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TRANSPORT AND ISOMORPHIC EQUILIBRIA

BY

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# Transport and Isomorphic Equilibria

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## ABSTRACT

It is shown that large classes of plasma equilibria can have identical drift orbits and associated transport. Such equilibria are named *isomorphic*. In particular, the neoclassical transport coefficients are given for all equilibria in which the magnetic field strength depends on one helicity.

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## I. INTRODUCTION

Plasmas in different magnetic equilibria can have identical drift orbits and associated transport.<sup>1</sup> In some sense, these equilibria have the same structure and we will call them isomorphic. An example of a set of isomorphic equilibria is the symmetric torus, the straight helix, and the straight elliptical cylinder. Generally, the isomorphism between two equilibria is apparent only in magnetic coordinates. Indeed, it is the lack of uniqueness of the magnetic coordinates<sup>2</sup> coupled with the fact that drift motion is essentially determined by the magnetic field strength<sup>3</sup> which makes isomorphic equilibria possible. In this paper a method of demonstrating isomorphisms in systems with two periodic coordinates is developed. A symmetric coordinate, like the  $Z$  symmetry of a cylinder, can be viewed as a periodic coordinate. A general treatment of drift orbits and transport in the long mean free path limit is also given for equilibria in which the magnetic field strength depends on only two variables.

## II. ISOMORPHIC TRANSFORMATION

The magnetic field associated with a general scalar pressure equilibrium can be written<sup>3</sup>

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta_0 \quad (1)$$

$$= \vec{\nabla}\chi + \beta \vec{\nabla}\psi \quad (2)$$

The plasma equilibrium is assumed to have two periodicities. Topologically, this is a torus. If we let  $2\pi\psi$  be the magnetic flux enclosed by a pressure surface, the toroidal flux, then poloidal and toroidal angles  $\theta$  and  $\phi$  can be

defined<sup>2</sup> so

$$\chi = g(\psi)\phi + I(\psi)\theta, \quad (3)$$

$$\theta = \theta_0 + \tau(\psi)\phi. \quad (4)$$

It can be shown<sup>2</sup> that  $cg/2$  is the poloidal current outside and  $cI/2$  the toroidal current inside a constant pressure surface. The rotational transform of the field lines is  $\tau$ .

The most important quantity for determining particle drift orbits is the magnetic field strength. In addition, the field strength is central to the spacial Jacobian for

$$(\vec{\nabla}\psi \times \vec{\nabla}\theta_0) \cdot \vec{\nabla}\chi = B^2, \quad (5)$$

$$(\vec{\nabla}\psi \times \vec{\nabla}\theta) \cdot \vec{\nabla}\phi = \frac{B^2}{g + \tau I}. \quad (6)$$

In showing two systems are isomorphic, the most important feature is showing that the magnetic field strength is in some sense the same function of the magnetic coordinates.

Due to the periodicities, one can write<sup>2</sup>

$$\frac{1}{B^2} = \frac{1}{B_0^2} \left\{ 1 + \sum'_{n,m} \delta_{nm} \cos(n\phi - m\theta + \lambda_{nm}) \right\}, \quad (7)$$

with  $B_0$ ,  $\delta_{nm}$ , and  $\lambda_{nm}$  functions of  $\psi$  and the prime on the sum implying the  $n = 0, m = 0$  term is eliminated. Suppose  $1/B^2$  contained only two helicities. Let

$$\bar{\theta} = m\theta - n\phi, \quad (8)$$

$$\bar{\phi} = N\phi - M\theta. \quad (9)$$

Then with only two helicities,  $1/B^2$  can be written

$$\begin{aligned} \frac{1}{B^2} &= \frac{1}{B_0^2} \left\{ 1 - 2 \sum_{j \neq 0} [\epsilon_j \cos(j\bar{\theta} + \lambda_j) + \delta_j \cos(j\bar{\phi} + \Lambda_j)] \right\} \\ &= \frac{1}{B_0^2} [1 - 2\epsilon(\psi) p(\psi, \bar{\theta}) - 2\delta(\psi) P(\psi, \bar{\phi})], \end{aligned} \quad (10)$$

with  $p(\psi, \bar{\theta})$  and  $P(\psi, \bar{\phi})$  periodic functions with a period  $2\pi$  in  $\bar{\theta}$  and  $\bar{\phi}$ . The field strength in Eq. (10) looks like that in a tokamak with one toroidal ripple. To show that all two helicity fields are reducible to this form, we must show that one can find a  $\bar{\psi}$ ,  $\bar{\theta}_0$ ,  $\bar{\chi}$ ,  $\bar{\theta}$ ,  $\bar{g}$ ,  $\bar{I}$ , and  $\bar{r}$  such that Eqs. (1) through (4) are satisfied in the bar variables. It is easily verified that Eqs. (1) through (4) are satisfied by

