BNL 6425

HH-3

ACCELERATOR DEPARTMENT
(AGS)
Internal Report

BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, L.I., N.Y.

NON-LINEAR BUNCH MOTION AT TRANSITION

Harald Hahn

December 4, 1961

1. Introduction

A summary of the dynamic behavior of the proton bunches in the Brookhaven Alternating Gradient Synchrotron (AGS) has been given in [1, 2]. In these reports, the usual linearization of the differential equations involved has been made and the theory was restricted to well bunched beams. The linearized approach is no longer valid at transition where the actual phase angle of the bunch can differ appreciably for a short time from the stable phase angle ϕ_o.

In this report the non-linearity of the differential equations for phase oscillations will no longer be neglected. At transition the beam is slow enough so that the electronics of the bootstrap system can be considered as being ideal and the radius servo loop can be characterized by one time constant. Under these assumptions the analysis can be carried out in a two-dimensional phase plane. The essential new result will be the short existence of a stable equilibrium point for the bunch motion not coinciding with ϕ_o. The results here derived have been tested experimentally and at least a qualitative agreement was found. However, the conclusions are no more valid if debunching takes place since we have still neglected the finite bunch width.

2. The Equations of Motion

The canonical differential equations for small-amplitude or paraxial motion of an individual particle with reference to the central equilibrium orbit (trajectory axis) for a circular machine with constant gradient was compiled in a previous report [HH-2]. The same set of equations remains valid in the AGS, if we redefine the coordinates (Δr, $\Delta\theta$) as "orbit coordinates" [3].

We consider only the case where a median plane exists at all times and the trajectories of the protons are confined to this median plane ($\Delta z = 0$). For a given magnetic field, the possible closed (equilibrium) orbits form a continuous mesh spanning the median plane. A particle with momentum p_o travels on the central equilibrium orbit of length C_o, which is chosen as reference orbit ($\Delta r = 0$); in the case of an ideal AG synchrotron it is the only orbit composed of straight lines and circular arcs with radius of curvature ρ_o.

$$C_o = 2\pi r_o = 2\pi\rho_o + \text{total length of straight sections.}$$

A particle with momentum $p_o + \Delta p$ travels on a different equilibrium orbit of length $C_o + \Delta C$; this orbit is labeled by the coordinate Δr

$$\Delta r = \frac{\Delta C}{2\pi} .$$

Planes normal to the central equilibrium orbit are labeled by the spatial distance s along this trajectory or equivalently by

$$\theta = \frac{s}{r_o} \quad \text{or} \quad \Delta\theta = \frac{\Delta s}{r_o} .$$

An equilibrium orbit Δr = const. intersects the plane θ = const. at a point whose spatial distance x from the trajectory axis is a function of θ. However the average distance $\bar{x}$ is related to the coordinate value Δr by:

$$\Delta r = \bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x \sqrt{1 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta \approx \frac{1}{2\pi} \int_0^{2\pi} x d\theta \, .$$

Furthermore $\bar{x}$ is a function of $\Delta p \approx \Delta p_\theta$; the linear approximation defines the momentum compaction factor α

$$\frac{\bar{x}}{r_o} = \frac{\Delta C}{C_o} = \alpha \frac{\Delta p}{p_o} \, .$$

The differential equations for the bunch motion are obtained by averaging over all particles. If the finite bunch width can be neglected, the equations for the motion of the center of charge are of the same form as for the individual particle. Consequently these equations are only capable of describing the bunch motion during transition if no debunching occurs.

Using the same notation as in previous reports, we have the following equations:

$$\frac{d}{dt} \langle \Delta p_i \rangle = \frac{eV_o}{C_o} \left(\sin \langle \phi_i \rangle - \sin \phi_o \right) \qquad (2.1)$$

Equation (2.1) is valid as long as $\langle \sin \phi_i \rangle \approx \sin \langle \phi_i \rangle$. The stable phase angle of the reference particle is

$$\phi_o = \begin{cases} 34^\circ & t < \text{transition} \\ 146^\circ & t > \text{transition} \end{cases}$$

$$\frac{\langle \bar{x}_i \rangle}{r_o} = \alpha \frac{\langle \Delta p_i \rangle}{p_o} \qquad (2.2)$$

In the AGS, the rf phase is given by:

$$\langle \phi_i \rangle = h (F_b - 1) \bullet \langle \theta_i \rangle + \phi_b - \phi_x \qquad (2.3a)$$

ϕ_b = quasistatic differential phase shift of the bootstrap system plus any programed phase jump at transition.

