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TITLE: EXACT INVARIANTS IN THE FORM OF MOMENTUM RESONANCES FOR
PARTICLE MOTION IN ONE-DIMENSIONAL, TIME-DEPENDENT POTENTIALS

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EXACT INVARIANTS IN THE FORM OF MOMENTUM RESONANCES
FOR PARTICLE MOTION IN ONE-DIMENSIONAL, TIME-DEPENDENT POTENTIALS*

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INTRODUCTION

Exact invariants for Hamiltonian systems can be very useful for obtaining insight into the nature of solutions of the equations of motion and they can be helpful in computing solutions numerically. The search for invariants for specific systems has a long history. As an example, we mention the gravitational three-body problem, which is relevant to the motion of celestial bodies and has been studied extensively.[1] In 1887 Bruns showed that the ten so-called classical integrals of the three-body problem are the only invariants that exist that are algebraic functions of the coordinates, momenta and time. In 1889 Poincaré showed for the restricted three-body problem that the only explicitly time-independent invariant that is periodic in the coordinates is an energy known as the Jacobian energy. These important results illustrate that an explicitly time-dependent invariant is likely to be outside the class of algebraic functions and that explicitly time-independent invariants are uncommon, even for autonomous systems.

In the search for exact invariants, attention traditionally has been concentrated on explicitly time-independent functions. Also, the systems considered have usually been autonomous; that is, the Hamiltonians usually have not depended on time explicitly. In this lecture, we consider the motion of a particle in a one-dimensional potential and allow the invariants as well as the potential to be explicitly time-dependent. For an autonomous one-dimensional Hamiltonian system, all invariants that are functionally independent of the energy are explicitly time-dependent. For any particular nonautonomous system, it may be that all invariants are explicitly

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time-dependent. In any event, any complete set of $2n$ invariants for an n -dimensional system must include at least one explicitly time-dependent invariant.

The equations of motion of any one-dimensional nonautonomous Hamiltonian system are equivalent to those of a corresponding autonomous two-dimensional Hamiltonian system. Therefore, it is always possible to treat one-dimensional nonautonomous systems by considering certain two-dimensional autonomous systems instead. The corresponding two-dimensional Hamiltonian depends linearly on a momentum variable that is associated with the energy of the one-dimensional system. In this regard, it should be noted that invariants derived by Holt [2], Hall [3] and others for particle motion in a two-dimensional, time-independent potential cannot be used to obtain invariants for one-dimensional, time-dependent potentials. The reason is that the Hamiltonian for the motion of a particle in a two-dimensional, time-independent potential depends quadratically on each momentum.

An important area in which explicitly time-dependent invariants for time-dependent potentials play a crucial role is the theory of collisionless plasmas.[4] When there is only one spatial dimension, the governing equations, which are the Vlasov-Poisson equations, describe a continuum of particles that move in the electric field generated by the particles themselves. The electric field is to be determined self-consistently along with the motion of the particles. The phase-space distribution function for the particles, which is a solution of the Vlasov equation, is a function of invariants of the motion of a single particle in the electric field. An exact or approximate invariant can be useful in connection with the Vlasov-Poisson equations if it applies to a class of electric fields that approximate the field associated with the exact solution.

Exact invariants for particle motion in classes of one-dimensional explicitly time-dependent potentials have been found for both linear and nonlinear equations of motion. For linear, arbitrarily time-dependent oscillators, invariants are known that are homogeneous quadratic forms in the coordinate and momentum.[5] For nonlinear equations of motion, an exact invariant is known that is quadratic in the momentum when the potential has a certain form that involves an arbitrary function of a time-dependent linear function of the spatial coordinate.[6,7] This invariant has been used [8] to find new exact solutions of the Vlasov-Poisson equations.

