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Strong Uniqueness of Best Complex Chebyshev Approximation to Analytic Perturbations of Analytic Function*

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Abstract¹

By studying the strong uniqueness properties of best complex Chebyshev approximations to perturbations of a special class of functions, we show that strong uniqueness is not a generic property when we restrict ourselves to "nonoscillatory" functions.

1 Introduction

Let $\mathcal{D}$ be a closed bounded domain in the complex plane $\mathbb{C}$ and $\mathcal{A}(\mathcal{D})$ be the space of complex-valued function analytic in $\mathcal{D}$'s interior (assumed nonempty) and continuous in $\mathcal{D}$. Let $n \geq 1$ be a fixed integer and let $\Phi_1, \Phi_2, \dots, \Phi_n \in \mathcal{A}(\mathcal{D})$ be functions that satisfy the Haar condition. It is well known ([10], [5]) that for any $F \in \mathcal{A}(\mathcal{D})$, there exist a unique

$$P^* \in \mathcal{P}\{c_1\Phi_1 + \dots + c_n\Phi_n \mid c_j \in \mathbb{C}\}$$

at which the norm $\|F - P^*\|$ is minimized thus:

$$\|F - P^*\| := \max_{z \in \mathcal{D}} |F(z) - P^*(z)| \leq \|F - P\| \quad \text{for all } P \in \mathcal{P}.$$

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In real approximation, best Chebyshev approximation from subspace that satisfies the Haar condition is always strongly unique [6]. Although the result does not hold in complex approximation [3], Blatt [1] shows that in the space of continuous (not analytic) function $\mathcal{C}(\mathcal{D})$, the subset of function whose best Chebyshev approximation is strongly unique is dense in $\mathcal{C}(\mathcal{D})$. This result, however, does not seem to well describe the situation in $\mathcal{A}(\mathcal{D})$ because examples of nonstrong uniqueness abound (cf. [1] and also [7], [2] in light of [3]).

In this paper, we study the strong uniqueness property of best approximation to analytic perturbations $F + \alpha G$ to a certain class of F 's in $\mathcal{A}(\mathcal{D})$ (cf. Definition 2 for details). Among other things, we show that regardless if F 's best approximation is strongly unique or not, such property remains unchanged by small perturbation (corollaries 1 and 2 of Theorem 2); and we prove that in "nonoscillatory" subspace of $\mathcal{A}(\mathcal{D})$, the set of F 's with strongly unique approximations is by no means dense (Corollary 3 of Theorem 2).

2 Notation and Formulation

In this section, we transform our complex approximation problem in complex parameters to a complex approximation problem in real parameters. Moreover, we set up the notation and definitions to be used throughout the paper and state our problem formally.

To state the approximation problem in terms of real parameters, we first apply the Maximum Modulus Theorem to reduce the domain of the problem from $\mathcal{D}$ to its boundary $\partial\mathcal{D}$. Consequently, we can replace our domain by $[0, 1]$ via a parametrization $\gamma : [0, 1] \rightarrow \partial\mathcal{D}$. Furthermore, we will assume that $\partial\mathcal{D}$ is a simple smooth curve, and that the second derivative γ'' is continuous. Thus, the function space can be replaced by

$$\mathcal{F} := \left\{ f : [0, 1] \rightarrow \mathbb{C} \mid f = F \circ \gamma, F \in \mathcal{A}(\mathcal{D}) \right\},$$

and the approximants by

$$\mathcal{P} := \left\{ \lambda_1 \varphi_1 + \cdots + \lambda_{2n} \varphi_{2n} \mid \lambda_j \in \mathbb{R}, j = 1, 2, \dots, 2n \right\},$$

where

$$\varphi_j := \Phi_j \circ \gamma, \quad \varphi_{n+j} := i\Phi_j \circ \gamma \quad \text{for } j = 1, 2, \dots, n.$$

Given $\lambda \in \mathbb{R}^{2n}$, we denote $\lambda_1 \varphi_1 + \cdots + \lambda_{2n} \varphi_{2n}$ by p_λ . Thus, given any $f \in \mathcal{F}$, there is a unique $\lambda^* \in \mathbb{R}^{2n}$ such that

$$\|f - p_{\lambda^*}\| := \max_{t \in [0, 1]} |f(t) - p_{\lambda^*}(t)| \leq \|f - p_\lambda\| \quad \text{for all } \lambda \in \mathbb{R}^{2n}.$$