F_b = dynamic transfer function of the bootstrap system.

It is worthwhile to note that the beam dynamics allows $\langle\phi_i\rangle$ to jump instantaneously, though not $\langle\theta_i\rangle$.

The measurements showed for F_b a band width of 15 kc. At transition, the bootstrap system is fast compared to the beam or the radius loop and we are allowed to make the idealization $F_b = 1$. Therefore

$$\langle\phi_i\rangle = \phi_b - \phi_x . \qquad (2.3b)$$

ϕ_x is produced by the radius control system

$$\phi_x = F_r * \langle\bar{x}_i\rangle. \qquad (2.4a)$$

In writing (2.4a), the radial pickup electrodes are assumed to be located at a place where $x = \bar{x}$ and to be centered at $\Delta r = 0$. The "radial offset" is not essential and will be neglected. The radius loop is then described sufficiently by

$$\phi_x + T_r \frac{d\phi_x}{dt} = \gamma \langle\bar{x}_i\rangle \qquad (2.4b)$$

with $T_r = 0.25$ ms

$$\gamma = \pm 0.5 \ 1/\text{cm} \qquad \begin{matrix} t < \text{sense reversal} \\ t > \text{sense reversal} \end{matrix}$$

γ reverses sign at the "sense reversal", which we assume to be abrupt. In practice, the sense reversal takes 1 ms.

After elimination of the variables $\langle\Delta p_i\rangle$ and ϕ_x, we obtain a system of two simultaneous first order non-linear differential equations in $\langle\phi_i\rangle$ and $\langle\bar{x}_i\rangle$ (for short ϕ and x), which are both readily accessible to measurement.

$$T_r \frac{d\phi}{dt} + \phi = -\gamma x + \phi_b \tag{2.5}$$

$$T\gamma \frac{dx}{dt} + \gamma x = \frac{G}{\cos\phi_o} (\sin\phi - \sin\phi_o) \, . \tag{2.6}$$

The time constant of the beam T can be assumed as constant for the considered period

$$T = \frac{p_o}{\dot{p}_o} = \frac{B_o}{\dot{B}_o} \approx 0.2 \text{ s} \, .$$

G characterizes the static open loop gain

$$G = \gamma \alpha r_o \operatorname{ctg} \phi_o$$

$$\left.\begin{array}{l} \alpha = 0.014 \\ r_o = 128 \text{ m} \end{array}\right\} \quad |G| \approx 120 \, .$$

Under these assumptions, elimination of time is possible

$$\frac{T\gamma}{T_r} \frac{dx}{d\phi} = \frac{\frac{G}{\cos\phi_o} (\sin\phi - \sin\phi_o) - \gamma x}{-\phi - \gamma x + \phi_b} \tag{2.7}$$

If ϕ_b is constant or a step function (phase-jump), the equation can be studied in a phase plane (ϕ, x) in terms of its singularities [4].

3. The Equilibrium Points

The equilibrium points (ϕ_e, x_e) are given by the singularities of (2.7):

$$-\phi_e - \gamma x_e + \phi_b = 0 \tag{3.1a}$$

$$\frac{G}{\cos\phi_o} (\sin\phi_e - \sin\phi_o) - \gamma x_e = 0 \, . \tag{3.1b}$$

In contrast to the linearised case, we obtain here two interesting equilibrium points. The numerical values for ϕ_b and G are such that an approximate analytical expression can be found.