In this lecture we describe our use of a momentum-resonance ansatz of Lewis and Leach [9] to study exact invariants for time-dependent, one-dimensional potentials. This ansatz provides a framework for finding invariants admitted by a larger class of time-dependent potentials than was known previously. For a potential that admits an exact invariant in this resonance form, we have shown how to construct the invariant as a functional of the potential in terms of the solution of a definite linear algebraic system of equations. We have found a necessary and sufficient condition on the potential for the existence of an invariant with a given number of resonances. There exist more potentials that admit invariants with two resonances than were previously known and we have found an example in parametric form of such a potential. We have also found examples of potentials that admit invariants with three resonances.

THE MOMENTUM-RESONANCE ANSATZ

We consider the Hamiltonian for a particle moving in a potential that depends on a coordinate q and time t ,

$$H = \frac{1}{2} p^2 + V(q,t) . \quad (1)$$

The resonance ansatz of Lewis and Leach [9] postulates that the momentum dependence of an invariant $I(q,p,t)$ be expressed in terms of simple poles in the complex momentum plane,

$$I(q,p,t) = c(q,t) + \sum_{n=1}^N \frac{v_n(q,t)}{p - u_n(q,t)} . \quad (2)$$

This ansatz includes the case of invariants that are polynomials in p because any function of an invariant is also an invariant. Lewis and Leach showed that any function, $f(p)$, with a finite number of singularities located at points $p = p_i$, $1 \leq i \leq N$, can be expressed as some function of another function, $g(p)$, where the singularities of $g(p)$ are simple poles located at the points p_i . The functions of position and time that appear in the expression for the invariant in

resonance form satisfy conditions that are decoupled to a remarkable degree. The condition that $I(q,p,t)$ be an invariant is

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + p \frac{\partial I}{\partial q} - \frac{\partial V}{\partial q} \frac{\partial I}{\partial p} = 0, \quad (3)$$

which implies that necessary and sufficient conditions on the functions c , v_n and u_n such that $I(q,p,t)$ be an invariant are

$$\begin{aligned} \frac{\partial c}{\partial q} &= 0, & \frac{\partial c}{\partial t} + \sum_{n=1}^N \frac{\partial v_n}{\partial q} &= 0, \\ \frac{\partial v_n}{\partial t} + \frac{\partial}{\partial q} (u_n v_n) &= 0, & \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial q} &= - \frac{\partial V}{\partial q}. \end{aligned} \quad (4)$$

All potentials that admit an invariant with one pole ($N=1$) were determined by Lewis and Leach [9]. They also found a class of potentials that admit an invariant with two poles. In that case, the potential involves an arbitrary function of a time-dependent linear function of the spatial coordinate.

THE DISCRETE-MOMENT FORMULATION

There is a formulation of the momentum-resonance ansatz in terms of certain discrete moments that is very useful. Any invariant that can be written in resonance form can be constructed explicitly from these discrete moments as a functional of the potential. In addition, a necessary and sufficient condition for a potential to admit an invariant with N poles can be expressed in terms of the discrete moments. The k^{th} moment $g_k(q,t)$ is defined by

$$g_k(q,t) = \sum_{n=1}^N u_n^k v_n. \quad (5)$$

If, for fixed q and t , we consider the quantities $v_n(q,t)$ to be the values of a function $v(q,p,t)$ that is defined at a discrete set of

values of p given by $p = u_n(q, t)$ for $1 \leq n \leq N$, then $g_k(q, t)$ is the k th moment of $v(q, p, t)$ in that discrete space of values of p . By direct manipulation of (4), it can be shown that these moments satisfy the differential recursion relation

$$\frac{\partial g_k}{\partial q} = - \frac{\partial g_{k-1}}{\partial t} - (k-1)g_{k-2} \frac{\partial v}{\partial q}, \quad k > 1 \quad (6a)$$

with the initial condition

$$g_0(q, t) = \alpha_1(t)q + \alpha_2(t), \quad (6b)$$

where $\alpha_1(t) = -dc/dt$. Equation (6a) is precisely the recursion relation satisfied by the momentum moments of (3).^{*} This equation defines the invariant $I(q, p, t)$ and, in the context of collisionless plasma physics, it is the Vlasov equation for the phase-space distribution function. Thus, it is natural to interpret $v(q, p, t)$ as the representation of an invariant or a phase-space distribution function in a discrete space of momentum values given by $p = u_n(q, t)$ for $1 \leq n \leq N$.

