MASTER

SOME NUMERICAL APPROACHES TO SOLVING ONE-DIMENSIONAL
INVERSE PROBLEMS

Frank Hagin
Department of Mathematics
University of Denver
Denver, CO 80208

This work was supported in part by the Department of Energy,
Contract No. DE-AC02-80ER10769.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

This report was prepared as an account of Government-sponsored work. Neither the United States, nor the Department of Energy nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of, any information, apparatus, method, or process disclosed in this report.

As used in the above, 'person acting on behalf of the Commission' includes any employee or contractor of the Administration or employee of such contractor, to the extent that such employee or contractor of the Administration employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Administration, or his employment with such contractor.

SOME NUMERICAL APPROACHES TO SOLVING ONE-DIMENSIONAL
INVERSE PROBLEMS

Frank Hagin
Department of Mathematics
University of Denver
Denver, CO 80208

DISCLAIMER

This book was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

This work was supported in part by the Department of Energy,
Contract No. DE-AC02-80ER10769.

ABSTRACT

A class of one-dimensional inverse scattering problems are studied, with the goal of reconstructing (say) propagation speed to moderate accuracy as inexpensively as possible. Three alternatives are discussed; all starting from a change to the 'travel-time' variable and converting the problem to integral equation form. The approaches are compared as to their economy of use and the problems for which they are effective. Several numerical examples illustrate these comparisons.

I. INTRODUCTION

We will be considering one-dimensional inverse problems of the following form:

$$y''(x) + \frac{\omega^2}{v(x)^2} y(x) = 0$$

$$y'(0) + \frac{i\omega}{v(0)} y(0) = 2i\omega$$

$$y'(X) - \frac{i\omega}{v(X)} y(X) = 0$$

where $X \leq \infty$. The goal is to reconstruct the unknown (say) velocity v from certain data prescribed at a set of ω values. The real parameter ω arise from an earlier Fourier transform of the original wave equation.

Similar problems have been studied by Gray [1], [2], Hagin [2], [3], Chen [4] and others. Our purpose here is to investigate the computational feasibility of some of these ideas and, in addition, to introduce another alternative (Sec. 3) that in some cases is very attractive.

In [1] the "travel time" variable change was shown to be effective in certain instances. Our investigations have confirmed that this is in general a good starting point; so we define

$$\tau = \int_0^x ds/v(s).$$

as our new independent variable and let $u(\tau) = y(x)$ and $c(\tau) = v(x)$. The boundary value problem now appears

$$u'' - c'/c u' + \omega^2 u(\tau) = 0$$

$$u'(0) + i\omega u(0) = c(0)2i\omega \quad (1)$$

$$u'(T) - i\omega u(T) = 0$$

where $T = \tau(X) \leq \infty$. Since we will soon truncate (if necessary) and discretize for numerical computations, let us assume that $T < \infty$. For convenience we will assume that both $T = 1$ and $c(0) = 1$ (which could be accomplished by scaling). Finally, it will be convenient to have the differential equation in self-adjoint form, and this leads to the statement

$$(u'/c)' + \omega^2 u/c = Lu = 0, \quad (2)$$

$$[u' + i\omega u](0) = 2i\omega, \quad [u' - i\omega u](1) = 0.$$

A central notion is that of an "incident wave", denoted herein by u_i . For example, our problem could be interpreted as the result of wave $u_i = e^{i\omega\tau}$ moving in from the region $\tau \leq 0$ in which the media has constant propagation velocity $c = 1$. More generally, from our viewpoint u_i will denote some first approximation to the solution $u(\tau; \omega)$ of (2) in which u_i does satisfy the boundary conditions.

To obtain an integral equation for the unknown c we proceed as follows. Applying the Lagrange identity to (2), using the fact $Lu = 0$, and defining $u_s = u - u_i$ we obtain

$$\begin{aligned} \int_0^1 (uLu_i - u_iLu) &= \int_0^1 uLu_i = \int_0^1 (u_i + u_s)Lu_i \\ &= [uu_i' - u'u_i]_0^1 = -2i\omega u_s(0; \omega). \end{aligned} \quad (3)$$

