, and in a region where $S = 0$. These solutions (the linearly independent, 'fundamental' solutions corresponding to a two-point boundary value or initial-value problem), together with an analysis of means for their accurate numerical valuation, are presented to provide a method of checking the so-called "numerical solutions" for these special conditions.

Differentiating (2) and substituting into (1), there results, for constant values of D, Σ_a, Σ_s, the defining equation for $\phi(E)$:

$$E \frac{d^2\phi}{dE^2} + ME \frac{d\phi}{dE} + N\phi = S_o \quad , \qquad (3)$$

where

$$M = \frac{1}{T}\left[1 + \frac{\epsilon}{\Sigma_s}(\Sigma_a + B^2 D) \right] ,$$

$$N = \frac{1}{T}\left[1 + \frac{\epsilon}{\Sigma_s}(\Sigma_a + B^2 D) - \frac{\Sigma_a + B^2 D}{\xi \Sigma_s} \right] ,$$

$$S_o = \frac{1}{T}\left[\frac{(\epsilon\xi - 1) S + \epsilon \xi E \dfrac{dS}{dE}}{\xi \Sigma_s} \right] .$$

In a region where $S = 0$ this becomes

$$E \frac{d^2\phi}{dE^2} + ME \frac{d\phi}{dE} + N\phi = 0 \quad . \qquad (4)$$

which is essentially the equation with which the remainder of this report is concerned.

Note here two other formulations of the problem, coupling the heavy gas thermalisation model with a synthetic kernel slowing down model (Thompson and Lawrence 1960; Thompson 1962). The respective expressions for q are

$$q = \xi(\Sigma_s + \epsilon \Sigma_a)\left[(E - T)\,\phi + ET \frac{d\phi}{dE} \right] - \epsilon \xi E S + \epsilon \xi E B^2 D \phi \qquad (5)$$

and

$$q = \xi \Sigma_s \left[(E - T)\,\phi + ET \frac{d\phi}{dE} \right] + \epsilon \xi \left[(E - T)(\Sigma_a \phi + B^2 D \phi - S) + ET \frac{d}{dE}(\Sigma_a \phi + B^2 D \phi - S) \right] . \qquad (6)$$

which give rise to the same Equation 3 with

$$M = \frac{1}{T}\left[1 + \frac{\epsilon B^2 D}{\Sigma_s + \epsilon \Sigma_a}\right],$$

$$N = \frac{1}{T}\left[1 + \frac{\epsilon B^2 D}{\Sigma_s + \epsilon \Sigma_a} - \frac{\Sigma_a + B^2 D}{\xi(\Sigma_s + \epsilon \Sigma_a)}\right],$$

$$S_o = \frac{1}{T}\left[\frac{S(\epsilon \xi - 1) + \epsilon \xi E \frac{dS}{dE}}{\xi(\Sigma_s + \epsilon \Sigma_a)}\right],$$

and

$$M = \frac{1}{T},$$

$$N = \frac{1}{T}\left[1 - \frac{\Sigma_a + B^2 D}{\xi \{\Sigma_s + \epsilon(\Sigma_a + B^2 D)\}}\right],$$

$$S_o = \frac{1}{T}\left[\frac{S(\epsilon \xi - 1) + \epsilon \xi E \frac{dS}{dE} + \epsilon \xi E T \frac{d^2 S}{dE^2}}{\xi \{\Sigma_s + \epsilon(\Sigma_a + B^2 D)\}}\right],$$

respectively.

We are therefore concerned with the fundamental solutions of the linear, second-order, homogeneous equation

$$\begin{cases} x \dfrac{d^2 y}{dx^2} + x \dfrac{dy}{dx} + \lambda y = 0, \\ x > 0,\ 0 < \lambda < 1. \end{cases} \tag{7}$$

2 DERIVATION OF THE DIFFERENT FORMS OF THE FUNDAMENTAL SOLUTIONS

Three basically different approaches can be made in attempting to obtain explicit solutions of Equation 7:

(i) power series solutions;

(ii) solutions in terms of definite and indefinite integrals;

(iii) continued fraction solutions for the logarithmic derivatives.

Perhaps the most convenient to calculate numerically are the series solutions.

2.1 Series Solutions

$x = 0$ is a regular singular point of (7) and so each of the fundamental solutions can be expressed as a power series in x. It turns out that because $0 < \lambda < 1$ we may use these series solutions to obtain accurate (1 part in 10^6 or better) values of $y(x)$ in the range $0 < x \leq 5$. If $x > 5$, asymptotic series solutions can be formulated which are also extremely accurate.

In either range, we can estimate the error involved in truncating a particular series at a given term (see Section 3).

The solution $y = \Sigma_{n=0}^{\infty} A_n x^{n+s}$ has associated with it the indicial equation:

$$s(s-1) = 0,$$

and a recursion relation:

$$(n+s)(n+s+1)A_{n+1} + (n+s+\lambda)A_n = 0 \quad , \qquad n = 0, 1, 2, \ldots \quad ,$$

and it is clear that only for $s = 1$ does there exist a (fundamental) solution of this form. Thus we have:

$$y_1 = \Sigma_{n=0}^{\infty} A_n x^{n+1} \tag{8}$$

$$\frac{A_n}{A_o} = (-1)^n \frac{(1+\lambda)(2+\lambda)\ldots(n+\lambda)}{\lfloor n \; \lfloor n+1} \quad , \qquad n = 1, 2, \ldots \quad . \tag{9}$$

The other fundamental solution is of the form:

$$y_2 = c(\log x)\, y_1 + \Sigma_{n=0}^{\infty} B_n x^n \tag{10}$$

where

$$0 = A_o c + \lambda B_o \tag{11}$$

and

$$(n+1)(n+2) B_{n+2} + (n+1+\lambda) B_{n+1} + c\,\{A_n + (2n+3) A_{n+1}\} = 0 \; . \tag{12}$$

If x is much greater than 5, y_1 and y_2 are represented by series which are too slowly convergent for convenient numerical computation. In this case, we resort to the use of asymptotic series.

Let the adjoint of y be y^*. Then if primes denote differentiation with respect to x, we have:

$$y = x\, e^{-x}\, y^* \; , \tag{13}$$

where y^* satisfies the confluent hypergeometric equation:

$$x\, y^{*\prime\prime} + (2-x)\, y^{*\prime} - (1-\lambda)\, y^* = 0 \; . \tag{14}$$

Now the asymptotic solutions of the confluent hypergeometric equation:

$$x\, y^{*\prime\prime} + (c-x)\, y^{*\prime} - a\, y^* = 0 \tag{14a}$$

are

$$y_1^* \cong x^{-a}\, \Sigma_{n=0}\, \alpha_n x^{-n} \; , \tag{15a}$$

and

$$y_2^* \cong e^x\, x^{a-2}\, \Sigma_{n=0}\, \beta_n x^{-n} \; , \tag{16a}$$

where:

$$\frac{\alpha_n}{\alpha_o} = (-1)^n \frac{\Pi_{k=0}^{k=n-1}(a+k)\; \Pi_{k=1}^{k=n}(a-c+k)}{\lfloor n} \quad , \tag{17a}$$

and

$$\frac{\beta_n}{\beta_o} = \frac{\Pi_{k=0}^{k=n-1}(c-a+k)\; \Pi_{k=1}^{k=n}(k-a)}{\lfloor n} \quad . \tag{18a}$$

Substituting $c = 2$, $a = 1 - \lambda$ in these equations we get:

$$y_1^* \cong x^{-(1-\lambda)}\, \Sigma_{n=0}\, \alpha_n x^{-n} \quad , \tag{15}$$

$$y_2^* \cong e^x\, x^{-(1+\lambda)}\, \Sigma_{n=0}\, \beta_n x^{-n} \quad , \tag{16}$$

where:

$$\frac{\alpha_n}{\alpha_o} = (-1)^{n+1} \frac{\lambda(n-\lambda)\, \Pi_{k=1}^{k=n-1}(k-\lambda)^2}{\lfloor n} \quad , \tag{17}$$

and

$$\frac{\beta_n}{\beta_o} = \frac{\lambda(n+\lambda)\, \Pi_{k=1}^{k=n-1}(k+\lambda)^2}{\lfloor n} \quad . \tag{18}$$

