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ACCELERATOR DEVELOPMENT DEPARTMENT

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Internal Report

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ON THE DESIGN OF QUADRUPOLE FOCUSING SYSTEMS

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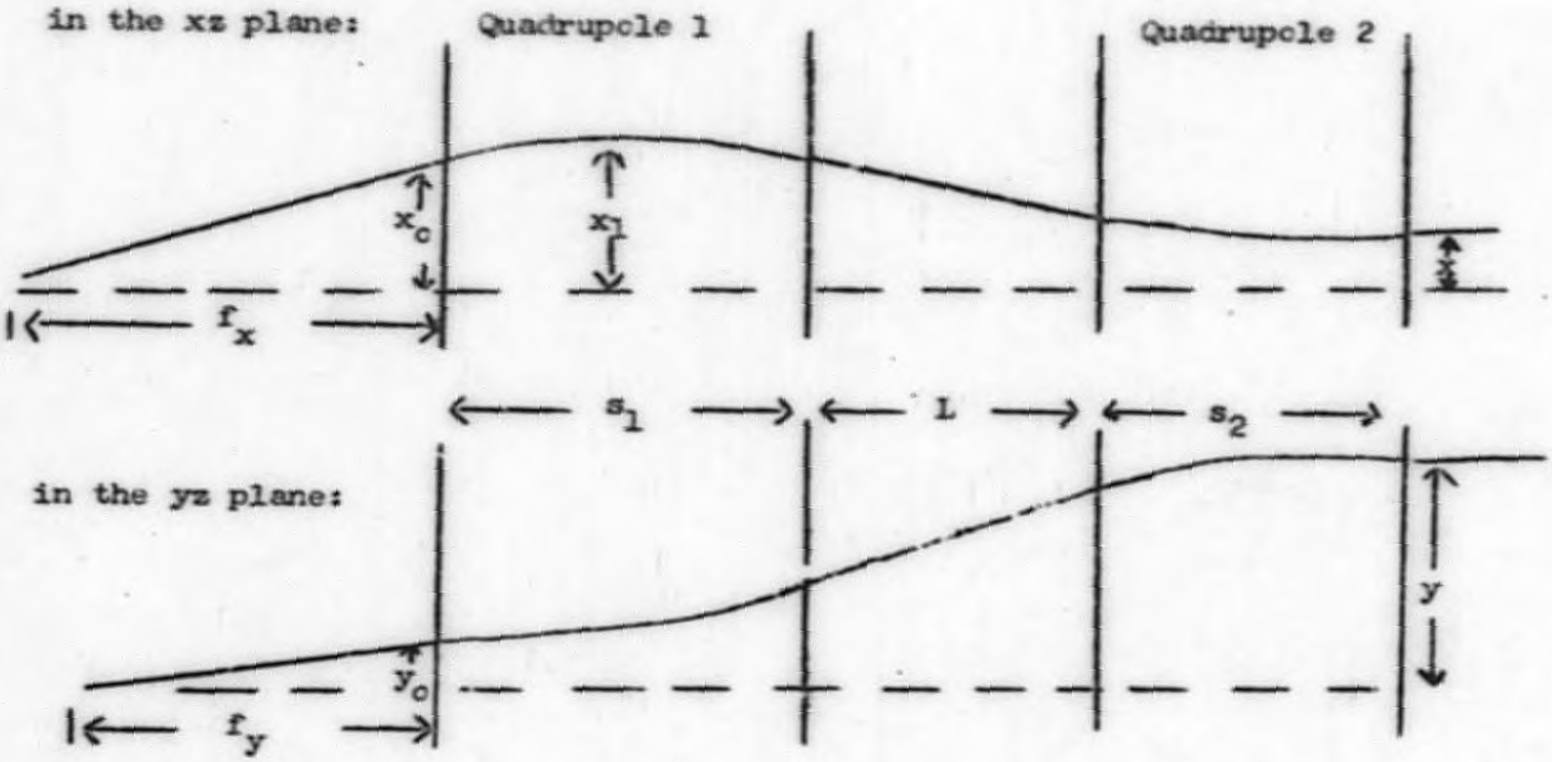
1. Introduction

In this report we present solutions of the design problem in which a system of quadrupole lenses is required to carry a particle beam from given focal lines in the x and y planes to other given focal lines. Particular attention will be given to the case of the anastigmatic lens system which takes a beam from one focal point to another focal point.