<u>The first equilibrium point</u> (ϕ_{e1}, x_{e1}) is in the vicinity of ϕ_o, the stable phase angle for the motion of the individual particle.

$$\phi_{e1} - \phi_o = \frac{1}{1+G} (\phi_b - \phi_o) \tag{3.2a}$$

$$\gamma x_{e1} = \frac{G}{1+G} (\phi_b - \phi_o) \tag{3.2b}$$

The locus of all first equilibrium points with ϕ_b as parameter is therefore a straight line:

$$\gamma x_{e1} = G (\phi_{e1} - \phi_o). \tag{3.3}$$

The use of $(\phi, |\gamma| x)$ rather than (ϕ, x) as coordinates provides a very convenient scale for the plot of the trajectories in the phase plane. This is within geometrical accuracy:

$$\phi_{e1} - \phi_o \approx 0$$

$$\gamma x_{e1} \approx (\phi_b - \phi_o) .$$

<u>The second equilibrium point</u> (ϕ_{e2}, x_{e2}) is in the vicinity of $\phi_o^* = 180^\circ - \phi_o$, the unstable phase angle.

$$\phi_{e2} - \phi_o^* = \frac{1}{1-G} (\phi_b - \phi_o^*) \approx 0 \tag{3.4a}$$

$$\gamma x_{e2} = \frac{-G}{1-G} (\phi_b - \phi_o^*) \approx (\phi_b - \phi_o^*) . \tag{3.4b}$$

The locus of the second equilibrium point is equally a straight line:

$$\gamma x_{e2} = - G (\phi_{e2} - \phi_o^*) . \tag{3.5}$$

Depending on the values for ϕ_o and γ we can obtain four different cases at transition:

Case (+, +)

below transition energy	$\mathrm{ctg}\ \phi_o > 0$	$G > 0$
before sense reversal	$\gamma > 0$	

Case (+, -)

below transition energy	$\mathrm{ctg}\ \phi_o > 0$	$G < 0$
after sense reversal	$\gamma < 0$	

Case (-, +)

above transition energy	$\mathrm{ctg}\ \phi_o < 0$	$G < 0$
before sense reversal	$\gamma > 0$	

Case (-, -)

above transition energy	$\mathrm{ctg}\ \phi_o < 0$	$G > 0$
after sense reversal	$\gamma < 0$	

At transition we have to come from case (+, +) to (-, -). The time at which transition energy is reached and the time of the sense reversal trigger jitter slightly (~ 1 ms) from pulse to pulse, and therefore the case (+, -) or (-, +) characterizes the bunch motion for a short interval. Actually, the timing system is set so that the sense reversal occurs always a few milliseconds before transition energy is reached.

Figures 1 and 2 show the plot for the equilibrium points under the four conditions.

It is desirable and possible to go through transition without the help of a phase jump (ϕ_b = const.) and yet without a change in the radial position, if $\phi_b \approx \frac{\pi}{2}$.

The electronic system of the AGS produces under present conditions, together with the sense reversal, a positive-going phase jump of approximately 50°, partially due to the radial offset of the detection diodes [5]. Nevertheless it is possible to go through transition without changing the radial position by a proper choice of the phase compensating cable [6].

4. Stability Analysis of the Equilibrium Points

The nature of solutions near a singularity may be explored by an expansion around this point:

$$\phi = \phi_e + \varphi$$
$$x = x_e + \xi .$$

First equilibrium point ($\phi_{e1} \approx \phi_o$):

The solutions depend on the equation

$$\frac{T}{T_r}\frac{\gamma\, d\xi}{d\varphi} = \frac{G\,\frac{\cos\phi_{e1}}{\cos\phi_o}\varphi - \gamma\xi}{-\varphi - \gamma\xi} . \qquad (4.1)$$