Moreover, the norm $\|\cdot\|$ when applied to any vector μ in $\mathbb{R}^{2n}$ means

$$\|\mu\| := \max_{1 \leq j \leq 2n} |\mu_j|.$$

Given any g in $\mathcal{F}$, we use $\text{ext}(g)$ to denote the set of g 's extrema in $[0, 1)$ thus:

$$\text{ext}(g) := \left\{ t \in [0, 1) \mid |g(t)| = \|g\| \right\}.$$

In this setting with real parameters, the usual definition of primitive extremal signature ([4], [9], [3]) can be reformulated as below:

Definition 1 For any $(t, \vartheta) \in [0, 1) \times [0, 2\pi)$, define the vector $\mathbf{v}(t, \vartheta)$ in $\mathcal{R}^{2n}$ by

$$\mathbf{v}^T(t, \vartheta) := \text{Re}(e^{-i\vartheta} [\varphi_1(t), \varphi_2(t), \dots, \varphi_{2n}(t)]).$$

A set $U \subset [0, 1) \times [0, 2\pi)$ is said to be a primitive extremal signature if and only if the followings hold:

1. The convex hull $\text{conv}(\mathcal{V})$ generated by

$$\mathcal{V} := \{ \mathbf{v}(t, \vartheta) \mid (t, \vartheta) \in U \}$$

contains the origin $\mathbf{o} \in \mathcal{R}^{2n}$.

2. The convex hull $\text{conv}(\mathcal{V}')$ for any proper subset $\mathcal{V}' \subset \mathcal{V}$ does not contain the origin.

Primitive extremal signature is useful in the characterization of best approximation ([9], [5], [4]): Given $f \in \mathcal{F}$, $p_{\lambda^\bullet}$ is the best approximation if and only if the set

$$\{(t, \vartheta) \mid t \in \text{ext}(f - p_{\lambda^\bullet}) \text{ and } \vartheta = \text{Arg}([f - p_{\lambda^\bullet}](t))\}$$

contains a primitive extremal signature.

In this paper, we study the best approximation of perturbations $f + \alpha g$ to a certain class of f 's which we now define.

Definition 2 Let $p_{\lambda^\bullet}$ best approximate f . Then f is said to be in $\mathcal{S}$ if and only if the following conditions hold:

1. The set $\{(t, \vartheta) \mid t \in \text{ext}(f - p_{\lambda^\bullet}) \text{ and } \vartheta = \text{Arg}([f - p_{\lambda^\bullet}](t))\}$ is a primitive extremal signature.
2. The second derivative $\frac{\partial^2}{\partial t^2} |f - p_{\lambda^\bullet}(t)|$ is nonzero (strictly negative) at each $t \in \text{ext}(f - p_{\lambda^\bullet})$.

It is well known (cf. [4]) that primitive extremal signatures have cardinality varying from $n + 1$ to $2n + 1$. Thus, whenever $f \in \mathcal{S}$,

$$n + 1 \leq |\text{ext}(f - p_{\lambda^\bullet})| \leq 2n + 1,$$

and we can make the definition below.

Definition 3 The deficiency d of a function $f \in \mathcal{S}$ is defined by

$$d := 2n + 1 - |\text{ext}(f - p_{\lambda^\bullet})|.$$

Gutknecht's result in [3], when applied to any function f in $\mathcal{S}$, says that the best approximation to f is strongly unique if and only if f 's deficiency d is zero. Our main result (Theorem 2) is that given $f \in \mathcal{S}$ with deficiency d , and any $g \in \mathcal{F}$, $\|g\| = 1$, there is an $\varepsilon > 0$ that depends only on f , $\mathcal{P}$, and $\|g'\|$ such that whenever $|\alpha| < \varepsilon$, $f + \alpha g$ remains in $\mathcal{S}$ with the same deficiency d of f .

3 Perturbation and Local Extrema

Our result relies heavily on how the extrema of the optimal error graph of $f + \alpha g$ varies as α changes. We study the change of local extrema in this section, and the change of global extrema in the next.

