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Analytic Approach for the Pion-Proton Scattering Phase Shifts

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ABSTRACT

A simple method of solving for the phase shifts of the pion-proton scattering is presented. The rapid solution afforded can be utilized as the Ashkin diagrams have been so employed to give starting values to an electronic computer or alternatively to analyze with more ease the variation of the phase shifts as a function of the input data in terms of the coefficients of the angular distributions. A new plot of a function of the total cross-section versus the pion energy is introduced. The near straight line resulting should help to evaluate the experimental data.

* On sabbatical leave from Brooklyn College.

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INTRODUCTION

The first analyses of pion-proton scattering were performed by Fermi,[1] et al., with an electronic computer. A thorough statistical investigation is necessary to extract the greatest accuracy and maximum consistency from the experimental information, but the essence of the physics is thereby obscured. Extensive calculations have shown that the phase shifts[2] δ_{11} and δ_{13} are small and erratic. A good assumption is then to take these phase shifts equal to zero.[3,4] We shall see that the remaining four phase shifts can be easily evaluated analytically. Increased insight into the nature of the solutions results as a consequence. Of course our conclusions do not differ essentially from those reached by others using fast digital calculations or Ashkin diagrams,[3] but we offer our method in the hope that its simplicity will help us understand the behavior of the pion-proton scattering.

1. Fermi, Metropolis, and Alei, Phys. Rev. 95, 1581 (1954)
2. We use the notation of Bethe and De Hoffman, Mesons and Fields, Vol. II (Row Peterson 1955). See also reference 5 below.
3. R. Martin, Phys. Rev. 95, 1606 (1954)
4. Harita and Serber: Fourth Annual Rochester Conference on High Energy Physics (1954)

POSITIVE PION-PROTON SCATTERING

We first develop our formulae for the case of the $\pi^+ - p$ scattering. Here our method is basically the transformation of the graphical or geometrical procedure of Ashkin to an algebraic guise.

Given the experimental data in terms of the coefficients[2,5] A_+, B_+, C_+ of the angular distribution[6]

$$\lambda\!\!\!^{-}{}^{-2}\frac{d\sigma}{d\Omega} = A_+ + B_+ \cos\theta + C_+ \cos^2\theta$$

we get for the S phase shift δ_3

$$\sin 2\delta_3 = -\mu(D - \Sigma^2)^{1/2} + (\Sigma - 2)\left(\frac{1}{L} - \mu^2\right)^{1/2} \qquad (1)$$

where $D = 4(A_+ + B_+ + C_+)$; $\Sigma = 2(A_+ + C_+/3)$

$L = D - 4\Sigma + 4$ and $\mu = (\Sigma - 2 - 2B_+)/L$

Again we may prefer to use the equivalent formulae

$$\cos(\beta - 2\delta_3) = (\Sigma - 2 - 2B_+)/L^{1/2} \qquad (2a)$$

$$\cos\beta = (\Sigma - 2)/L^{1/2} \qquad (2b)$$

The multiplicity of the allowable solutions comes from the ambiguity of the signs of the square roots in Eq. (1) or alternatively the choice of the branches of the cosine functions in Eqs. (2a) and (2b).

5. De Hoffman, Metropolis, Alei and Bethe, Phys. Rev. 95, 1586 (1954)
6. Barita, Phys. Rev. 99, 630 (A) (1955)

We illustrate our procedure with the data[5] at 120 Mev where

$$A_+ = .200; \quad B_+ = -.360 \text{ and } C_+ = 1.040$$

We get

$$D = 3.520; \quad \Sigma = 1.093; \quad L = 3.147$$

$$(D - \Sigma^2)^{1/2} = 1.525; \quad \mu = -.0593$$

$$(1/L - \mu^2)^{1/2} = .5606; \quad \Sigma - 2 = -.9067$$

$$\sin 2\delta_3 = -.4178 \text{ and } \delta_3 = -1235^\circ$$

If we use Eqs. (2), we have $\cos(\beta - 2\delta_3) = -.1052$ and $\cos\beta = -.5111$; $\beta - 2\delta_3 = -96.04^\circ$ and $\beta = -120.74^\circ$ and $\delta_3 = -12.35^\circ$ as before.