$$\begin{aligned} \frac{d\bar{\psi}}{d\psi} &= \frac{N - rM}{mN - Mn} & \bar{\theta}_0 &= \frac{mN - Mn}{N - rM} \theta_0 \\ \bar{\chi} &= \chi & \bar{\theta} &= \frac{mN - Mn}{N - rM} \theta \\ & & & (11) \end{aligned}$$

$$\bar{g} = \frac{nI + mg}{mN - Mn} \quad \bar{I} = \frac{NI + Mg}{mN - Mn}$$

$$\bar{r} = \frac{rM - n}{N - rM}$$

The field strength written in terms of the bared variables will be called the standard form. Since all two helicity fields can be written in one standard form, they are all potentially isomorphic. With three or more helicities, any two helicities can be reduced to the standard form, but not all poloidal and toroidal mode numbers can be eliminated. Consequently, isomorphisms become rare in systems with more than two helicities.

Two special cases of the isomorphic transformations, Eq. (11), are the identity transformation ( $N = 1, M = 0, n = 0, m = 1$ ) and the transformation which switches toroidal and poloidal quantities ( $N = 0, M = -1, n = -1, m = 0$ ).

### III. ONE-HELICITY SYSTEMS

To demonstrate the power of the concept of isomorphisms, we will study systems with one helicity. Systems with one helicity include the torus, the elliptical cylinder, and the helix. The field strength for all one-helicity systems can be written as

$$\frac{1}{B^2} = \frac{1}{B_0^2} [1 + 2\epsilon p(\bar{\theta})] , \quad (12)$$

with  $\bar{\theta} = m\theta - n\phi$ . We assume  $p(\bar{\theta})$  is normalized so

$$\int_0^{2\pi} p^2(\bar{\theta}) d\bar{\theta} = \pi . \quad (13)$$

The  $\psi$  dependence has been suppressed since it will be found to play more the role of a parameter than a variable. Since we have only one helicity, the transformation equations can be simplified by letting  $N = 1, M = 0$ ;

so  $\phi = \bar{\phi}$ . It is assumed  $m \neq 0$ . If  $m$  were zero, the poloidal and toroidal coordinates must be switched first. The transformation then simplifies to

$$\begin{aligned}\bar{\psi} &= \frac{\psi}{m} & \bar{\theta}_0 &= m\theta_0 \\ \bar{\chi} &= \chi & \bar{\beta} &= m\beta \\ \bar{g} &= g + \frac{n}{m} I & \bar{I} &= \frac{I}{m} \\ \bar{r} &= rm - n.\end{aligned}\tag{14}$$

To simplify the notation, bars will be dropped with the assumption the field has been transformed to the standard form.

The first result we wish to establish is that all systems with one helicity have a constant of the motion analogous to  $P_\phi$  conservation for toroidally symmetric systems. It should be noted that although all systems with a symmetry direction or axis have only one helicity in the representation for  $1/B^2$ , the converse is not generally valid.

The drift equations in magnetic coordinates are<sup>3</sup>

$$\frac{d\theta_0}{dt} = -v_\parallel B \left( \frac{\partial \rho_\parallel}{\partial \psi} - \frac{\partial \beta \rho_\parallel}{\partial \chi} \right), \tag{15}$$

$$\frac{d\psi}{dt} = v_\parallel B \frac{\partial \rho_\parallel}{\partial \theta_0}, \tag{16}$$

$$\frac{d\chi}{dt} = v_\parallel B \left( 1 - \frac{\partial \beta \rho_\parallel}{\partial \theta_0} \right), \tag{17}$$

$$\frac{d\rho_I}{dt} = v_I B \left[ \frac{\partial \rho_I}{\partial \chi} + \rho_I \left( \frac{\partial \rho_I}{\partial \theta} \frac{\partial \theta}{\partial \chi} - \frac{\partial \rho_I}{\partial \chi} \frac{\partial \theta}{\partial \theta} \right) \right] , \quad (18)$$

with  $\rho_I = v_I / (eB/mc)$  a periodic function of  $\theta$ . The quantity  $B$  can be shown<sup>2</sup> to equal