Therefore the asymptotic forms of the solutions of (7) may be written:

$$\bar{y}_1 \cong x e^{-x} y_1^* = e^{-x} x^{\lambda} \Sigma_{n=0} \alpha_n x^{-n} , \tag{19}$$

$$\bar{y}_2 \cong x e^{-x} y_2^* = x^{-\lambda} \Sigma_{n=0} \beta_n x^{-n} . \tag{20}$$

2.2 Solutions in Terms of Definite and Indefinite Integrals

A solution of the adjoint Equation 14 is:

$$y_1^* = \int_0^{\infty} e^{-xt} \left(\frac{1+t}{t}\right)^{\lambda} dt , \tag{21}$$

so that a solution of (7) is:

$$y_1 = x e^{-x} \int_0^{\infty} e^{-xt} \left(\frac{1+t}{t}\right)^{\lambda} dt. ; \tag{22}$$

another linearly independent solution is given by

$$y_2 = \int_0^{x} e^{-t} \left(\frac{t}{x-t}\right)^{\lambda} dt . \tag{23}$$

Equation 23 may be obtained formally by applying the Laplace transform to (7). Writing

$$L \{y(t)\} \equiv \int_0^{\infty} e^{-st} y(t) dt \equiv f(s) ,$$

we have

$$L \{t y''(t) + t y'(t) + \lambda y(t)\} = 0 ;$$

this gives:

$$s(1+s) f^{(1)} + (1 + 2s - \lambda) f = 0 ,$$

that is,

$$f(s) = \frac{h}{s^{1-\lambda} (1+s)^{1+\lambda}} , \tag{24}$$

where h = constant. Now

$$\left.\begin{aligned} L^{-1}\left\{\frac{\Gamma(1+\lambda)}{(1+s)^{1+\lambda}}\right\} &= e^{-t} t^{\lambda} \\ L^{-1}\left\{\frac{\Gamma(1-\lambda)}{s^{1-\lambda}}\right\} &= t^{-\lambda} \end{aligned}\right\} s > 1 .$$

Therefore the convolution theorem gives:

$$\text{constant} \cdot y = L^{-1} \left\{\frac{\Gamma(1+\lambda)\,\Gamma(1-\lambda)}{s^{1-\lambda}(1+s)^{1+\lambda}}\right\} = \int_0^{t} e^{-u} \left(\frac{u}{t-u}\right)^{\lambda} du .$$

To make the derivation of (23) rigorous, we must establish the existence of the transforms of y, y', and y". We have, assuming all the integrals exist,

$$L\{y'(t)\} = [e^{-st}\,y(t)]_o^\infty + \frac{1}{s}L\{y(t)\},$$

$$L\{y''(t)\} = [e^{-st}\,y'(t)]_o^\infty + \frac{1}{s}L\{y'(t)\}.$$

The asymptotic solutions (19) and (20) clearly show that $L\{y(t)\}$ must exist and that moreover $[e^{-st}y(t)]_o^\infty = 0$ if $y(0+0) = 0$; hence the existence of $L\{y'(t)\}$ is also demonstrated. The existence of $L\{y''(t)\}$ by establishing that of $[e^{-st}y'(t)]_o^\infty$ may be shown by examining the asymptotic solutions of the second order equation satisfied by $Y(t) \equiv y'(t)$. In Equation 7, put $Y(X) \equiv y'(X)$, $\chi \equiv -x$ to give, after differentiation,

$$\chi Y'' + (1-\chi)Y' - (1+\lambda)Y = 0.$$

The asymptotic solutions of this equation are:

$$\overline{Y}_1 \simeq \chi^{-(1+\lambda)}\Sigma_{n=o}\,\alpha_n\chi^{-n} = (-x)^{-(1+\lambda)}\,\Sigma_{n=o}(-1)^n\,\alpha_n(-x)^{-n},$$

and

$$\overline{Y}_2 \simeq e^{\chi}\chi^{\lambda}\,\Sigma_{n=o}\,\beta_n\,\chi^{-n} = e^{-x}(-x)^{\lambda}\,\Sigma_{n=o}\,\beta_n(-x)^{-n},$$

so that

$$[e^{-st}\,y'(t)]_o^\infty = -y'(0+0).$$

Thus y_2 is a valid solution of (7); but we must prove it to be linearly independent of y_1 by employing a reductio ad absurdum method as follows:

If y_1 and y_2 are linearly dependent, put

$$y_1 = k\,y_2, \tag{25}$$

where k is a constant. This implies:

$$\frac{x\,e^{-x}\int_o^\infty e^{-xt}\left(\frac{1+t}{t}\right)^\lambda dt}{\int_o^x e^{-t}\left(\frac{t}{x-t}\right)^\lambda dt} = k. \tag{26}$$

Now choose $A > 1$, $0 < \epsilon < 1$ such that $\frac{\epsilon e^A}{\epsilon + A} > 1$ and put $x = A + \epsilon$, $x = \epsilon$ successively in (26). After equating the two expressions for k and multiplying up, we get

$$\int_o^\infty e^{-\epsilon t}\left[e^{-At}\left(\frac{1+t}{t}\right)^\lambda\right]dt\int_o^\epsilon e^{-t}\left(\frac{t}{\epsilon-t}\right)^\lambda dt$$

$$= \frac{\epsilon e^A}{\epsilon + A}\int_o^{A+\epsilon} e^{-t}\left(\frac{t}{A+\epsilon-t}\right)^\lambda dt\int_o^\infty e^{-\epsilon t}\left(\frac{1+t}{t}\right)^\lambda dt,$$

from which it follows that:

$$\int_o^\epsilon e^{-t}\left(\frac{t}{\epsilon-t}\right)^\lambda dt > \frac{\epsilon e^A}{\epsilon + A}\int_o^{A+\epsilon} e^{-t}\left(\frac{t}{A+\epsilon-t}\right)^\lambda dt. \tag{27}$$

But

$$\int_o^\epsilon e^{-t}\left(\frac{t}{\epsilon-t}\right)^\lambda dt < \int_o^\epsilon \frac{dt}{(\epsilon-t)^\lambda} = \frac{\epsilon^{1-\lambda}}{1-\lambda}, \tag{28}$$

and

$$\int_o^{A+\epsilon} e^{-t}\left(\frac{t}{A+\epsilon-t}\right)^\lambda dt > \int_o^{\frac{1}{2}(A+\epsilon)} e^{-t}\,dt = 1 - e^{-\frac{1}{2}(A+\epsilon)}. \tag{29}$$

Equation 27 holds, by virtue of the assumed linear dependence of y_1 on y_2, for all values of A and $\epsilon > 0$. But for a given λ, we may choose an $\epsilon << 1$ and an $A >> 1$ such that (28) and (29) taken together contradict (27); thus (25) is false and y_1 and y_2 are linearly independent solutions of (7).