For the rest of the presentation, f is a fixed function in $\mathcal{S}$ with deficiency d . Without loss of generality, we assume that the best approximation to f is the zero polynomial and denote the extrema of the optimal error graph by

$$\text{ext}(f) := \{t_1^*, t_2^*, \dots, t_{2n+1-d}^*\}.$$

In this section, let g be a fixed function in $\mathcal{F}$, $\|g\| = 1$ and $\|g'\| \leq M$ for some $M > 0$.

Lemma 1 There exist positive constants R_1 and R_2 that depends only on f , $\mathcal{P}$, and M such that for any $\alpha \in \mathcal{R}$ and $\mu \in \mathcal{R}^{2n}$ satisfying

$$|\alpha| < R_2 \quad \|\mu\| < R_2,$$

the function $|f + \alpha g - p_\mu|$ would have exactly one local extremum $\hat{t}$,

$$\frac{d}{dt} |(f + \alpha g - p_\mu)(\hat{t})| = 0,$$

in each of the $2n + 1 - d$ intervals $(t_j^* - R_1, t_j^* + R_1)$, $j = 1, 2, \dots, 2n + 1 - d$.

The proof Lemma 1 is a straightforward application of the following constructive form of the Implicit Function Theorem.

Theorem 1 Let $\mathbf{H} : U \subseteq \mathcal{R}^\ell \times \mathcal{R}^m \rightarrow \mathcal{R}^\ell$ be a function such that $\mathbf{H}(\mathbf{x}^0, \mathbf{y}^0) = \mathbf{0}$ at some $(\mathbf{x}^0, \mathbf{y}^0) \in U$. Moreover, at $(\mathbf{x}^0, \mathbf{y}^0)$ (1) the Jacobian $\mathbf{H}_{\mathbf{x}}$ of $\mathbf{H}$ with respect to $\mathbf{x}$ is nonsingular, and (2) both $\mathbf{H}$ and $\mathbf{H}_{\mathbf{x}}$ are Lipschitz continuous with Lipschitz constant L . Then, there exist positive constants σ_1 and σ_2 that depend only on L and $\|\mathbf{H}_{\mathbf{x}}^{-1}(\mathbf{x}^0, \mathbf{y}^0)\|$ such that for each $\mathbf{y}$, $\|\mathbf{y} - \mathbf{y}^0\| < \sigma_2$, there is a unique $\mathbf{x}$, $\|\mathbf{x} - \mathbf{x}^0\| < \sigma_1$, satisfying $\mathbf{H}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

Proof of Theorem 1: The proof is a straightforward modification of the one given in 5.1.9 and 5.2.4 of [8]. QED

Proof of Lemma 1: Define the function $\mathbf{H} : [0, 1)^{2n+1-d} \times \mathcal{R}^{2n+1} \rightarrow \mathcal{R}^{2n+1-d}$ by

$$\mathbf{H}_i(t_1, \dots, t_{2n+1-d}, \mu_1, \dots, \mu_{2n}, \alpha) := \frac{d}{dt} \left\| f + \alpha g - p_{\mu} \right\|_{t=t_i}, \quad i = 1, 2, \dots, 2n+1-d.$$

Clearly, $\mathbf{H} = \mathbf{o}$ at

$$(\mathbf{t}^*, \mathbf{o}) := (t_1^*, t_2^*, \dots, t_{2n+1-d}^*, 0, 0, \dots, 0).$$

Moreover, the Jacobian of $\mathbf{H}$ with respect to the $\mathbf{t}$ at $(\mathbf{t}^*, \mathbf{o})$ is a diagonal matrix with the j -th diagonal entry equals $\frac{d^2}{dt^2} \left\| f(t_j^*) \right\|$ which is nonzero by assumption. Hence $\mathbf{H}_{\mathbf{t}}(\mathbf{t}^*, \mathbf{o})$ is nonsingular and $\|\mathbf{H}_{\mathbf{t}}^{-1}(\mathbf{t}^*, \mathbf{o})\|$ depends only on f . Furthermore, the function $\mathbf{H}$ and $\mathbf{H}_{\mathbf{t}}$ are clearly Lipschitz at $(\mathbf{t}^*, \mathbf{o})$ with Lipschitz constants that depend only on f , $\mathcal{P}$, and M . Consequently, the lemma can be established by applying Theorem 1. QED

4 Perturbation and Global Extrema

In this section, we show that for α small enough, the error graph $f + \alpha g - p_{\mu}$ corresponding to the best approximation p_{μ} of the function $f + \alpha g$ has global extrema close to those of f . The first lemma shows that the best approximation polynomial varies continuously with the function it best approximates.