$$B = -\frac{dg}{d\psi} \phi - \frac{dI}{d\psi} \theta + B_*, \quad (19)$$

with  $B_*$  a periodic function of  $\theta$ . Using Eqs. (3) and (4) one finds

$$\left. \frac{\partial \rho_I}{\partial \theta} \right|_{\theta_0} = \frac{g}{g + \kappa I} \left. \frac{\partial \rho_I}{\partial \theta} \right|_{\phi} \quad \left. \frac{\partial \rho_I}{\partial \chi} \right|_{\theta_0} = -\frac{\kappa}{g + \kappa I} \left. \frac{\partial \rho_I}{\partial \theta} \right|_{\phi}$$

$$\left. \frac{\partial B}{\partial \theta} \right|_{\theta_0} = \frac{g}{g + \kappa I} \left. \frac{\partial B_*}{\partial \theta} \right|_{\phi} + \frac{g'I - I'g}{g + \kappa I} ,$$

$$\left. \frac{\partial B}{\partial \chi} \right|_{\theta_0} = -\frac{\kappa}{g + \kappa I} \left. \frac{\partial B_*}{\partial \theta} \right|_{\phi} + \frac{g' + \kappa I'}{g + \kappa I} .$$

So one has

$$\frac{d\rho_I}{dt} = -\frac{v_I B}{g + \kappa I} (\kappa - g\rho_I) \frac{\partial \rho_I}{\partial \theta} , \quad (20)$$

$$\frac{d\psi}{dt} = \frac{v_I B}{g + \kappa I} g \frac{\partial \rho_I}{\partial \theta} . \quad (21)$$

Consider the quantity

$$\psi_* = \psi_p - g\rho_I , \quad (22)$$

with the poloidal flux  $\psi_p$  defined by  $d\psi_p/d\psi = \kappa$ . The time derivative of  $\psi_*$  is zero using Eqs. (20) and (21) for

$$\frac{d\psi_*}{dt} = (x - g\rho_1) \frac{d\psi}{dt} - g \frac{d\rho_1}{dt} .$$

Consequently,  $\psi_*$  is a constant of the motion.

The drift orbits in a one-helicity fluid are given by Eq. (22) plus energy  $E$  and magnetic moment  $\mu$  conservation

$$E = \frac{1}{2} m v_i^2 + \mu B + e\phi , \quad (23)$$

with  $\phi$  the electrostatic potential. In the special case  $g = 0$ , particles remain precisely on a flux surface in the drift approximation. Equilibria with this property are called omnigenous.<sup>4,5</sup> It is easily shown that all omnigenous toroidal systems, in the sense of Ref. 5, are one helicity systems. If  $g$  is not zero, the drift orbits are qualitatively similar to those of a tokamak. The passing particle drift orbits are crudely circles displaced by the approximate distance  $\epsilon(B_T/B_p)\rho$  with  $B_T = |g\vec{\nabla}\phi|$ ,  $B_p = |\nabla\phi \times \nabla\psi_p|$ , and  $\rho = v/\omega_c$ , the gyroradius. The trapped particle orbits are banana shaped of width  $\epsilon^{1/2} (B_T/B_p)\rho$ . It should be stressed  $\epsilon$  measures the field strength variation of the surface and is not, in general, closely related to the inverse aspect ratio.

Let us now consider transport in a one-helicity system. It is customary to give transport results in term of the average number of particles or the average amount of heat crossing a flux surface per unit area. In complex geometries, this convention is cumbersome. Here we will give the total particle flux  $\Gamma_t$  and heat flux  $q_t$  crossing the surface. As in toroidal symmetry, the transport coefficients change markedly depending on the ratio of the mean free path  $v/v$  and the connection length  $L$ . The

connection length is the typical distance scale, along a field line, for field strength variations. For one-helicity systems, this characteristic distance is

$$L = \frac{q + \frac{1}{2}I}{\pi B_0}, \quad (24)$$

which reduces to  $qR$  for a tokamak. For  $v/v \ll L$ , the plasma is sufficiently collisional for fluid equations to be valid which is known as the Pfirsch-Schluter regime. For  $v/v \gg L/\varepsilon^{3/2}$ , even trapped particles can complete their drift orbits. This is known as the banana regime. The intermediate collisionality regime is known as the plateau.