The character of the solutions is independent of ϕ_b as long as the approximation holds

$$\frac{\cos\phi_{e1}}{\cos\phi_o} \approx 1 .$$

The characteristic equation is then

$$\lambda^2 + \left(\frac{1}{T_r} + \frac{1}{T}\right)\lambda + \frac{1+G}{T\,T_r} = 0 \qquad (4.2)$$

with the two characteristic roots

$$\lambda = -\frac{1}{2T_r}\left(1 \pm \sqrt{1 - 4\,\frac{G\,T_r}{T}}\right) \qquad (4.3)$$

where we made use of

$$T \gg T_r,\ |G| \gg 1,\ 4\,|G|\,\frac{T_r}{T} \approx 0.5 .$$

As expected, the first equilibrium point is a stable node for $G > 0$ which can be seen from the numerical values for

$$\lambda_1 = - \frac{0.3}{2T_r} \qquad \lambda_2 = - \frac{1.7}{2T_r} .$$

Solution curves in normal form are given by

$$y_2 = \text{const.}\ y_1^{\frac{\lambda_2}{\lambda_1}} = \text{const.}\ y_1^{5.7} . \tag{4.4}$$

A linear transformation gives the solution curves in general form:

$$\varphi = y_1 + y_2 \tag{4.5a}$$

$$\gamma \xi = - 0.85\ y_1 - 0.15\ y_2 \tag{4.5b}$$

Figure 3 shows solution curves around a node for machine parameters assuming $\gamma > 0$. The curves for $\gamma < 0$ are mirror symmetric to the axis $\xi = 0$.

A larger radial time constant T_r or a higher gain would change the stable node into a less desirable stable focus. The gain setting is mainly determined by requirements at injection and we find thus an upper limit for T_r.

On the other hand, the first equilibrium point is an unstable saddle for $G < 0$, as seen from the values for

$$\lambda_1 = \frac{0.22}{2\ T_r} \qquad \lambda_2 = - \frac{2.22}{2\ T_r} .$$

This situation is currently encountered when the sense reversal occurs before transition energy is reached.

Solution curves in normal form are given by:

$$y_2\ y_1^{10} = \text{const.} \tag{4.6}$$

A linear transformation gives the solution curves in general form:

$$\varphi = y_1 + y_2 \tag{4.7a}$$

$$\gamma \xi = - 1.11\ y_1 + 0.11\ y_2 \tag{4.7b}$$

Figure 4 shows solution curves around a saddle for $\gamma > 0$. The curves for $\gamma < 0$ are mirror symmetric to the axis $\xi = 0$.

<u>Second equilibrium point</u> ($\phi_{e2} \simeq \phi_o^*$):

The solutions depend on the equation

$$\frac{T}{T_r}\frac{\gamma\, d\xi}{d\varphi} = \frac{G \dfrac{\cos \phi_{e2}}{\cos \phi_o}\varphi - \gamma \xi}{-\varphi - \gamma \xi} \cong \frac{-G\varphi - \gamma\xi}{-\varphi - \gamma\xi} .$$

The second equilibrium point is an unstable saddle for $G > 0$ and a stable node for $G < 0$. The solution curves near this equilibrium point are identical to those shown in Figures 3 and 4.

When the sense reversal occurs before transition energy is reached, we find - due to the non-linearity of the equations - a new stable equilibrium point for the bunch motion near ϕ_o^*. The motion of the individual particle in the bunch, however, is unstable and a gradual debunching occurs. The experience has shown that an acceleration on this stable second equilibrium point is possible for several milliseconds (~ 5 ms) without detectable particle losses.

<u>Stability of Motion</u>

Case	First equilibrium point ϕ_o		Second equilibrium point $\phi_o^* = \pi - \phi_o$	
	Bunch	Particle	Bunch	Particle
(+, +)	Stable	Stable	Unstable	Unstable
(+, -)	Unstable	Stable	Stable	Unstable
(-, +)	Unstable	Stable	Stable	Unstable
(-, -)	Stable	Stable	Unstable	Unstable

Acknowledgements

The experimental study of the machine behavior during transition has been carried out together with H.J. Halama and E.C. Raka, and most of the ideas contained in this report resulted from discussions with them.

References

[1] E.D. Courant
Internal Report EDC-40, BNL, October 1960.

[2] E.C. Raka
Internal Report ECR-5, BNL, August 1961.

[3] L.C. Teng
Report ANLAD-60, May 1960.

[4] W.J. Cunningham
Introduction to Non-Linear Analysis, Chapter 5, McGraw-Hill 1958.