The values $u_s(0;\omega)$ for some ω -set is our data and will be assumed known for the ω values of our choice. This integral equation for c is still not convenient since it involves the unknown u or u_s (although in Sec. 4 we will attack it via iteration). The problem is much more manageable if we are justified in assuming that u_s is small in comparison to u_i so that the u_s part of (3) can be dropped to obtain

$$\int_0^1 u_i L u_i = -2i\omega u_s(0;\omega). \quad (4)$$

In (4) u_i will be a known estimate and $L u_i$ will involve the unknown c ; thus we obtain a linear integral equation for c . In the next two sections we turn to two natural selections for u_i ; $e^{i\omega\tau}$ and $\sqrt{c} e^{i\omega\tau}$. We shall see that if c'/c is relatively small, the first of these leads to an exceptionally inexpensive method for reconstructing c (as in [1]). If c'/c is not small enough then $u_i = e^{i\omega\tau}$ is not an adequate estimate for u and hence the neglected u_s term in (3) is significant. In Sec. 3 we show that in this case, and if c is sufficiently well-behaved, then the WKB approximate $\sqrt{c} e^{i\omega\tau}$ is a practical choice which leads to a fairly inexpensive algorithm for reconstructing c . Finally, in Sec. 4 we turn to the more general situation in which we return to (3) and through iteration involve the u_s term neglected in (4). Although this process is considerably more expensive than the earlier alternatives, it is certainly not prohibitive; moreover, it is probably unavoidable in many problems.

II. RESULTS USING $u_i = e^{i\omega\tau}$

For this choice for u_i , it will be convenient to introduce the operator L_0 given by $L_0 v = v'' + \omega^2 v$. Note that $L_0 u_i = 0$. Proceeding as in the derivation of (3), we obtain

$$\begin{aligned} \int_0^1 [u_i L_0 u - u L_0 u_i] &= \int_0^1 u_i L_0 u = \int_0^1 e^{i\omega\tau} (c'/c) u' \\ &= \int_0^1 e^{i\omega\tau} [i\omega e^{i\omega\tau} + u_s'] (c'/c) d\tau = 2i\omega u_s(0; \omega). \end{aligned} \quad (5)$$

In [2] it was shown that if $\gamma \equiv c'/c = O(\epsilon) < 1$, in the L^1 norm, then the term in (5) involving u_s' is $O(\epsilon^3)$. Hence if γ is indeed small, one is justified in dropping that term to get

$$\int_0^1 e^{2i\omega\tau} \gamma(\tau) d\tau \approx 2u_s(0; \omega). \quad (6)$$

Clearly (6) can be solved easily for $\gamma = c'/c$ via Fourier methods assuming data $u_s(0; \omega)$ is available for the required ω values. Suppose, for example, we have reason to believe that γ can be well represented by its Fourier sine series (e.g. $\gamma(0) = 0 = \gamma(1)$). By taking the imaginary part of (6) and setting $\omega = \omega_k = k\pi/2$ we have

$$\int_0^1 \sin k\pi\tau \gamma(\tau) d\tau = 2 \operatorname{Im} u_s(0; \omega_k) = b_k/2 \quad (7)$$

where b_k are the (approximate) Fourier coefficients of γ .

Finally we have

$$\begin{aligned} \gamma(\tau) &= \sum_{k=1}^{\infty} b_k \sin k\pi\tau \approx \sum_{k=1}^N b_k \sin k\pi\tau \\ &= 4 \sum_{k=1}^N \operatorname{Im} u_s(0; \omega_k) \sin k\pi\tau. \end{aligned}$$

Since $\gamma = c'/c$ we have from integration

$$c(\tau) = \exp\left[4 \sum_{k=1}^N \operatorname{Im} u_s(0; \omega_k) (1 - \cos k\pi\tau)/k\pi\right]. \quad (8)$$

(Formula (8) amounts to a discrete version of the approach of [1] in which the $T = \infty$ case was treated using Fourier transform.)

In [2] it was shown that if (6) is solved for γ the error due to ignoring the u_s term in (5) is approximately $d^3/4\pi$, where d is the L^1 norm of γ . For example, if we are seeking two figure accuracy, this suggests that the relative error (in L^1 norm) should be less than .01; i.e. $d^2/4\pi \approx d^2/10 < .01$ or

$$d = \|\gamma\| < .3. \quad (9)$$

At least (9) should serve as a rule-of-thumb for how large $\gamma = c'/c$ can be for (8) to produce two figure results. In the following example we investigate the situation numerically.