The Pfirsch-Schluter transport for the simple Ohm's law  $\vec{E} + \vec{v} \times \vec{B}/c = \vec{\eta} \cdot \vec{j}$ , has been found for arbitrary toroidal scalar pressure equilibria.<sup>2</sup> Writing

$$\Gamma_t = -D \frac{dP}{d\psi}, \quad (25)$$

with  $P$  the pressure, one finds for the one-helicity field

$$D = 2\pi^2 \eta_i c^2 \frac{q + \frac{1}{2}I}{B_0^4} \left( \frac{qR}{L} \right)^2, \quad (26)$$

Transport in the long mean free path limit is a complicated process in fully three dimensional geometries. However, in systems with one helicity, the problem can be treated analytically. In the analytic theory it is customary to assume that  $\varepsilon \ll 1$  and that  $p(\theta) = \cos \theta$ . Although we will make these assumptions for simplicity, a more general form for  $p(\theta)$  can be easily included.<sup>1</sup> The drift kinetic equation is

$$v_{\parallel} \vec{B} \cdot \vec{\nabla} \theta \frac{\partial f}{\partial \theta} + \vec{v} \cdot \vec{\nabla} \psi \frac{\partial f_M}{\partial \psi} + e v_{\parallel} \mathcal{E}_{\parallel} \frac{\partial f_M}{\partial E} = C(f) , \quad (27)$$

with the distribution function  $F = f + f_M$  with  $f_M$  a Maxwellian. Using Eq.

(21) for  $d\psi/dt = \vec{v} \cdot \vec{\nabla} \psi$  and

$$\vec{B} \cdot \vec{\nabla} \theta = r \frac{B^2}{g + rI} , \quad (28)$$

one finds

$$r \frac{v_{\parallel} B}{g + rI} \left[ \frac{\partial f}{\partial \theta} + \frac{\partial}{\partial \theta} \left( \frac{g}{r} \rho_{\parallel} \right) \frac{\partial f_M}{\partial \psi} \right] + e \mathcal{E}_{\parallel} \frac{\partial f_M}{\partial E} = C(f) . \quad (29)$$

The particle flux is given by<sup>6</sup>

$$\begin{aligned} \Gamma_c &= \int_{\psi} d\vec{S} \cdot \left( \frac{\vec{B} \times \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \frac{1}{\omega_c} \int v_{\parallel} C(f - f_S) d^3 v \right) \\ &= - \frac{g}{r} (g + rI) \int \frac{r \partial d\psi}{B^2} \left[ \int \rho_{\parallel} C(f - f_S) d^3 v \right] , \end{aligned} \quad (30)$$

with the Spitzer distribution<sup>7</sup> function  $f_S$  the solution of

$$e v_{\parallel} \mathcal{E}_{\parallel} \frac{\partial f_M}{\partial E} = C(f_S) . \quad (31)$$

Following the standard neoclassical calculations<sup>8</sup> or almost by examining

Eqs. (29) and (30), one can show the total particle flux across a surface is

$$\begin{aligned} \Gamma_c &= -(2\pi)^2 \frac{g + rI}{B_o^2} g^2 \sqrt{\epsilon} \frac{\rho}{r^2} \frac{1}{\tau_e} \left[ a_1 \frac{dn}{d\psi} + a_2 \frac{n}{T_e} \frac{dT_e}{d\psi} + a_3 \frac{n}{T_i} \frac{dT_i}{d\psi} \right] \\ &= (2\pi)^2 a_4 \frac{g + rI}{B_o^2} \sqrt{\epsilon} \frac{g c}{B_o r} n \mathcal{E}_{\parallel} , \end{aligned} \quad (32)$$

with  $a_1 \dots a_4$  numerical constants which we will determine by comparing with the well-known results for the large aspect ratio tokamak, in particular, those of Rosenbluth, Hazeltine, and Hinton.<sup>8</sup> The electron gyroradius, collision time, and temperature are  $\rho_e$ ,  $\tau_e$ , and  $T_e$ . The ion temperature is  $T_i$ ,  $n$  is the density, and  $c$  is the speed of light. The flux of Rosenbluth et al.,  $\Gamma_R$ , is related to  $\Gamma_c$  by

$$\Gamma_R = \Gamma_c / (2\pi r)(2\pi R), \quad (33)$$

with  $r$  the local minor radius and  $R$  the major radius. One also has

$$\frac{dn}{d\psi} = \frac{1}{rB_\phi} \frac{dn}{dr}, \quad g = R B_\phi, \quad I = r B_\theta, \quad x = \left(\frac{R}{r}\right) \left(\frac{B_\theta}{B_\phi}\right), \quad (34)$$

with these substitutions

$$\begin{aligned} \Gamma_R = & - \left(\frac{B_\phi}{B_\theta}\right)^2 \sqrt{\epsilon} \frac{c_e^2}{\tau_e} \left[ a_1 \frac{dn}{dr} + a_2 \frac{n}{T_e} \frac{dT_e}{dr} + a_3 \frac{n}{T_i} \frac{dT_i}{dr} \right] \\ & - a_4 c \sqrt{\epsilon} \left(\frac{B_\phi}{B_\theta}\right) \frac{\mathcal{E}_1}{B_\theta}. \end{aligned} \quad (35)$$