Since the general problem is almost impossibly complicated a simplification is introduced by breaking the lens system into two parts. The first part of the lens system is required to bring the initial beam to the state where it is parallel to the z axis in both planes. The second part carries the initially parallel beam to the required final condition. Each part will involve two quadrupoles so that the complete system will consist of four quadrupoles; usually, however, the field gradients in the second and third quadrupoles can be made identical so that these quadrupoles can be combined into one and the system becomes a three quadrupole system.

The configuration of the lens element will be as shown in the figures below. These figures indicate also the general character of the beam path in the two planes.



2. Symbols

The pertinent lengths (all to be measured in meters) are indicated in the figures in the preceding section.

The characteristics of the particle beam are as follows:

velocity = v meters per second

kinetic energy = T electron volts

total energy = W electron volts

Quadrupole 1 has magnetic field gradients $G_1 = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$.

Quadrupole 2 has magnetic field gradients $-G_2$.

We shall make extensive use of the symbols a_1 and a_2 where

$$a_1 = \sqrt{\frac{G_1 c^2}{vW}} \quad \left(= \sqrt{\frac{G_1 v}{2T}} \right. \text{ for non-relativistic particles)} ,$$

$$a_2 = \sqrt{\frac{G_2 c^2}{vW}} \quad \left(= \sqrt{\frac{G_2 v}{2T}} \right. \text{ for non-relativistic particles)} .$$

c is the velocity of light = 2.998×10^8 meters per second.

For calibration of future results we tabulate a few α's for field gradients of 1000 gauss per centimeter: (the particles are assumed to be protons)

Particle energy:	750 kev	50 Mev	3 Bev	10Bev	25 Bev
α :	8.9	3.1	0.87	0.52	0.34

The initial conditions for the particle beam as it enters Quadrupole 1 are (as indicated in the figures):

$$x = x_o \qquad \dot{x}/v = x_o/f_x$$

$$y = y_o \qquad \dot{y}/v = y_o/f_y \quad .$$

We shall also make use of the following abbreviations:

$$C_1 = \cot \alpha_1 s_1 \qquad C_2 = \cot \alpha_2 s_2$$

$$K_1 = \coth \alpha_1 s_1 \qquad K_2 = \coth \alpha_2 s_2$$

3. Exit Parameters from Quadrupole 2

The displacement and velocity components in the x plane as the particle leaves Quadrupole 2 are given by

$$\begin{pmatrix} x \\ \dot{x}/v \end{pmatrix} = \begin{pmatrix} \cosh \alpha_2 s_2 & \frac{1}{\alpha_2} \sinh \alpha_2 s_2 \\ \alpha_2 \sinh \alpha_2 s_2 & \cosh \alpha_2 s_2 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha_1 s_1 & \frac{1}{\alpha_1} \sin \alpha_1 s_1 \\ -\alpha_1 \sin \alpha_1 s_1 & \cos \alpha_1 s_1 \end{pmatrix} \begin{pmatrix} x_o \\ x_o/f_x \end{pmatrix}$$

$$= \frac{x_o}{f_x} \sinh \alpha_2 s_2 \sin \alpha_1 s_1 \begin{pmatrix} (f_x+L)K_2C_1 + (\frac{1}{\alpha_1} - \alpha_1 f_x L)K_2 + \frac{C_1}{\alpha_2} - \frac{\alpha_1 f_x}{\alpha_2} \\ K_2C_1 - \alpha_1 f_x K_2 + \alpha_2(f_x+L)C_1 + \frac{\alpha_2}{\alpha_1} - \alpha_1\alpha_2 f_x L \end{pmatrix} .$$