[5] H.J. Halama
Internal Report HJH-5, BNL, April 1961

[6] M. Plotkin
1960 I.R.E. National Convention Record, Part 9.

HH/yew
12/6/61

Distribution: AD B1, B2

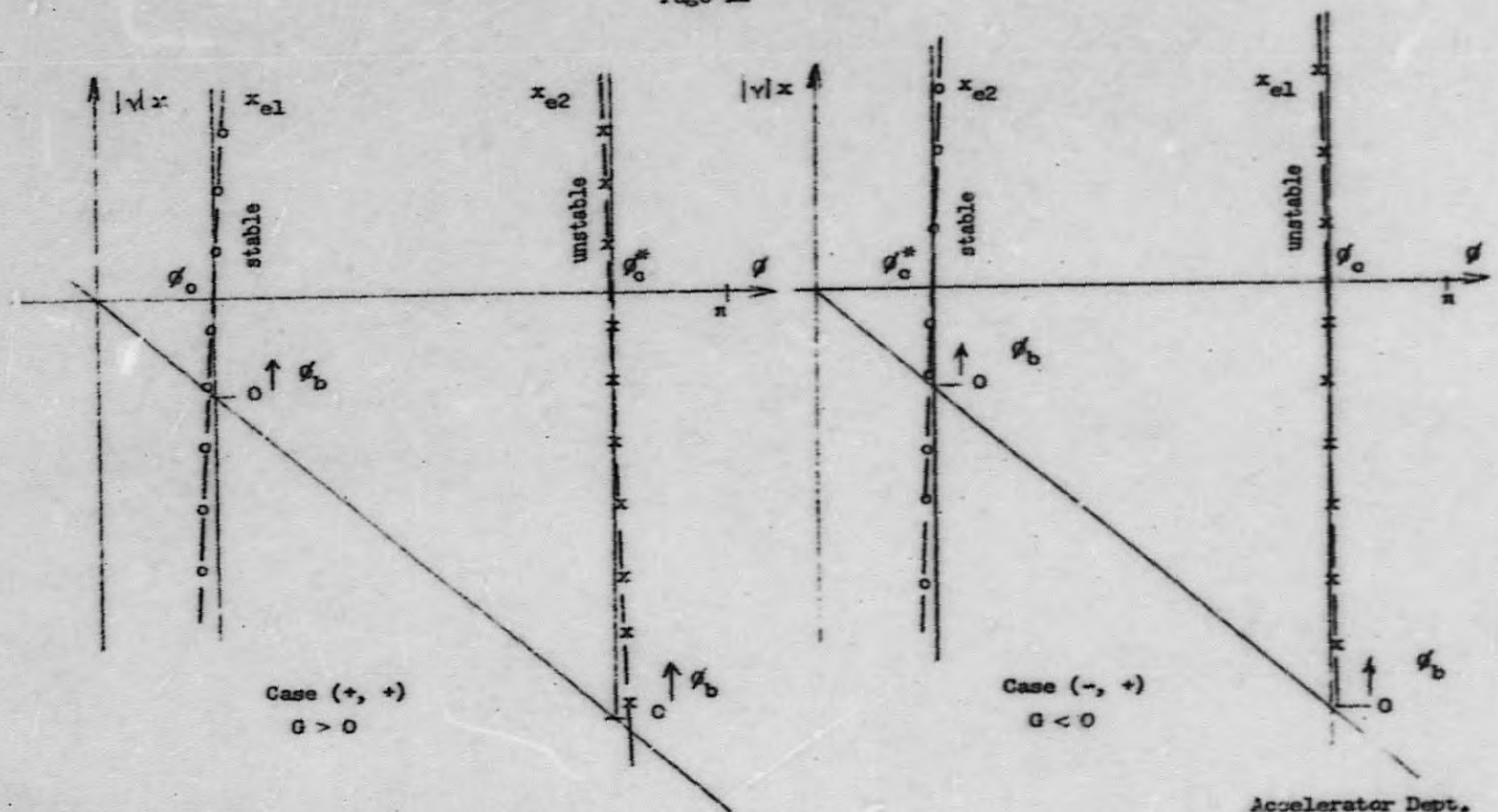


Fig. 1. Locus of equilibrium points ($\gamma > 0$)