Except for factors of  $B_\phi/B$  which were set equal to unity, but are important for the reversed field pinch,<sup>9</sup> this is the flux given by Rosenbluth et al. with

$$\begin{aligned} a_1 &= 1.12 \left(1 + \frac{T_i}{T_e}\right), & a_2 &= -0.43, & a_3 &= -0.19 \frac{T_e}{T_i}, \\ a_4 &= 2.44. \end{aligned} \quad (36)$$

The total electron and ion heat fluxes can be similarly evaluated and are of the same form as the particle flux. The parallel current is

$$j_{\parallel} = \sigma_{NC} \mathcal{E}_{\parallel} - \sqrt{\epsilon} \frac{gc}{r} \frac{e}{B_0} \left[ a_5 \frac{dn}{d\psi} + a_6 \frac{n}{T_e} \frac{dT_e}{d\psi} + a_7 \frac{n}{T_i} \frac{dT_i}{d\psi} \right]. \quad (37)$$

Reduction to the large aspect ratio tokamak case give

$$j_{\parallel} = \sigma_{NC} \mathcal{E}_{\parallel} - \sqrt{\epsilon} \left( \frac{B_0}{B_0} \right) \frac{cT_e}{B_0} \left[ a_5 \frac{dn}{dr} + a_6 \frac{n}{T_e} \frac{dT_e}{dr} + a_7 \frac{n}{T_i} \frac{dT_i}{dr} \right], \quad (38)$$

with

$$\sigma_{NC} = \left[ 0.51 \frac{n_e}{e n T_e} \right]^{-1} [1 - 1.95 \epsilon^{1/2}], \quad (39)$$

$$a_5 = 2.44 (1 + T_i/T_e), \quad a_6 = 0.69, \quad a_7 = -0.42 \frac{T_i}{T_e}.$$

To advance the density in time one uses

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \vec{F} = S, \quad (40)$$

with  $S$  the sources of particles. Integrating this equation over the volume and using  $n = n(\psi)$  one has

$$\frac{\partial n}{\partial t} + \frac{B_0^2}{g + \pi i} \frac{\partial}{\partial \psi} \left( \frac{\Gamma_t}{(2\pi)^2} \right) = \bar{S}, \quad (41)$$

with  $\bar{S}$  the surface averaged source of particles. There is a similar equation for the time rate of change of the electron and ion temperatures.

#### IV. CONCLUSIONS

Plasmas with different physical shapes can have identical drift orbits and associated transport. Such systems are called isomorphic. The most important feature of a plasma confinement geometry, for the evaluation of drift orbits, is the magnetic field strength expressed in magnetic coordinates. In plasma systems with closed magnetic surfaces, there are two periodicities. Such systems are topologically toroidal. Drift orbits in topologically toroidal systems depend on three functions of the radial coordinate, the toroidal flux  $2\pi\psi$ , in addition to the field strength. However for good confinement, drift orbits must be localized in  $\psi$ ; so  $\psi$  generally plays the role of a parameter rather than a variable. The three functions of  $\psi$  which enter the solution for the drift orbits are  $g$  and  $I$ , which are within a factor  $c/2$  the total poloidal current outside and toroidal current inside a magnetic surface, and  $\lambda$  the rotational transform.

Systems with closed magnetic surfaces can be decomposed in a Fourier series in the two periodic coordinates  $\theta$  and  $\phi$ . A typical term in this decomposition depends on  $n\phi - m\theta$ . Any other term, which depends on  $r(n\phi - m\theta)$  with  $r$  a rational number, is said to be of the same helicity. If the Fourier decomposition of the field strength contains only one helicity, there is a constant of the motion, which is canonical momentum conservation in toroidal symmetry. In addition, one can evaluate the transport coefficients in a form that is valid for all one-helicity systems. In effect, all one-helicity systems are isomorphic although they may naturally fall at different places in  $g$ ,  $I$ ,  $\lambda$  parameter spaces. Examples of one-helicity systems are the symmetric tokamak, the straight stellarator, and the elliptical cylinder.

The Fourier decomposition of a field strength with two helicities can

be reduced to a standard form. Consequently, all two-helicity systems are isomorphic in the same sense as all one-helicity systems are. Physical realizations of two-helicity systems are the tokamak with toroidal ripple and a simple toroidal stellarator. The drift orbits and transport in two-helicity fields are far more complex and subtle than in one-helicity systems due to the absence of a local constant of the motion of the canonical momentum type. However, a combination of analytic and numerical<sup>10</sup> work could give a general treatment of transport in systems with two helicities.

Systems with three or more helicities can not all be reduced to a standard form. In such systems the relative mode numbers of the different helicities are additional parameters in a transport theory.

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