.............(1)

In the y plane, the position and velocity are given by changing all α's to $j\alpha$'s and all x's to y's in the above transformation so that

$$\begin{pmatrix} y \\ \dot{y}/v \end{pmatrix} = \frac{y_o}{f_y} \sinh \alpha_1 s_1 \sin \alpha_2 s_2 \begin{pmatrix} (f_y+L)K_1C_2 + (\frac{1}{\alpha_1} + \alpha_1 f_y L)C_2 + \frac{K_1}{\alpha_2} + \frac{\alpha_1 f_y}{\alpha_2} \\ K_1C_2 + \alpha_1 f_y C_2 - \alpha_2(f_y+L)K_1 - \frac{\alpha_2}{\alpha_1} - \alpha_1\alpha_2 f_y L \end{pmatrix} . \qquad \text{.........(2)}$$

4. Relations between the Lens Parameters

The relations between parameters which give a lens section yielding a beam parallel to the axis in both planes will be obtained by setting $\dot{x} = \dot{y} = 0$. This equation can be written

$$L = \frac{1}{\alpha_1}\left(\frac{\alpha_1 f_x C_1 + 1}{\alpha_1 f_x - C_1}\right) - \frac{K_2}{\alpha_2} = \frac{C_2}{\alpha_2} - \frac{1}{\alpha_1}\left(\frac{\alpha_1 f_y K_1 + 1}{\alpha_1 f_y + K_1}\right) . \qquad \text{........(3)}$$

If we eliminate L between the equations (3) we obtain

$$\frac{K_2 + C_2}{\alpha_2} = \frac{1}{\alpha_1}\left\{\frac{\alpha_1 f_x C_1 + 1}{\alpha_1 f_x - C_1} + \frac{\alpha_1 f_y K_1 + 1}{\alpha_1 f_y + K_1}\right\} . \qquad \text{........(4)}$$

If equations (3) are satisfied, our lens has the desired property. In this case it is fairly easy to show that the displacements as the particle leaves Quadrupole 2 are given by

$$x = \frac{x_o(\alpha_1 f_x \sin \alpha_1 s_1 - \cos \alpha_1 s_1)}{\alpha_2 f_x \sinh \alpha_2 s_2} , \qquad \text{........(5)}$$

$$y = \frac{y_o(\alpha_1 f_y \sinh \alpha_1 s_1 + \cosh \alpha_1 s_1)}{\alpha_2 f_y \sin \alpha_2 s_2} . \qquad \text{........(6)}$$

The latter expression, for the displacement in the initially defocusing plane, gives the maximum displacement anywhere in the system and will be the expression which defines the aperture for the whole system.

For completeness we include the expression for the maximum displacement in Quadrupole 1 in the initially focusing plane. It is

$$x_1 = \frac{x_o}{\alpha_1 f_x} \sqrt{1 + \alpha_1^2 f_x^2} \quad . \qquad \ldots\ldots\ldots(7)$$

This expression defines the aperture required in Quadrupole 1.

5. Combination of Lens Elements to form a complete Lens

When a double quadrupole system has been designed by satisfying equations(3) (for quick methods see Section 8), it can be combined with another combination satisfying the same equations to give a complete lens system. In this fashion, a beam emerging from focal lines distant f_x and f_y from the entry point of the first quadrupole can be restored to focal lines distant f_x' and f_y' from the exit of the last quadrupole. The system will now look like this

Quadrupole 1   Quadrupole 2   Quadrupole 2'   Quadrupole 1'

Generally it will be possible to set $\alpha_2 = \alpha_2'$. If this is done, Quadrupoles 2 and 2' can be combined into one quadrupole so that the complete lens consists of three quadrupoles in all.