Lemma 2 *Let $g \in \mathcal{F}$, $\|g\| = 1$, be fixed. For any given $\varepsilon > 0$, there is a $\delta > 0$ that depends only on f and $\mathcal{P}$ such that $\|\mu\| < \varepsilon$ whenever p_{μ} best approximates $f + \alpha g$, where $|\alpha| < \delta$.*

Proof of Lemma 2: By the results in [9] and [3], there are positive constants c and c' that depend only on f and $\mathcal{P}$ such that whenever $\|\mu\| < c'$,

$$\|f - p_{\mu}\| \geq \|f\| + c\|\mu\|^{\beta}, \quad \begin{cases} 1 & \text{if } d = 0 \\ \frac{1}{2} & \text{if } d > 0 \end{cases},$$

where d is the deficiency of f . Now, let p_{μ} best approximate $f + \alpha g$.

$$\begin{aligned} \|\mu\|^{\beta} &\leq \frac{1}{c} (\|f - p_{\mu}\| - \|f\|) \\ &\leq \frac{1}{c} (\|f + \alpha g - p_{\mu}\| - \|f\| + \|\alpha g\|) \\ &\leq \frac{1}{c} (\|f + \alpha g\| - \|f\| + \|\alpha g\|) \\ &\leq \frac{1}{c} (\|f\| + \|\alpha g\| - \|f\| + \|\alpha g\|) \\ &\leq \frac{2}{c} |\alpha| \|g\|. \end{aligned}$$

Thus, $\|\mu\| \leq \text{constant} \cdot |\alpha|^{1/\beta}$ where “constant” depends only on f and $\mathcal{P}$. This completes the proof. QED

Lemma 3 *Given $\varepsilon > 0$, there exists a $\delta > 0$ that depends only on f such that for any $h \in \mathcal{F}$ satisfying $\|h\| < \delta$,*

$$\text{ext}(f + h) \subseteq \bigcup_{j=1}^{2n+1-d} (t_j^* - \varepsilon, t_j^* + \varepsilon).$$

Proof of Lemma 3: The lemma holds because $f + h$ converges to f uniformly as $\|h\| \rightarrow 0$ and f has only $2n + 1 - d$ (a finite number) extrema. QED

Lemma 4 *Let $g \in \mathcal{F}$, $\|g\| = 1$, be fixed. Then for any given $\varepsilon > 0$, there exist a $\delta > 0$ that depends only on f and $\mathcal{P}$ such that whenever p_μ best approximates $f + \alpha g$ where $|\alpha| < \delta$,*

$$\text{ext}(f + \alpha g - p_\mu) \subseteq \bigcup_{j=1}^{2n+1-d} (t_j^* - \varepsilon, t_j^* + \varepsilon).$$

Proof of Lemma 4: Observe that for any α and μ ,

$$\|\alpha g - p_\mu\| \leq |\alpha| + K\|\mu\|,$$

where K depends only on $\mathcal{P}$. With this observation, the lemma follows from Lemma 2 and Lemma 3. QED

5 Perturbation and Strong Uniqueness

In this section, we show that

$$|\text{ext}(f + \alpha g - p_\mu)| = 2n + 1 - d$$

whenever α is small enough. The results we have already established imply (see Lemma 5 below)

$$|\text{ext}(f + \alpha g - p_\mu)| \leq 2n + 1 - d.$$

We will establish that equality holds for the above (Theorem 2) after Lemma 6 is proved.

Lemma 5 *Let $g \in \mathcal{F}$, $\|g\| = 1$ and $\|g'\| \leq M$, be fixed. Then there exist $\varepsilon > 0$ that depends only on f , $\mathcal{P}$, and M such that whenever p_μ best approximates $f + \alpha g$ where $|\alpha| < \varepsilon$,*

$$|\text{ext}(f + \alpha g - p_\mu)| \leq 2n + 1 - d.$$

Proof of Lemma 5: Let R_1 and R_2 be the two positive numbers (depending only on f , $\mathcal{P}$, and M) given in Lemma 1. By Lemma 4, there is a $\varepsilon_1 > 0$ depending only on f , $\mathcal{P}$, and M such that whenever p_μ best approximates $f + \alpha g$, $|\alpha| < \varepsilon_1$, then

$$\text{ext}(f + \alpha g - p_\mu) \subseteq \bigcup_{j=1}^{2n+1-d} (t_j^* - R_1/2, t_j^* + R_1/2).$$

Next, by Lemma 2, there exist $\varepsilon_2 > 0$ that depends only on f and $\mathcal{P}$ such that whenever $|\alpha| < \varepsilon_2$, $\|\mu\| < R_2$.