Example 1. Consider the function

$$c(\tau) = 1 + \epsilon \sin^2(\pi\tau/2) \quad \text{for } \tau \text{ in } [0,1].$$

Since $\gamma = c'/c \approx c' = \epsilon\pi/2 \sin \pi\tau \approx \epsilon$, (9) suggests that we should be able to achieve two figure results with ϵ roughly as large as .3. The data $u_s(0; \omega_k)$ was created artificially for this and subsequent examples by using the exact c in (2) to generate numerically $u(\tau; \omega_k)$ and then $u_s = u - u_i$. Then c is reconstructed by using, in this case, (8) above with $N = 10$. Table I summarizes the numerical results. For several values of ϵ ranging between .3 and 2.0 two types of

errors are listed: the 'maximum error' over the ten computed points and the 'mean error' (which is the discrete version of the L^1 error). Note that we achieved the desired two figure accuracy for $\varepsilon < .8$ and lose it for $\varepsilon > .8$, hence the indication given by (9) above is somewhat conservative in this instance. The results are illustrated graphically in Fig. 1.

In the next section we refine the procedure just discussed in order to handle some cases in which $\gamma = c'/c$ is not "small".

III. RESULTS USING $u_i = \sqrt{c} e^{i\omega\tau}$

One way of viewing the results of the last section is: if c varies little (c'/c small) then $u_i = e^{i\omega\tau}$ is an effective estimate for $u(\tau; \omega)$; hence dropping the $u_s = u - u_i$ term in (3) does little harm. We now look at a procedure which is effective in at least some cases in which c'/c is not small. We choose as u_i the WKB approximation

$$u_i = \sqrt{c} e^{i\omega\tau}.$$

As interpreted physically in Bremmer [5], if a wave $e^{i\omega\tau}$ for $\tau < 0$ enters the medium at $\tau = 0$, then $\sqrt{c} e^{i\omega\tau}$ is the approximation to u obtained by considering only the primary reflections and transmissions due to the inhomogeneous medium (c non-constant). From a mathematical point of view, $\sqrt{c} e^{i\omega\tau}$ represents the first two terms in a large ω expansion of solution u ; whereas $e^{i\omega\tau}$ is the first term (e.g. see [6]). This comparison can be illustrated as follows. Consider the

operator L of (2); recall that we seek a solution to $Lu = 0$.

It is easy to show that

$$\begin{aligned} L(e^{i\omega\tau}) &= -i\omega\gamma/c e^{i\omega\tau}, \\ L(\sqrt{c} e^{i\omega\tau}) &= 1/2 c^{-1/2}(c''/c - 3/2 (c'/c)^2)e^{i\omega\tau}. \end{aligned} \quad (10)$$

Viewing the right sides in (10) as "error", we see that $L(e^{i\omega\tau}) = O(\omega)$ while $L(\sqrt{c} e^{i\omega\tau}) = O(1)$ in ω . In practice $\sqrt{c} e^{i\omega\tau}$ is an excellent approximation to u for large ω and is often a surprisingly good estimate for moderate and even "small" ω . Equally important in the right sides of (10) are the expressions involving c . In the first of these note that $\gamma = c'/c$ appears, supporting the "small" γ assumption of the last section. In the second equation γ^2 appears as does a c'' term. In particular the c'' forewarns us that c may have to be quite smooth and that c'' cannot be too large. This is born out in the examples below.

To obtain an integral equation we use (4) with $u_i = \sqrt{c} e^{i\omega\tau}$ and (10),

$$\int_0^1 u_i L u_i = 1/2 \int_0^1 e^{2i\omega\tau} (c''/c - 3/2 \gamma^2) d\tau = -2i\omega u_s(0; \omega).$$

Analogous to the approach of Sec. 2, we choose $\omega = \omega_k = k\pi/2$ for $k = 1, 2, \dots$ and take the real part to obtain

$$\int_0^1 \cos k\pi\tau (c''/c - 3/2 \gamma^2) d\tau = 2k\pi \operatorname{Im} u_s(0; \omega_k) = 1/2 a_k \quad (11)$$

where the a_k are the Fourier coefficients in

$$c''/c - 3/2 (c'/c)^2 = \sum_0^\infty a_k \cos k\pi\tau = S_\infty(\tau). \quad (12)$$

Perhaps the most straight-forward way to proceed is to truncate the series in (12) to give a trigonometric polynomial $S_N(\tau)$ and solve the resulting second order differential equation numerically for c . In practice, however, this does not work unless c is exceptionally smooth. Note that the series in (12) is for a function involving c'' , hence unless c is very smooth the series will converge very slowly, if at all. Ideally one would like to integrate the series in (12) once or twice term-by-term in order to improve its behavior. We now move in that direction.