Once a lens system has been set up, it is possible to change the location of the focal lines within wide limits by changing α_1 and α_2 without making any changes in lens geometry.

6. Anastigmatic Lenses

If the beam emerges from a focal point instead of from two focal lines, the design formulae (3) are modified only by setting $f_x = f_y = f$ where f is the distance of the focal point from the entry into the first quadrupole. The element so derived can be combined with another element having a focal distance f' to restore the beam to another focal point distant f' from the point of exit.

It should be emphasized that this is not a true lens since it will not restore a beam from another point on the axis to a new focal point. As the point of origin of the beam is displaced, the final focal point will move to different points in the xz and yz planes.

To illustrate this and other properties of anastigmatic lenses we consider the special case of a symmetric lens designed to have f = f' = 1 meter. This lens is to be used with 750 kev protons and has a gradient of 200 gauss per cm. Consequently $\alpha = 4$. We allow a space of 10 cm between lens elements. From the tables of Section 9 we find $\alpha s_1 = 0.618$ and $\alpha s_2 = 0.502$, whence $s_1 = 15.4$ cm and $s_2 = 12.5$ cm. The whole lens will thus consist of initial and final quadrupoles 15.4 cm long and a central quadrupole 25 cm long. Including the two 10 cm spaces, the total length of the lens is 75.8 cm.

First we use equations (3) to study the motion of the point at which the beam is parallel to the axis as the point of origin is displaced from f = 1 meter. By some graphical manipulation we can rather easily derive the shift in the final focal points in the two planes. The results are summarized in the following table.

Distance of initial focal point from point of entry into first quadrupole (meters)	Distance from point of exit of final focus in xz plane (meters)	Distance from point of exit of final focus in yz plane (meters)
0.70	1.41	1.53
0.80	1.25	1.30
0.90	1.11	1.13
1.00	1.00	1.00
1.10	0.91	0.92
1.20	0.83	0.85
1.30	0.77	0.80

These figures illustrate the splitting of the final focus as the initial focal point is displaced from the design point.

Once a lens has been set up, it is of interest to determine whether it can be used as an anastigmatic system for different focal distances merely by changing the field gradients without changing quadrupole lengths or positions. It can be shown, using equations (3) that this is indeed possible. We illustrate this point by tabulating the necessary gradients in the lens just discussed for various focal distances. It is assumed that we wish to keep the lens symmetric so that the initial focal distance f remains equal to the exit focal distance f'. The results are as follows:

f (meters)	Gradient in first and final quadrupoles (gauss/cm)	Gradient in central quadrupole (gauss/cm)
0.6	268	243
0.8	227	217
1.0	200	200
1.2	180	186
1.4	165	174
1.6	154	166

7. Choice of Lens Parameters

Obviously equations (3) can be satisfied by many combinations of parameters. The final choice will be governed by such practical considerations as aperture available, maximum field gradient attainable and so forth.

If we are interested in minimum aperture, the particles must be deflected as rapidly as possible with a minimum drift distance in which, after defocusing, the particle can drift away from the axis. This means that high field gradients should be used and the α's should be as large as possible. At the same time L should be kept as small as possible.

8. Procedure for Lens Design

In the next section we have tabulated numerical solutions of equations (3) for the special case where $f_x = f_y = f$ and $\alpha_1 = \alpha_2 = \alpha$. If an anastigmatic lens of this type is desired, its parameters can be derived directly from the table.