In addition to the recursion relation (6a), the moments also satisfy the algebraic recursion relation^{**}

$$g_k = - \sum_{n=1}^N a_n g_{k-n}, \quad k > N, \quad (7)$$

where the quantities a_n and u_n are related by

$$u_n^N + \sum_{k=1}^N a_k u_n^{N-k} = 0. \quad (8)$$

This recursion relation is completely equivalent to the definition of

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^{*}We thank Daniel C. Barnes for mentioning this connection.

^{**}We thank Gene H. Golub for pointing out that the definition of the discrete moments implies the existence of an algebraic recursion relation among them.

the moments given by (5), a fact that plays an important role in the numerical analysis of an ordinary differential equation with constant coefficients.

We now make two remarkable observations. The moments can be calculated as functionals of the potential by means of (6a); and then the results can be used in N of the recursion equations given by (7) to obtain a system of linear algebraic equations that determine the coefficients a_n . The second observation is that (2), which expresses the invariant in terms of the u_n and v_n , can be rewritten to express the invariant completely in terms of the moments and the coefficients a_n [13]:

$$I(q,p,t) = c(t) + \frac{\sum_{n=1}^N p^{N-n} \sum_{k=1}^n a_{k-1} g_{n-k}}{p^N + \sum_{n=1}^N a_n p^{N-n}}, \quad (9)$$

where we have defined

$$a_0 \equiv 1. \quad (10)$$

The expressions for the invariant given by (2) and (9) are identically equal because of the algebraic recursion relation (7). These two observations prove the following linearization theorem:

THEOREM: If $V(q,t)$ admits an invariant with N resonances, this invariant is given by equation (9) in terms of moments g_k and coefficients a_n . The g_k can be expressed as functionals of $V(q,t)$ by means of (6) and the a_n are determined from the g_k through the linear algebraic system of equations (7).

In addition to the linearization theorem, we have derived [13] a condition for a potential to admit an invariant in resonance form:

NECESSARY and SUFFICIENT CONDITION on $V(q,t)$: The expression given by (9) in terms of the moments is an invariant ($dI/dt = 0$) if, and only if, the determinant

$$\Delta = \begin{vmatrix} g_0 & g_1 & \cdot & \cdot & \cdot & g_N \\ g_1 & g_2 & \cdot & \cdot & \cdot & g_{N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_N & \cdot & \cdot & \cdot & \cdot & g_{2N} \end{vmatrix} = 0 \quad (11)$$

is satisfied. The matrix in this condition is a Hankel matrix [10-12]

SOME PRELIMINARY APPLICATIONS

A direct application of the condition (11) for the case $N = 1$ leads to a first-order linear differential equation for $V(q,t)$ that can be completely integrated to yield the previously known result for potentials that admit an invariant with only one resonance. For $N = 2$, Eq. (11) is a nonlinear integro-differential equation. In the special case for which $g_0 = 0$, the condition again becomes a linear partial differential equation for the potential that can be integrated. The result is the class of potentials found by Lewis and Leach [9]. However, condition (11) allows a wider class of potentials that admit an exact invariant with two resonances. In a future publication [13], we shall exhibit further solutions of (11) with $N = 2$ and $g_0 \neq 0$. They are obtained by recasting the necessary and sufficient condition with $N = 2$ into a form different than (11).

To illustrate the use of our condition (11), we present two potentials for which invariants with three resonances exist and we construct them by using (9). These examples are the result of a preliminary study of the $N = 3$ case.

