HAUSDORFF DIMENSION OF SHRINKING-TARGET SETS UNDER NON-AUTONOMOUS SYSTEMS

Marco Antonio Lopez

Dissertation Prepared for the Degree of
DOCTOR OF PHILOSOPHY

UNIVERSITY OF NORTH TEXAS
August 2018

APPROVED:

Mariusz Urbański, Major Professor
William Cherry, Committee Member
Lior Fishman, Committee Member
Charles Conley, Chair of the Department of Mathematics
Su Gao, Dean of the College of Science
Victor Prybutok, Dean of the Toulouse Graduate School

For a dynamical system on a metric space a shrinking-target set consists of those points whose orbit hit a given ball of shrinking radius infinitely often. Historically such sets originate in Diophantine approximation, in which case they describe the set of well-approximable numbers. One aspect of such sets that is often studied is their Hausdorff dimension. We will show that an analogue of Bowen's dimension formula holds for such sets when they are generated by conformal non-autonomous iterated function systems satisfying some natural assumptions.
Copyright 2018
by
Marco Antonio Lopez
ACKNOWLEDGEMENTS

First I would like to thank Mexico’s National Science and Technology Council (CONACYT) for their financial support for four years, and the Warsaw Center of Mathematics and Computer Science for their financial support during the Spring semester of 2016. In particular, I would like to thank Krzysztof Barański and Anna Zdunik for your guidance.

Thanks to Thomas Kennedy and Douglas Mupasiri for your support and the opportunity to pursue research projects as an undergraduate. I am highly indebted to Bill Vélez: You have made an enormous positive impact on hundreds of students, including me. I would like to acknowledge the great formative experience at Northern Arizona University; in particular John Hagood, Nándor Sieben, and James Swift. I have your rigorous instruction to thank for helping me mature and transition successfully from undergraduate to graduate education.

I would also like to thank Douglas Brozovic for his mentorship during my early years in the doctoral program, and the committee members, William Cherry and Lior Fishman for their feedback. Thanks also to my fellow graduate students at UNT, with whom I had extensive discussions about mathematics, specially Jason Atnip, Tim Wilson, and James Reid.

I would like to express enormous gratitude to my advisor, Mariusz Urbański, whose mentorship was immensely valuable. Thank you for pushing my knowledge and capabilities towards the edge, while in return I only did the same for your patience. I always left every meeting more enriched and invigorated towards mathematical research.

To Grace, whose constant presence for the last 4 years has been my main support pillar.

Finalmente unas palabras en español para mi familia. A mis hermanos y hermana: como su hermano menor, mi único mérito es haber sacado provecho de su ejemplo y experiencia. Mis mejores cualidades son solo copias a medias de las suyas. Y finalmente, a mis papás. Ambos son para mí ejemplos canónicos de amor y perseverancia. Cualquier logro mío es también suyo.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS iii</td>
</tr>
<tr>
<td>CHAPTER 1 INTRODUCTION 1</td>
</tr>
<tr>
<td>CHAPTER 2 PRELIMINARIES FROM GEOMETRIC MEASURE THEORY 4</td>
</tr>
<tr>
<td>2.1. Hausdorff Measure and Dimension 4</td>
</tr>
<tr>
<td>2.2. The Inverse Frostman Lemma 8</td>
</tr>
<tr>
<td>2.3. Packing Measure and Dimension 8</td>
</tr>
<tr>
<td>2.4. Ahlfors Measures 10</td>
</tr>
<tr>
<td>CHAPTER 3 ITERATED FUNCTION SYSTEMS 11</td>
</tr>
<tr>
<td>3.1. Autonomous Iterated Function Systems 11</td>
</tr>
<tr>
<td>3.2. Non-Autonomous Iterated Function Systems 12</td>
</tr>
<tr>
<td>3.3. Conformality and Bounded Distortion Property 13</td>
</tr>
<tr>
<td>3.4. Topological Pressure 14</td>
</tr>
<tr>
<td>3.5. Attractors of Iterated Function Systems 15</td>
</tr>
<tr>
<td>3.5.1. Bowen’s Dimension Formula for Non-Autonomous I.F.S. 16</td>
</tr>
<tr>
<td>CHAPTER 4 SHRINKING-TARGET SETS 21</td>
</tr>
<tr>
<td>4.1. Shrinking-Target Sets for Dynamical Systems 21</td>
</tr>
<tr>
<td>4.2. Shrinking-Target Sets for I.F.S. 21</td>
</tr>
<tr>
<td>4.3. Pressure for Shrinking-Target Sets 23</td>
</tr>
<tr>
<td>CHAPTER 5 SHRINKING-TARGET SETS FOR NON-AUTONOMOUS I.F.S. 25</td>
</tr>
<tr>
<td>5.1. A Class of Examples of Shrinking-Target Sets for Non-Autonomous I.F.S. 25</td>
</tr>
<tr>
<td>5.2. Bowen’s Formula for Shrinking Targets and Non-Autonomous I.F.S. 26</td>
</tr>
<tr>
<td>5.2.1. Upper Bound 28</td>
</tr>
<tr>
<td>5.2.2. Preliminaries for the Lower Bound 29</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

The main object of study in this dissertation are certain dynamically-defined sets referred to as shrinking-target sets. Such sets originate historically in Diophantine approximation.

In classical Diophantine approximation the set of $\alpha$-well approximable numbers are

$$D_{\alpha} = \bigcap_{n \geq 1} \bigcup_{q \geq n} \bigcup_{0 \leq p \leq q} \left\{ \theta \in [0, 1] \setminus \mathbb{Q} : \left| \theta - \frac{p}{q} \right| \leq \frac{1}{q^{\alpha}} \right\}.$$ 

It is well known that if $0 \leq \alpha \leq 2$ this set is $[0, 1] \setminus \mathbb{Q}$. For $\alpha > 2$, $D_\alpha$ is a set of Lebesgue measure zero. Thus, for such sets a natural question is what is the Hausdorff dimension of $D_\alpha$. Hausdorff dimension is defined in Section 2.1. Jarnik [9] and Besicovitch [1] both proved that $\text{HD} (D_{\alpha}) = \frac{2}{\alpha}$.

One can also express $D_\alpha$ in the following way. Observe that

$$\left| \theta - \frac{p}{q} \right| \leq q^{-\alpha} \iff |q\theta - p| \leq q^{1-\alpha}$$

$$\iff \|q\theta\| \leq q^{1-\alpha},$$

where $\|\cdot\|$ denotes distance to the nearest integer. Under this metric we may view $[0, 1]$ as $\mathbb{R}/\mathbb{Z}$. If we define $T_q : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $x \mapsto qx$ we get that

$$D_{\alpha} = \bigcap_{m \geq 1} \bigcup_{q \geq m} \bigcup_{0 \leq p \leq q} \left\{ \theta \in \mathbb{R}/\mathbb{Z} : \left| \theta - \frac{p}{q} \right| \leq q^{-\alpha} \right\}$$

$$= \bigcap_{m \geq 1} \bigcup_{q \geq m} \left\{ \theta \in \mathbb{R}/\mathbb{Z} : \|q\theta\| \leq q^{1-\alpha} \right\}$$

$$= \bigcap_{m \geq 1} \bigcup_{q \geq m} \left\{ \theta \in \mathbb{R}/\mathbb{Z} : \|T_q (\theta)\| \leq q^{1-\alpha} \right\}$$

$$= \bigcap_{m \geq 1} \bigcup_{q \geq m} \left\{ \theta \in \mathbb{R}/\mathbb{Z} : T_q (\theta) \in B(0, q^{1-\alpha}) \right\},$$

Although the sequence $\{T_1, T_2, \ldots\}$ does not form a semigroup under composition it does so under addition; indeed, $T_{q+r} = T_q + T_r$. 

1
In this spirit, when we consider a map $T : X \to X$ on a metric space $X$ and the semigroup $\{\text{id}, T, T \circ T, \ldots\}$ under composition, we may define a shrinking-target set as

$$\mathcal{D} = \bigcap_{m \geq 1} \bigcup_{n \geq m} \{x \in X \mid T^m(x) \in B(x_n, r_n)\},$$

where $(x_n) \in X^\mathbb{N}$ and $r_n \downarrow 0$.

If we endow the metric space $X$ with a Borel measure, in particular if $X \subseteq \mathbb{R}^n$ and we consider $n$-dimensional Lebesgue measure, we may ask about the “size” of $\mathcal{D}$ from the measure-theoretic point of view. Indeed, if $T$ is expanding and

$$\sum_{n \geq 1} r_n < \infty,$$

then it follows from the Borel-Cantelli lemma that $\mathcal{D}$ is a Lebesgue null-set, in which scenario the question of “size” from the fractal-geometric point of view becomes non-trivial.

In Chapter 3 we will give an overview of iterated function systems and in Section 4.2 we will define a shrinking-target set for them. Such sets will be the main object of study in this dissertation, in particular their Hausdorff dimension.

In [4], R. Bowen used an appropriate definition of pressure $P$ to prove that the Hausdorff dimension $d$ of quasi-circles (i.e., images of circles under quasi-conformal maps) is given by the unique solution $\delta$ to the equation $P(\delta t) = 0$.

Since Bowen’s pioneering work, his technique has been expanded to several contexts for dynamically-defined sets. For an overview of dimension formulas for limit sets of both random and deterministic systems one may consult [12].

With regards to shrinking-target sets the technique was first applied by R. Hill and S. Velani. For instance, in [8], the authors prove a Bowen formula for the dimension of a shrinking-target set generated by an expanding rational map of the Riemann sphere, restricted to its Julia set.

In the case of non-autonomous iterated function systems, Bowen’s formula has been expanded to shrinking-target sets for a specific class of examples of iterated function systems.
on $X = [0, 1]$ in [7], to the same class of examples in the random setting in [17], and to limit sets for more general conformal iterated function systems in [15].

It should be pointed out that the collection of references mentioned above does not come close to being exhaustive on the topic of shrinking targets or Bowen’s formula.
CHAPTER 2
PRELIMINARIES FROM GEOMETRIC MEASURE THEORY

In this chapter we will give an overview of some geometric measure theory that will be used later.

2.1. Hausdorff Measure and Dimension

Suppose $(X,d)$ is a metric space, $A \subseteq X$ and $\delta > 0$. A countable collection $\mathcal{C} \subseteq 2^X$ is said to be a $\delta$-cover of $A$ if $A \subseteq \bigcup \mathcal{C}$ and

$$\sup_{C \in \mathcal{C}} \{\text{diam } (C)\} \leq \delta.$$  

For $t \geq 0$ and $\delta > 0$ define the set function $H^t_\delta : 2^X \rightarrow [0, \infty]$ as

$$H^t_\delta (A) := \inf \left\{ \sum_{C \in \mathcal{C}} [\text{diam } (C)]^t \mid \mathcal{C} \text{ is a } \delta\text{-cover of } A \right\}.$$  

Let us prove that $H^t_\delta$ is an outer measure.

**Proposition.** For every $\delta > 0$ and every $t \geq 0$, $H^t_\delta$ is an outer measure.

**Proof.** Note that $\text{diam } (\emptyset) = 0$, so $H^t_\delta (A) = 0$.

Suppose $A \subseteq B \subseteq X$. Note that every $\delta$-cover of $B$ is also a $\delta$-cover of $A$. Since the collection of covers of $B$ is contained in the collection of covers of $A$ it follows that $H^t_\delta (A) \leq H^t_\delta (B)$.

Now consider a sequence $(A_k)_{k \geq 1} \in (2^X)^\mathbb{N}$, and let $A = \bigcup_{k \geq 1} A_k$. We have to show that

$$H^t_\delta (A) \leq \sum_{k \geq 1} H^t_\delta (A_k).$$

If the right-hand side is infinite, then the inequality holds. Now assume that the right-hand side is finite. Let $\epsilon > 0$. For every $k \in \mathbb{N}$ we can choose a $\delta$-cover $(C_j^{(k)})_{j \geq 1}$ of $A_k$ such that

$$\sum_{j \geq 1} \left[ \text{diam } (C_j^{(k)}) \right]^t \leq H^t_\delta (A_k) + \frac{\epsilon}{2^k} < \infty.$$  

4
Since \((C_j^{(k)})_{j,k\geq 1}\) is a \(\delta\)-cover of \(A\) it follows that

\[
H_\delta^t(A) \leq \sum_{k \geq 1} \sum_{j \geq 1} \left[ \text{diam} \left( C_j^{(k)} \right) \right]^t \\
\leq \sum_{k \geq 1} \left[ H_\delta^t(A_k) + \frac{\epsilon}{2^k} \right] \\
= \sum_{k \geq 1} H_\delta^t(A_k) + \epsilon.
\]

Since \(\epsilon > 0\) was chosen arbitrarily, the result follows. \(\square\)

**Definition 1.** The \(t\)-dimensional Hausdorff measure of \(A \subseteq X\) is then defined as

\[
H^t(A) := \lim_{\delta \to 0^+} H_\delta^t(A).
\]

Since a \(\delta\)-cover of \(A\) is also a \(\delta'\)-cover of \(A\) whenever \(\delta < \delta'\) it follows that

\[
H^t(A) = \sup_{\delta > 0} H_\delta^t(A).
\]

**Definition 2.** The Hausdorff dimension of a set \(A \subseteq X\) is defined as

\[
\text{HD}(A) := \sup \{ t \geq 0 \mid H^t(A) > 0 \}.
\]

At this point we would like to point out that the definition of Hausdorff dimension does not give us a straightforward way to calculate the dimension of some set. Thus, finding a formula to compute the dimension of some large class of sets is a question of interest. We will address this question for a certain class of sets.

Now we present some properties of Hausdorff dimension. Properties (a)-(c) will be proven. Property (d) will not be needed in subsequent material and its proof will be omitted for conciseness. A proof can be found in [5].