2.3 Continued Fraction Solutions

Consider the linear, second-order, homogeneous equation in the form:

$$y'' + A_1 y' + A_0 y = 0 \quad , \tag{30}$$

where A_1 and A_0 are infinitely differentiable functions of x in a particular region $x_1 \leq x \leq x_2$. Then it is possible to obtain a formal continued fraction solution in the form:

$$\frac{y'}{y} = \frac{1}{a_0} + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \dots \quad , \tag{31}$$

as follows:

$$y = a_0 y' + b_1 y'', \qquad a_0 = \frac{-A_1}{A_0}, \; b_1 = \frac{-1}{A_0} \; ;$$

a second differentiation gives:

$$y' = a_1 y'' + b_2 y''', \qquad a_1 = \frac{a_0 + b_1'}{1 - a_0'}, \; b_2 = \frac{b_1}{1 - a_0'} \quad .$$

Combining these last two equations gives:

$$\frac{y}{y'} = a_0 + \left\{ \frac{b_1}{a_1} + \frac{b_2}{a_2} + \dots \right\} \quad ,$$

where

$$a_0 = \frac{-A_1}{A_0}, \; b_1 = \frac{-1}{A_0} \quad , \tag{32}$$

$$a_n = \frac{a_{n-1} + b_n'}{1 - a_{n-1}'} \quad , \qquad \text{and} \tag{33}$$

$$b_n = \frac{b_{n-1}}{1 - a_{n-2}'} \tag{34}$$

Applying this algorism in turn to (7) and its adjoint, (14), we get:

$$\frac{y_1'}{y_1} = \frac{1}{a_0} + \frac{b_1}{a_1} + \dots \quad ; \tag{35}$$

$$a_0 = \frac{-x}{\lambda} = b_1 \quad , \tag{36}$$

$$a_n = -\frac{n+x}{n+\lambda} \quad , \tag{37}$$

$$b_n = -\frac{x}{n-1+\lambda} \; ; \tag{38}$$

and

$$\frac{y_2'}{y_2} = \frac{1}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \dots \quad ; \tag{39}$$

$$\alpha_0 = \frac{2-x}{k} \quad , \tag{40}$$

$$\beta_1 = \frac{x}{k} \quad , \tag{41}$$

$$\alpha_n = \frac{n+2-x}{n+k} \quad (42)$$

$$\beta_n = \frac{x}{n-1+k} \quad . \quad (43)$$

$$k = 1-\lambda \quad . \quad (44)$$

Obviously, there will be a certain value of n, say N, beyond which α_n and β_n are always positive; for $x \leq 10$ we may take $N = 8$. Assuming y_1 and $y_2 = x e^{-x} y_1^*$ to be linearly independent (see Section 3) the logarithmic derivatives of the fundamental solutions of (7) may be written:

$$\frac{y_1'}{y_1} = \frac{1}{a_0} + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \ldots \quad . \quad (35)$$

$$\frac{y_2'}{y_2} = (\frac{1}{x} - 1) + \frac{1}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \ldots \quad . \quad (45)$$

The conditions for the existence of these continued fraction solutions presuppose that $y_i \neq 0$ $(i = 1, 2)$ in (x_1, x_2), and that the limits

$$\lim_{n\to\infty} a_n \quad , \quad \lim_{n\to\infty} b_n$$

actually exist and are finite.

The coefficients in (7) and (14) are such as to ensure the existence of the corresponding continued fraction expansions (35) and (39). (See, for a list of these existence conditions, Erdélyi, A. (editor) – "Higher Transcendental Functions" – volume 3).

3. NUMERICAL EVALUATION OF THE SOLUTIONS

Having derived the mathematical solutions (i), (ii), and (iii) above we turn to the problem of their numerical evaluation. The author used an "I.B.M. 1620" type digital computer to calculate numerical values in all three cases.

In what follows, the accuracy of these numerical evaluations will be discussed.

3.1 Series Solutions

From (8), $y_1(x)$ is obviously represented in the range $0 < x < \infty$ by a strictly alternating series for which

$$\left| \frac{A_{n+1} \; x^{n+2}}{A_n x^{n+1}} \right| < 1 \quad \text{for some value of } n \geq N .$$

(In the range $0 < x \leq 5$, the output from the co puter programmes designed to calculate $y_1(x)$ for various values of λ with $A_0 = 1$, indicated that we may take $N = 12$).

With regard to the second solution $y_2(x)$, we note first that none of A_0, B_0, c is zero; their otherwise arbitrary values are limited by the condition imposed by (18). Next, putting $B_1 = 0$, we get:

$$\frac{B_2}{A_0 c} = \frac{1+3\lambda}{4} < 1 \quad ,$$

$$\left| \frac{B_3}{A_0 c} \right| = \left| \frac{1}{36} (5 + 15\lambda + 7\lambda^2) \right| < \frac{3}{4} \quad ,$$

$$\left| \frac{B_3}{A_0 c} \right| - \frac{B_2}{A_0 c} = -\frac{1}{36}(4 + 12\lambda - 7\lambda^2) < 0 \; ,$$

so that

$$\left| B_3 \right| < \left| B_2 \right| \quad .$$

Now $\frac{1}{A_0}$ $\{A_n + (2n+3) A_{n+1}\} \lessgtr 0$ according as n is even or odd respectively. Therefore, setting

$$(n+1)(n+2) \frac{B_{n+2}}{A_0 c} = -\left[(n+1+\lambda)\frac{B_{n+1}}{A_0 c} + \frac{1}{A_0}\{A_n + (2n+3) A_{n+1}\}\right] ,$$

and remembering that $\frac{B_2}{A_0 c} > 0$, it follows by induction that

$$\frac{B_{2k}}{A_0 c} > 0, \; \frac{B_{2k+1}}{A_0 c} < 0, \qquad k = 1, 2, \ldots \quad ,$$

and

$$\Sigma_{n=0}^{\infty} \; B_n x^n$$

is thus a strictly alternating series.

To complete the analysis of the convergence rate of this series, we prove the following:

"If $\left|B_{N+1}\right| > T\left|B_{N+2}\right|$, $\qquad T < N$

then $\left|B_{N+k}\right| \geq T \left|B_{N+k+1}\right|$, $\qquad k = 1, 2, \ldots$ "

<u>PROOF</u>: We have:

$$\left|\frac{B_{N+2}}{A_0 c}\right| = \frac{1}{(N+1)(N+2)}\left\{(N+1+\lambda)\left|\frac{B_{N+1}}{A_0 c}\right| + \left[\frac{(1+\lambda)\ldots(N+\lambda)}{\underline{|N+1} \;\; \underline{|N+2}}\right]\left[(N+1)^2 + (2N+3)\lambda\right]\right\} ;$$

then if $\left|B_{N+1}\right| > T\left|B_{n+2}\right|$, it follows that

$$\left\{\frac{T\left[(N+1)^2 + (2N+3)\lambda\right]}{(N+1)(N+2) - T(N+1+\lambda)}\right\}\left\{\frac{(1+\lambda)(2+\lambda)\ldots(N+\lambda)}{\underline{|N+1} \;\; \underline{|N+2}}\right\} < \left|\frac{B_{N+1}}{A_0 c}\right| \quad .$$

and, conversely, if this last relation holds, then $\left|B_{N+1}\right| > T\left|B_{N+2}\right|$. Now from the assumed relationship between $|B_{N+2}|$ and $|B_{N+1}|$ we have

$$\left|\frac{B_{N+2}}{A_0 c}\right| > \left\{\frac{(N+1)^2 + (2N+3)\lambda}{(N+1)(N+2)}\right\}\left\{1 + \frac{T(N+1+\lambda)}{(N+1)(N+2) - T(N+1+\lambda)}\right\}\left\{\frac{(1+\lambda)(2+\lambda)\ldots(N+\lambda)}{\underline{|N+1} \;\; \underline{|N+2}}\right\}$$

$$= \frac{N+2}{N+1+\lambda}\left\{\frac{(N+3)\left[(N+1)^2 + (2N+3)\lambda\right]}{(N+1)(N+2) - T(N+1+\lambda)}\right\}\left\{\frac{(1+\lambda)(2+\lambda)\ldots(N+1+\lambda)}{\underline{|N+2} \;\; \underline{|N+3}}\right\}$$

$$> \frac{(N+3)\left[(N+1)^2 + (2N+3)\lambda\right]}{(N+1)(N+2) - T(N+1+\lambda)} \cdot \frac{(1+\lambda)(2+\lambda)\ldots(N+1+\lambda)}{\underline{|N+2} \;\; \underline{|N+3}} \quad .$$

The proof is completed by showing that:

$$\frac{(N+3)\left[(N+1)^2 + (2N+3)\lambda\right]}{(N+1)(N+2) - T(N+1+\lambda)} \geq \frac{T\left[(N+2)^2 + (2N+5)\lambda\right]}{(N+2)(N+3) - T(N+2+\lambda)} \quad .$$

and with some rearrangement, this can be proved when $T \leqslant N$.

Thus, using this lemma, we can set an upper bound to the magnitude of the truncation error obtained by terminating the series at the term in x^n, $n \geq N$, provided that $x \leq N$. (This upper bound is of course the magnitude of the first neglected term of the series).