To calculate δ_{33} we can use

$$\cos(2\delta_{33} - \theta) = \frac{|b+3|^2 + 9 + 4C_+ - |b|^2}{6|b+3|} \tag{3}$$

where $b+3 = (-\Sigma + L - \cos 2\delta_3) + i\left((D - \Sigma^2)^{1/2} - \sin 2\delta_3\right)$

$$\equiv |b+3|(\cos\theta + i\sin\theta) \equiv X + iY$$

For the 120 Mev data, we have

$$X = 1.9981; \quad Y = 1.9425; \quad |b+3|^2 = 7.7657$$

$$|b+3| = 2.7867; \quad \cos\theta = .7170; \quad \theta = 44.19^\circ$$

$$|b|^2 = (X-3)^2 + Y^2 = 4.777; \quad \cos(2\delta_{33} - \theta) = .9658$$

$$2\delta_{33} - \theta = \pm 15.03^\circ; \quad \delta_{33} = 29.61^\circ \text{ (Fermi) and } \delta_{33} = 14.58^\circ \text{ (Yang)}$$

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Thus our method gives both the Fermi and Yang solutions at the same time. It is interesting to note that the Fermi solution (δ_3, δ_{33}, δ_{31}) and the Yang solution (α_3, α_{33}, α_{31}) are related to each other by the following equations:

$$\alpha_3 = \delta_3 \tag{4 a}$$

$$\alpha_{31} - \alpha_{33} = \delta_{33} - \delta_{31} \tag{4 b}$$

$$\tan (\alpha_{33} - \delta_{31}) = \tfrac{1}{3} \tan (\delta_{33} - \delta_{31}) \tag{4 c}$$

The special case when $\delta_{31} = 0$ was derived by De Hoffman[5] et al.

To obtain δ_{31} we use

$$\cos 2 \delta_{31} = X - 2 \cos 2 \delta_{33} \tag{5}$$

At 120 Mev. we have for the Fermi solution

$$\cos 2 \delta_{31} = .9746; \quad \delta_{31} = 6.47^\circ$$

Our values check those quoted by De Hoffman[5] et al.

POSITIVE AND NEGATIVE PION-PROTON SCATTERING

We first observe that in general

$$\sin^2 \delta_1 + 2 \sin^2 \delta_{13} + \sin^2 \delta_{11} = \frac{(3\sigma_- - \sigma_+)}{8\pi \lambda\!\!\!^{-}{}^2} \tag{6}$$

Thus $\sin^2 \delta_1 \geq \dfrac{3\sigma_- - \sigma_+}{\sigma_0}$ where $\sigma_0 \equiv 8\pi \lambda\!\!\!^{-}{}^2$

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If we now make the explicit assumption that $\delta_{11} = \delta_{13} = 0$, we get

$$\sin^2 \delta_1 = \frac{3\sigma_- - \sigma_+}{\sigma_0} \tag{7}$$

In principle, we can compute $|\delta_1|$ from just the measurements of the total cross-sections alone. The error in this method is, however, large because there occurs the difference of two large numbers with sizable errors.

In order to make a self-consistent calculation,[4] we have to examine the relations that the coefficients A_+, B_+, C_+, A_o, B_o, C_o and A_-, B_-, C_- must obey under our condition of $\delta_{11} = \delta_{13} = 0$. By making use of the equations of reference 5, we can show that

$$3\,(B_- + B_o) = B_+ \tag{8 a}$$

$$3\,(C_- + C_o) = C_+ \tag{8 b}$$

$$9\,C_o = 18\,C_- = 2\,C_+ \tag{8 c}$$

If our data satisfy Eqs. (8) within the experimental error we can be assured that δ_{11} and δ_{13} are small and can therefore be set equal to zero.

We now explore two additional methods of computing δ_1. They are based on Eqs. (9)

$$2A_- - A_o = \tfrac{1}{6}\left(|s|^2 + s\,a^* + s^*a\right) \tag{9 a}$$

$$2B_- - B_o = \tfrac{1}{6}\left(s\,b^* + s^*b\right) \tag{9 b}$$

where $s = \exp(2i\delta_1) - 1$; $a = \exp(2i\delta_3) - 1$
and $b = 2\exp(2i\delta_{33}) + \exp(2i\delta_{31}) - 3$

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Another equation which may be useful is

$$\sin^2\delta_1 = 1/2 \left(3 \left(A_- + A_0 \right) - A_+ \right) \qquad (10)$$

We will explain our procedure with the recent accurate data obtained at Chicago[7] at 189 Mev.

Using a least squares fit, we first choose the best values of B_+, C_+ which satisfy Eqs. (8). In Table I, we enter our results as (c). For comparison, we also give (a) the original coefficients of Anderson[7] et al., and (b) a more precise set computed from their final phase shifts.

Table I: Positive Angular Distribution Coefficients

	A_+	B_+	C_+
(a)	.960 ± .101	.121 ± .172	3.395 ± .345
(b)	.917 ± .073	.345 ± .145	3.340 ± .181
(c)	.960 ± .101	.273 ± .141	3.365 ± .305

(a) Anderson[7] et al: original data and (b) from their final phase shifts (c) from our least squares fit.