**Proposition.** Under the standard metric in \(\mathbb{R}^d\), Hausdorff dimension satisfies the following properties for all subsets \(A, B \subseteq \mathbb{R}^d\).

(a) *(monotonicity)* \(\text{HD}(A) \leq \text{HD}(B)\) whenever \(A \subseteq B\).
(b) *(Lipschitz invariance)* If \( f : A \rightarrow B \) is Lipschitz, then
\[
\text{HD} (f (A)) \leq \text{HD} (A).
\]
Moreover, if \( f \) is bi-Lipschitz, then
\[
\text{HD} (f (A)) = \text{HD} (A).
\]

(c) *(\( \sigma \)-stability)* If \( \{A_k\}_{k \geq 1} \) is a countable collection of subsets of \( \mathbb{R}^d \) then
\[
\text{HD} \left( \bigcup_{k \geq 1} A_k \right) = \sup_{k \geq 1} \text{HD} (A_k).
\]

(d) *(Open sets)* If \( U \subseteq \mathbb{R}^d \) is open, then \( \text{HD} (U) = d \).

**Proof.**

(a) Note that if \( A \subseteq B \) then every \( \delta \)-cover of \( B \) is also a \( \delta \)-cover of \( A \). It follows that
\[
\inf \left\{ \sum_{k \geq 1} \text{diam} (C_k)^t \mid \{C_k\}_{k \geq 1} \text{ is a } \delta \text{-cover of } A \right\}
\]
\[
\leq \inf \left\{ \sum_{k \geq 1} \text{diam} (D_k)^t \mid \{D_k\}_{k \geq 1} \text{ is a } \delta \text{-cover of } B \right\}
\]

for every \( \delta > 0 \) and every \( t \geq 0 \). Thus, \( H^t (A) \leq H^t (B) \) for every \( t \). This implies that \( \text{HD} (A) \leq \text{HD} (B) \).

(b) Recall that \( f : A \rightarrow \mathbb{R}^d \) is Lipschitz if there exists a number \( c > 1 \) such that
\[
|f (x) - f (y)| \leq c |x - y|
\]
for all \( x, y \in A \), and \( f \) is bi-Lipschitz if \( f \) is a bijection and \( f^{-1} \) is Lipschitz.

To prove that \( t: = \text{HD} (A) \geq \text{HD} (f (A)) \) it is enough to show that \( H^{t+\epsilon} (f (A)) = 0 \)
for arbitrary \( \epsilon > 0 \). Since \( H^{t+\epsilon} (A) = 0 \) then for every \( \delta > 0 \) and every \( \eta > 0 \) we can choose a \((c^{-1}\delta)\)-cover \( \{C_k\}_{k \geq 1} \) of \( A \) such that
\[
\sum_{k \geq 1} [\text{diam} (C_k)]^{t+\epsilon} < \eta.
\]
Since $f$ is Lipschitz it follows that

$$\sum_{k \geq 1} [\text{diam } (f(C_k))]^{t+\epsilon} \leq c^{t+\epsilon} \sum_{k \geq 1} [\text{diam } (C_k)]^{t+\epsilon} \leq c^{t+\epsilon} \eta.$$ 

Note that $(f(C_k))_{k \geq 1}$ is a $\delta$-cover of $f(A)$. Since the inequality above holds for arbitrary $\delta > 0$ and $\eta > 0$ it follows that $H^{t+\epsilon} (f(A)) = 0$.

It remains to show that $s := \text{HD} (A) \leq \text{HD} (f(A))$. Since $f^{-1}$ is Lipschitz it follows from the proof above that

$$\text{HD} (A) = \text{HD} (f^{-1} (f(A))) \leq \text{HD} (f(A)).$$

This proves the desired equality.

(c) Let $A := \bigcup_{k \geq 1} A_k$. From part ((a)) it follows that

$$\sup_{k \geq 1} \text{HD} (A_k) \leq \text{HD} (A).$$

Now let $s := \sup_{k \geq 1} \text{HD} (A_k)$. To show that $\text{HD} (A) \leq s$ it suffices to show that $H^{s+\epsilon} (A) = 0$ for all $\epsilon > 0$. Indeed, by the subadditivity of the outer measure $H^{s+\epsilon}$ we have that

$$H^{s+\epsilon} (A) \leq \sum_{k \geq 1} H^{s+\epsilon} (A_k) = 0,$$

where the equality follows from

$$\sup_{k \geq 1} \text{HD} (A_k) = s \implies \text{HD} (A_k) \leq s \text{ for all } k \geq 1 \implies H^{s+\epsilon} (A_k) = 0 \text{ for all } \epsilon > 0 \text{ and all } k \geq 1.$$

This completes the proof of $\sigma$-stability.
2.2. The Inverse Frostman Lemma

It should be remarked that in order to obtain an upper bound (resp. lower bound) on the Hausdorff dimension, i.e., an inequality of the form $\text{HD} (A) \leq c$ (resp. $\text{HD} (A) \geq c$), it is enough to show for every $\epsilon > 0$ and for every $\delta > 0$ small enough that $H^{c+\epsilon}_\delta (A) = 0$ (resp. $H^{c-\epsilon}_\delta > 0$) holds.

Given the definition of Hausdorff measure the upper bound follows from showing that \textit{for some} $\delta$-cover of $A$, $H^{c+\epsilon}_\delta (A) = 0$ holds. In contrast, the lower bound follows from showing that \textit{for every} $\delta$-cover of $A$ ($\delta$ small enough), the condition $H^{c-\epsilon}_\delta (A) > 0$ holds.

Due to the universal quantifier in the statement for the lower bound, one would expect proofs of the lower bound to be much more difficult to obtain. One well-known technique to obtain a lower bound is the inverse Frostman lemma, also known as the mass distribution principle, which reduces the condition for the lower bound to the existence of a measure with an appropriate scaling property.

**Proposition 3.** (Inverse Frostman Lemma) Let $A \subseteq X$ and suppose that $\mu$ is a Borel probability measure with $\text{supp} (\mu) \subseteq A$. If there exist constants $C, t > 0$ such that for every $x \in A$ and every $r > 0$

$$
\mu \left( B (x, r) \right) \leq Cr^t,
$$

then $\text{HD} (A) \geq t$.

\textit{The proof of the inverse Frostman lemma can be found in [6].}

2.3. Packing Measure and Dimension

In this section we will give an introduction to packing measures and dimensions. Instead of the coverings used in the definition of Hausdorff measure, packing measures will be defined in terms of a packing.

**Definition 4.** Let $(X, d)$ be a metric space and $A \subseteq X$. A countable collection $(x_i, r_i)_{i \geq 1} \in (X \times [0, \infty))^\mathbb{N}$ is said to be a packing of $A$ if $x_i \in A$ for all $i \geq 1$ and if $d (x_i, x_j) \geq r_i + r_j$ for all $i \neq j$.

Moreover, if $r \geq r_i$ for all $i$, we say that $(x_i, r_i)_{i \geq 1}$ is an $r$-packing.
Before we're able to define packing measure, we need to define a precursor to it.

**Definition 5.** For a subset $A \subseteq X$ and $t \geq 0$ define

$$P^s_t(A) := \sup \left\{ \sum_{i \geq 1} (2r_i)^t \mid (x_i, r_i)_{i \geq 1} \text{ is an } r\text{-packing of } A. \right\},$$

and

$$P^s(A) := \lim_{r \to 0} P^s_t(A).$$

Since an $r$-packing of $A$ is also an $r'$-packing of $A$ whenever $r' > r$ it follows that

$$\lim_{r \to 0} P^s_t(A) = \inf_{r > 0} P^s_t(A).$$

It can be shown that $P^s_t$ is not an outer measure, as it does not satisfy countable subadditivity (for an example see [5]). The way to obtain an outer measure in terms of packing is the following.

**Definition 6.** For $A \subseteq X$ and $t \geq 0$ define

$$P^t(A) := \inf \left\{ \sum_{i \geq 1} P^s_t(A_i) \mid (A_i)_{i \geq 1} \text{ is a countable cover of } A. \right\}$$

The proof that $P^t$ is an outer measure is similar to that for $H_{\delta}^t$ and will be omitted.

The packing dimension is now defined just like the Hausdorff dimension with the corresponding measure

**Definition 7.** The packing dimension of $A \subseteq X$ is defined to be

$$\text{PD}(A) := \sup \left\{ t \geq 0 \mid P^t(A) > 0 \right\}.$$ 

Packing dimension shares many of the basic properties of Hausdorff dimension.

**Proposition.** If $A, B \subseteq \mathbb{R}^n$ and $\mathbb{R}^n$ is endowed with the standard metric, then

(a) (*monotonicity*) If $A \subseteq B$, then $\text{PD}(A) \leq \text{PD}(B)$

(b) (*bi-Lipschitz invariance*) If $f : A \to \mathbb{R}^n$ is Lipschitz, then $\text{PD}(A) = \text{PD}(B)$.

(c) (*σ-stability*) If $(A_k)_{k \geq 1} \in (2^X)^N$, then $\text{PD}(\bigcup_{k \geq 1} A_k) = \sup_{k \geq 1} \text{PD}(A_k)$

(d) (*open sets*) If $U$ is an open subset of $\mathbb{R}^n$ then $\text{PD}(U) = n$. 


2.4. Ahlfors Measures

When constructing an appropriately measure to invoke Frostman’s lemma it is often useful to use an auxiliary measure with nice geometric properties. One such measure that will be indispensable in later proofs is that of Ahlfors measure.

**Definition 8.** A finite Borel measure $\mu_h$ supported on a set $A \subseteq X$ is said to be an $h$-Ahlfors measure if $h > 0$ and there exist numbers $C, r_0 > 1$ such that for every $x \in A$ and $0 < r < r_0$ we have that

$$C^{-1}r^h \leq \mu_h(B(x, r)) \leq Cr^h. \quad (2.1)$$

In Section 3.6 we will prove existence results for Ahlfors measures related to non-autonomous iterated function systems.
3.1. Autonomous Iterated Function Systems

**Definition 9.** Let $X \subseteq \mathbb{R}^d$ be a compact, convex subset with non-empty interior. An autonomous iterated function system $\Phi$ on $X$ is a countable collection of maps $\varphi_a: X \to X$ indexed by a set $I$ and satisfying the following properties:

- (Open Set Condition, OSC) For every $a, b \in I$ distinct,
  $$\varphi_a(\text{int}X) \cap \varphi_b(\text{int}X) = \emptyset.$$  

- (Uniform Contraction Condition, UCC) There exists a number $\theta > 0$ such that
  $$\sup_{a \in I} \text{diam} \left( \varphi_a(A) \right) \leq e^{-\theta} \text{diam} \left( A \right)$$
  for all $A \subseteq X$.

The term “iterated function system” will be abbreviated as “i.f.s.”

The index set $I$ is referred to as the alphabet, its elements are referred to as letters, and elements in $I^n := \prod_{k=1}^{n} I$ are referred to as words of length $n$. If $\#I < \infty$ we say that $\Phi$ is finite, otherwise we say it is infinite. We also define

$$I^\ast := \bigcup_{n \geq 1} I^n$$

and refer to it as the set of finite words. The set of infinite words is

$$I^\infty := I^\mathbb{N}.$$

If $\omega \in I^n$ we put $n = |\omega|$. If $\omega \in I^n$ and $\tau \in I^m$ then the concatenation $\omega\tau \in I^{n+m}$ is defined coordinate-wise as

$$((\omega\tau)_k) = \begin{cases} 
\omega_k & \text{if } 1 \leq k \leq n \\
\tau_{k-n} & \text{if } n+1 \leq k \leq n+m.
\end{cases}$$
If $\omega \in I^n$ and $k \in \{1, \ldots, n - 1\}$ then the truncation of $\omega$ up to length $k$ is

$$\omega|_k := (\omega_1, \ldots, \omega_k) \in I^k,$$

and the $k$-shift of $\omega$ is

$$\sigma^k \omega := (\omega_{k+1}, \ldots, \omega_n) \in I^{n-k}.$$

We will be interested in the dynamics on $X$ given by compositions of maps in $\Phi$. For $\omega \in I^n$ we define

$$\varphi^n_\omega := \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n}.$$

**Example 3.1.** A simple, classical example of an iterated function system is that which generates Cantor’s middle-third set. Let $X = [0, 1]$, $I = \{0, 2\}$ and define

$$\varphi_k(x) = \frac{1}{3}(x + k).$$

Note that Cantor’s middle-third Cantor set is given by

$$\bigcap_{n \geq 1} \bigcup_{\omega \in I^n} \varphi^n_\omega([0, 1]).$$

The set above is known as the limit set, or attractor, of the i.f.s. We will address the question of the dimension of such sets in section 3.5.

### 3.2. Non-Autonomous Iterated Function Systems

In this dissertation we will be primarily interested in certain sets dynamically-defined via non-autonomous i.f.s.

**Definition 10.** Let $X$ be a metric space. A non-autonomous iterated function system on $X$ is a sequence $(\Phi^{(n)})_{n \geq 1}$ of iterated function systems on $X$.

The alphabet encoding the ifs $\Phi^{(n)}$ will be denoted by $I^{(n)}$ and

$$I^{(m,n+1)} := \prod_{k=m}^{n+1} I^{(k)}$$

will consist of words of length $n$ coming from alphabets $I^{(m)}, \ldots, I^{(n+1)}$. The set $I^{(1,n)}$ will be denoted simply by $I^n$. 
Maps from $\Phi^{(n)}$ will be denoted by $\varphi^{(n)}_a$, $a \in I^{(n)}$, and

$$\varphi^n_\omega := \varphi^{(1)}_\omega \circ \cdots \circ \varphi^{(n)}_\omega.$$ 

Notice that if $\Phi^{(n)} = \Phi^{(m)}$ for all $m, n \in \mathbb{N}$, then the i.f.s. reduces to an autonomous system. Thus, when we refer to a non-autonomous i.f.s., we include the possibility that such an i.f.s. is an autonomous one.