In the range $0 < x \leq 5$, at most fifteen terms were required to obtain numerical values of y_1 and y_2 accurate to better than 1 part in 10^6.

From (17) and (18):

$$\lambda(1-\lambda)(2-\lambda)\ldots(n-1-\lambda) > \left|\frac{\alpha_n}{\alpha_0}\right| > \frac{\lambda}{n}(1-\lambda)(2-\lambda)\ldots(n-1-\lambda)$$

$$\left|\frac{\alpha_n x^{-n}}{\alpha_{n-1}x^{-(n-1)}}\right| < 1 , \quad x \geq n-1 , \tag{46}$$

$$\lambda(n+\lambda)(1+\lambda)(2+\lambda)\ldots(n-1+\lambda) > \frac{\beta_n}{\beta_0} > \lambda(1+\lambda)(2+\lambda)\ldots(n-1+\lambda)$$

$$\frac{\beta_n x^{-n}}{\beta_{n-1}x^{-(n-1)}} < 1 , \quad x \geq n+1 . \tag{47}$$

Equations 46 and 47 fix the end terms of the asymptotic series (19) and (20) respectively, and therefore the corresponding error terms may be calculated by taking them to be at most the absolute value of the following term in the appropriate series. For x moderately larger than 5, the errors incurred by using the asymptotic forms (19) and (20) are negligible.

3.2 Solutions in Terms of Definite and Indefinite Integrals

The solutions y_1 and y_2 of (22) and (23) [which are not necessarily proportional to the corresponding y_1 and y_2 of (8) and (10)] may be numerically computed using well-known quadrature formulae.

In the range $0 < x \leq 5$ the series solutions (Section 3.1) and integral solutions (Section 3.2) of boundary value problems of the type:

$$y(a) = A , \quad y(b) = B , \quad a < b , \tag{48}$$

were compared and shown to agree. It is worth noting that the solution of (7) in terms of (22) and (23) is not limited numerically to the relatively small range, $0 < x \leq 5$.

3.3 Continued Fraction Solutions for $\frac{y'}{y}$

The continued fraction solutions were computed numerically for x in the range $0 < x \leqslant 12$ and the convergence rate was very high in both cases.

The analysis of the truncation error of the continued fraction form of $\frac{y_2'}{y_2}$ is very simple for x in the above range:

For $x = 20$, we calculate the first eighteen convergents, and write:

$$\mathfrak{Y} = \frac{1}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \ldots + \frac{\beta_{18}}{\alpha_{18} + P_{18}} . \tag{49}$$

where:

$$P_{18} = \frac{\beta_{18}}{\alpha_{18}} + \frac{\beta_{20}}{\alpha_{20}} + \ldots \frac{\beta_n}{\alpha_n} + \frac{\beta_{n+1}}{\alpha_{n+1}} + \ldots \tag{50}$$

and all the β_i's and α_i's are positive for $i \geqslant 19$.

Now P_{18} lies between the values of successive convergents so that we may calculate P_{18} as accurately as we please. Hence we can also calculate $\mathfrak{Y}$ and therefore $\frac{y_2'}{y_2}$ as accurately as required.

The analysis of the truncation error incurred by terminating the continued fraction in (35) is not quite so simple. We write:

$$F = \cfrac{1}{a_0 + \cfrac{b_1}{a_1 + R_1}} . \tag{51}$$

where:

$$R_1 = \frac{b_2}{a_2} + \frac{b_3}{a_3} + \dots$$

$$= \frac{|b_2|}{|a_2|} - \frac{|b_3|}{|a_3|} - \dots \quad ,$$

that is,

$$R_1 = \frac{\underset{\sim}{b}_1}{\underset{\sim}{a}_1} - \frac{\underset{\sim}{b}_2}{\underset{\sim}{a}_2} - \dots \quad , \tag{52}$$

where:

$$\underset{\sim}{b}_n = |b_{n+1}| \, , \; \underset{\sim}{a}_n = |a_{n+1}| \quad .$$

In the sequel we will need certain general lemmas and theorems associated with infinite continued fractions. These properties are now listed, together with their proofs, before continuing with the error analysis of the numerical evaluation of R_1.

Let V be the infinite continued fraction

$$V = \frac{w_1}{v_1} + \frac{w_2}{v_2} + \dots \quad , \tag{53}$$

and let $V_n = \frac{p_n}{q_n}$ be the n^{th} convergent to V.

Then the following propositions are true.

<u>PROP. I</u>

$$\left.\begin{aligned} \text{"}p_n &= w_n p_{n-1} + v_n p_{n-2} \quad , \\ q_n &= w_n q_{n-1} + v_n q_{n-2} \quad ; \quad n = 2, 3, \dots \end{aligned}\right\} , \tag{54}$$

where $p_0 = 1, \; p_1 = w_1 , \; q_0 = 1, \; q_1 = v_1$;"

[this is easily established by induction] .

<u>PROP. II</u> "$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} w_1 w_2 \dots w_n$" (55)

[for, $p_n q_{n-1} - p_{n-1} q_n = (-1) w_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$,

therefore $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} w_n w_{n-1} \dots w_2 (p_1 q_0 - p_0 q_1)$,

and the required result follows] .

<u>COROLLARY</u>

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \frac{w_1 w_2 \dots w_n}{q_n q_{n-1}} \quad . \tag{56}$$

<u>PROP. III</u>

"The continued fraction (53) is equivalent to

$$\overline{V} = \frac{k_1 w_1}{k_1 v_1} + \frac{k_1 k_2 w_2}{k_2 v_2} + \dots + \frac{k_{n-1} k_n w_n}{k_n v_n} + \dots \tag{57}$$

($K_i \neq 0$, i = 1, 2, ...) in the sense that $\overline{V}_n = V_n$ for all n".

[Put $\overline{V}_n = \frac{\overline{p}_n}{\overline{q}_n}$: $\quad \overline{V}_1 = \frac{w_1}{v_1} = V_1$,

$$\overline{V}_2 = \frac{k_1 k_2 v_2 w_2}{k_1 k_2 v_1 v_2 + k_1 k_2 w_2} = V_2 \quad , \quad \text{and}$$

we can establish by induction that:

$$\bar{p}_n = (k_1 k_2 \ldots k_n)\, p_n \quad ,$$

$$\bar{q}_n = (k_1 k_2 \ldots k_n)\, q_n \quad ,$$

so the proposition is proved].

(We have already used this result to derive (52)).

<u>PROP. IV</u>

"If the w_i and v_i of the continued fraction (53) are all positive, then the value of the continued fraction V lies between the values of successive convergents. Also, each convergent is nearer than the preceding convergent to the value of the continued fraction".

[Convert the continued fraction V into its equivalent continued fraction $\bar{V}$ (Equation 57). Choose the k_i such that $k_i v_i > 1$.

Then $\bar{q}_n > \bar{q}_{n-1} > 0$, since $\bar{q}_n = k_n v_n \bar{q}_{n-1} + k_{n-1} k_n w_n \bar{q}_{n-2}$. The proof is completed by applying (56) and noting that

$$\left|\left(\frac{\bar{p}_n}{\bar{q}_n} - \frac{\bar{p}_{n-1}}{\bar{q}_{n-1}}\right)\Big/\left(\frac{\bar{p}_{n+1}}{\bar{q}_{n+1}} - \frac{\bar{p}_n}{\bar{q}_n}\right)\right| = \frac{\bar{q}_{n+1}}{(k_n k_{n+1} w_{n+1})\, \bar{q}_{n-1}} > 1 \;] .$$

This too was used in the discussion of the truncation error associated with the continued fraction P_{18} of (50).

<u>PROP. V</u>

"The continued fraction V is equivalent to

$$Y = \frac{1}{g_1} + \frac{1}{g_2} + \ldots + \frac{1}{g_n} + \ldots \quad , \tag{58}$$

where: $$g_{2k+1} = \frac{w_2 w_4 w_6 \ldots w_{2k}\, v_{2k+1}}{w_1 w_3 w_5 \ldots w_{2k-1} w_{2k+1}} \quad , \tag{59}$$

$$g_{2k} = \frac{w_1 w_3 w_5 \ldots w_{2k-1}\, a_{2k}}{w_2 w_4 w_6 \ldots \; w_{2k}} \quad \text{"} \; ; \tag{60}$$

[this is easily shown by induction, or by applying proposition III].