We consider $N = 3$ and choose $g_0 = g_1 = g_3 = 0$ and $g_2 = 1$. This is consistent with (6). All other g_k can be calculated from (6a). The result is

$$g_4 = -3V(q,t) - \frac{3}{2} v_1(t) , \quad (12a)$$

$$g_5 = 3 \frac{\partial}{\partial t} \int V(q,t) dq + \frac{3}{2} \frac{dv_1}{dt} + v_2(t) , \quad (12b)$$

$$g_6 = -3 \frac{\partial^2}{\partial t^2} \iint V(q,t) dq^2 - 3 \frac{d^2 v_1}{dt^2} q^2 - \frac{dv_2}{dt} q - v_3(t) + \frac{15}{2} v_1 V(q,t) + \frac{15}{2} v^2(q,t) , \quad (12c)$$

where $v_1(t)$, $v_2(t)$ and $v_3(t)$ are unspecified functions of time. Condition (11) for this case is simply $g_6 - g_4^2 = 0$, which is the following integro-differential equation for the potential:

$$v^2(q,t) + v_1(t) v(q,t) + 2 \frac{\partial^2}{\partial t^2} \iint V(q,t) dq^2 = -\phi(q,t) , \quad (13)$$

where

$$\phi(q,t) = 2 \frac{d^2 v_1}{dt^2} q^2 + 2 \frac{dv_2}{dt} q + 2v_3 + \frac{3}{2} v_1^2 . \quad (14)$$

The search for solutions is simplified by starting with the equation

$$2 \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v^2}{\partial q^2} + v_1(t) \frac{\partial^2 v}{\partial q^2} = -4 \frac{d^2 v_1}{dt^2} , \quad (15)$$

which is obtained by taking the second spatial derivative of (13). Any solution of (15) can then be substituted into (13), which becomes an equation for the remaining unknowns. Particular solutions of (13-15) can be found easily. Two immediate solutions are:

$$V(q,t) = At\sqrt{q} , \quad v_1 = 0 , \quad v_2 = -\frac{1}{6} A^2 t^2 + B , \quad v_3 = 0 , \quad (16a)$$

and

$$V(q,t) = -\frac{A}{2} \pm \sqrt{(A^2 + 4C + 4Bq)} ,$$

$$V_1 = A , \quad V_2 = -\frac{1}{2}Bt + D , \quad V_3 = -\frac{3}{4}A^2 - \frac{1}{2}C , \quad (16b)$$

where A, B, C and D are arbitrary constants. Using those solutions in (12) to determine the g_k and, with the g_k calculating the a_n from (7), we obtain the corresponding invariants:

$$I = \frac{1}{p^3 + 3Apt\sqrt{q} - 2Aq\sqrt{q} + \frac{1}{2}A^2t^3 - 3B} , \quad (17a)$$

and

$$I = \frac{1}{p^3 \pm 3p\sqrt{(A^2 + 4C + 4Bq)} + \frac{3}{2}Bt} , \quad (17b)$$

respectively. It is interesting to notice that although both invariants are explicitly time-dependent, the first system is nonautonomous, but the second is autonomous. The latter possesses the energy invariant in addition to I and, therefore, we have a complete set of two invariants. The trajectories could be determined by, for example, solving the energy invariant for the momentum p and substituting this into I.

FINAL REMARKS

The momentum-resonance ansatz has been developed into an alternative form which, in an initial study, has exhibited additional advantages. We were able to show that an invariant in resonance form can be determined by only solving linear algebraic equations. We derived a closed condition that determines all potentials that admit invariants in resonance form. First applications of this condition

reproduce previously known cases of exact invariants for one-dimensional potentials. In addition, we have found potentials that have three-resonance invariants.

The possibilities for applying our formalism are certainly not exhausted. A detailed study of the $N > 2$ cases is of immediate interest, including the meaning of the limit $N \rightarrow \infty$. Extensions of our analysis to include systems with more than one spatial dimension, systems with magnetic forces and quantum mechanical systems should be investigated.

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