3.3. Conformality and Bounded Distortion Property

We will be particularly interested in conformal, non-autonomous i.f.s. where the alphabet $I$ is finite. Here we briefly recall the definition of conformality and some basic properties.

**Definition.** If $V \subseteq \mathbb{R}^d$ is an open subset and $f : V \to V$ is differentiable, we say that $f$ is conformal if the derivative $Df(x) : V \to \mathbb{R}^d$ is a similarity map for every $x \in V$.

**Remark.** When we refer to a conformal i.f.s. $\Phi^{(n)}$ on (the closed set) $X$ we will assume that each map $\varphi_a^{(n)}$ extends to a diffeomorphism on some open set $V$ containing $X$, $\varphi_a^{(n)}(V) \subseteq V$ and $\varphi_a^{(n)}$ is conformal on $V$. In what follows, we will concern ourselves with conformal i.f.s.

Conformality is closely related to the bounded distortion property

**Definition.** An i.f.s. $(\Phi^{(n)})_{n \geq 1}$ on $X$ has the bounded distortion property (BDP) if there exists $K > 1$ such that for every $x, y \in X$, every $n \in \mathbb{N}$, and every $\omega \in I^n$ we have that

$$|D\varphi^n_\omega(x)| \leq K |D\varphi^n_\omega(y)|.$$

It should be noted that a sufficient condition for BDP, one in terms of the maps $\varphi_a^{(n)}$ and not in terms of the composition $\varphi^n_\omega$, is if there exists $\alpha > 0$ such that

$$\left|\frac{D\varphi_a^{(n)}(x)}{D\varphi_a^{(n)}(y)} - 1\right| \leq K |x - y|^\alpha,$$

for all $x, y \in X$, all $n \in \mathbb{N}$, and all $a \in I^{(n)}$. 

13
One geometric consequence of BDP is that for every ball $B(x, r) \subseteq X$, for all $n \in \mathbb{N}$, and for all $\omega \in I^n$, we have that

$$B\left(\varphi^n_\omega(x), K^{-1} \|D\varphi^n_\omega\| r\right) \subseteq \varphi^n_\omega\left(B(x, r)\right) \subseteq B\left(\varphi^n_\omega(x), K \|D\varphi^n_\omega\| r\right).$$

For a proof of this fact see, for instance, [11].

We remark that conformality implies the Bounded Distortion Property whenever $d \geq 2$. For $d = 2$ this follows from Koebe’s distortion theorem [13], and for $d \geq 3$ it is a consequence of Liouville’s theorem for conformal maps [2].

3.4. Topological Pressure

One important concept in thermodynamic formalism is that of topological pressure. The definition of topological pressure can be found in the classical text [3]. For a modern treatment in terms of topological covers one may consult [14] or [10]; and in terms of spanning sets one may consult [16].

Here we briefly discuss the topological pressure as it pertains a non-autonomous conformal i.f.s. In [15] the authors define the upper and lower pressure.

**Definition 11.** Consider an i.f.s. $(\Phi^{(n)})$. The upper topological pressure $P : [0, \infty) \to \mathbb{R}$ is defined as

$$P_{RU}(t) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \|D\varphi^n_\omega\|^t.$$

The lower topological pressure is defined similarly, by replacing the limit superior with a limit inferior.

One important property of the upper and lower pressure functions is their strict monotonicity. Again in [15] the authors prove

**Lemma 12.** If $t_1 < t_2$ then $P_{RU}(t_2) \leq P_{RU}(t_1)$. Moreover, the inequality is strict if $P_{RU}(t_1) < \infty$. 

14
Proof. Since $\|D\varphi^n_\omega\| \leq e^{-n\theta} < 1$ it follows that
\[
\sum_{\omega \in I^n} \|D\varphi^n_\omega\|^2 \leq \sum_{\omega \in I^n} \|D\varphi^n_\omega\|^{t_1} \|D\varphi^n_\omega\|^{t_2-t_1} \leq \sum_{\omega \in I^n} \|D\varphi^n_\omega\|^{t_1} e^{-n\theta(t_2-t_1)},
\]
so
\[
\frac{1}{n} \log \sum_{\omega \in I^n} \|D\varphi^n_\omega\|^2 \leq \frac{1}{n} \log \left[ \sum_{\omega \in I^n} \|D\varphi^n_\omega\|^{t_1} e^{-n\theta(t_2-t_1)} \right] = \frac{1}{n} \log \sum_{\omega \in I^n} \|D\varphi^n_\omega\|^{t_1} - \theta(t_2-t_1).
\]
This implies that $P_{RU}(t_2) \leq P_{RU}(t_1) - \theta(t_2-t_1)$. Since $t_2 > t_1$, it follows that $P_{RU}(t_2) < P_{RU}(t_1)$ whenever $P_{RU}(t_1) < \infty$. □

The same statement holds when we consider the lower pressure.

3.5. Attractors of Iterated Function Systems

As we saw in example 3.1, iterated function systems may generate interesting sets which are in some sense irregular.

Definition 13. The attractor, or limit set, $J$ of an i.f.s. on $X$ with alphabets $(I^{(n)})$ is defined as
\[
J = \bigcap_{n \geq 1} \bigcup_{\omega \in I^n} \varphi^n_\omega(X).
\]

We will often express the limit set in terms of a projection map $\pi$ on $I^\infty$.

Notice that $(\varphi^k_{\xi_{\ell_k}}(X))_{k \geq 1}$ is a decreasing sequence of nested compact sets, whose intersection is a singleton. For every $n \in \mathbb{N}$ we define $\pi_n : I^{(n+1,\infty)} \to X$ such that $\pi_n(\xi)$ is the element inside the singleton
\[
\bigcap_{k \geq 1} \varphi^{(n+1,n+k)}_{\xi_{\ell_k}}(X).
\]
We will write $\pi$ for $\pi_0$. In terms of $\pi$, the limit set can also be written as
\[
J = \pi(I^\infty).
\]
3.5.1. Bowen’s Dimension Formula for Non-Autonomous I.F.S.

As it was mentioned in section 2.2, obtaining a lower bound for the Hausdorff dimension of a set is generally non-trivial. In 1979, Rufus Bowen exhibited a connection between the Hausdorff dimension of limit sets with the topological pressure of the system which defines such a set.

Monotonicity of the pressure function (Lemma 12) is the main property that allows one to define the Bowen parameter, or Bowen dimension.

**Definition.** The Bowen parameter of a non-autonomous i.f.s. \( \Phi \) is defined as

\[
B(\Phi) := \sup \{ t \geq 0 \mid P_{RU}(t) > 0 \} = \inf \{ t \geq 0 \mid P_{RU}(t) < 0 \}.
\]

The principal objective of this dissertation is to bridge the results of [7] and [15] to prove Bowen’s formula for shrinking target sets for conformal i.f.s. in any dimension satisfying certain conditions.


In this Section we will prove some auxiliary results that will be used in our main theorems.

Let us define for every \( n \in \mathbb{N} \),

\[
\rho_n = \max_{a, b \in I^{(n)}} \left\| D\varphi_a^{(n)} \right\|/\left\| D\varphi_b^{(n)} \right\|,
\]

and

\[
Z_n(t) = \sum_{\omega \in I^n} \| D\varphi_{\omega}^n \|^t.
\]

Following the analysis in [15] we obtain the following result.

**Theorem 14.** If the sequences \((\#I^{(n)})_{n \geq 1}, (\rho_n)_{n \geq 1}, (Z_n(h))_{n \geq 1}, \) and \((Z_n^{-1}(h))_{n \geq 1}\) are bounded, then there exists an \( h \)-Ahlfors measure supported on \( J \).
In the proof of Theorem 3.2 in [15] the authors choose a sequence $(\mu^{(n)})_{n \geq 1}$ of Borel probability measures with
\[ \text{supp} (\mu^{(n)}) = \bigcup_{\omega \in I^n} \varphi^n_\omega (X) \]
satisfying
\[ (3.1) \quad \mu^{(n)} (\varphi^n_\omega (X)) = \frac{\|D\varphi^n_\omega\|^h}{Z_n(h)} \]
for all $\omega \in I^n$.

Letting $\mu$ be a weak limit of $(\mu^{(n)})_{n \geq 1}$ they prove that for every $t \geq 0$ such that
\[ \liminf_{n \to \infty} \frac{Z_{n-1} (t) \cdot (\# I^{(n)})^{1/d}}{1 + \log \left[ \max_{j \leq n} \rho_j \right]} \min_{a \in I^{(n)}} \left\| D\varphi_a^{(n)} \right\|^t > 0, \]
then the measure $\mu$ satisfies
\[ \mu (B (x, r)) \leq Cr^t \]
for every $x \in X$ and $r > 0$. The constant $C$ is independent of $x$ and $r$.

We claim that the measure $\mu$ is $h$-Ahlfors. In order to prove the upper bound in the Ahlfors condition it suffices to show that the limit inferior above is positive for $t = h$.

Let $B$ be a bound for all the sequences in the hypothesis of the theorem. Since $\# I^{(n)} \geq 2$ and $\rho_n \leq B$, it suffices to show that
\[ \liminf_{n \to \infty} Z_{n-1} (h) \min_{a \in I^{(n)}} \left\| D\varphi_a^{(n)} \right\|^h > 0. \]

Note that since the sequence $(Z_n^{-1} (h))_{n \geq 1}$ is bounded abounded above by $B$ we have that the sequence $(Z_n (h))_{n \geq 1}$ is bounded below by $B^{-1} > 0$.

So it suffices to show the following

Claim 3.2. The sequence
\[ \left( \min_{a \in I^{(n)}} \left\| D\varphi_a^{(n)} \right\|^h \right)_{n \geq 1} \]
is bounded below by a positive number.
Proof of Claim 15. Note that

\[ B^{-1} \leq Z_{n+1}(h) \]

\[ = \sum_{\tau \in \mathcal{I}^{n+1}} \| D\varphi_\tau^{n+1} \|^h \]

\[ = \sum_{\omega \in \mathcal{I}^n} \sum_{a \in \mathcal{I}(n+1)} \| D\varphi_\omega^{n+1} \|^h \]

\[ \leq \sum_{\omega \in \mathcal{I}^n} \sum_{a \in \mathcal{I}(n+1)} \| D\varphi_\omega \|^h \| D\varphi_a^{(n+1)} \|^h \]

\[ = Z_n(h) \sum_{a \in \mathcal{I}(n+1)} \| D\varphi_a^{(n+1)} \|^h \]

\[ \leq Z_n(h) \left( \# \mathcal{I}(n+1) \right) \max_{a \in \mathcal{I}(n+1)} \| D\varphi_a^{(n+1)} \|^h \]

\[ = Z_n(h) \left( \# \mathcal{I}(n+1) \right) \rho_{n+1} \min_{a \in \mathcal{I}(n+1)} \| D\varphi_a^{(n+1)} \|^h \]

Since the product \( Z_n(h) \left( \# \mathcal{I}(n+1) \right) \rho_{n+1} \) is uniformly bounded above, the claim follows. \( \Box \)

To prove that \( \mu(B(x, r)) \geq C^{-1} r^t \) we shall now consider an arbitrary \( 0 \leq r < \text{diam}(J) \) and \( x \in J \). Note that

\[ x = \pi(\xi) \]

for some \( \xi \in \mathcal{I}^\infty \). Define

\[ n = \max \{ k \in \mathbb{N} \mid \text{diam}(\varphi_{\xi\mid_k}^k(X)) \geq r \} . \]

It follows that

\[ \mu(B(x, r)) \geq \mu(\varphi_{\xi\mid_{n+1}}^{n+1}(X)) \]

since \( \varphi_{\xi\mid_{n+1}}^{n+1}(X) \subseteq B(x, r) \).

We make the following

**Claim 3.3.** For all \( n \in \mathbb{N} \) and every \( \omega \in \mathcal{I}^n \) the measure \( \mu \) satisfies

\[ \mu(\varphi^n_\omega(X)) \geq K^{-h} Z_n^{-1}(h) \| D\varphi^n_\omega \|^h , \]

where \( K \geq 1 \) is the distortion constant.
Proof of Claim 16. From (3.1) we have that for every \( q \in \mathbb{N} \) and every \( \omega \in I^n \)

\[
\mu^{(n+q)}(\varphi^n_\omega (X)) = \mu^{(n+q)}(\varphi^n_\omega (X) \cap \text{supp} (\mu^{(n+q)}))
\]

\[
= \mu^{(n+q)} \left( \bigcup_{\gamma \in I^{(n+1,n+q)}} \varphi^{n+q}_{\omega \gamma} (X) \right)
\]

\[
= \sum_{\gamma \in I^{(n+1,n+q)}} \mu^{(n+q)}(\varphi^{n+q}_{\omega \gamma} (X))
\]

\[
= Z_{n+q}^{-1} (h) \sum_{\gamma \in I^{(n+1,n+q)}} \| D \varphi_{\omega \gamma}^{n+q} \|^h
\]

\[
\geq Z_{n+q}^{-1} (h) \sum_{\gamma \in I^{(n+1,n+q)}} K^{-h} \| D \varphi_{\omega}^{n} \|^h \| D \varphi_{\gamma}^{(n+1,n+q)} \|^h,
\]

where the last inequality follows from the BDP.