<u>PROP. VI</u>

"Let $$V = \frac{w_1}{v_1} + \frac{w_2}{v_2} + \ldots + \frac{w_N}{v_N} + \frac{w_{N+1}}{v_{N+1}} + \ldots$$

and let $$R = \frac{w_{N+1}}{v_{N+1}} + \frac{w_{N+2}}{v_{N+2}} + \ldots$$ be convergent.

Then, if $$V_n = \frac{p_N}{q_N} = \frac{w_1}{v_1} + \frac{w_2}{v_2} + \ldots + \frac{w_N}{v_N} \quad ,$$

$$V = \frac{w_1}{v_1} + \frac{w_2}{v_2} + \ldots + \frac{w_N}{v_{N+R}} \quad ,$$

provided that $q_n + R_N q_{N-1} \neq 0$".

[$V = \lim_{n \to \infty} V_n$ by definition.

Now let $|V - V_M| < \epsilon$ where ϵ is an arbitrarily small positive number, and $|V - V_n| < \epsilon$ whenever $n \geq M$. Then if M is large enough, $|R - R_n| < \epsilon$, where we write R_n for

$$\frac{w_{N+1}}{v_{N+1}} + \frac{w_{N+2}}{v_{N+2}} + \ldots + \frac{w_{N+n}}{v_{N+n}} \quad ,$$

and the inequality holds for $N + n \geqslant M$.

Now $$V_n = \frac{w_1}{v_1} + \frac{w_2}{v_2} + \dots + \frac{w_N}{v_N + R_n} = \frac{p_N + R_n\, p_{N-1}}{q_N + R_n\, q_{N-1}} \quad ;$$

thus $$\lim_{n\to\infty} V_n = V = \frac{p_N + R\, p_{N-1}}{q_N + R\, q_{N-1}} \quad ,$$

provided that the denominator $\neq 0$].

This result is used continually in all our discussions concerning the continued fractions F and $\mathfrak{F}$.

PROP. VII

"Let $$V = \frac{w_1}{v_1} - \frac{w_2}{v_2} - \dots - \frac{w_n}{v_n} - \dots \quad , \tag{61}$$

where $v_i, w_i > 0$, $i = 1, 2, \dots$.

Then if $v_n \geqslant 1 + w_n$ for all values of n, we have:

(i) V is convergent,

(ii) if $v_n = 1 + w_n$ for every n, and the series $1 + b_1 + b_1 b_2 + \dots + (b_1 b_2 \dots b_n) + \dots$ converges to a sum s, then $V = 1 - \frac{1}{s}$, but if the series diverges, $V = 1$,

(iii) if $v_n \geqslant 1 + w_n$ with the inequality holding for at least one value of n, then $0 < V < 1$".

[We first show that p_n and q_n are positive and increase with n:

$$p_n - p_{n-1} = (v_n - 1)\, p_{n-1} - w_n\, p_{n-2} \geqslant w_n (p_{n-1} - p_{n-2}) \quad ;$$

therefore $$p_n - p_{n-1} \geqslant w_n w_{n-1} \dots w_2 (p_1 - p_0) \geqslant w_n w_{n-1} \dots w_1 \ ,$$

also, $$q_n - q_{n-1} \geqslant w_n w_{n-1} \dots w_2 (q_1 - q_0) \geqslant w_n w_{n-1} \dots w_2 (v_1 - 1) \ ,$$
$$\geqslant w_n w_{n-1} \dots w_2 w_1 \ ,$$

so $p_n > p_{n-1}$ and $q_n > q_{n-1}$.

Again, the convergents themselves form an increasing sequence of positive numbers, because:

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{w_1 w_2 \dots w_n}{q_n q_{n-1}} > 0 \ .$$

Now it has been shown that:

$$p_n - p_{n-1} \geqslant w_1 w_2 \dots w_n \quad ,$$

and $$q_n - q_{n-1} \geqslant w_1 w_2 \dots w_n \quad ;$$

hence $$p_n \geqslant w_1 + w_1 w_2 + \dots + (w_1 w_2 \dots w_n) \quad , \tag{a}$$

$$q_n \geqslant 1 + w_1 + \dots + (w_1 w_2 \dots w_n) \ . \tag{b}$$

If $v_n = 1 + w_n$ for all n, then from (a) and (b):

$$q_n - p_n = 1 \ , \ \frac{p_n}{q_n} = 1 - \frac{1}{q_n} \quad .$$

Thus $\lim_{n\to\infty} \frac{p_n}{q_n} = 1$ or $1 - \frac{1}{s}$ according as the series

$$1 + w_1 + w_1 w_2 + \dots + (w_1 w_2 \dots w_n)$$

converges to a sum s or diverges. This proves the second part of the proposition.

Now let N be the least value of n for which

$$v_n > 1 + w_n; \quad \text{put } v_N - (1 + w_N) = \eta \text{ and}$$

let $\quad x_n = (1-k)\, q_n - p_n$, where $0 < k < \dfrac{\eta}{q_N - q_{N-1}}$;

then from (a) and (b) we have, for $n \leqslant (N-1)$:

$$x_n = 1 - k\, q_n \quad . \tag{c}$$

Now:

$$q_N = v_N q_{N-1} - w_N q_{N-2} > (v_N - w_N) q_{N-1} > (\eta + 1)\, q_{N-1} \text{ so}$$

$$0 < k < \frac{\eta}{q_N - q_{N-1}} < \frac{1}{q_{N-1}} \quad ;$$

thus, $\quad x_{N-1} > 0$,

and $\quad x_N - x_{N-1} = (v_N - 1)\, x_{N-1} - w_N x_{N-2} = \eta\, x_{N-1} + w_N (x_{N-1} - x_{N-2})$,

that is, $\quad x_N - x_{N-1} = \eta - k\,(\eta\, q_{N-1} + w_N q_{N-1} - w_N q_{N-2})$, by (c)

$$= \eta - k\, \{(v_N - 1)\, q_{N-1} - w_N q_{N-2}\}$$

$$= \eta - k\, \{q_N - q_{N-1}\}$$

$$> 0 \; .$$

Also $\quad x_n - x_{n-1} \geqslant w_n (x_{n-1} - x_{n-2})$, so $x_n > x_{N-1} > 0$ for $n \geqslant N$. Therefore $(1-k)\, q_n > p_n$ for $n \geqslant N$; hence $\frac{p_n}{q_n} < (1-k)$. But since $\frac{p_n}{q_n}$ increases with n and is positive it must tend to a positive limit less than unity].

PROP. VIII

"Let $\quad V = \dfrac{w_1 x}{v_1} - \dfrac{w_2 x}{v_2} - \dots - \dfrac{w_n x}{v_n} - \dots \quad (62)$

where $v_n, w_n > 0$ for all n and $0 < x \leqslant 1$. Then if:

(i) $v_{2n-1} \geqslant w_{2n-1} + w_{2n}$ and $v_{2n} \geqslant 2$ for all n ,

(ii) $\dfrac{w_1}{w_2} + \dfrac{w_1 w_3}{w_2 w_4} + \dots + \dfrac{w_1 w_3 \dots w_{2n-1}}{w_2 w_4 \dots w_{2n}} + \dots$ converges to a sum s ,

then

$$0 < V \leqslant \frac{2 x s}{1 + 2 s} \quad .$$

If the series (ii) diverges, then $0 < V \leqslant x$".

$$[\; q_2 - q_1 = (v_2 - 1)\, q_1 - w_2\, x\, q_0 \geqslant q_1 - w_2 \geqslant w_1, \text{ so}$$

$$q_2 > q_1 > w_2 q_0 > 0 \quad .$$

Suppose that $q_{2n-2} > q_{2n-3} > w_{2n-2}\, q_{2n-4} > 0$. Then

$$q_{2n-1} = v_{2n-1}\, q_{2n-2} - w_{2n-1} x\, q_{2n-3} \geqslant (w_{2n-1} + w_{2n})\, q_{2n-2} - w_{2n-1}\, q_{2n-3} ;$$

therefore $$q_{2n-1} - w_{2n}\, q_{2n-2} \geqslant w_{2n-1}(q_{2n-2} - q_{2n-3}) > 0 . \qquad \text{(a)}$$

Again, $$q_{2n} = v_{2n}\, q_{2n-1} - w_{2n} x\, q_{2n-2} \geqslant 2\, q_{2n-1} - w_{2n}\, q_{2n-2} ,$$ and so

$$q_{2n} - q_{2n-1} \geqslant q_{2n-1} - w_{2n}\, q_{2n-2} > 0 . \qquad \text{(b)}$$

Therefore $q_{2n} > q_{2n-1} > w_{2n}\, q_{2n-2} > 0$ and it follows (by induction) that all the above relations hold for every n.