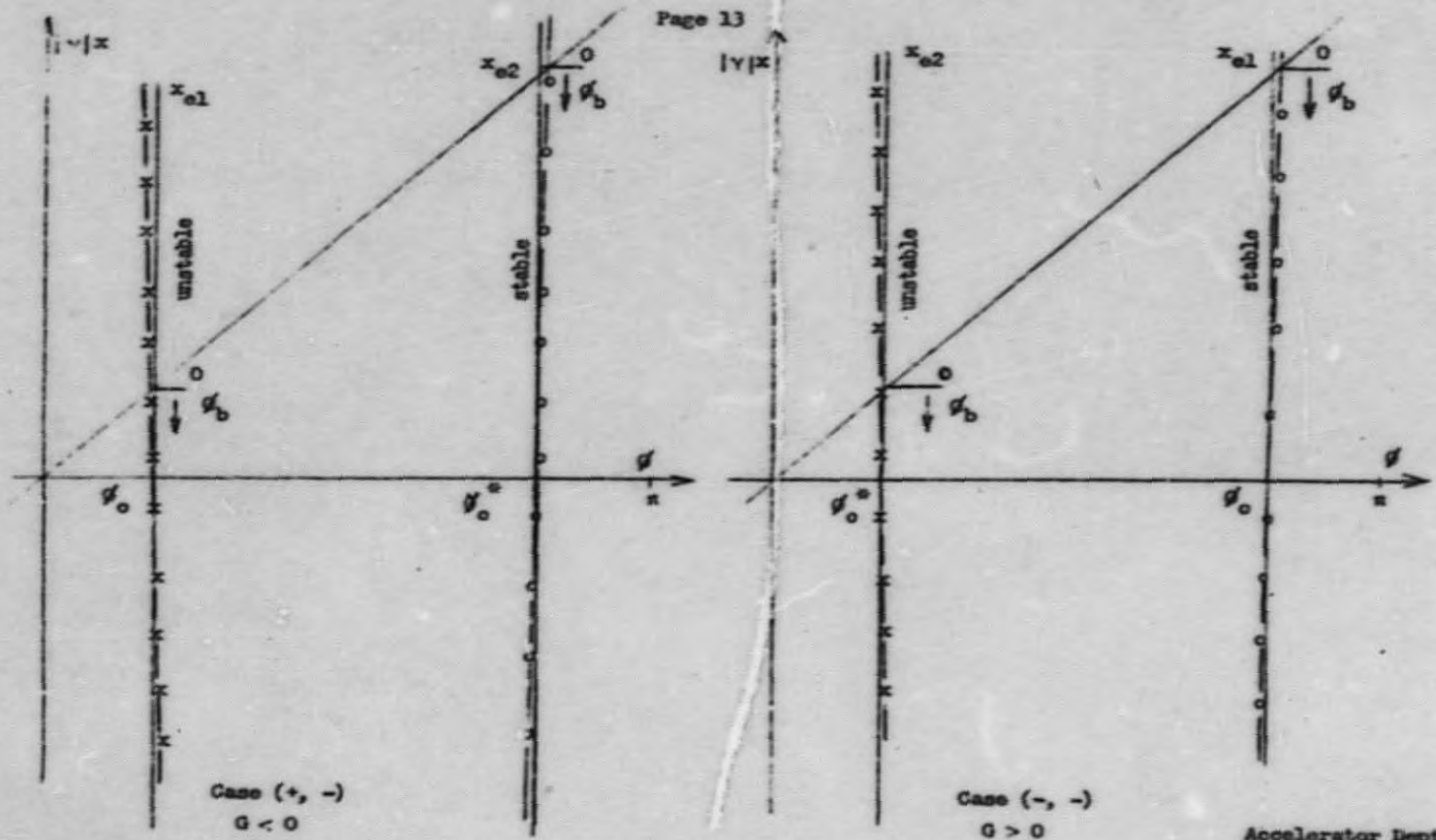