Furthermore, the inequality

\[
Z_{n+q}^{-1} (h) \sum_{\gamma \in I^{(n+1,n+q)}} \| D \varphi_{\gamma}^{(n+1,n+q)} \|^h \geq Z_{n}^{-1} (h)
\]

follows from noting that

\[
Z_{n+q} (h) = \sum_{r \in I^{n+q}} \| D \varphi_{r}^{n+q} \|^h
\]

\[
= \sum_{\omega' \in I^n} \sum_{\gamma \in I^{(n+1,n+q)}} \| D \varphi_{\omega' \gamma}^{n+q} \|^h
\]

\[
\leq \sum_{\omega' \in I^n} \sum_{\gamma \in I^{(n+1,n+q)}} \| D \varphi_{\omega'}^{n} \|^h \| D \varphi_{\gamma}^{(n+1,n+q)} \|^h
\]

\[
= \sum_{\omega' \in I^n} \| D \varphi_{\omega'}^{n} \|^h \sum_{\gamma \in I^{(n+1,n+q)}} \| D \varphi_{\gamma}^{(n+1,n+q)} \|^h
\]

\[
= Z_{n} (h) \sum_{\gamma \in I^{(n+1,n+q)}} \| D \varphi_{\gamma}^{(n+1,n+q)} \|^h.
\]

This proves that

\[
\mu_{n+q} (\varphi^n_\omega (X)) \geq Z_{n}^{-1} (h) K^{-h} \| D \varphi_{\omega}^{n} \|^h
\]

for all \( q \in \mathbb{N} \). Taking the limit as \( q \to \infty \) proves the claim.
From Claim 3.3 it follows now that
\[
\mu(B(x, r)) \geq \mu\left(\varphi_{\xi_{n+1}}^{n+1}(X)\right)
\geq C^{-1} Z_{n+1}^{-1}(h) \left\| D\varphi_{\xi_{n+1}}^{n+1} \right\|^h
\geq C^{-1} Z_{n+1}^{-1}(h) K^{-h} \left\| D\varphi_{\xi_{n+1}}^n \right\|^h \left\| D\varphi_{\xi_{n+1}}^{(n+1)} \right\|^h,
\]
where the last inequality follows from BDP.

By the mean value inequality we have that
\[
\left\| D\varphi_{\xi_{n+1}}^n \right\| \text{diam}(X) \geq \text{diam}(\varphi_{\xi_{n+1}}^n(X)) \geq r.
\]
Redefining \( C \) we obtain that
\[
\mu(B(x, r)) \geq C^{-1} Z_{n+1}^{-1}(h) r^h \left\| D\varphi_{\xi_{n+1}}^{(n+1)} \right\|^h.
\]
From the hypothesis and Claim 3.2 the product \( Z_{n+1}^{-1}(h) \left\| D\varphi_{\xi_{n+1}}^{(n+1)} \right\|^h \) is uniformly bounded below by a positive number. This allows us to redefine \( C \), independent of \( x \) and \( r \), to obtain
\[
\mu(B(x, r)) \geq C^{-1} r^h,
\]
as desired. \( \square \)
CHAPTER 4

SHRINKING-TARGET SETS

4.1. Shrinking-Target Sets for Dynamical Systems

Consider a map $T: X \to X$. For the dynamics under iteration of this map one may consider sets of points whose orbits share some interesting property. Such as the case of shrinking-target sets.

**Definition 15.** Let $(x_n)_{n \geq 1} \in X^\mathbb{N}$ and $(r_n)_{n \geq 1} \in (0, 1)^\mathbb{N}$ such that $r_n \searrow 0$. The shrinking-target set associated to the sequence of balls (or “shrinking targets”) $Q := (B(x_n, r_n))_{n \geq 1}$ is defined as

$$
\mathcal{Q}_Q := \bigcap_{m \geq 1} \bigcup_{n \geq m} \{x \in X \mid T^n(x) \in B(x_n, r_n)\}.
$$

4.2. Shrinking-Target Sets for I.F.S.

Loosely speaking, one may think of the maps in an i.f.s. on $X$ and with limit set $J$ as the inverse branches of a non-invertible map $T: J \to J$.

**Example.** Consider again the i.f.s. in Example 3.1. It can be shown that its limit set $C$ is completely invariant under the map $T: [0, 1] \to [0, 1]$ defined as

$$
T(x) = \begin{cases} 
3x & \text{if } 0 \leq x \leq \frac{1}{3} \\
3x - 1 & \text{if } \frac{1}{3} < x < \frac{2}{3} \\
3x - 2 & \text{if } \frac{2}{3} < x \leq 1,
\end{cases}
$$

i.e.,

$$
T^{-1}(C) = C.
$$

Thus, one can restrict $T$ to $C$, and its inverse branches are the maps $\varphi_j$, $j \in \{0, 2\}$. 

21
EXAMPLE. Consider the map Gauss map $T: (0, 1] \to (0, 1]$ defined by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$ 

Note that if $T_n$ is the restriction of $T$ to the interval $(\frac{1}{n+1}, \frac{1}{n}]$ we have that $T_n(x) = \frac{1}{x} - n$ and $T_n$ is invertible. A simply calculation yields that $T_n^{-1}(x) = \frac{1}{x+n}$. These inverse branches form an infinite i.f.s. on $[0, 1]$ with alphabet $I = \mathbb{N}$, where

$$\varphi_n(x) = \frac{1}{x+n}.$$ 

In this case the limit set is

$$J = [0, 1].$$

If an i.f.s. satisfies the bounded distortion property, then

$$\bigcap_{m \geq 1} \bigcup_{n \geq m} \{x \in X \mid T^n(x) \in B(x_n, r_n)\} = \bigcap_{m \geq 1} \bigcup_{n \geq m} \{x \in X \mid x \in T^{-n}(B(x_n, r_n))\}$$

$$= \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\omega \in I^n} \{x \in X \mid x \in \varphi^n_\omega(B(x_n, r_n))\}$$

$$\subseteq \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\omega \in I^n} \{x \in X \mid x \in B(\varphi^n_\omega(x_n), r_n K \|D\varphi^n_\omega\|)\}$$

$$\subseteq \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\omega \in I^n} \{x \in X \mid x \in B(\varphi^n_\omega(x_n), r_n K e^{-n\theta})\}.$$ 

Putting

$$B_\omega := B(\varphi^n_\omega(x_n), r_n K e^{-n\theta}),$$

we will define a shrinking target set for the i.f.s. as

$$\mathcal{Q} = \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\omega \in I^n} B_\omega.$$
4.3. Pressure for Shrinking-Target Sets

Bowen’s dimension formula was first established for shrinking-target sets by R. Hill and S. Velani. In [8] the authors consider an expanding rational map $T$ of the Riemann sphere with Julia set $J$, and a Hölder continuous function $f: J \to (0, \infty)$ satisfying $f \geq \log |T'|$.

For a fixed $z_0 \in J$, the shrinking-target set is then defined as

$$D_{z_0} := \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{y \in T^{-n}(z_0)} B(y, e^{-S_n f(y)}) .$$

The authors prove that the Hausdorff dimension of $D_{z_0}$ is given by the unique zero of a certain pressure function.

The pressure function in [8] is defined as

$$\overline{P}(t) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \sup \left\{ \sum_{z \in F_n(\epsilon)} e^{-t S_n f(z)} \mid F_n(\epsilon) \subseteq J \text{ is an } (n, \epsilon)\text{-separated set} \right\} .$$

The upper pressure function corresponding to non-autonomous i.f.s. will be defined in a similar spirit in Section 5.2.

We will briefly note one important feature of the shrinking-target set $D_{z_0}$ (and that also concerns the pressure $\overline{P}$), specifically of the expression $e^{-S_n f(z)}$. From the assumption $f \geq \log |T'|$ one obtains that

$$S_n f(z) = \sum_{k=0}^{n-1} f(T^k(z)) \geq \sum_{k=0}^{n-1} \log |T'(T^k(z))| = \log \prod_{k=0}^{n-1} |T'(T^k(z))| = \log |(T^n)'(z)| .$$

Hence, for some $0 < \epsilon < 1$ independent of $n$ and $z \in J$,

$$e^{-S_n f(z)} \leq \left| (T^n)'(z) \right|^{-1} \leq (1 - \epsilon)^n ,$$

23
where the last inequality follows from the assumption that $T$ is expanding. Thus, the condition that $f \geq \log |T'|$ implies that the shrinking-targets shrink in radius at an exponential rate. This feature will re-appear in our setting for i.f.s.
CHAPTER 5

SHRINKING-TARGET SETS FOR NON-AUTONOMOUS I.F.S.

5.1. A Class of Examples of Shrinking-Target Sets for Non-Autonomous I.F.S.

In [7], the authors established a Bowen-type dimension formula for shrinking targets generated in the following way.

Consider a sequence $T_n$ of interval maps $T_n : [0, 1) \rightarrow [0, 1)$ defined by

$$T_n(x) = q_n x \mod 1,$$

where $(q_n)_{n \geq 1} \in \mathbb{N}_{\geq 2}$ is a sequence of integers no smaller than 2.

To define a shrinking-target set the authors consider an arbitrary sequence $Q = (\alpha_n)_{n \geq 1} \in (0, \infty)^\mathbb{N}$. Putting $\alpha(n) := \alpha_1 + \cdots + \alpha_n$ and

$$T^n = T_n \circ \cdots \circ T_1$$

they define

$$\mathcal{D}_Q = \bigcap_{m \geq 1} \bigcup_{n \geq m} \left\{ x \in [0, 1) : \|T^m(x)\| \leq e^{-\alpha(n)} \right\},$$

where $\|x\|$ is the distance of $x$ to the nearest integer.

The corresponding upper pressure function is defined to be

$$P(t) = \limsup_{n \to \infty} \frac{1}{n} \left[ (1 - t) \log (q_1 \cdots q_n) - t \alpha(n) \right].$$

The upper pressure function above is strictly decreasing and has a unique number $b \in [0, 1]$ such that

$$b = \sup \left\{ t \geq 0 \mid P(t) > 0 \right\} = \inf \left\{ t \geq 0 \mid P(t) < 0 \right\}.$$

The authors prove that $HD(\mathcal{D}_Q) = b$.

While the authors analyze the problem from the perspective of the dynamics by the maps $T_n$, an alternative formulation of the problem, following the discussion in Section 4.2, is in terms of a non-autonomous i.f.s.
For every \( n \in \mathbb{N} \) let \( I^{(n)} := \{0, \ldots, q_n - 1\} \) and for every \( a \in I^{(n)} \) let \( \varphi_a^{(n)} : [0, 1) \rightarrow [0, 1) \) be defined as
\[
\varphi_a^{(n)}(x) = q_n^{-1}(x + a).
\]
Then
\[
\mathcal{D}_Q = \bigcap_{m \geq 1} \bigcup_{n \geq m} \{ x \in [0, 1) : \|T^n(x)\| \leq e^{-\alpha(n)} \}
\]
\[
= \bigcap_{m \geq 1} \bigcup_{n \geq m} \{ x \in [0, 1) : T^n(x) \in B(0, e^{-\alpha(n)}) \}
\]
\[
= \bigcap_{m \geq 1} \bigcup_{n \geq m} \{ x \in [0, 1) : x \in T^{-n}(B(0, e^{-\alpha(n)})) \}
\]
\[
= \bigcup_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\omega \in I^n} \{ x \in [0, 1) : x \in B(\varphi^{(n)}_{\omega} (0), (q_1 \cdots q_n)^{-1} e^{-\alpha(n)}) \}
\]
\[
= \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\omega \in I^n} B_{\omega}.
\]

5.2. Bowen’s Formula for Shrinking Targets and Non-Autonomous I.F.S.

In this section we will state and prove our main results. We will consider a finite i.f.s. \( \Phi \) consisting of conformal contractions; there exists \( \theta > 0 \) such that for all \( n \in \mathbb{N} \) and all \( j \in I^{(n)} \) we have that
\[
\kappa^{(n)} := \max_{j \in I^{(n)}} \|D\varphi_j^{(n)}\| \leq e^{-\theta},
\]
and \( D\varphi_j^{(n)} \) is a similarity.

To define a shrinking target set we fix a sequence \( (\beta_n)_{n \geq 1} \) of functions \( \beta_n : I^{(n, \infty)} \rightarrow (0, \infty) \). Let \( S_n\beta : I^\infty \rightarrow (0, \infty) \) be defined by
\[
S_n\beta(\xi) = \beta_1(\xi) + \beta_2(\sigma\xi) + \cdots + \beta_n(\sigma^{n-1}\xi).
\]
The quantity above will determine the rate at which the radii of the shrinking-targets shrink to zero in the following way: Fix a sequence \( (\xi^{(n)}) \) where \( \xi^{(n)} \in I^{(n+1, \infty)} \), and let us define sequence \( (x^{(n)}) \in X^\mathbb{N} \) as \( x^{(n)} := \pi_n(\xi^{(n)}) \). For every \( \omega \in I^n \) we define the shrinking-targets as
\[
B_\omega = B(\varphi^{(n)}_{\omega} (x^{(n)}), e^{-S_n\beta(\omega\xi^{(n)})})
\]
The shrinking-target set is then defined as
\[ D = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{\omega}. \]

Now for \( t \geq 0 \) we define the upper pressure
\[ \overline{P}_\beta (t) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} e^{-tS_n \beta(\omega \xi^{(n)})}. \]
(5.1)

The lower pressure \( P_\beta (t) \) is defined similarly by taking a limit inferior instead of a limit superior. If \( \overline{P}_\beta (t) = P_\beta (t) \) holds, we denote this common value by \( P_\beta (t) \).

Now we briefly explore certain properties of the pressure functions. Note that for \( \epsilon > 0 \) we have that
\[ \overline{P}_\beta (t + \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \left( e^{-tS_n \beta(\omega \xi^{(n)})} e^{-\epsilon S_n \beta(\omega \xi^{(n)})} \right) \]
\[ \leq \limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sum_{\omega \in I^n} e^{-tS_n \beta} \right) \right] \]
\[ = \overline{P}_\beta (t), \]
so the upper (as well as lower) pressure function is non-increasing. We say that the sequence \( (\beta_n) \) is tame if the the upper pressure is strictly decreasing. Furthermore, assuming \( \# I^{(k)} \geq 2 \) for all \( k \) it is immediate that \( \overline{P}_\beta (0) \geq \log (2) \).