Now from (a) and (b),

$$q_{2n} - q_{2n-1} \geqslant w_{2n-1}(q_{2n-2} - q_{2n-3}) ;$$

hence $$q_{2n} - q_{2n-1} \geqslant w_{2n-1}\, w_{2n-3} \ldots w_3\, (q_2 - q_1) . \qquad \text{(c)}$$

Put $$P_n = w_1 w_3 \ldots w_{2n-1} , \qquad Q_n = w_2 w_4 \ldots w_{2n} .$$

Then because $q_2 - q_1 \geqslant w_1$ we have, from (a) and (c):

$$\left.\begin{aligned} q_{2n} - q_{2n-1} &\geqslant P_n \\ q_{2n-1} - w_{2n}\, q_{2n-2} &\geqslant P_n \end{aligned}\right\} ,$$

whence by addition $q_{2n} - w_{2n}\, q_{2n-2} \geqslant 2 P_n$,

and $$\frac{q_{2n}}{Q_n} - \frac{q_{2n-2}}{Q_{n-1}} \geqslant \frac{2 P_n}{Q_n} .$$

Now $\frac{q_2}{Q_1} \geqslant 1 + 2 \frac{w_1}{w_2}$, so

$$\frac{q_{2n}}{Q_n} \geqslant 1 + 2 s_n , \qquad \text{(d)}$$

where:

$$s_n = \frac{w_1}{w_2} + \frac{w_1 w_3}{w_2 w_4} + \ldots + \frac{w_1 w_3 \ldots w_{2n-1}}{w_2 w_4 \ldots w_{2n}} .$$

Put $y_n = q_n - \frac{p_n}{x}$; then $y_n = v_n\, y_{n-1} - w_n x\, y_{n-2}$ and $y_0 = 1$;

also, $y_1 = v_1 - w_1 x \geqslant w_2$, $y_2 - y_1 = (v_2 - 1)\, y_1 - w_2 x\, y_0 \geqslant 0$, so

$$y_2 \geqslant y_1 \geqslant w_2\, y_0 ,$$

and by inductive reasoning exactly the same as that which led to (a) and (b) we can show that $y_{2n} - y_{2n-1} \geqslant y_{2n-1} - w_{2n}\, y_{2n-2} \geqslant 0$. Therefore $y_{2n} \geqslant w_{2n}\, y_{2n-2}$, and so $y_{2n} \geqslant Q_n$.

Hence $$q_{2n} - \frac{p_{2n}}{x} \geqslant Q_n ,$$

and, using (d) and the fact that $\frac{p_n}{q_n}$ is a positive monotonic increasing function of n, we have:

$$\frac{p_{2n-1}}{q_{2n-1}} < \frac{p_{2n}}{q_{2n}} \leqslant x\, (1 - \frac{Q_n}{q_{2n}}) \leqslant x\, (1 - \frac{1}{1 + 2 s_n}) .$$

Therefore, if the series $\Sigma \frac{P_n}{Q_n}$ converges to a sum s, and

$$h = \frac{2 x s}{1 + 2 s} , \text{ then } \frac{p_n}{q_n} \leqslant h , \text{ so } \lim_{n \to \infty} \frac{p_n}{q_n} = k \leqslant h .$$

If on the other hand, $\Sigma \frac{P_n}{Q_n}$ diverges, $\frac{P_n}{q_n} < x$ and $\lim_{n\to\infty} \frac{P_n}{q_n} = k < x$] .

Returning to the problem of evaluating the truncation error associated with the continued fraction

$$R_1 = \frac{\underset{\sim}{b}_1}{\underset{\sim}{a}_1} - \frac{\underset{\sim}{b}_2}{\underset{\sim}{a}_2} - \dots , \qquad (52)$$

using PROP. V and PROP. III, we convert R_1 into the equivalent fraction:

$$R_1 = \frac{1}{g_1} - \frac{1}{g_2} - \frac{1}{g_3} - \dots , \qquad (52^1)$$

where
$$g_{2k-1} = \frac{\underset{\sim}{b}_2 \underset{\sim}{b}_4 \underset{\sim}{b}_6 \dots \underset{\sim}{b}_{2k-2} \underset{\sim}{a}_{2k-1}}{\underset{\sim}{b}_1 \underset{\sim}{b}_3 \underset{\sim}{b}_5 \dots \underset{\sim}{b}_{2k-3} \underset{\sim}{b}_{2k-1}} ,$$

and
$$g_{2k} = \frac{\underset{\sim}{b}_1 \underset{\sim}{b}_3 \underset{\sim}{b}_5 \dots \underset{\sim}{b}_{2k-1} \underset{\sim}{a}_{2k}}{\underset{\sim}{b}_2 \underset{\sim}{b}_4 \underset{\sim}{b}_6 \dots \underset{\sim}{b}_{2k}} ,$$

for $k = 1, 2, \dots$.

Now
$$g_{2k-1} = g_{2k-1}(k, x, \lambda) = \frac{(1+\lambda)(3+\lambda)(5+\lambda)\dots(2k-1+\lambda)}{(2+\lambda)(4+\lambda)(6+\lambda)\dots(2k+\lambda)} \left[1 + \frac{2k}{x}\right] ,$$

and
$$g_{2k} = g_{2k}(k, x, \lambda) = \frac{(2+\lambda)(4+\lambda)(6+\lambda)\dots(2k+\lambda)}{(1+\lambda)(3+\lambda)(5+\lambda)\dots(2k-1+\lambda)} \left[\frac{2k+1+x}{2k+1+\lambda}\right] .$$

For given k, x, since $\frac{\partial g_{2k-1}}{\partial \lambda} > 0$ and $\frac{\partial g_{2k}}{\partial \lambda} < 0$, we have

$$[g_{2k-1}]_{min} = \frac{1.3.5.\dots(2k-1)}{2.4.6.\dots 2k} \left[1 + \frac{2k}{x}\right] > \frac{1.3.5.\dots(2k-1)}{2.4.6.\dots 2k} \left[\frac{2k}{x}\right] ,$$

$$[g_{2k-1}]_{max} = \frac{2.4.6\dots(2k)}{3.5.7\dots(2k+1)} \left[1 + \frac{2k}{x}\right] > \frac{2.4.6.\dots(2k)}{3.5.7\dots(2k+1)} \left[\frac{2k}{x}\right] ;$$

and
$$[g_{2k}]_{min} = \frac{3.5.7\dots(2k+1)}{2.4.6\dots(2k)} \left[\frac{2k+1+x}{2k+2}\right] ,$$

$$[g_{2k}]_{max} = \frac{2.4.6\dots(2k)}{1.3.5\dots(2k-1)} \left[\frac{2k+1+x}{2k+1}\right] .$$

Now

$\frac{g_{2k-1}}{g_{2k+1}}$ and $\frac{g_{2k}}{g_{2k+2}}$ rapidly approach unity as $k \to \infty$, and both g_{2k-1} and $g_{2k} \to \infty$ with k.

Now $g_{2k}(k, x, \lambda) > \frac{3.5.7\dots(2k+1)}{2.4.6\dots(2k)} \left[\frac{2k+1}{2k+2}\right] = \phi(k)$, say, where $\phi(k)$ is a strictly increasing function of k .