If a lens is to be designed for focal distances f_x and f_y which are different in the two planes, the tables can still be used in a rapid successive approximation method. This method makes use of the fact that the ratio $(\alpha_1 f_y K_1 + 1)/(\alpha_1 f_y + K_1)$ which appears in equations (3) does not usually differ very much from unity. The quantity f_y does not appear anywhere else in these equations; consequently, to a first order, the parameters of the lens are independent of f_y and the lens parameters taken from the table for the case $f =$ the desired value of f_x, will be a first order approximation to the parameters of the desired lens. To proceed from this approximation to the case of a lens having focal distances f_x and f_y and having the same gradients in both lens sections (so $\alpha_1 = \alpha_2 = \alpha$) the method is as follows:

We use the formulae (derived by rewriting equations (3)):

$$C_2 = \frac{\alpha f_y K_1 + 1}{\alpha f_y + K_1} + \alpha L \quad , \qquad \text{........(8)}$$

and

$$C_1 = \frac{\alpha f_x K_2 + \alpha^2 f_x L - 1}{\alpha(f_x + L) + K_2} \quad . \qquad \text{.......(9)}$$

The following steps are necessary:

a) Choose a value of α consistent with the particle energy, estimated aperture and quadrupole pole-tip field considered desirable.

b) Choose a value of L. If small aperture is important, L should be barely long enough to allow space for the end windings of the quadrupoles.

c. In the tables of Section 9 look up the value of αs_1 for the given values of αL and αf_x .

d. Look up K_1 in a table of hyperbolic cotangents (all of the necessary tables can be found in the Chemical Rubber Publishing Company's "Handbook of Chemistry and Physics").

e) From equation (8) derive C_2.

f) Look up αs_2 in a table of cotangents.

g) Look up K_2 in a table of hyperbolic cotangents.

h) From equation 9 derive C_1.

i) From a cotangent table find the new value of αs_1.

j) Repeat (d) to (i) until the process converges - usually about three times will be sufficient.

k) Compute the maximum aperture from equation (6).

This whole procedure takes about a half hour. As an example we derive the parameters of a lens to focus a beam, radiating from two focal lines in the two planes, to a single point focus. The beam will consist of 50 Mev protons. The focal line in the xz plane will be at z = 0, the focal line in the yz plane will be at z = 1 meter, and the focal point to which the beam is to be restored will be at z = 6 meters. We shall study the case where the beam dimensions are such that a field of 900 gauss/cm gradient is appropriate so that α (from Section 2) is 3.0. We start the lens arbitrarily roughly midway between the focal points at z = 3.0 m. so that $\alpha f_x = 9.0$ and $\alpha f_y = 6.0$. Between lens elements we leave a space of 25 cm for end windings; thus αL is 0.75.

From the tables of Section 9 we find a starting value of αs_1 for $\alpha f = 9.0$ and $\alpha L = 0.75$. This value is $\alpha s_1 = 0.362$. The equations (8) and (9) for this case are:

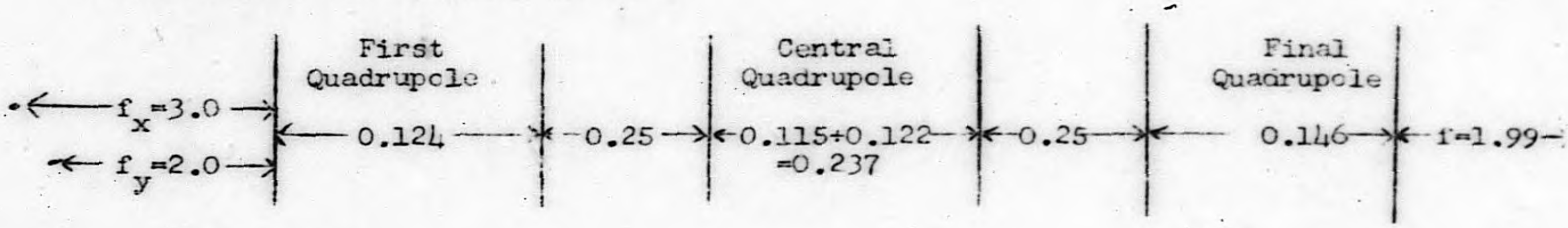
$$C_2 = \frac{6.0K_1 + 1}{6.0 + K_1} + 0.75 = \frac{6.75K_1 + 5.50}{6.0 + K_1} , \quad \text{........ (8a)}$$

$$C_1 = \frac{9.0K_2 + 5.75}{9.75 + K_2} . \quad \text{........(9a)}$$

We now carry through the procedure (d) to (i) with the following results:

	First Approximation	Second Approximation	Third Approximation
αs_1 =	0.362	0.370	0.372
K_1 =	2.87	2.82	2.81
C_2 =	2.80	2.78	2.78
αs_2 =	0.343	0.345	
K_2 =	3.03	2.99	
C_1 =	2.58	2.56	

The final result is $\alpha s_1 = 0.372$ and $\alpha s_2 = 0.345$ whence $s_1 = 12.4$ cm and $s_2 = 11.5$cm. From equation 6 the maximum aperture in this section is given by $y/y_c = 1.65$. This completes the design of the first half of the lens which now extends to $z = 3.0 + 0.124 + 0.25 + 0.115 = 3.489$ m. The second half of the lens can be taken directly from the tables of Section 9. It must meet the requirement that $s'_1 + L' + s'_2 + f' = 6.0 - 3.489 = 2.511$ m. We use the same α and L as before. By interpolation in the table we find for this lens $\alpha s'_1 = 0.437$ and $\alpha s'_2 = 0.367$; hence $s'_1 = 14.6$ cm, $s_2 = 12.2$ cm and $f' = 1.99$ m. The whole lens system is: (all lengths are shown in meters)



9. Tabulation of Parameters for Anastigmatic Lenses

The writer is indebted to Mr. K. Jellett who has computed the essential parameters tabulated below for a wide range of possible anastigmatic quadrupole lens elements. Since focal distances are usually given and field gradients and quadrupole spacings are determined by practical considerations, the numerical data are presented in terms of the parameters αf and αL. All cases tabulated are for equal gradients in both quadrupoles. For each combination of αf and αL three numbers are tabulated. The first number is αs_1, the second is αs_2 and the third is the ratio y/y_o which is a measure of the maximum necessary aperture.

For changing the focal distance in a quadrupole already constructed and in position it is not possible (as we have shown in Section 6) to keep α_1 equal to α_2. No quick method for deriving the appropriate changes has yet been evolved. Consequently it would be desirable to have tabulations of $\alpha_1 s_1$, $\alpha_2 s_2$ and aperture for other values than unity of the ratio α_1/a_2 . With these additional tabulations the appropriate changes in gradient for changing focal distances could be derived quickly by interpolation. Tabulations of the required form are now in process of preparation for values of the ratio α_1/a_2 of 0.8, 0.9, 1.1 and 1.2 . It is hoped that they can be available during the next few months.

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Parameters for Anastigmatic Quadrupole Lenses