Observe that

$$c''/c - 3/2 (c'/c)^2 = (c'/c)' - 1/2 (c'/c)^2 = \gamma' - 1/2 \gamma^2.$$

Using this in (11) we can write

$$\begin{aligned} \int_0^1 \gamma' \cos k\pi\tau \, d\tau &= 1/2 \left[a_k + \int_0^1 \gamma^2 \cos k\pi\tau \, d\tau \right] \\ &= 1/2 A_k \quad \text{for } k = 1, 2, \dots \end{aligned} \tag{13}$$

where $a_k = 4k\pi \operatorname{Im} u_S(0; \omega_k)$ as in (11) and A_k are the coefficients in

$$\gamma' = \sum_{k=1}^{\infty} A_k \cos k\pi\tau. \tag{14}$$

The coefficient $A_0 = \int \gamma' = \gamma(1) - \gamma(0)$ requires special consideration, so to simplify the presentation we will, as we did in Sec. 2, assume that $c'(0) = c'(1) = 0$ and hence that $A_0 = 0$. We now define our procedure for iteratively computing the A_k and ultimately c itself. We first compute γ'_0 by ignoring the a_k correction in (13) and proceed as follows:

$$\begin{aligned}
\gamma'_0(\tau) &= \sum_{k=1}^N a_k \cos k\pi\tau, \\
\gamma_0(\tau) &= \sum_{k=1}^N a_k/k\pi \sin k\pi\tau, \\
c_0(\tau) &= \exp\left[\sum_{k=1}^N a_k/(k\pi)^2 (1 - \cos k\pi\tau)\right].
\end{aligned} \tag{15}$$

A comparison with the results of Sec. 2 shows that c_0 is precisely that c computed by the $u_i = e^{i\omega\tau}$ assumption.

Using this as a starting point we iterate as follows:

1. Using current estimate for γ in the right side of (13), compute the A_k for $k = 1, 2, \dots, N$.
2. Using the updated A_k , compute γ' , γ and c by (15) with A_k replacing the a_k .

Steps 1 and 2 are repeated as necessary to achieve convergence of c to the desired accuracy. For example, when seeking two figure results we typically repeated these steps twice. Although a rigorous proof of convergence would no doubt require that γ^2 be "small" compared to γ' , we have had success when these two quantities were about the same size.

We now consider two examples in which c will be reconstructed by the algorithm just described. As before, the data $u_s(0; \omega_k)$ was constructed artificially by using the known c in (2). In the first of these two examples we again use the c of Example 1 above for comparison.

Example 2. Consider the c of Example 1

$$c(\tau) = 1 + \epsilon \sin^2(\pi\tau/2)$$

Recall in Sec. 2 we were able to reconstruct c to only one figure (about 10% error) as ϵ approached 2. Table II compares the results of Sec. 2 (producing c_0) to the results of two corrections (iterations) as outlined above (producing c_2). In particular, we are able to achieve two figure accuracy for ϵ up to 2. In this example and the next we only corrected the first four a_k in (13) as an economy measure (clearly for a reasonably smooth function the first several Fourier coefficients are the most significant). The computing times on a Burroughs 6800 was .17 seconds for c_0 and 1.72 seconds for c_2 .

A third set of figures are included in Table II under the heading "undercorrecting in (13)" for the following reason. In our numerical work we have observed that in "correcting" the coefficients via (13), there has been a tendency to over-correct. The actual coefficients are consistently somewhere between a_k and A_k ; typically about $.2 a_k + .8 A_k$. For that reason we can improve our results (plus save computing time) by putting a factor of, say, .8 in front of the correcting integral in (13) and only making one iteration. Since this phenomenon is not fully understood, we only make this passing remark and illustrate the numerical results. (It seems to suggest that for some reason a weighted average of $e^{i\omega t}$ and $\sqrt{c} e^{i\omega t}$ is a superior u_1 for these purposes.)

Recall that in this development the factor c plays a central role and, as pointed out earlier, this forewarns one that c will have to remain relatively small. In spite of this, we are able to handle the following c which is

continuous but c' is discontinuous; hence technically c'' is a delta function.