Therefore $\phi(k) > 2$ for all $k \geqslant 4$ and so $g_{2k} > 2$ for all $k \geqslant 4$.

Now

$$g_{2k-1} > [g_{2k-1}]_{min} > \frac{1.3.5 \dots (2k-1)}{2.4.6 \dots 2k} \cdot \frac{2k}{x} = \frac{3.5 \dots (2k-1)}{2.4 \dots (2k-2)} \cdot \left[\frac{1}{x}\right].$$

and certainly $g_{2k-1} > 2$ if $2x < \frac{3.5 \dots (2k-1)}{2.4 \dots (2k-2)}$.

so for $x \leqslant 1$, $k \geqslant 5$, $g_{2k-1} > 2$.

Therefore, by PROPOSITION VII or VIII we can put

$$\left\{ \begin{array}{l} R_1 = \frac{1}{g_1} - \frac{1}{g_2} - \dots - \frac{1}{g_7 - \varepsilon} = \frac{p_7 - \varepsilon p_6}{q_7 - \varepsilon q_6} \\ \text{where} \quad 0 < \varepsilon < 1, \quad 0 < x \leqslant 1 \end{array} \right\} . \tag{63}$$

Of course, we can evaluate ε quite accurately. We have

$$\varepsilon = \frac{1}{g_8} - \frac{1}{g_9} - \frac{1}{g_{10}} - \dots = \frac{1}{f_1} - \frac{1}{f_2} - \frac{1}{f_3} - \dots , \tag{64}$$

where we have written $f_n = g_{n+7}$. From the proof of PROP. VII we showed that $p_n > w_1 w_2 \dots w_n + p_{n-1}$, $q_n > w_1 w_2 \dots w_n + q_{n-1}$; hence applying this to (64) we get:

$$q_n > 1 + q_{n-1}, \quad p_n > 1 + p_{n-1} \quad ;$$

but $q_1 = f_1 > 2$, so $q_2 > 3$, $q_3 > 4$, and generally, $q_n > n + 1$. Thus the difference between the value of the N^{th} convergent to ε and ε is less than the quantity

$$s = \frac{1}{(N+1)(N+2)} + \frac{1}{(N+2)(N+3)} + \dots = \frac{1}{N+1} .$$

This concludes the analysis of the convergence note of the continued fraction (52) for x in the range $0 < x \leqslant 1$. For $x > 1$ this technique is a little awkward because of the far larger number of convergents needed before we can introduce ε. To handle this case we write $a_{2k} = 2\mu_{2k}$ and express R_1 as the equivalent continued fraction

$$R_1 = \frac{\underset{\sim}{b}_1}{\underset{\sim}{a}_1} - \frac{\underset{\sim}{b}_2/\mu_2}{2} - \frac{\underset{\sim}{b}_3/\mu_2}{\underset{\sim}{a}_3} - \dots - \frac{\underset{\sim}{b}_{2k}/\mu_{2k}}{2} - \frac{\underset{\sim}{b}_{2k+1}/\mu_{2k}}{\underset{\sim}{a}_{2k+1}} - \dots \tag{65}$$

Now the relation $\quad a_{2n-1} \geqslant \frac{\underset{\sim}{b}_{2n-1}}{\mu_{2n-2}} + \frac{\underset{\sim}{b}_{2n}}{\mu_{2n}}$

implies

$$n \geqslant x \left\{ \frac{2n+\lambda}{2n+x-1} + \frac{2n+1+\lambda}{2n+1+x} - \frac{1}{2} \right\} . \tag{66}$$

and the converse.

Suppose the relation (66) holds for a particular value of n; then, for $x > 1$,

$$n+1 \geqslant \frac{n+1}{n} \left\{ \frac{2n+1}{2n+x-1} + \frac{2n+1+\lambda}{2n+1+x} - \frac{1}{2} \right\} x . \tag{67}$$

Now the relation $\quad n+1 \geqslant \left\{ \frac{2n+2+\lambda}{2n+1+x} + \frac{2n+3+\lambda}{2n+3+x} - \frac{1}{2} \right\} x \quad$ (68)

implies $\quad \underset{\sim}{a}_{2n+1} \geqslant \underset{\sim}{b}_{2n+1}/\mu_{2n} + \underset{\sim}{b}_{2n+2}/\mu_{2n+2} . \quad$ (69)

Therefore, if we can show that for a certain value of n, (66) is true, and

$$\frac{n+1}{n} \left\{ \frac{2n+1}{2n+x-1} + \frac{2n+1+\lambda}{2n+1+x} - \frac{1}{2} \right\} \geqslant \left\{ \frac{2n+2+\lambda}{2n+1+x} + \frac{2n+3+\lambda}{2n+3+x} - \frac{1}{2} \right\} . \tag{70}$$

we will have shown that for this and all subsequent values of n,

$$\underset{\sim}{a}_{2n-1} \geqslant \frac{\underset{\sim}{b}_{2n-1}}{\mu_{2n-2}} + \frac{\underset{\sim}{b}_{2n}}{\mu_{2n}} \quad . \tag{71}$$

Now (70) is equivalent to:

$$\frac{(n+1)(2n+\lambda)}{2n+x-1} \geqslant \frac{n(2n+3+\lambda)}{2n+3+x} + \frac{x-(1+2\lambda)}{2(2n+1+x)} \quad ,$$

or,
$$\frac{6n^2+(9+6\lambda-x)n+\lambda(3+x)}{(2n+x-1)(2n+x+3)} \geqslant \frac{x-(1+2\lambda)}{2(2n+1+x)}$$

If $x \leqslant 1+2\lambda$, we may take $n \geqslant 2$ in (71). Otherwise, we have to show that:

$$\psi(x,n,\lambda) = \frac{2[6n^2+(9+6\lambda-x)n+\lambda(3+x)]\,[2n+1+x]}{(2n+x-1)(2n+3+x)\,[x-(1+2\lambda)]} \geqslant 1$$

where n is the particular value which makes (71) true.

Now $\dfrac{\partial\psi}{\partial\lambda} > 0$,

so
$$\psi > \frac{2(2n+1+x)\,[6n^2+(9-x)n]}{x(2n+x-1)(2n+x+3)} = \psi(x,n,0) \quad .$$

Also, (66) holds for $n = 7$ and $1 < x \leqslant 7$, while $\psi(x,7,0) > 1$ for $1 < x \leqslant 7$. Hence (71) holds for $n \geqslant 7$, $1 < x \leqslant 7$, and we may put:

$$R_1 = \frac{\underset{\sim}{b}_1}{\underset{\sim}{a}_1} - \frac{\underset{\sim}{b}_2/\mu_2}{2} - \dots - \frac{\underset{\sim}{b}_{12}/\mu_{12}}{2-\epsilon'} = \frac{p_{12}-\epsilon' p_{11}}{q_{12}-\epsilon' q_{11}} \quad ,$$

and $0 < \epsilon' < 1$, by PROPOSITION VIII.

When estimating ϵ' it is to be noted that the successive convergents are positive and increase with n. Also, if we write:

$$\epsilon' = \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{2} - \dots = \frac{\underset{\sim}{b}_{13}/\mu_{12}}{\underset{\sim}{a}_{13}} - \frac{\underset{\sim}{b}_{14}/\mu_{14}}{2} - \dots \quad ,$$

where
$$\left\{\begin{aligned} \beta_{2k-1} &= \underset{\sim}{b}_{11+2k}/\mu_{10+2k} = \frac{2x}{2k+11+x} \\ \beta_{2k} &= \underset{\sim}{b}_{12+2k}/\mu_{12+2k} = \frac{2k+13+\lambda}{2k+12+\lambda}\cdot\frac{2x}{2k+13+x} \\ \alpha_{2k-1} &= \frac{2k+12+x}{2k+12+\lambda}\ ,\ \alpha_{2k} = \frac{2k+13+x}{2k+13+\lambda} \end{aligned}\right\} k = 1, 2, \dots \quad ,$$

then from the proof of PROPOSITION VIII we have

$$q_{2n} > \beta_{2n}\, q_{2n-2} \ ;$$

therefore
$$\frac{\beta_{2n}}{q_{2n}} < \frac{1}{q_{2n-2}} \ ; \text{ hence } \frac{\beta_1\beta_2\dots\beta_{2n}}{q_2\ q_{n-1}} < \frac{\beta_1\beta_2\dots\beta_{2n-1}}{q_{2n-1}\,q_{2n-2}} \quad .$$

and so
$$0 < \left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right) < \left(\frac{p_{2n-1}}{q_{2n-1}} - \frac{p_{2n-2}}{q_{2n-2}}\right) \qquad \text{for all } n.$$

Similarly, the relation $\left(\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}}\right) < \left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right)$ implies

$\frac{\beta_{2n+1}}{q_{2n+1}} < \frac{1}{q_{2n-1}}$ or $\beta_{2n+1}\, q_{2n-1} < q_{2n+1}$, and the converse.