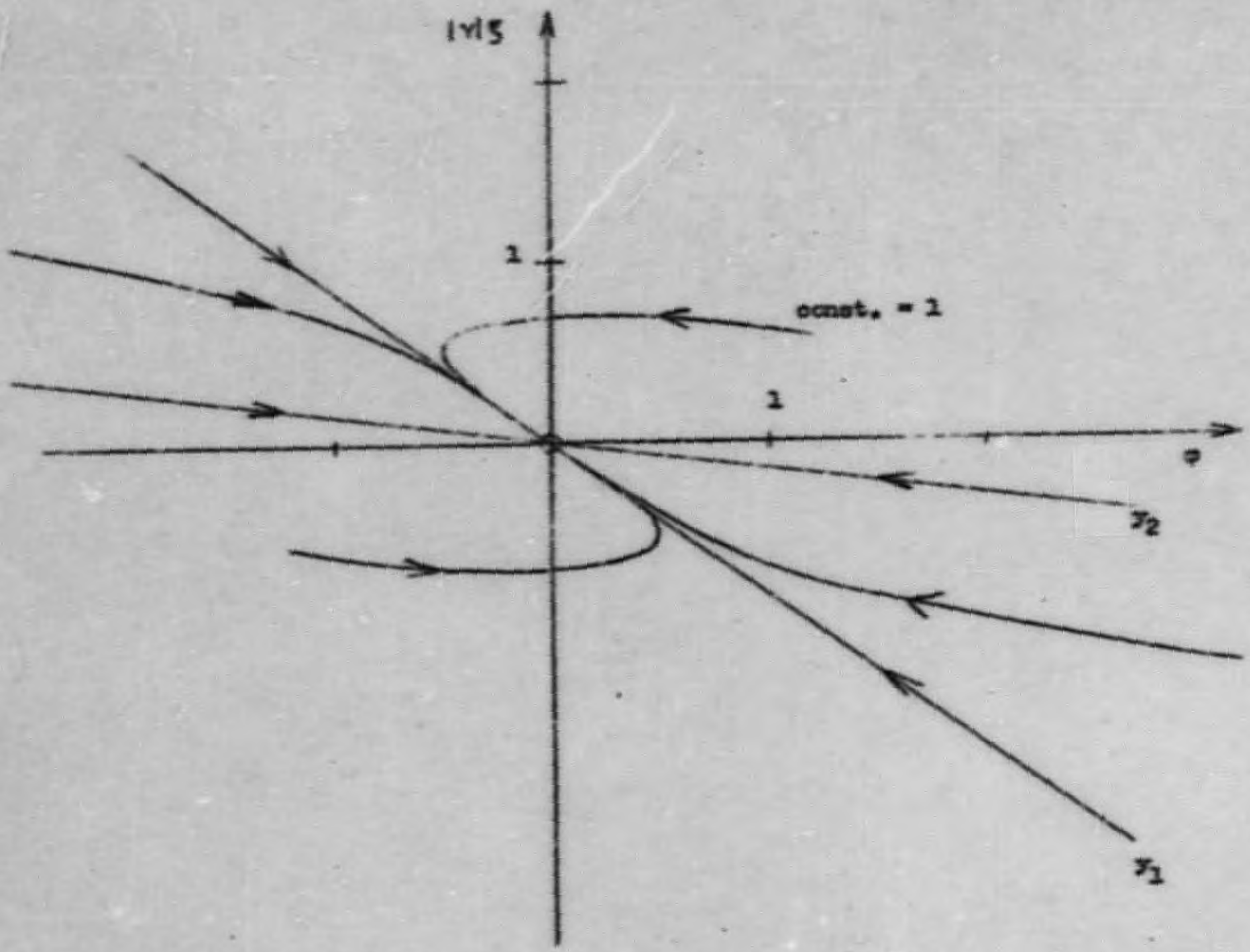