Now, define

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, R_2/2\}.$$

Then, whenever $|\alpha| < \varepsilon$, we must have

$$|\alpha| < R_2 \quad \text{and} \quad \|\mu\| < R_2.$$

Thus by Lemma 1, $f + \alpha g - p_\mu$ has at most one local extrema in each of the $2n + 1 - d$ intervals

$$(t_j^* - R_1, t_j^* + R_1), \quad j = 1, 2, \dots, 2n + 1 - d.$$

Consequently,

$$|\text{ext}(f + \alpha g - p_\mu)| \leq 2n + 1 - d.$$

QED

Finally, we must prove that

$$|\text{ext}(f + \alpha g - p_\mu)| = 2n + 1 - d.$$

The rough idea is as follows. Since p_μ best approximates $f + \alpha g$, the set

$$U_\alpha := \left\{ (t, \vartheta) \mid t \in \text{ext}(f + \alpha g - p_\mu), \quad \vartheta = \text{Arg}([f + \alpha g - p_\mu](t)) \right\}$$

must contain a primitive extremal signature. Moreover, whenever α is small, each $(t, \vartheta) \in U_\alpha$ must be close to an element in the set

$$U^* := \left\{ (t_j^*, \vartheta_j^*) \mid \vartheta_j^* = \text{Arg}(f(t_j^*)), \quad j = 1, 2, \dots, 2n + 1 - d \right\},$$

which is a primitive extremal signature by assumption. Consequently, U_α must contain at least $2n + 1 - d$ elements.

Lemma 6 *There exists an R that depends only on f and $\mathcal{P}$ such that whenever a primitive extremal signature*

$$U := \left\{ (t_1, \vartheta_1), (t_2, \vartheta_2), \dots, (t_m, \vartheta_m) \right\}, \quad m \leq 2n + 1 - d,$$

satisfies the condition

$$\min_{1 \leq j \leq 2n+1-d} (|t_i - t_j^*| + |\vartheta_i - \vartheta_j^*|) < R \quad \text{for } i = 1, 2, \dots, m,$$

then $m = 2n + 1 - d$.

Proof of Lemma 6: Recall the definition

$$\mathbf{v}^T(t, \vartheta) := \operatorname{Re} \left(e^{-i\vartheta} [\varphi_1(t), \varphi_2(t), \dots, \varphi_{2n}(t)] \right),$$

and define

$$\mathcal{V}^* := \left\{ \mathbf{v}(t_j^*, \vartheta_j^*) \mid j = 1, 2, \dots, 2n+1-d \right\}.$$

Since $\mathcal{U}^*$ is a primitive extremal signature, therefore $\mathbf{o} \in \operatorname{conv}(\mathcal{V}^*)$ and that

$$\mathbf{o} \notin \operatorname{conv}(\mathcal{V}) \quad \text{for any proper subset } \mathcal{V} \subset \mathcal{V}^*.$$

Thus, the Euclidean distance from $\mathbf{o} \in \mathcal{R}^{2n}$ to the convex hull $\operatorname{conv}(\mathcal{V})$ generated by any one of the finite number of proper subsets $\mathcal{V} \subset \mathcal{V}^*$ is positive. Thus, there exist a $\delta > 0$ such that $\mathbf{o} \notin \operatorname{conv}(\mathcal{W})$ for any set $\mathcal{W}$ in $\mathcal{R}^{2n}$ that satisfies

$$\operatorname{dist}(\mathcal{W}, \mathcal{V}) < \delta, \quad \text{for some proper subset } \mathcal{V} \subset \mathcal{V}^*.$$