We use Eq. 2 and find $\cos(\beta - 2\delta_3) = .6751$; $\cos\beta = .9030$

$$\beta - 2\delta_3 = 47.51^\circ \text{ and } \beta = 25.41^\circ$$

$$\delta_3 = -11.05^\circ$$

At 189 Mev, we note that the signs of both angles $\beta - 2\delta_3$ and β have changed from their assignment at 120 Mev. Of course we have to determine as a function of the meson energy E_L when $\cos(\beta - 2\delta_3)$ and $\cos\beta$ go through 1 in order to get a continuous or here an analytical change in δ_3 vs E_L. By tracking or following the cosine function

7. Anderson, Davidson, Glickman and Kruse, Phys. Rev. 100, 279 (1955).

we find the critical region when cos $(\beta - 2\delta_3)$ and cos β go through 1 is for both of them about 169 Mev. This behavior accounts for the large number of solutions found by De Hoffman[5] et al. at this energy. The multiplicity of solutions arises from the various choices of the sign for β and $\beta - 2\delta_3$.

The value of $\delta_3 = -11.1°$ is the Fermi solution, i.e. the continuous extension of the solution at low energies. Anderson[7] et al. give $\delta_3 = -11.3° \pm 3.2°$.

With our choice of δ_3, we proceed to δ_{33}. We have $\theta = 210.92°$ and $2\delta_{33} - \theta = \pm 24.77°$

$$\delta_{33} = 93.08° \text{ (Fermi)}; \quad \delta_{33} = 117.85° \text{ (Yang)}$$

The function $(\Omega - \Sigma^2)^{1/2}$ has changed sign as it becomes zero when Im (a + b) = 0 and this happens at about 177 Mev.

The behavior of Eq. (2) as a function of energy is smooth and no new branching of solutions occurs. Of course these conclusions can be obtained equally well and were so arrived at with the use of Ashkin diagrams.[3]

The use of Eq. (5) gives $\delta_{31} = -13.01°$.

To complete our phase shift analysis we have to determine δ_1. Eq. (7) gives $|\delta_1| = 10.8° \pm 9.5°$. Eq. (10) has a resulting value of $|\delta_1| = 13.8° \pm 12.4°$.

We rewrite Eq. (9a) in the form

$$2A_- - A_0 = 1/3\left(\cos 2(\delta_1 - \delta_3) - 2\cos 2\delta_1 - \cos 2\delta_3 + 2\right) \quad (11)$$

Clementel[8] et al. have developed a similar point of view.

8. Clementel, Poiani and Villi, Nuovo Cimento 2, 352 (1955); 2, 389 (1955)

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For $2\,A_- - A_0 = -\,.044 \pm .080$, we get $\delta_1 = 10^\circ \pm 15^\circ$

Our most reliable determination of δ_1 comes from Eq. (9 b) which we rewrite as

$$2\,B_- - B_0 = 1/3 \Big(2 \cos 2\,(\delta_{33} - \delta_1) + \cos 2\,(\delta_{31} - \delta_1) - 3 \cos 2\delta_1 - 2 \cos 2\delta_{33} - \cos 2\delta_{31} + 3 \Big) \tag{12}$$

Then $\delta_1 = 17.1^\circ$ from $2\,B_- - B_0 = .113$. In Table II we summarize our results and compare them to those of Anderson[7] et al. and Orear.[9]

Table II: 189 Mev Phase Shifts

	Anderson et al.	Orear	Ours
δ_3	$-11.3^\circ \pm 3.2^\circ$	-10.3°	$-11.1^\circ \pm 1.8^\circ$
δ_{33}	$98.8^\circ \pm 3.6^\circ$	89°	$93.1^\circ \pm 9.4^\circ$
δ_{31}	$-11.6^\circ \pm 5.1^\circ$	0	$-13.0^\circ \pm 5.5^\circ$
δ_1	$-2.8^\circ \pm 4.5^\circ$	15°	$17.1^\circ \pm 8.0^\circ$
δ_{13}	$-2.1^\circ \pm 3.8^\circ$	0	0
δ_{11}	$-2.6^\circ \pm 7.5^\circ$	0	0

ENERGY DEPENDENCE OF THE CROSS-SECTION.

A new plot of a function of the total cross-section versus the energy will be introduced. The resulting near straight line should aid in evaluating the experimental data. We now derive the relation for σ_+ vs. ω the center of mass energy of the pion. We avail ourselves

9. Orear, Phys. Rev. 100, 288 (1955)

of the equation given by Chew and Low[10] that $\frac{k^3}{\omega^*} \cot \delta_{33}$ vs ω^* is almost a straight line. k is the momentum of the meson and ω^* is ω plus the kinetic energy of the proton. Serber and Lee[11] and Dyson, Castillejo and Dalitz[12] have shown that in general

$$\frac{k^3}{\omega^*} \cot \delta_{33} - \frac{1}{\omega^*} \text{ is an analytic function of } \omega^* .$$