Now, if we assume that \( B > 0 \) such \( \# I^{(n)} \leq B \) for all \( n \in \mathbb{N} \), and that (5.3) holds then
\[ \overline{P}_\beta (d) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} e^{-dS_n \beta(\omega \xi^{(n)})} \]
\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( B^n \max \left\{ e^{-dS_n \beta(\omega \xi^{(n)})} : \omega \in I^n \right\} \right) \]
\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( B^n \pi_d^n e^{-nd\alpha} \right) \]
\[ \leq \log B - d\alpha + \limsup_{n \to \infty} \left( \frac{d}{n} \log \pi_n \right) \]
\[ \leq \log B - d\alpha - d\theta \]

It follows that \( \overline{P}_\beta (d) \leq 0 \) if \( B \leq e^{d(\theta + \alpha)}. \)
We observe that if $P_\beta$ is strictly decreasing, and $P_\beta(0) \cdot P_\beta(d) < 0$, then there exists a unique number $0 < b < d$ such that

$$b = \inf \{ t \geq 0 \mid P_\beta(t) < 0 \} = \sup \{ t \geq 0 \mid P_\beta(t) > 0 \}.$$  

Note that such a unique number $b$ still exists in $[0, \infty]$ when only assuming condition (5.3). We refer to such number as the Bowen parameter. The main objective of our analysis is to establish conditions under which $HD(\mathcal{D}) = b$.

5.2.1. Upper Bound

We can establish an upper bound for the Hausdorff dimension of $\mathcal{D}$ under quite general conditions.

**Theorem 16.** For any shrinking target set $\mathcal{D}$ originating from a non-autonomous i.f.s. and a tame sequence $(\beta_n)$, we have that $HD(\mathcal{D}) \leq b$.

**Proof.** Let $t > b$. We will show that $H^t(\mathcal{D}) = 0$. Note that for any $N \geq 1$ the collection $(\bigcup_{\omega \in I^n} B_\omega)_{n \geq N}$ covers $\mathcal{D}$, so

$$H^t(\mathcal{D}) \leq \sum_{n \geq N} \sum_{\omega \in I^n} [\text{diam}(B_\omega)]^t = 2^t \sum_{n \geq N} \sum_{\omega \in I^n} e^{-tS_n(\omega \xi(n))}.$$  

Since $t > b$ and $(\beta_n)$ is tame we have that $P_\beta(t) < 0$. Thus, for large enough $M$,

$$n \geq M \implies \frac{1}{n} \log \sum_{\omega \in I^n} e^{-tS_n(\omega \xi(n))} < \frac{1}{2} P_\beta(t) \leq 0.$$  

Hence,

$$\sum_{\omega \in I^n} e^{-tS_n(\omega \xi(n))} < e^{n P_\beta(t)} < 1.$$  

Thus,

$$\sum_{n \geq N} \sum_{\omega \in I^n} [\text{diam}(B_\omega)]^t \leq 2^t \sum_{n \geq N} e^{n P_\beta(t)}.$$  

28
The right hand side of the inequality above is the tail of a converging geometric series. After fixing $\epsilon > 0$ we can choose $N$ large enough so that

$$\sum_{n \geq N} \sum_{\omega \in I^n} [\text{diam } (B_\omega)]^t < \epsilon.$$  

This shows that $H^t (\emptyset) < \epsilon$. Since $\epsilon > 0$ and $t > b$ were chosen arbitrarily, we have that $\text{HD} (\emptyset) \leq b$. \hfill $\square$

5.2.2. Preliminaries for the Lower Bound

For the proof of the lower bound we will need to impose some restrictions on our i.f.s. First we establish some preliminary definitions and results.

We define

\[ \kappa(n) = \min_{j \in I(n)} \inf_{x \in X} |D\varphi_j^n(x)|, \]
\[ \bar{\kappa}(n) = \max_{j \in I(n)} \sup_{x \in X} |D\varphi_j^n(x)|, \]
\[ \kappa_n = \min_{\omega \in I^n} \inf_{x \in X} |D\varphi_j^n(x)|, \]
\[ \bar{\kappa}_n = \max_{\omega \in I^n} \sup_{x \in X} |D\varphi_j^n(x)|. \]

It is easy to check that

\[(5.2) \quad \prod_{k=1}^n \kappa(k) \leq \kappa_n \leq \bar{\kappa}_n \leq \prod_{k=1}^n \bar{\kappa}(k). \]

Let $J$ be the limit set (attractor) of the i.f.s., i.e.,

$$J = \bigcap_{n \geq 1} \bigcup_{\omega \in I^n} \varphi_\omega^n \big( X \big).$$

Consider the projection map $\pi_n : I^{(n+1, \infty)} \to X$ where $\pi_n (\xi)$ is defined as the element in the singleton set

$$\bigcap_{k \geq 1} \varphi_{\xi(k)}^{(n+1, n+k)} (X).$$

Note that $\pi_0 = \pi$. We also consider a sequence of dynamical-defined sets $J_n$,

$$J_n = \pi_n \big( I^{(n+1, \infty)} \big).$$
We note that for every $n \in \mathbb{N}$ and every $\omega \in I^n$, $\varphi^n_\omega (J_n) \subseteq J$; indeed,

\[
\varphi^n_\omega (J_n) = \varphi^n_\omega \left( \bigcup_{\xi \in I^{(n+1, \infty)}} \bigcap_{k \geq 1} \varphi^{(n+1,n+k)}_{\xi|k} (X) \right) \\
\subseteq \bigcup_{\xi \in I^{(n+1, \infty)}} \bigcap_{k \geq 1} \left( \varphi^n_\omega \left( \varphi^{(n+1,n+k)}_{\xi|k} (X) \right) \right) \\
= \bigcap_{k \geq 1} \bigcup_{\xi \in I^{(n+1, \infty)}} \varphi^{n+k}_{\omega \xi_{n+k}} (X) \\
= \bigcap_{k \geq 1} \bigcup_{\xi \in I^{(n+1,n+k)}} \varphi^{n+k}_{\omega \xi_{n+k}} (X) \\
\subseteq \bigcup_{k \geq n+1} \bigcup_{\tau \in I^k} \varphi^n_{\tau} (X) \\
= J.
\]

For every $n \in \mathbb{N} \cup \{0\}$ we fix $\xi^{(n)} \in I^{(n+1, \infty)}$ and from this we define a sequence $x^{(n)} \in J_n$ as $x^{(n)} = \pi_n \left( \xi^{(n)} \right)$. This implies that the balls $B_\omega = B \left( \varphi^n_\omega \left( x^{(n)} \right), e^{-S_n \beta} \right)$ are centered at a point in $J$.

Furthermore, in addition to the OS and UC conditions introduced in Definition 9 we make the following assumptions:

- For all $n \in \mathbb{N}$ and all $j \in I^{(n)}$, $\varphi_j^{(n)}$ is injective.

- **Exponentially shrinking condition (ESC):** We assume that there exist numbers $\alpha$ and $\bar{\alpha}$ such that

\[
0 < \alpha \leq \beta_k (\xi) + \log \kappa_k (\xi) \leq \beta_k (\xi) + \log \bar{\kappa}_k (\xi) \leq \bar{\alpha},
\]

for all $k$ and all $\xi \in I^{(k, \infty)}$. It is easy to check that

\[
0 < n \alpha \leq S_n \beta (\xi) + \log \kappa_n \leq S_n \beta (\xi) + \log \bar{\kappa}_n \leq n \bar{\alpha},
\]

for all $n$ and all $\xi \in I^\infty$. 

30
• **Non-empty quasi middle (NEQ):** Recall that for a set \( A \) in a metric space and \( \epsilon > 0 \), the \( \epsilon \)-thickening of \( A \) is

\[
B(A, \epsilon) = \bigcup_{x \in A} B(x, \epsilon).
\]

Now let

\[
X_{\epsilon} := X \setminus B(\mathbb{R}^d \setminus X, \epsilon).
\]

We assume that there exists \( \epsilon > 0 \) for which

\[
(5.4) \quad J_n \cap X_{\epsilon} \neq \emptyset, \text{ for all } n.
\]

Hence, assuming the NEQ condition we can choose the point \( x^{(n)} \) appearing in the definition of the balls \( B_\omega \) to be in \( J_n \cap X_{\epsilon} \).

• **Linear Variation Condition (LVC):** The sequence \( (\beta_n) \) is said have the linear variation condition if

\[
\lim_{n \to \infty} \frac{1}{n} \left( \sup_{\xi \in I_n} S_n \beta(\xi) - \inf_{\xi \in I_n} S_n \beta(\xi) \right) = 0.
\]

Such a condition holds if, for example, each function \( \beta_n \) is constant or, more generally, if

\[
\lim_{n \to \infty} \sup_{\xi, \bar{\xi} \in I(n, \infty)} (\beta_n(\xi) - \beta_n(\bar{\xi})) = 0.
\]

We note that this condition implies that for all \( \epsilon > 0 \) there exists \( N_\epsilon \geq 1 \) such that for all \( n \geq N_\epsilon \) and all \( \xi, \bar{\xi} \in I^\infty \) we have that

\[
(5.5) \quad \exp \left\{ -S_n \beta(\xi) - \epsilon n \right\} \leq \exp \left\{ -S_n \beta(\bar{\xi}) \right\} \leq \exp \left\{ -S_n \beta(\xi) + \epsilon n \right\}.
\]

Let us now examine some consequences of a conformal nonautonomous i.f.s. having these properties. First we note that ESC and UCC imply that the radii of \( B_\omega \) decay exponentially fast; Indeed \( e^{-S_n \beta(\omega^{(n)})} \leq \kappa_n e^{-n \theta} \).

One geometric consequence of BDP is that for every ball \( B(x, r) \subseteq X \), for all \( n \in \mathbb{N} \), and for all \( \omega \in I^n \), we have that

\[
B \left( \varphi^n_\omega(x), K^{-1} \|D\varphi^n_\omega\| r \right) \subseteq \varphi^n_\omega(B(x, r)) \subseteq B \left( \varphi^n_\omega(x), K \|D\varphi^n_\omega\| r \right).
\]
For a proof of this fact see, for instance, [11].

We remark that conformality implies the Bounded Distortion Property whenever \( d \geq 2 \). For \( d = 2 \) this follows from Koebe’s distortion theorem [13], and for \( d \geq 3 \) it is a consequence of Liouville’s theorem for conformal maps [2].

Another consequence of ESC, NEQ, and BDP is the following

**Claim 5.6.** For all \( \omega \in I^n \), and all \( n \) large enough, we have that \( B_\omega \subset \varphi^n_\omega (X) \).

**Proof.** Notice that the center of the ball \( B_\omega \) is contained in \( \varphi^n_\omega (X_\varepsilon) \), by condition NEQ. Now,

\[
\varphi^n_\omega (X) \supset \varphi^n_\omega (B (x^{(n)}, \varepsilon)) \\
\supset B (\varphi^n_\omega (x^{(n)}), K^{-1} \|D \varphi^n_\omega \| \varepsilon) \\
\supset B (\varphi^n_\omega (x^{(n)}), K^{-1} \kappa_n \varepsilon) .
\]

Thus, it suffices to show that \( K^{-1} \kappa_n \varepsilon \geq e^{-S_n \beta (\omega \xi^{(n)})} \) for all \( \omega \in I^n \). Given condition ESC notice that the desired inequality holds for all \( n \geq K (\varepsilon \alpha)^{-1} \). \( \square \)

5.2.3. Lower Bound for Bounded Derivative Ratios

First we prove

**Theorem 17.** Let \( \Phi \) be a nonautonomous conformal i.f.s. on a compact, convex set \( X \subseteq \mathbb{R}^d \) with nonempty interior satisfying OSC, ESC, UCC, LVC, and NEQ conditions. Suppose that the sequence \( \left( \frac{\alpha_n}{\alpha_n} \right) \) is bounded, that \((\beta_n)\) is tame, and that there exists an \( h\)-Ahlfors measure, \( \mu_h \), where \( h = HD (J) \), and \( \text{supp} (\mu_h) = J \). Then \( HD (\mathcal{D}) = b \).

**Proof.** Recall that \( HD (\mathcal{D}) \leq b \) has been proven in Theorem 16. Let \( 0 < t < b \). Our strategy consists of constructing a measure \( m \) supported on a set \( K \subseteq \mathcal{D} \) satisfying the hypothesis of the Frostmann Lemma with exponent \( t \). Choose an increasing sequence \( (n_l) \in \mathbb{N}^N \) such that

\[
(5.7) \quad \mathcal{P}_\beta (t) = \lim_{l \to \infty} \frac{1}{n_l} \log \sum_{\omega \in I^n} e^{-tS_{n_l} \beta (\omega \xi^{(n)})}.
\]
If necessary, we refine our subsequence so that it satisfies the following inequality for all \( l \):

\[
(5.8) \quad n_{t+1} \geq \frac{4h}{P(t)} \left( \text{const} + \bar{\alpha} \sum_{k=1}^{t} n_k \right).
\]

Now define \( R_1 = I^{n_1} \). Assuming \( R_l \subseteq I^{n_l} \) has been defined, for every \( \omega \in R_l \) let

\[
R_{l+1}(\omega) := \{ \tau \in I^{n_{l+1}} \mid B_\tau \subset B_\omega \}.
\]

Now define

\[
R_{l+1} := \bigcup_{\omega \in R_l} R_{l+1}(\omega).
\]

Now we will focus on obtaining a lower bound on the cardinality of the sets \( R_{l+1}(\omega) \).

We denote \( B(\varphi^{n_{l+1}}_\omega (x^{(n)}) , \frac{1}{2} e^{-S_{n_l} \beta(\omega^{(n)})}) \) by \( \frac{1}{2} B_\omega \).