By continuity of $\mathbf{v}(t, \vartheta)$ in t and ϑ , there exist a positive number R that depends only on f and $\mathcal{P}$ such that the inequality

$$|t - t_j^*| + |\vartheta - \vartheta_j^*| < R$$

implies

$$\|\mathbf{v}(t, \vartheta) - \mathbf{v}(t_j^*, \vartheta_j^*)\| < \delta$$

for $j = 1, 2, \dots, 2n+1-d$. Now, let

$$U := \left\{ (t_1, \vartheta_1), (t_2, \vartheta_2), \dots, (t_m, \vartheta_m) \right\}, \quad m \leq 2n+1-d,$$

be a primitive extremal signature that satisfies the condition

$$\min_{1 \leq j \leq 2n+1-d} (|t_i - t_j^*| + |\vartheta_i - \vartheta_j^*|) < R \quad \text{for } i = 1, 2, \dots, m,$$

and consider the set

$$\mathcal{W} := \left\{ \mathbf{v}(t, \vartheta) \mid (t, \vartheta) \in U \right\}.$$

Then $m < 2n+1-d$ would imply

$$\operatorname{dist}(\mathcal{W}, \mathcal{V}) < \delta \quad \text{for some proper subset } \mathcal{V} \subset \mathcal{V}^*.$$

This in turn implies $\mathbf{o} \notin \operatorname{conv}(\mathcal{W})$, which is a contradiction. Thus, we must have $m = 2n+1-d$ as claimed. QED

We have now gathered all the facts that will allow us to prove our main theorem.

Theorem 2 *Given any $f \in \mathcal{S}$ with deficiency d and any $g \in \mathcal{F}$, $\|g\| = 1$ and $\|g'\| \leq M$, there is an $\varepsilon > 0$ that depends only on f , $\mathcal{P}$, and M such that whenever $|\alpha| < \varepsilon$, $f + \alpha g$ remains in $\mathcal{S}$ with the same deficiency d of f .*

Proof of Theorem 2: By Lemma 5, there is an $\varepsilon_1 > 0$ such that whenever $|\alpha| < \varepsilon_1$,

$$|\text{ext}(f + \alpha g - p_\mu)| \leq 2n + 1 - d.$$

Let R be the number specified by Lemma 6. Then by Lemma 2, there exists an $\varepsilon_2 > 0$ such that whenever $|\alpha| < \varepsilon_2$,

$$\min_{1 \leq j \leq 2n+1-d} (|t - t_j^*| + |\vartheta - \vartheta_j^*|) < R$$

for any pair (t, ϑ) , $t \in \text{ext}(f + \alpha g - p_\mu)$ and $\vartheta = \text{Arg}([f + \alpha g - p_\mu](t))$. Now, define

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}.$$

By Lemma 6, we must have

$$|\text{ext}(f + \alpha g - p_\mu)| = 2n + 1 - d.$$

That completes the proof. QED

Corollary 1 *Let $f \in S$ have a strongly unique best approximation, and let $g \in \mathcal{F}$ be chosen arbitrarily. Then there exist $\varepsilon > 0$ that depends only on f , $\mathcal{P}$, and $\|g'\| / \|g\|$ such that the best approximation to $f + \alpha g$ for any $|\alpha| < \varepsilon$ is strongly unique.*

Corollary 2 *Let $f \in S$ have a nonstrongly unique best approximation, and let $g \in \mathcal{F}$ be chosen arbitrarily. Then there exist $\varepsilon > 0$ that depends only on f , $\mathcal{P}$, and $\|g'\| / \|g\|$ such that the best approximation to $f + \alpha g$ for any $|\alpha| < \varepsilon$ is nonstrongly unique.*

Corollary 3 *Let $f \in S$ have a nonstrongly unique best approximation. Consider the subset*

$$\mathcal{G} := \left\{ g \in \mathcal{F} \mid \frac{\|g'\|}{\|g\|} \leq \frac{\|f'\| + 1}{\|f\|} \right\}$$

of $\mathcal{F}$. (In particular $f \in \mathcal{G}$.) Then strong uniqueness is not generic in $\mathcal{G}$.

Proof of Corollary 3: Define $B := \{g \in \mathcal{G} \mid \|g\| \leq 1\}$ and define $M := (\|f'\| + 1) / \|f\|$. Then, $\|g'\| \leq M\|g\|$ for all $g \in B$. Thus, there exist an $\varepsilon > 0$ such that the best approximation of $f + \alpha g$ is nonstrongly unique for all $g \in B$ and for all $|\alpha| < \varepsilon$. Therefore, the set of functions in $\mathcal{G}$ with strongly unique approximations is by no means dense. QED

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