As $\sigma_+ = \sigma_0 \left(\sin^2 \delta_{33} + 1/2\,(\sin^2 \delta_3 + \sin^2 \delta_{31})\right)$ and $\sin^2 \delta_3 + \sin^2 \delta_{31}$ is small and believed to vary slowly with energy, we can surmise that

$$\frac{\sigma_+ - (\sigma_+ - \sigma_0)_r}{\sigma_0} = \sin^2 \delta_{33} + \mathcal{E}(\omega) \text{ where}$$

$\mathcal{E}(\omega)$ is small and slowly varying. Further as $k = \lambda\!\!\!^{-\,-1} \sim \sigma_0^{-1/2}$, we can transform the Chew-Low equation into

$$y \equiv \frac{1}{\omega \sigma_0} \left(\frac{\sigma_0 - \sigma_c}{\sigma_c\, \sigma_0} \right)^{1/2} \text{ vs } \omega \text{ should be nearly a straight}$$

line. Here $\sigma_c = \sigma_+ - (\sigma_+ - \sigma_0)_r$ and the subscript r refers to the resonance energy ω_r. The advantage of this representation is that we can test our data for smoothness without the intermediary or knowledge of the angular distribution. Incidentally as y varies chiefly as $(\sigma_0 - \sigma_c)^{1/2}$ we can use this equation as a convenient interpolation formula. Also we can see from this expression that y is more sensitive to errors in ω and σ_+ near than away from resonance.

10. Chew and Low: Fifth Annual Rochester Conference (Interscience Publishers, Inc., 1955).
11. Serber and Lee: Quoted in Reference 12.
12. Dyson, Castillejo and Dalitz (preprint).

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The recent Brookhaven[13] data has been analyzed according to the above prescription. A least squares fit first to a linear and then a cubic function of ω was made. Moreover the 181 and 189 Mev. data were considered (a) as above resonance as preferred by Lindenbaum and Yuan[13] and (b) as below resonance as advocated by Bethe[5] et al. We enter our results in Table III.

Table III: Comparison of the (a) Brookhaven (b) Bethe Assumptions.

		R.M.S.		$\frac{\Delta y}{\varepsilon}$			
		St. Line	Cubic	St. Line		Cubic	
				181 Mev	189 Mev	181 Mev	189 Mev
(a)	Brookhaven	.738	.699	7.3	4.1	6.4	3.2
(b)	Bethe	.478	.462	-2.2	-3.2	-1.7	-2.8
(c)	Russian σ_+	.353	.336				
(d)	Russian σ_-	.661					

In Table III, $\Delta y \equiv$ deviation of y from the least squares fit. $\varepsilon \equiv$ experimental error in y due to $\Delta\sigma_+$. We ignored the error due to $\Delta\omega$. A slightly greater weight is thus given to the data around resonance. The R.M.S. $\equiv \frac{1}{n}\sqrt{\sum_1^n \left|\frac{\Delta y}{\varepsilon}\right|^2}$ is taken as a measure of the least squares fit.

A similar analysis was performed on the Russian data[14] for from 140 to 229 Mev. We summarize our results in Table IV. $E_r \equiv$ laboratory resonance energy. The $\pi^{\pm}P$ Russian data[14] from 140 to 335 Mev (we excluded the data at 363 and 393 Mev) was studied only for the straight line case. $\sigma_c = 3\sigma_- - (3\sigma_- - \sigma_0)_n$ or we have $\sigma_+ \rightarrow 3\sigma_-$ in our formulae. For comparison we recall that Bethe[5] et al. give for E_r 195 Mev.

13. Lindenbaum and Yuan, Phys. Rev. 100, 306 (1955).
14. Ignatenko, Muchin, Ozerov and Pontecorvo, Doklady, Akad. Nauk, SSSR 103, 45 (1955)

Table IV: Resonance Parameters

	E_γ (Mev)	$(\sigma_+ - \sigma_0)_r$ (mb)
(a) Brookhaven	188	7.5
(b) Bethe	194	7.5
(c) Russian σ_+	198	16.1
(d) Russian σ_-	196	32.9

Using the interpolation formula $y = (\sigma_0 - \sigma_c)^{1/2}$ as linear in ω we obtained Table V from Eq. (7). Here η is the pion momentum in units of $m_\pi c$. If we assume that $\delta_1 \sim \eta$ then $\delta_1/\eta = 6.7^\circ$. Orear[9] gives 9.2° for this quantity. In Table V, δ_1^l is the lower limit and δ_1^u the upper limit of δ_1.

Table V: δ_1 Russian Data[14].

E_L (Mev)	η	δ_1	δ_1^l	δ_1^u
140	1.354	0°	0	11.9°
184	1.376	4.3°	0	13.4°
197	1.650	17.9°	11.7°	22.7°
216	1.742	13.0°	0°	19.5°
226	1.789	21.3°	12.6°	26.0°

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END