**Claim 5.9.** Let \( \tau \in I^{n_{l+1}} \) and \( \omega \in I^{n_l} \). If \( n_{l+1} \geq \theta^{-1} \left[ \log (2) + n_l (\bar{\alpha} + \theta) \right] \) then either

\[
\varphi^{n_{l+1}}_\tau (X) \cap \frac{1}{2} B_\omega = \emptyset
\]

or

\[
\varphi^{n_{l+1}}_\tau (X) \subseteq B_\omega.
\]

**Proof of Claim.** Assume \( \varphi^{n_{l+1}}_\tau (X) \cap \frac{1}{2} B_\omega \neq \emptyset \). It suffices to show that \( 4|\varphi^{n_{l+1}}_\tau (X)| \leq |B_\omega| \). Indeed,

\[
4|\varphi^{n_{l+1}}_\tau (X)| \leq |B_\omega| \iff 2\kappa_{n_{l+1}} \leq e^{-S_{n_l} \beta(\omega^{(n)})}
\]

\[
\iff S_{n_l} \beta(\omega^{(n)}) + \log \kappa_{n_{l+1}} \leq - \log 2
\]

\[
\iff S_{n_l} \beta(\omega^{(n)}) + \log \kappa_{n_l} + \sum_{j=n_l+1}^{n_{l+1}} \log \kappa_{(j)} \leq - \log 2
\]

\[
\iff n_l \bar{\alpha} + \sum_{j=n_l+1}^{n_{l+1}} \log \kappa_{(j)} \leq - \log 2
\]

\[
\iff n_l \bar{\alpha} - \sum_{j=n_l+1}^{n_{l+1}} \theta \leq - \log 2
\]

\[
\iff n_l \bar{\alpha} - (n_{l+1} - n_l) \theta \leq - \log 2
\]

\[
\iff n_{l+1} \geq \theta^{-1} \left[ \log (2) + n_l (\bar{\alpha} + \theta) \right],
\]

33
where the 3rd, 4th, and 5th implications follow from (5.2), ESC, and UCC, respectively.

This proves the Claim.

From the Ahlfors property of $\mu_h$ we get that for all $\omega \in R_t$

$$C^{-1} \left( \frac{1}{2} e^{-S_{n_j} \beta (\omega \xi (n_i))} \right)^h \leq \mu_h \left( \frac{1}{2} B_\omega \right)$$

$$\leq \# \left\{ \tau \in I_{n+l} \mid \varphi^{n+i} (\omega X) \cap \frac{1}{2} B_\omega \neq \emptyset \right\} \max_{\tau \in I^l} \mu_h (\varphi^{n+i} (\omega X))$$

$$= \# \left\{ \tau \in I_{n+l} \mid \varphi^{n+i} (\omega X) \subset B_\omega \right\} \max_{\tau \in I^l} \mu_h (\varphi^{n+i} (\omega X))$$

$$\leq \# R_{n+l} (\omega) \max_{\tau \in I^l} \mu_h (\varphi^{n+i} (\omega X))$$

$$\leq \# R_{n+l} (\omega) C \kappa_{n+l}^h,$$

where the equation above follows from Claim 5.9. Therefore, we obtain that

$$\# R_{n+l} (\omega) \geq C^{-2} \left( \frac{e^{-S_{n_j} \beta (\omega \xi (n_i))}}{2 \kappa_{n+l}} \right)^h.$$ 

By redefining the constant $C$ we will write

$$\# R_{n+l} (\omega) \geq C^{-1} \left( \frac{e^{-S_{n_j} \beta (\omega \xi (n_i))}}{\kappa_{n+l}} \right)^h.$$ 

(5.10)

Notice that $R_{n+l} (\omega) \neq \emptyset$ if we choose our subsequence $(n_l)$ to increase rapidly enough; indeed,

$$\# R_{n+l} (\omega) \geq 1 \iff C^{-1} \left( \frac{e^{-S_{n_j} \beta (\omega \xi (n_i))}}{\kappa_{n+l}} \right)^h \geq 1$$

$$\iff \kappa_{n+l} \leq C^{-1/h} e^{-S_{n_j} \beta (\omega \xi (n_i))}$$

$$\iff \kappa_{n+l} \prod_{k=n+l}^{n+l} \kappa (k) \leq C^{-1/h} e^{-S_{n_j} \beta (\omega \xi (n_i))}$$

$$\iff e^{-(n+l-n_i) \theta} \leq C^{-1/h} e^{-S_{n_j} \beta (\omega \xi (n_i))}$$

$$\iff e^{-(n+l-n_i) \theta} \leq C^{-1/h} e^{-n_l \alpha}$$

$$\iff (n+l-n_i) \theta \geq \frac{1}{h} \log (C) + n_l \alpha$$
\[
\iff n_{t+1} \geq \frac{1}{\theta} \left[ \frac{1}{h} \log(C) + n_t(\theta + \alpha) \right] .
\]

Now for every \( \omega \in R_1 \) define
\[
m_1(B_\omega) = (\#R_1)^{-1} .
\]
Assuming that \( m_l(B_\omega) \) has been defined for every \( \omega \in R_l \) we now define for every \( \tau \in R_{t+1}(\omega) \)
\[
m_{t+1}(B_\tau) = \frac{m_l(B_\omega)}{\#R_{t+1}(\omega)}
\]
\[
= \left[ \prod_{k=1}^l (\#R_{k+1}(\omega|n_k))^{-1} \right] (\#R_1)^{-1} .
\]

We can extend the functions \( m_l \) to a measure on \( X \) and let us take a weak limit \( m \) of the sequence \( (m_l) \). The function \( m \) is then a Borel probability measure. Furthermore, notice that \( \text{supp}(m) \subset \text{supp}(m_l) = \bigcup_{\omega \in R_l} B_\omega(X) \) for all \( l \). This implies that
\[
K := \text{supp}(m)
\]
\[
= \bigcap_{l \geq 1} \bigcup_{\omega \in R_l} B_\omega(X) \subset \mathcal{D} .
\]

Hence, for \( \tau \in R_{t+1} \) we have that \( m(B_\tau) = m_{l+1}(B_\tau) \). Furthermore, from \( R_l \neq \emptyset \) it follows that \( K \neq \emptyset \).

For \( \tau \in R_{t+1}(\omega) \), the inequality (5.10) yields the following estimate for \( m(B_\tau) \):
\[
m(B_\tau) \leq \left[ \prod_{k=1}^l (\#R_{k+1}(\omega|n_k))^{-1} \right] (\#R_1)^{-1}
\]
\[
\leq \prod_{k=1}^l C \left( \frac{e^{-S_{n_k} \beta(\omega|n_k \xi(n_k))}}{\kappa_{nk+1}} \right)^{-h}
\]
\[
= C^l \pi_{nk+1}^h e^{hS_{nk} \beta(\omega|n_k \xi(n_k))} .
\]

Now consider \( x \in K \) and a number \( r \) such that \( 0 < r < \kappa_{n_1} e^{-n_1 \pi} \leq \min \{ e^{-S_{n_1} \beta(\xi(n_1))} : \omega \in I^{n_1} \} \).

Let
\[
\ell(r) := \min_{l \in \mathbb{N}} \left\{ l \mid \max_{\tau \in R_{t+1}} e^{-S_{n_1} \beta(\tau \xi(n_1))} \leq r \right\} ,
\]
and

\[ \# \ell(r+1) := \# \{ \tau \in R_{\ell(r)+1} \mid B_\tau \cap B(x, r) \neq \emptyset \} . \]

Since \( x \in K \subset \bigcup_{\tau \in R_{\ell(r)+1}} B_\tau \) it follows that \( |x - \phi^{n_{\ell(r)+1}}(\tau(x(n)))| \leq e^{-S_{n_{\ell(r)+1}}(\tau(x(n)))} \leq r \) for some \( \tau \in R_{\ell(r)+1} \). This implies that \( \phi^{n_{\ell(r)+1}}(\tau(x(n))) \in B(x, r) \) and it follows that \( \# \ell(r+1) \geq 1 \).

Recall that \( m \) is supported on \( K \subset \bigcup_{\tau \in R_{\ell(r)+1}} B_\tau \) and that \( m(B_\omega) = m(B_\bar{\omega}) \) for all words \( \omega, \bar{\omega} \in R_{l} \) of the same length, so for all \( \tau \in R_{\ell(r)+1} \) we have that

\[
m(B(x, r)) \leq \# \ell(r+1) \max_{\tau \in R_{\ell(r)+1}} m(B_\tau)
\]

\[
= \# \ell(r+1) m(B_\tau)
\]

\[
\leq \# \ell(r+1) C^{\ell(r)} \prod_{k=1}^{\ell(r)} \frac{h}{n_{k+1}} e^{h S_{n_k} \beta(\tau|_{n_k} \xi^{(n_k)})}
\]

\[
= \# \ell(r+1) C^{\ell(r)} \exp \left\{ h \sum_{k=1}^{\ell(r)} S_{n_k} \beta(\tau|_{n_k} \xi^{(n_k)}) \right\} \prod_{k=1}^{\ell(r)} \frac{h}{n_{k+1}}.
\]

We will use the following upper bound for \( \# \ell(r+1) \).

**Claim 5.13.** \( \# \ell(r+1) \leq C \left( \frac{r}{S_{n_{\ell(r)+1}}} \right)^h \).

**Proof of Claim.** Notice that if \( B_\tau \cap B(x, r) \neq \emptyset \) we have that \( B_\tau \subset B(x, 2r) \) since

\[
e^{-S_{n_{\ell(r)+1}} \beta(\tau(x(n)))} \leq r.
\]

From the Ahlfors condition (2.1) and from Claim 5.6 we get that

\[
C r^h \geq \mu_h(B(x, r))
\]

\[
\geq \# \{ \tau \in I^{n_{\ell(r)+1}} : \phi^{n_{\ell(r)+1}}(\tau(x(n))) \cap B(x, r) \neq \emptyset \} \min_{\tau \in I^{n_{\ell(r)+1}}} \mu_h(\phi^{n_{\ell(r)+1}}(\tau(x(n)))
\]

\[
\geq \# \{ \tau \in R_{\ell(r)+1} : \phi^{n_{\ell(r)+1}}(\tau(x(n))) \cap B(x, r) \neq \emptyset \} \min_{\tau \in I^{n_{\ell(r)+1}}} \mu_h(\phi^{n_{\ell(r)+1}}(\tau(x(n))))
\]

\[
\geq \# \{ \tau \in R_{\ell(r)+1} : B_\tau \cap B(x, r) \neq \emptyset \} \min_{\tau \in I^{n_{\ell(r)+1}}} \mu_h(\phi^{n_{\ell(r)+1}}(\tau(x(n))))
\]

36
By Frostman’s lemma it follows that there exists $\tau$ such that $\mu_h(\varphi^\tau_\ell(X))$ and $C^{-1}\#\ell(r)+1 \Sigma_{n(\ell(r)+1)}^h \mu_h$. The result follows by solving for $\#\ell(r)+1$.

From the previous claim we obtain that

$$m(B(x, r)) \leq C^{\ell(r)} (\frac{r}{n(\ell(r)+1)})^h \exp \left\{ h \sum_{k=1}^{\ell(r)} S_{n_k} \beta \left( \tau |_{n_k} \xi^{(n_k)} \right) \right\} \prod_{k=1}^{\ell(r)} \kappa_{n_{k+1}}^r \leq \text{const} \cdot r^\ell(r)$$

By Frostman’s lemma it is enough to show that there exists $\tau \in R_{\ell(r)+1}$ for which

$$C^{\ell(r)} \left( \frac{\tau}{n(\ell(r)+1)} \right)^h \exp \left\{ h \sum_{k=1}^{\ell(r)} S_{n_k} \beta \left( \tau |_{n_k} \xi^{(n_k)} \right) \right\} \prod_{k=1}^{\ell(r)-1} \kappa_{n_{k+1}}^r \leq \text{const} \cdot r^{\ell(r)}$$

holds for some $\tau \in R_{\ell(r)+1}$. From the definition of $\ell_r$ it follows that $\exp \left\{ -S_{n_{\ell(r)}} \beta \left( \tau |_{n_{\ell(r)}} \xi^{(n_{\ell(r)})} \right) \right\} > r$ for some $\tau \in R_{\ell(r)+1}$. By comparing (3.1) and (5.1) we see that $t < b \leq \text{HD}(J) = h$, so that $\frac{t}{h} < 1$. Hence, we have that

$$\exp \left\{ \left( 1 - \frac{t}{h} \right) S_{n_{\ell(r)}} \beta \left( \tau |_{n_{\ell(r)}} \xi^{(n_{\ell(r)})} \right) \right\} < r^\ell(r),$$

for some $\tau \in R_{\ell(r)+1}$. So it suffices to show that

$$C^{\ell(r)} \left( \frac{\kappa_{n_{\ell(r)+1}}^r}{n(\ell(r)+1)} \right)^h \exp \left\{ h \sum_{k=1}^{\ell(r)} S_{n_k} \beta \left( \tau |_{n_k} \xi^{(n_k)} \right) \right\} \prod_{k=1}^{\ell(r)-1} \kappa_{n_{k+1}}^r \leq \text{const} \cdot \exp \left\{ \left( 1 - \frac{t}{h} \right) S_{n_{\ell(r)}} \beta \left( \tau |_{\ell(r)} \xi^{(\ell(r))} \right) \right\}$$

for some $\tau \in R_{\ell(r)+1}$, which is equivalent to showing that

$$C^{\ell(r)} \left( \frac{\kappa_{n_{\ell(r)+1}}^r}{n(\ell(r)+1)} \right)^h \exp \left\{ h \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \left( \tau |_{n_k} \xi^{(n_k)} \right) \right\} \prod_{k=2}^{\ell(r)} \kappa_{n_k} \leq \text{const} \cdot \exp \left\{ -\frac{t}{h} S_{n_{\ell(r)}} \beta \left( \tau |_{\ell(r)} \xi^{(n_{\ell(r)})} \right) \right\}$$