Fig. 2 Locus of equilibrium points (γ < 0)

Accelerator Dept.
HH-3 12/4/61



<u>Fig. 3. Solution curves for stable node</u> ($\nu > 0$)

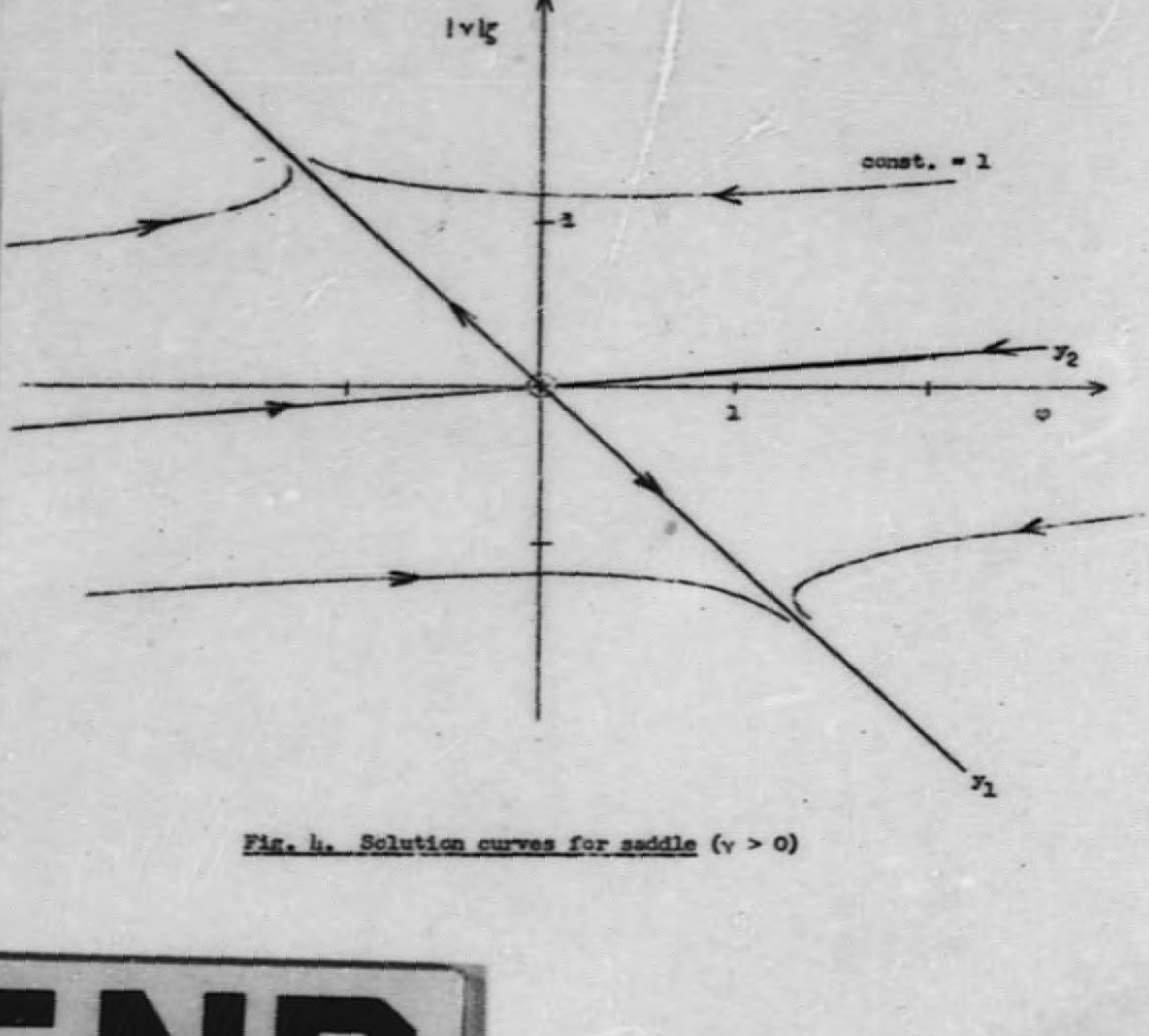


Fig. 4. Solution curves for saddle ($\gamma > 0$)