Example 3. Consider the "ramp" function:

$$c(\tau) = \begin{cases} 1 & \text{for } 0 \leq \tau < .2 \\ 1 + \frac{5}{2} (\tau - .2) & \text{for } .2 \leq \tau < .8 \\ 2.5 & \text{for } .8 \leq \tau \leq 1.0 \end{cases} \quad (16)$$

In spite of the "corners" in this function, the first ten terms of the Fourier expansion is again adequate for our purposes. Here we are able to achieve almost two figure results with two corrections (or with under-correcting once). The results are summarized in Table III and shown graphically in Fig. 2.

The above example pushes the algorithm of this section to its limit. For example, if c were discontinuous or if it were smooth but such that c'' is large (say greater than 10) the numerical results are typically little better than those obtained by the simpler method of Sec. 2. We shall give examples of this in the next section. Moreover, if more than two figure accuracy is needed the approaches of the last two sections are usually not adequate. In such cases there seems to be little alternative to some form of iteration. We now turn to a straight-forward iteration scheme.

IV. ITERATION

In preparation for iteration we return to (5) and put the u'_S term on the right;

$$i\omega \int_0^1 e^{2i\omega\tau} \gamma \, d\tau = 2i\omega u_S(0;\omega) - \int_0^1 \gamma u'_S(\tau;\omega) e^{i\omega\tau} \, d\tau.$$

Once again we will assume, for simplicity, that $c'(0) = c'(1) = 0$ so that the sine series of γ is appropriate. Hence in the above equality if we divide out the $i\omega$, take the imaginary part and set $\omega = \omega_k = k\pi/2$ we get

$$\begin{aligned} \int_0^1 \gamma \sin k\pi\tau \, d\tau &= 2 \operatorname{Im} u_S(0;\omega_k) + \frac{1}{\omega_k} \operatorname{Re} \int_0^1 \gamma u'_S(\tau;\omega_k) e^{ik\pi\tau/2} \, d\tau \\ &= 1/2 b_k + \frac{1}{\omega_k} \operatorname{Re} \int_0^1 \gamma u'_S(\tau;\omega_k) e^{ik\pi\tau/2} \, d\tau \quad (17) \\ &= 1/2 B_k ; \quad k = 1, 2, \dots \end{aligned}$$

In (17) the b_k are the approximate Fourier coefficients of Sec. 2 and the B_k represent the exact coefficients in

$$\gamma(\tau) = \sum_{k=1}^{\infty} B_k \sin k\pi\tau.$$

Of course the B_k are not directly available since the integral in the right side of (17) involves the unknown γ (both explicitly and implicitly through $u'_S = u'_S(\tau;\omega_k;c)$); hence it will be necessary to iterate. We use the γ and c of Sec. 2 as our starting values; i.e. first one computes

$$\begin{aligned}\gamma_0(\tau) &= \sum_1^N b_k \sin k\pi\tau = 4 \sum_1^N \operatorname{Im} u_s(0; \omega_k) \sin k\pi\tau, \\ c_0(\tau) &= \exp \left[\frac{1}{\pi} \sum_1^N b_k (1 - \cos k\pi\tau)/k \right].\end{aligned}\tag{18}$$

One can then use (17) for iteration, particularly if c is very smooth. However, generally it is better to remove the $\gamma = c'/c$ from the computation in the right side by integration-by-parts. Writing the integrand as $c'[(u'_s/c) e^{i\omega\tau}]$ and integrating c' and differentiating the second term leads to, after considerable simplification, this replacement of (17):

$$\begin{aligned}\int_0^1 \gamma \sin k\pi\tau \, d\tau &= 1/2 B_k = -k\pi \int_0^1 \ln c \cos k\pi\tau \, d\tau \\ &+ [(\operatorname{Im} u + \operatorname{Re} u'/\omega_k) \cos \omega_k + (\operatorname{Re} u - \operatorname{Im} u'/\omega_k) \sin \omega_k]_{\tau} = 1.\end{aligned}\tag{19}$$

The iteration procedure is therefore:

1. Using most recent estimate for c and with $\omega_k = k\pi/2$ for $k = 1, 2, \dots, N$; numerically solve, for $\tau \leq 1$, the initial value problems:

$$\begin{aligned}(u'/c)' + \frac{\omega_k^2}{c} u &= 0 \\ u(0; \omega_k) &= u_s(0; \omega_k) + 1\end{aligned}\tag{20}$$

$$u'(0; \omega_k) = u'_s(0; \omega_k) + i\omega_k = i\omega_k [1 - u_s(0; \omega_k)].$$

Compute B_k using this c and the computed values of u and u' (at $\tau = 1$) in the right side of (19).