holds for some $\tau \in R_{\ell(r)+1}$.
Since $\mathcal{P}_\beta(t) > 0$ we have (by choosing $n_1$ large enough if necessary) that
\[
\frac{1}{n_{\ell(r)}} \log \sum_{\omega \in I_{\ell(r)}} \exp \left\{ -tS_{n_{\ell(r)}} \beta \left( \omega \xi^{(n_{\ell(r)})} \right) \right\} \geq \frac{3}{4} \mathcal{P}_\beta(t),
\]
which implies that
\[
\sum_{\omega \in I_{\ell(r)}} \exp \left\{ -tS_{n_{\ell(r)}} \beta \left( \omega \xi^{(n_{\ell(r)})} \right) \right\} \geq \exp \left\{ \frac{n_{\ell(r)}}{2} \mathcal{P}_\beta(t) \right\}.
\]
By defining $n_1$ to be large enough if necessary it follows from inequality (5.5) that for any $\tau \in R_{\ell(r)+1}$
\[
\#I_{n(\ell(r))} \exp \left\{ -tS_{n_{\ell(r)}} \beta \left( \tau|I_{\ell(r)} \right) \xi^{(n_{\ell(r)})} \right\} \geq \sum_{\omega \in I_{\ell(r)}} \exp \left\{ -tS_{n_{\ell(r)}} \beta \left( \omega \xi^{(n_{\ell(r)})} \right) \right\}.
\]
Combining the last two inequalities we get that it suffices to show that
\[
\exp \left\{ -\frac{t}{\hbar} S_{n_{\ell(r)}} \beta \left( \tau|I_{\ell(r)} \right) \xi^{(n_{\ell(r)})} \right\} \geq (\#I_{n(\ell(r))})^{-1/\hbar} \exp \left\{ -\frac{t}{\hbar} \varepsilon n_{\ell(r)} \right\} \exp \left\{ \frac{3n_{\ell(r)}}{4\hbar} \mathcal{P}_\beta(t) \right\}.
\]
This estimate yields the further sufficient condition
\[
C^{\ell(r)} \left( \frac{K_{n_{\ell(r)}+1}}{K_{n_{\ell(r)}+1}} \right) \exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \right\} \prod_{k=2}^{\ell(r)} K_{n_k} \leq \const \cdot \exp \left\{ \frac{3n_{\ell(r)}}{4\hbar} \mathcal{P}_\beta(t) \right\} \exp \left\{ -\frac{t}{\hbar} \varepsilon n_{\ell(r)} \right\} (\#I_{n(\ell(r))})^{-1/\hbar}
\]
for some $\tau \in R_{\ell(r)+1}$.

If we choose $\varepsilon$ such that $0 < \varepsilon < \frac{\mathcal{P}_\beta(t)}{4\hbar}$ then it suffices to show that
\[
C^{\ell(r)} \left( \frac{K_{n_{\ell(r)}+1}}{K_{n_{\ell(r)}+1}} \right) \exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \right\} \prod_{k=2}^{\ell(r)} K_{n_k} \leq \const \cdot \exp \left\{ \frac{n_{\ell(r)}}{2\hbar} \mathcal{P}_\beta(t) \right\} (\#I_{n(\ell(r))})^{-1/\hbar}
\]
for some $\tau \in R_{\ell(r)+1}$.

Now, since supp $(\mu_h) = J \subseteq \bigcup_{\omega \in I_{\ell(r)}} \varphi_\omega(X)$ we have that
\[
1 = \sum_{\omega \in I_{\ell(r)}} \mu_h \left( \varphi_\omega(X) \right) \geq C^{-1} (\#I_{n(\ell(r))}) K_{n(\ell(r))}^h,
\]
which yields the inequality
\[
(\#I_{n(\ell(r))})^{-1/\hbar} \geq C^{-1/\hbar} K_{n(\ell(r))}^h.
\]
Hence, it is enough to show that for some $\tau \in R_{\ell(r)+1}$

\begin{equation}
C^{\ell(r)} \left( \frac{\overline{r}_{n_{\ell(r)+1}}}{\overline{r}_{n_{\ell(r)+1}}} \right) \exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \left( \tau |_{n_k} \zeta^{(n_k)} \right) \prod_{k=2}^{\ell(r)-1} \kappa_{n_k} \right\} \leq \text{const} \cdot \exp \left\{ \frac{n_{\ell(r)} P_\beta}{2h} (t) \right\}.
\end{equation}

Since the sequence $\left( \frac{\overline{r}_n}{\kappa_n} \right)$ is bounded, this inequality follows by showing

\begin{equation}
C^{\ell(r)} \exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \left( \tau |_{n_k} \zeta^{(n_k)} \right) \prod_{k=2}^{\ell(r)-1} \kappa_{n_k} \right\} \leq \text{const} \cdot \exp \left\{ \frac{n_{\ell(r)} P_\beta}{4h} (t) \right\} \cdot \exp \left\{ \frac{n_{\ell(r)} P_\beta}{4h} (t) \right\}.
\end{equation}

Furthermore, it is enough to show that

\begin{equation}
\exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \left( \tau |_{n_k} \zeta^{(n_k)} \right) \prod_{k=2}^{\ell(r)-1} \kappa_{n_k} \right\} \leq \text{const} \cdot \exp \left\{ \frac{n_{\ell(r)} P_\beta}{4h} (t) \right\}
\end{equation}

for some $\tau \in R_{\ell(r)+1}$ and that

\begin{equation}
C^{\ell(r)} \leq \exp \left\{ \frac{n_{\ell(r)} P_\beta}{4h} (t) \right\}.
\end{equation}

The first inequality is satisfied given condition (5.8). The second inequality is satisfied by choosing our rapidly increasing sequence $(n_l)$ to satisfy $n_l \gg l$. This completes the proof. \hfill \Box

5.2.4. Lower Bound for Subexponential Growth of Derivative Ratios

If we also assume continuity of the pressure function at $\delta$, we can relax the condition that the sequence $\left( \frac{\overline{r}_n}{\kappa_n} \right)$ is bounded.

**Theorem 18.** Let $\Phi$ be a conformal nonautonomous i.f.s. satisfying the OSC, ESC, UCC, LVC, and NEQ conditions. If the sequence $(\beta_n)$ is tame, there exists an $h$-Ahlfors measure supported on $J$, $\overline{P}(t) = P(t)$ on a neighborhood of $b$, and

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \log \frac{\kappa_n}{\overline{\kappa}_n} = 0,
\end{equation}

then $HD(\mathcal{D}) = b$. 

39
Proof. As before, we choose 0 \leq t < b in the neighborhood of b where \( P_\beta \) exists. It suffices to show that inequality (5.14) holds. This will follow from showing that the following three inequalities hold for some \( \tau \in R_{\ell(r)+1} \):

\[
(5.16) \quad C^{\ell(r)} \exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \left( \tau | n_k \xi^{(n_k)} \right) \right\} \prod_{k=2}^{\ell(r)-1} \kappa_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n_{\ell(r)}}{6h} P_\beta (t) \right\},
\]

\[
\frac{\kappa_{n_{\ell(r)}}}{\kappa_{n_{\ell(r)}}} \leq \exp \left\{ \frac{n_{\ell(r)}}{6h} P_\beta (t) \right\},
\]

and

\[
\frac{\kappa_{n_{\ell(r)+1}}}{\kappa_{n_{\ell(r)+1}}} \leq \exp \left\{ \frac{n_{\ell(r)}}{6h} P_\beta (t) \right\}.
\]

The second inequality is equivalent to the inequality

\[
\frac{1}{n_{\ell(r)}} \log \frac{\kappa_{n_{\ell(r)}}}{\kappa_{n_{\ell(r)}}} \leq \frac{P_\beta (t)}{6h},
\]

which is satisfied simply by choosing \( n_1 \) large enough. This can be achieved without loss of generality since \( P_\beta (t) > 0 \) and by assumption (5.15).

To prove the third inequality we first note that it is equivalent to

\[
\frac{n_{\ell(r)+1}}{n_{\ell(r)}} \frac{1}{n_{\ell(r)+1}} \log \frac{\kappa_{n_{\ell(r)}}}{\kappa_{n_{\ell(r)}}} \leq \frac{P_\beta (t)}{6h}.
\]

Let

\[
0 < A \leq \frac{P_\beta (t)}{6h} \alpha (\log C + \alpha)^{-1},
\]

where \( C \) is the same constant as in (2.1). Since \( \overline{P}_\beta (t) = P_\beta (t) \) we have that (5.7) holds for every increasing sequence \( (n_l) \).

Consider in particular an increasing sequence with the property

\[
(5.17) \quad n_{l+1} = \min \{ n \in \mathbb{N} \mid nA \geq \alpha (n_1 + \cdots + n_l) \},
\]

for all \( l \in \mathbb{N} \).

Such a sequence satisfies the following claim.
CLAIM 5.18. The following inequality holds:

\[ A^{-1} \alpha \leq \frac{n_{t+1}}{n_t} \leq A^{-1} \alpha + 2. \]

PROOF. Condition 5.17 implies that

\[ A(n_{t+1} - 1) \leq \alpha (n_1 + \cdots + n_l) \leq An_{t+1}. \]

Therefore,

\[ A(n_{t+1} - n_l - 1) \leq \alpha n_l \leq A(n_{t+1} - n_l + 1) \]

Re-arranging terms algebraically we get

\[ \alpha n_l A^{-1} - 1 \leq n_{t+1} - n_l \leq \alpha n_l A^{-1} + 1, \]

\[ n_l + \alpha n_l A^{-1} - 1 \leq n_{t+1} \leq n_l + \alpha n_l A^{-1} + 1, \]

\[ \frac{\alpha A^{-1} + 1 - \frac{1}{n_l}}{n_l} \leq \frac{n_{t+1}}{n_l} \leq \frac{\alpha A^{-1} + 1 + \frac{1}{n_l}}{n_l}, \]

\[ \frac{\alpha A^{-1}}{n_l} \leq \frac{n_{t+1}}{n_l} \leq \alpha A^{-1} + 2. \]

This proves the claim. □

Since our sequence is chosen so that \( \frac{n_{t+1}}{n_l} \) is uniformly bounded, the desired inequality

\[ \frac{n_{\ell(r)+1}}{n_{\ell(r)}} \frac{1}{n_{\ell(r)+1}} \log \frac{\kappa_{n_{\ell(r)+1}}}{\kappa_{n_{\ell(r)+1}}} \leq \frac{P_\beta (t)}{6h} \]

follows again by choosing \( n_1 \) large enough.

The remaining inequality

\[ C^{\ell(r)} \exp \left\{ \sum_{k=1}^{\ell(r)-1} S_{n_k} \beta \left( \tau | n_k \xi^{(n_k)} \right) \right\} \prod_{k=2}^{\ell(r)-1} \kappa_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n_{\ell(r)}}{6h} P_\beta (t) \right\} \]

is equivalent to showing

\[ \ell (r) \log (C) + \sum_{k=1}^{\ell(r)-1} \left( S_{n_k} \beta \left( \tau | n_k \xi^{(n_k)} \right) + \log \kappa_{n_k} \right) \leq \text{const} + \frac{n_{\ell(r)}}{6h} P_\beta (t). \]
Given ESC, it suffices to show
\[ \ell(r) \log(C) + \sum_{k=1}^{\ell(r)-1} n_k \alpha \leq \text{const} + \frac{n(\ell(r))}{6h} P_\beta(t) . \]

Since \( \text{const} > 0 \), this inequality follows from showing
\[ n(\ell(r)) \geq \frac{6h}{P_\beta(t)} \left[ \ell(r) \log(C) + \sum_{k=1}^{\ell(r)-1} n_k \alpha \right] . \]

In view of Claim 5.18, we have that \( n_l \geq A^{-1} \alpha (n_1 + \cdots + n_{l-1}) \) for all \( l \). Now, condition (5.19) holds if
\[ A^{-1} \alpha \sum_{k=1}^{l-1} n_k \geq \frac{6h}{P_\beta(t)} \left[ l \cdot \log(C) + \sum_{k=1}^{l-1} n_k \alpha \right] . \]
or, re-arranging terms, if
\[ A^{-1} \geq \frac{6h}{P_\beta(t)} \left[ \frac{l}{n_1 + \cdots + n_{l-1}} \cdot \frac{\log(C)}{\alpha} + \frac{\alpha}{\alpha} \right] . \]

Since the sequence \( (n_l) \) is increasing and assuming without loss of generality that \( n_1 \geq 2 \), we have that \( l \leq n_1 + \cdots + n_{l-1} \) for all \( l \). Hence, it suffices to show that
\[ A^{-1} \geq \frac{6h}{P_\beta(t)} \left[ \frac{\log(C) + \alpha}{\alpha} \right] . \]

This follows from our choice of \( A \) above.

Since all three inequalities in (5.16) hold, this completes the proof. \( \square \)

5.3. Perturbations in Dimension 1

Let \( X = [0, 1] \) and consider a piecewise linear nonautonomous i.f.s. \( \Phi = \{ \varphi^{(n)}_e \}_{n \in \mathbb{N}, e \in I^{(n)}} \).