Now $q_{2n+1} = \alpha_{2n+1} q_{2n} - \beta_{2n+1} q_{2n-1}$ and therefore the inequality $q_{2n+1} > \beta_{2n+1} q_{2n-1}$ is equivalent to $\alpha_{2n+1} q_{2n} > 2\beta_{2n+1} q_{2n-1}$.

But $2\beta_{2n+1} = \frac{4x}{2n+13+x} \leqslant \frac{4x}{27+x}$ for $n \geqslant 7$, and if we confine x to the region $1 < x \leqslant 7$,

then $2\beta_{2n+1} \leqslant \frac{14}{17} < 1 < \alpha_{2n+1}$.

Also, because $q_{2n} > q_{2n-1}$, we have finally:

$$\alpha_{2n+1}\, q_{2n} > 2\beta_{2n+1}\, q_{2n-1}, \text{ and so } 0 < \left(\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}}\right) < \left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right).$$

Thus, $\frac{p_n}{q_n}$ is closer to $\frac{p_{n-1}}{q_{n-1}}$ than the latter is to $\frac{p_{n-2}}{q_{n-2}}$.

Again,

$$\left(\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}}\right) \Big/ \left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right) = \beta_{2n+1}\frac{q_{2n-1}}{q_{2n+1}}$$

$$= \frac{1}{\frac{q_{2n}}{q_{2n-1}}\frac{\alpha_{2n+1}}{\beta_{2n+1}} - 1}$$

$$< \tfrac{1}{2} \text{ for } x < 1 \leqslant 7,\ n \geqslant 13.$$

Also,

$$\left(\frac{p_{2n+2}}{q_{2n+2}} - \frac{p_{2n+1}}{q_{2n+1}}\right) \Big/ \left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right) = \beta_{2n+1}\beta_{2n+2}\frac{q_{2n-1}\, q_{2n}}{q_{2n+1}\, q_{2n+2}}$$

$$< \frac{\beta_{2n+1}\, q_{2n-1}}{q_{2n+2}}$$

$$= \frac{\beta_{2n+1}\, q_{2n-1}}{\alpha_{2n+2}\,\alpha_{2n+1}\, q_{2n} - (\alpha_{2n+2}\,\beta_{2n+1}\, q_{2n-1} + \beta_{2n+2}\, q_{2n})}$$

$$= \frac{\beta_{2n+1}}{[\alpha_{2n+1}\alpha_{2n+2} - \beta_{2n+2}]\frac{q_{2n}}{q_{2n-1}} - \alpha_{2n+2}\beta_{2n+1}}.$$

and for $n \geqslant 15$, this last ratio is $< \frac{2}{3}$, provided $1 < x \leqslant 7$.

Thus if we put $\Delta_n = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$ we have:

$$\Delta_{32} < \tfrac{2}{3}\Delta_{30}, \quad \Delta_{34} < (\tfrac{2}{3})^2\Delta_{30}, \quad \Delta_{36} < (\tfrac{2}{3})^3\Delta_{30}, \quad \dots;$$

$$\Delta_{31} < \tfrac{1}{2}\Delta;$$

and

$$\Delta_{33} < \frac{\Delta_{30}}{3}, \quad \Delta_{35} < \tfrac{2}{3}\left(\frac{\Delta_{30}}{3}\right), \quad \Delta_{37} < (\tfrac{2}{3})^2\frac{\Delta_{30}}{3}, \quad \dots;$$

Therefore

$$\varepsilon' - \frac{p_{29}}{q_{29}} = \Delta_{30} + \Delta_{31} + \Delta_{32} + \dots$$

$$< \tfrac{9}{2}\Delta_{30}.$$

Obviously, if $\frac{\Delta_{30}}{p_{29}/q_{29}} \ll 1$ then $\frac{p_{29}}{q_{29}}$ will be an accurate estimate of ε'. In practice, $\frac{p_{29}}{q_{29}}$ is extremely close to $\frac{p_N}{q_N}$, where $N < 10$, and this value differs very little from ε'.

For $7 < x \leqslant 12$ asymptotic methods are probably better and certainly more convenient. However, by writing $g_n = \frac{3}{2}\ \theta_n$ in (52^1 , page 15) we have:

$$R_1 = \frac{1/\theta_1}{3/2}\ - \ \frac{1/\theta_1\theta_2}{3/2}\ - \ \ldots \quad .$$

and we can show that $\frac{3}{2} > 1 + 1/\theta_n\theta_{n+1}$ if $n \geqslant 27$, $7 < x \leqslant 12$.

Thus:

$$R_1 = \frac{1/\theta_1}{3/2}\ - \ \frac{1/\theta_1\theta_2}{3/2}\ - \ldots - \frac{1/\theta_{28}\theta_{28}}{\frac{3}{2} - \varepsilon''} = \frac{p_{28} - \varepsilon'' p_{28}}{q_{28} - \varepsilon'' q_{28}} \quad .$$

where $0 < \varepsilon'' < 1$.

Before concluding we note that the continued fractions

(35)

$$\frac{y_1'}{y_1} = \frac{1}{a_0 +}\ \frac{b_1}{a_1 +} \ldots \quad ,$$

and (39) ,

$$\frac{y_2^{*\prime}}{y_2^*} = \frac{1}{\alpha_0 +}\ \frac{\beta_1}{\alpha_1 +} \ldots \quad ,$$

must be linearly independent because $\beta_n \neq b_n$ and $\alpha_n \neq a_n$ for all n, and $\frac{a_0}{\alpha_0} \neq \frac{b_1}{\beta_1}$.

Hence

$$\frac{y_1'}{y_1} \text{ and } \frac{y_2'}{y_2} = \left(\frac{1}{x} - 1\right) + \frac{1}{\alpha_0 +}\ \frac{\beta_1}{\alpha_1 +} \ldots$$

are linearly independent solutions for the logarithmic derivative $\frac{y'}{y}$.

4. CONCLUSION

For $0 < x \leqslant 7$ we may calculate the fundamental solutions of Equation 7,

$$x\ y'' + x\ y' + \lambda y = 0 ,$$

using either infinite series or a combination of definite and indefinite integrals. If $x > 7$ we can obtain accurate asymptotic series or we may still use the "mixed integral" form of the solution.

For $0 < x \leqslant 12$ there exist rapidly convergent continued fraction expansions of the logarithmic derivatives of the fundamental solutions, and their accuracy can be checked fairly easily by converting them into equivalent forms.

In terms of the original physical model, this means that we can accurately evaluate the solution of the system represented by Equations 1 and 2 for the special case of constant Σ_a, Σ_s, D and zero source term S.

Thus we have established a criterion for checking the accuracy of "numerical" solutions of the system (1) and (2) for this model; solutions which are generally much easier to programme on a computer than "analytical" solutions.

5. REFERENCES

Rubbra, F.T., and Pollard, J.P., (1962). – Studies in thermalisation – A modified heavy gas equation. AAEC/TM129.

Thompson, J.J., (1962). – The second order mass ratio expansion of the neutron slowing down density in a monatomic gaseous moderator. AAEC/TM170.

Thompson, J.J., and Lawrence, B.R., (1960). – A formulation of age diffusion theory for thermal – intermediate reactors. Internal Report AAEC/K352.