2. Compute a new estimate of c and γ by:

$$\gamma(\tau) = \sum_{k=1}^N B_k \sin k\pi\tau \quad (21)$$

$$c(\tau) = \exp \left[\frac{1}{\pi} \sum_{k=1}^N B_k (1 - \cos k\pi\tau) / k \right].$$

Note in (19) the data $u_s(0; \omega_k)$ no longer appears explicitly; however, it is involved in the values of u and u' at $\tau = 1$ since the data defines the initial values of u in (20). We have recommended the solution of the initial value problem (20) rather than the boundary value problem (2) for reason of economy. In practice we have typically needed to repeat Steps 1 and 2 three or four times in seeking two figure accuracy in c (unless $\gamma = c'/c$ is small as in Sec. 2, in which case iteration is unnecessary).

The following examples illustrate the effectiveness of the above procedure for reconstructing c . Both examples are such that the less expensive methods of Sections 1 and 2 are not adequate.

Example 4. Consider

$$c(\tau) = \begin{cases} 1 + 2 \sin^2 2\pi(\tau - .25) & \text{for } .25 \leq \tau \leq .75 \\ 1 & \text{elsewhere.} \end{cases}$$

Although this c is relatively smooth, it does peak rather sharply near $\tau = .5$. Hence it causes difficulty for the approach of Sec. 2 since $c' = 4\pi \sin 4\pi(\tau - .25)$ and consequently $\gamma = c'/c$ is not "small". This is born out in computation; e.g. $c_0(.5) = 2.77$ is off by about 8%. The method

of Sec. 3 is not appropriate since $c'' = 16\pi^2 \cos 4\pi(\tau - .25)$ is too large for the effective use of that approach; see Fig. 3.

If we iterate as outlined above, (19) - (21), we are able to achieve the desired two figure accuracy by taking $N = 20$ and doing three iterations. Again for efficiency we only correct the first five B_k in Step 1. The results are illustrated in Fig. 3. Computing time for this example was: 12.7 seconds (as compared to about .2 seconds to get c_0 and 2 seconds for the method of Sec. 3 when these methods are applicable).

The final example offers a severe test of the robustness of the iteration scheme; c has a large discontinuity, hence c' is a delta function. The Fourier series for c is extremely slow to converge (and the series for γ is divergent; hence the abandoned iteration form (17) is especially unattractive). The proof of convergence of the iteration given in [2] does not apply in this case. Moreover, the Gibbs phenomenon in the series for such a c will eliminate any chance for a good uniform approximation to c . In spite of all this, we shall see that a reasonable reconstruction of c is still possible.

Example 5. Consider the step function

$$c(\tau) = \begin{cases} 1 & \text{for } \tau < .5 \\ 3 & \text{for } \tau \geq .5 \end{cases} .$$

The iteration (19) - (21) was applied with the following simple modification. In solving the differential equations in

(20) some consideration should be given to the fact that the "old" c used is actually a truncated Fourier series of an approximation to the step function above. In particular, it will have unwelcome oscillations, especially near $\tau = .5$, which will give the differential equation solver some difficulty. We have found that the computer time can be reduced significantly (by over 60% in this example) and somewhat more accurate results can be obtained if the old c is smoothed before it is used in (20) and (19). The following simple smoothing scheme has proved very effective:

$$c(\tau_j) \leftarrow \frac{1}{4} [c(\tau_{j-1}) + 2c(\tau_j) + c(\tau_{j+1})]; \quad j = 2, \dots, N-1. \quad (22)$$

Figure 4 illustrates the results of taking four iterations with $N = 40$ (but once again we only correct $B_1, \dots, B_5$). The final c (denoted c_4) was not smoothed in Fig. 4. Smoothing (22) can also be used to make the final c a bit more attractive when graphed; this is shown in Fig. 5. This smoothing can be demonstrated numerically as follows: consider the sub-intervals $[0, .4]$ and $[.6, 1]$ of $[0, 1]$ which exclude the discontinuity at $\tau = .5$. On these sub-intervals the average (absolute value) error in the unsmoothed c_4 (of Fig. 4) is .025 whereas the average error for the smoothed c_4 (of Fig. 5) is .011. In short, smoothing allows us to reconstruct c to two figures (away from the discontinuity). The computer time was 12.0 seconds for the results shown in Fig. 4 or Fig. 5; without any smoothing the time was 31.6 seconds.