Now consider a nonlinear perturbative system \( \tilde{\Phi} = \{ \tilde{\varphi}^{(n)}_e \}_{n \in \mathbb{N}, e \in I^{(n)}} \) satisfying
\[ \tilde{\varphi}^{(n)}_e(x) = \varphi^{(n)}_e(u^{(n)}_e) + \left( \varphi^{(n)}_e \right)' \int_{u^{(n)}_e}^x \left( 1 + \gamma^{(n)}_e(t) \right) dt , \]
where \( u^{(n)}_e \in [0, 1] \) and \( \gamma^{(n)}_e : [0, 1] \to (-\varepsilon_n, \varepsilon_n) \) is Hölder continuous and \( \varepsilon_n > 0 \) is independent of \( e \in I^{(n)} \). Our goal is to establish sufficient conditions on the functions \( \gamma^{(n)}_e \) for which the system \( \{ \tilde{\varphi} \} \) satisfies the hypothesis of Theorem 17 or 18.
Observe that

\begin{equation}
\left| (\tilde{\varphi}^{(n)}_e)'(x) \right| = \left| (\varphi^{(n)}_e)' \left| 1 + \gamma^{(n)}_e(x) \right| \right.,
\end{equation}

and

\begin{align*}
\left| (\tilde{\varphi}^{(n)}_\omega)'(x) \right| &= \prod_{k=1}^{n} (\tilde{\varphi}^{(k)}_\omega)'(\tilde{\varphi}^{(k+1, n)}(x)) \\
&= \left| (\varphi^{(n)}_\omega)' \prod_{k=1}^{n} \left| 1 + \gamma^{(k)}(\varphi^{(k+1, n)}_{\sigma_k \omega}(x)) \right| \right. \\
&\leq \kappa_n \prod_{k=1}^{n} \left| 1 + \gamma^{(k)}(\varphi^{(k+1, n)}_{\sigma_k \omega}(x)) \right|.
\end{align*}

Now define

\[
\overline{\gamma}^{(n)} = \max_{e \in I^{(n)}} \sup_{x \in [0, 1]} \{ |\gamma^{(n)}_e(x)| \}.
\]

Then we have that for all \( x \in [0, 1] \)

\[
\kappa_n \prod_{k=1}^{n} \left[ 1 - \overline{\gamma}^{(k)} \right] \leq \left| (\tilde{\varphi}^{(n)}_\omega)'(x) \right| \leq \kappa_n \prod_{k=1}^{n} \left[ 1 + \overline{\gamma}^{(k)} \right].
\]

Now we impose some conditions on \( \epsilon \) that will guarantee \( \{ \tilde{\varphi} \} \) to satisfy the OSC. Let \( g^{(n)} \) be the size of the smallest “gap” between images under the unperturbed system \( \Phi \) at level \( n \), i.e.,

\begin{equation}
(5.2) \quad g^{(n)} := \min \left\{ \left| \varphi^{(n)}_j(x) - \varphi^{(n)}_i(y) \right| : x, y \in [0, 1] ; j, i \in I^{(n)}, j \neq i \right\}.
\end{equation}

We will assume \( \Phi \) has the strong separation condition, i.e., that \( g^{(n)} > 0 \) for all \( n \).

**Lemma 19.** If \( 0 < \varepsilon_n < \frac{g^{(n)}}{2\kappa^{(n)}} \) for all \( n \), then \( \tilde{\Phi} \) has the strong separation condition.

**Proof.** Observe that

\[
\left| \varphi^{(n)}_j(x) - \varphi^{(n)}_j(x) \right| \\
= \left| \varphi^{(n)}_j(u^{(n)}_j) + \left( (\varphi^{(n)}_j)' \right) \int_{u^{(n)}_j}^{x} \left( 1 + \gamma^{(n)}_j(t) \right) dt - \varphi^{(n)}_j(u^{(n)}_j) - \int_{u^{(n)}_j}^{x} \left( (\varphi^{(n)}_j)'(t) \right) dt \right| \\
\leq \left| \left( (\varphi^{(n)}_j)' \right)(1 + \varepsilon_n)(x - u^{(n)}_j) - (\varphi^{(n)}_j)'(x - u^{(n)}_j) \right| \\
\leq (1 + \varepsilon_n) \left| (\varphi^{(n)}_j)' \right| \left| x - u^{(n)}_j \right|.
\]

43
\[
\leq \varepsilon_n \left| \left( \varphi_j^{(n)} \right)' \right| \\
\leq \varepsilon_n \kappa_{(n)} \\
< \frac{g(n)}{2}.
\]

Note that the right hand side is independent of \( j \in I^{(n)} \). Now, it is an elementary fact in analysis that \( |a + b + c| \geq |a| - |b| - |c| \) for all \( a, b, c \in \mathbb{R} \). Using this inequality we show that

\[
\left| \tilde{\varphi}_j^{(n)}(x) - \tilde{\varphi}_i^{(n)}(y) \right| = \left| \varphi_j^{(n)}(x) - \varphi_i^{(n)}(y) \right| - \left| \tilde{\varphi}_j^{(n)}(x) - \tilde{\varphi}_i^{(n)}(y) \right| \\
\geq \varphi_j^{(n)}(x) - \varphi_i^{(n)}(y) - \varphi_j^{(n)}(x) + \varphi_i^{(n)}(y) \\
> g(n) - \frac{g(n)}{2} - \frac{g(n)}{2} \\
= 0.
\]

for all \( j \neq i \) and all \( x, y \in [0, 1] \). \( \square \)

Furthermore, define

\[
\bar{\kappa}_{(n)} = \max_{e \in I^{(n)}} \sup_{x \in (0, 1)} \left| (\tilde{\varphi}_e^{(n)})'(x) \right| \\
\tilde{\kappa}_{(n)} = \min_{e \in I^{(n)}} \inf_{x \in (0, 1)} \left| (\tilde{\varphi}_e^{(n)})'(x) \right| \\
\kappa_{(n)} = \max_{\omega \in I^n} \sup_{x \in (0, 1)} \left| (\varphi_\omega^{(n)})'(x) \right| \\
\tilde{\kappa}_{n} = \min_{\omega \in I^n} \inf_{x \in (0, 1)} \left| (\varphi_\omega^{(n)})'(x) \right|.
\]

Observe that

\[
\prod_{j=1}^n \tilde{\kappa}_{(j)} \leq \tilde{\kappa}_{n} \leq \kappa_{n} \leq \prod_{j=1}^n \kappa_{(j)}.
\]

Now,

\[
\frac{\bar{\kappa}_{n}}{\tilde{\kappa}_{n}} \leq \frac{\kappa_{n}}{\tilde{\kappa}_{n}} \prod_{k=1}^n \frac{1 + \gamma(k)}{1 - \gamma(k)}.
\]

44
Since \(1 - \gamma^{(k)} \geq 1 - \epsilon_k > 0\) it follows that \(\tilde{\kappa}_{(n)} > 0\) for all \(n\). From this estimate we see that the sequence \(\left(\frac{\tilde{\kappa}_n}{\xi_n}\right)\) is bounded if \(\left(\frac{\tilde{\kappa}_n}{\xi_n}\right)\) is bounded and if \(\sup_{n \geq 1} \prod_{k=1}^{n} \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}} < \infty\).

Note that

\[
\sup_{n \geq 1} \prod_{k=1}^{n} \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}} < \infty \iff \sum_{k \geq 1} \log \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}} < \infty
\]

\[
\iff \sum_{k \geq 1} \gamma^{(k)} < \infty,
\]

where the last step follows from the limit comparison test in calculus.

Now we turn to the ESC.

**Lemma.** If \(\Phi\) has the ESC with lower constant \(\alpha > 0\) and if \(\sup_{n \geq 1} \varepsilon_n < 1 - e^{-2}\), then \(\tilde{\Phi}\) satisfies the ESC.

**Proof.** Define \(\varepsilon := \sup_{n \geq 1} \varepsilon_n\) and assume \(\varepsilon < 1 - e^{-2}\). This implies that \(\alpha + \log (1 - \varepsilon) > 0\).

Choose \(0 < \tilde{\alpha} \leq \alpha + \log (1 - \varepsilon)\). We will show that

\[
\tilde{\alpha} \leq \beta_n (\xi) + \log \tilde{\kappa}_{(n)}
\]

for all \(n \in \mathbb{N}\) and all \(\xi \in I^{(n, \infty)}\). Indeed,

\[
\tilde{\alpha} \leq \alpha + \log (1 - \varepsilon)
\]

\[
\leq \beta_n (\xi) + \log [(1 - \varepsilon) \tilde{\kappa}_{(n)}]
\]

\[
\leq \beta_n (\xi) + \log [(1 - \varepsilon_n) \tilde{\kappa}_{(n)}]
\]

\[
\leq \beta_n (\xi) + \log \tilde{\kappa}_{(n)},
\]

where the last inequality follows from (5.1).

Hence, we have the following

**Theorem 20.** Suppose that \(\Phi\) is a nonautonomous i.f.s. of linear functions satisfying the hypothesis of Theorem 17 and that \(\left(\gamma^{(n)} (x)\right)_{n \in \mathbb{N}, e \in I^{(n)}}\) is a sequence of Hölder-continuous
functions from $[0, 1]$ into $(-\varepsilon, \varepsilon)$ for some $\varepsilon < 1 - e^{-\alpha}$. Furthermore, let $\tilde{\Phi}$ be a nonlinear perturbation of $\Phi$ defined as

$$\tilde{\varphi}_e^{(n)}(x) = \varphi_e^{(n)}(u_e^{(n)}) + \left((\varphi_e^{(n)})'\right) \int_{u_e^{(n)}}^x (1 + \gamma_e^{(n)}(t)) \, dt.$$ 

If

$$\sum_{k \geq 1} \gamma^{(k)} < \infty,$$

and either

(11a) $\tilde{\varphi}_e^{(n)}([0, 1]) \subset \varphi_e^{(n)}([0, 1])$, or

(11b) $0 < \gamma^{(n)} < \frac{g(n)}{2\gamma^{(n)}}$, 

then $\tilde{\Phi}$ satisfies the hypothesis of Theorem 17.

We wish to formulate a similar theorem for perturbed systems corresponding to Theorem 18. If we now assume that the ESC and (5.15) hold, then we see that

$$0 \leq \frac{1}{n} \log \frac{\bar{K}_n}{\underline{K}_n} \leq \frac{1}{n} \log \frac{\bar{K}_n}{\underline{K}_n} + \frac{1}{n} \log \prod_{k=1}^n \left(1 + \gamma^{(k)}\right) - \frac{1}{n} \log \prod_{k=1}^n \left(1 - \gamma^{(k)}\right) \sim \frac{1}{n} \log \frac{\bar{K}_n}{\underline{K}_n} + \frac{1}{n} \sum_{k=1}^n \gamma^{(k)}.$$

From (5.15) it suffices to have $\frac{1}{n} \sum_{k=1}^n \gamma^{(k)} = 0$, which holds whenever

$$\lim_{k \to \infty} \gamma^{(k)} = 0.$$

**Theorem 21.** Suppose that $\Phi$ is a nonautonomous i.f.s. of linear functions satisfying the hypothesis of Theorem 18 and that $\left(\gamma_e^{(n)}(x)\right)_{n \in \mathbb{N}}$ is a sequence of Hölder-continuous functions from $[0, 1]$ into $(-\varepsilon, \varepsilon)$ for some $\varepsilon < 1 - e^{-\alpha}$. Furthermore, let $\tilde{\Phi}$ be a nonlinear perturbation of $\Phi$ defined as

$$\tilde{\varphi}_e^{(n)}(x) = \varphi_e^{(n)}(u_e^{(n)}) + \left((\varphi_e^{(n)})'\right) \int_{u_e^{(n)}}^x (1 + \gamma_e^{(n)}(t)) \, dt.$$ 

If

$$\frac{1}{n} \sum_{k=1}^n \gamma^{(k)} \to 0,$$

in particular, if $\lim_{k \to \infty} \gamma^{(k)} = 0$, and either
(13a) \( \tilde{\varphi}_e^{(n)}([0,1]) \subset \varphi_e^{(n)}([0,1]) \), or

(13b) \( 0 < \gamma(n) < \frac{g(n)}{2\rho(n)} \),

then \( \tilde{\Phi} \) satisfies the hypothesis of Theorem 18.

**Remark.** We emphasize that if \( \Phi \) is a linear autonomous system satisfying the hypothesis of Theorem 17 or Theorem 18, then we can construct via small perturbations an abundance of non-autonomous systems satisfying our hypotheses.

5.4. Perturbations in Dimension 2

In a way similar to Section 5.3 we will now consider perturbations of similarities in dimension 2.

Consider a closed, convex subset \( X \subseteq \mathbb{R}^2 \) with non-empty interior. Let \( \Phi = \left( \varphi_a^{(n)} \right)_{n \in \mathbb{N}, a \in I^{(n)}} \) be an i.f.s. where each \( \varphi_a^{(n)} \) has a holomorphic extension to some open set \( V \) containing \( X \) and such that \( \varphi_a^{(n)}(V) \subseteq V \).

Now we define a perturbed i.f.s. \( \tilde{\Phi} = \left( \tilde{\varphi}_a^{(n)} \right)_{n \in \mathbb{N}, a \in I^{(n)}} \) in the following way. For each \( n \in \mathbb{N} \) and \( a \in I^{(n)} \) we consider a holomorphic function \( \gamma_a^{(n)} : X \to (-\varepsilon_n, \varepsilon_n) \) and \( u_a^{(n)} \in X \).

The perturbed system \( \tilde{\Phi} \) is then defined by

\[
\tilde{\varphi}_a^{(n)}(x) = \varphi_a^{(n)}(u_a^{(n)}) + \int_\Gamma \left[ 1 + \gamma_a^{(n)}(z) \right] D\varphi_a^{(n)}(z) \, dz,
\]

where \( \Gamma \) is any rectifiable path in \( X \) from \( u_a^{(n)} \) to \( x \). Since \( X \) is assumed to be convex we may take \( \Gamma(t) = (1-t)u_a^{(n)} + tx \).

As before,

\[
\left| D\tilde{\varphi}_e^{(n)}(x) \right| = \left| D\varphi_e^{(n)} \right| \left| 1 + \gamma_a^{(n)}(x) \right|.
\]

Defining \( g(n) \) as in (5.2) it is easy to check, following the proof of Lemma 19, that \( \tilde{\Phi} \) has the strong separation condition if \( 0 < \varepsilon_n < \frac{g_n}{2\rho(n) \text{diam}(X)} \).

The rest of the analysis follows exactly as in the case of dimension 1, which gives us analogs to Theorems 20 and 21 for the case \( X \subseteq \mathbb{R}^2 \).
REFERENCES


[12] Yakov Pesin and Howard Weiss, *On the dimension of deterministic and random Cantor-