αf	$\alpha \ell$ =	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.2	αs_1 =	1.960	1.775	1.655	1.599	1.566	1.541	1.518
	αs_2 =	0.799	0.601	0.471	0.387	0.328	0.284	0.251
	y/y_o =	30.0	31.9	35.8	40.6	46.0	51.6	57.3
0.4	αs_1 =	1.775	1.592	1.489	1.424	1.382	1.353	1.333
	αs_2 =	0.798	0.599	0.470	0.386	0.327	0.283	0.250
	y/y_o =	14.6	15.5	17.4	19.6	22.2	24.9	27.7
0.6	αs_1 =	1.616	1.431	1.327	1.262	1.221	1.194	1.173
	αs_2 =	0.796	0.596	0.468	0.384	0.326	0.282	0.248
	y/y_o =	9.48	10.06	11.26	12.78	14.42	16.19	17.97
0.8	αs_1 =	1.478	1.294	1.190	1.128	1.086	1.061	1.042
	αs_2 =	0.791	0.594	0.467	0.382	0.324	0.280	0.247
	y/y_o =	6.98	7.43	8.32	9.39	10.58	11.86	13.26
1.0	αs_1 =	1.367	1.185	1.080	1.015	0.975	0.948	0.928
	αs_2 =	0.786	0.589	0.464	0.380	0.322	0.278	0.246
	y/y_o =	5.50	5.91	6.58	7.43	8.38	9.38	10.41
1.2	αs_1 =	1.272	1.091	0.989	0.924	0.885	0.857	0.838
	αs_2 =	0.778	0.584	0.460	0.377	0.320	0.276	0.244
	y/y_o =	4.63	4.88	5.47	6.19	6.98	7.78	8.64
1.4	αs_1 =	1.194	1.011	0.910	0.847	0.806	0.780	0.761
	αs_2 =	0.770	0.578	0.456	0.374	0.317	0.274	0.242
	y/y_o =	4.01	4.23	4.73	5.34	6.00	6.69	7.41
1.6	αs_1 =	1.129	0.947	0.843	0.783	0.744	0.717	0.695
	αs_2 =	0.761	0.570	0.450	0.369	0.314	0.270	0.238
	y/y_o =	3.56	3.76	4.20	4.72	5.29	5.90	6.51
1.8	αs_1 =	1.073	0.891	0.794	0.731	0.692	0.667	0.648
	αs_2 =	0.752	0.562	0.442	0.364	0.308	0.267	0.234
	y/y_o =	3.21	3.40	3.80	4.26	4.77	5.32	5.86
2.0	αs_1 =	1.026	0.842	0.745	0.687	0.649	0.620	0.600
	αs_2 =	0.742	0.551	0.433	0.357	0.303	0.263	0.232
	y/y_o =	2.97	3.13	3.48	3.90	4.37	4.84	5.33
3.0	αs_1 =	0.864	0.684	0.591	0.534	0.498	0.472	0.453
	αs_2 =	0.697	0.510	0.400	0.331	0.284	0.248	0.220
	y/y_o =	2.25	2.37	2.61	2.90	3.21	3.53	3.86
4.0	αs_1 =	0.768	0.591	0.503	0.449	0.415	0.391	0.373
	αs_2 =	0.657	0.475	0.370	0.308	0.265	0.233	0.208
	y/y_o =	1.92	2.02	2.21	2.44	2.68	2.92	3.17
5.0	αs_1 =	0.703	0.529	0.443	0.393	0.360	0.335	0.318
	αs_2 =	0.623	0.444	0.348	0.289	0.248	0.217	0.194
	y/y_o =	1.73	1.82	1.99	2.17	2.37	2.57	2.80
6.0	αs_1 =	0.655	0.483	0.402	0.353	0.322	0.299	0.282
	αs_2 =	0.595	0.418	0.326	0.272	0.234	0.206	0.184
	y/y_o =	1.62	1.70	1.84	2.00	2.17	2.34	2.51
7.0	αs_1 =	0.618	0.448	0.370	0.322	0.293	0.271	0.257
	αs_2 =	0.517	0.400	0.309	0.258	0.222	0.195	0.176
	y/y_o =	1.53	1.60	1.74	1.88	2.02	2.18	2.33
8.0	αs_1 =	0.588	0.420	0.344	0.300	0.270	0.250	0.234
	αs_2 =	0.550	0.379	0.296	0.246	0.212	0.186	0.168
	y/y_o =	1.47	1.54	1.66	1.79	1.92	2.06	2.18
9.0	αs_1 =	0.564	0.401	0.326	0.280	0.252	0.233	0.218
	αs_2 =	0.533	0.364	0.284	0.236	0.202	0.178	0.162
	y/y_o =	1.43	1.49	1.60	1.72	1.84	1.96	2.08
10.0	αs_1 =	0.543	0.380	0.307	0.265	0.238	0.219	0.204
	αs_2 =	0.516	0.350	0.272	0.226	0.194	0.172	0.155
	y/y_o =	1.39	1.45	1.55	1.66	1.77	1.88	[illegible]

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