V. SUMMARY

We have discussed three related but significantly different schemes for reconstructing c from the original problem (2) and data $u_s(0;\omega)$. In Section 2 we used $u_i = e^{i\omega\tau}$ as our first estimate of u , solution to (2); and demonstrated that if c does not vary much (c'/c small) then c can be reconstructed to perhaps two figures very cheaply (typically .1 to .2 seconds on a fairly slow large scale computer). In Sec. 3 we demonstrated that if c were smooth and if c' did not vary too much (c'' moderate) then the WKB estimate $u_i = \sqrt{c} e^{i\omega\tau}$ makes an effective first estimate. When applicable, using that approach one can reconstruct c to, typically, two figure accuracy at modest cost (say 1 to 2 seconds) even if c'/c is not small.

Unless the variance of c is quite small, both of the above procedures are unlikely to produce more than two figure results since they do ignore a term of some significance in integral equation (3). In Section 4 we discussed an iterative scheme which is more generally applicable. It is considerably more expensive, with two figure results typically taking 10 to 15 seconds (on problems out of the range of the other methods). Moreover, the iteration method is on a sound mathematical foundation as shown in [2] where convergence of the scheme was established. However, to achieve, say, six or eight figure results would mean overcoming some computational hurdles that we have not dealt with. For example, this would entail solving (20) for large N and thus demand

the numerical solution of a large number of highly oscillatory differential equation. We have not pursued these matters since in our applications (and most others) the data is not accurate enough to merit seeking such precision.

REFERENCES

1. S. Gray, SIAM J. Appl. Math., to appear.
2. S. Gray and F. Hagin, SIAM J. Appl. Math., submitted.
3. F. Hagin, SIAM J. Appl. Math., to appear.
4. Y.M. Chen and D.S. Tsien, J. Comput. Phys. 25, No. 4 (1977), 366.
5. H. Bremmer, "The Theory of Electromagnetic Waves" (edited by M. Kline), Dover, New York, 1965.
6. F.W.J. Olver, "Asymptotics and Special Functions," Academic, New York, 1974.

TABLE I

Errors in solving for $c = 1 + \epsilon \sin^2(\pi\tau/2)$ via (8)

Time: .17 seconds per ϵ

ϵ	Max Error	Mean Error
.3	0.001	0.0003
.7	0.011	0.0039
.8	0.016	0.0057
1.0	0.030	0.0106
2.0	0.194	0.0715

TABLE II

Errors in solving for $c = 1 + \epsilon \sin^2(\pi\tau/2)$ via:

Two corrections via (13)-(15)

Undercorrecting in (13)

Time: 1.72 seconds per ϵ

.98 seconds per ϵ

ϵ	Max Error	Mean Error	Max Error	Mean Error
.8	0.009	0.0045	0.006	0.0030
1.0	0.013	0.0061	0.008	0.0037
2.0	0.025	0.0143	0.012	0.0063

TABLE III

Errors in solving for the ramp function (16):

c_0 via (15)

Correcting via (13)-(15)

Undercorr. in (13)

Time: .17 sec.

3.87 sec.

2.00 sec.

Max Err	Mean Err	Max Err	Mean Err	Max Err	Mean Err
0.129	0.051	0.038	0.019	0.031	0.014

Fig. 1. Graphs of $c = 1 + \epsilon \sin^2(\pi x/2)$ for $\epsilon = .7$ and 2 ; and their approximations due to (8) denoted by $+$.

Fig. 2. Graph of the ramp function (16) and its approximation due to two corrections by (13)-(15) denoted by +; and by (8) denoted by . Errors in (13)-(15) solution are: max error = 0.055 and mean error = 0.025.

Fig. 3. Graph of $c = 1 + 2 \sin^2 2\pi(\tau - .25)$ and its approximation due to (19)-(21) denoted by +; and by (13)-(15) denoted by *. Errors in (19)-(21) solution are: max error = .039 and mean error = .016.

Fig. 4. Graph of step function and its approximation due to four iterations (19)-(21). Final iteration, denoted by +, was not smoothed.

Fig. 5. Graph of step function and its approximation due to four iterations (19)-(21). Final iteration, denoted by +, was smoothed by (22).