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Abstract

INTERFACING THEORY AND EXPERIMENT IN POLARIZATION STUDIES *

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The results of this paper pertain to any particle reaction in atomic, nuclear, or high energy physics, and deal with some of the general properties of such reactions. A formalism for discussing polarization experiments in such reactions is suggested in which appropriately chosen reaction amplitudes are directly related to quantities actually measured in experiments, thus making the connections between theory and experiment as simple as possible and eliminating intermediary quantities between theory and experiment. It is proven that such direct experimental observables along two quantization directions always suffice in completely determining the amplitudes from experimental data. It is also shown that given a specific theoretical model prediction for the reaction amplitudes in any formulation, one can always find a quantization direction such that the measurement of the number of particles in one single projection state with respect to that quantization direction forms a null-experiment to test the theoretical prediction. Three examples of particle reactions are worked out in detail to illustrate the problem and the suggested formalism. In the concluding section the main results are summarized and their implications are discussed with respect to the design of experiments and the interfacing of theories with experimental verification.

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I. INTRODUCTION

The use of polarization experiments as a source of information about particle reactions in atomic, nuclear, or high energy physics has been increasing substantially in recent years.^{1,2} The types of information sought include tests of symmetries, phenomenological determinations of reaction amplitudes, tests of specific theoretical dynamical models, etc. In all these objectives, one faces the problem of deriving information about the reaction amplitudes from measurements of polarization quantities.

The procedure, as done in the past, involves three steps: The choice of some set of reaction amplitudes, the choice of some set of polarization quantities, and the determination of these polarization quantities from the actual measurements.

The criteria for performing these three tasks have been different. Amplitudes are generally chosen on the basis of a theoretically or phenomenologically "natural" expression for the M-matrix. The polarization quantities have been chosen on the basis of traditional conventions which appear "natural" within the framework of some formal phenomenology of polarizations. Finally, the quantities actually measured in experiments have of course been determined by the practical convenience set by the way polarized beams, targets and end products are prepared and/or detected.

This hybrid set of criteria has resulted in a far less than ideal coordination, efficiency and effectiveness in the use of polarization experiments. Since often the polarization quantities are, on the one hand, unnecessarily complicated in terms of the amplitudes chosen, and on the other hand are complicated in terms of the actual experimental results, there are two interfaces which impede the process of getting information about amplitudes from measurements.

In this paper we would like to attempt a formulation of a smooth interfacing of these various stages. In particular, our aim will be to choose amplitudes which are "natural" from the point of view of the polarization measurements, and to eliminate, to the maximal possible extent, the intermediate step of defining polarization quantities, thus relating amplitudes directly to measurements.

The rationale behind this method is that it is not worthwhile to choose amplitudes which are pretty by some abstract theoretical or general phenomenological standard, since all complete sets of amplitudes are equivalent and the transformation from one to another is merely a matter of algebra, involving purely linear relations, and no accumulation of errors, uncertainties, or unwanted dependences. Furthermore, we believe that the introduction of polarization quantities (various entities designated as polarization, asymmetry, spin correlation tensors, depolarization, polarization transfer tensors, etc.) is altogether an unnecessary step which can only complicate the relations and compound errors and uncertainties.

The paper will consist of five parts. After this introduction, three examples will be presented in detail, illustrating the type of simplifications we hope to obtain from our formulation. This is followed by the proof of a theorem about the minimum number of polarization directions in which measurements must be carried out in order to determine the amplitudes. The fourth section then gives a general discussion of the way we propose to define amplitudes so they are directly and simply connected to polarization measurements. The last section contains some concluding remarks.

II. THREE EXAMPLES

In this section three specific types of reactions will be discussed to illustrate the differences between various ways of coordinating amplitudes,

polarization quantities, and experiments. The three reactions are $0 + \frac{1}{2} \rightarrow 0 + \frac{1}{2}$, $1 + 0 \rightarrow 0 + 0$, and $2 + 0 \rightarrow 0 + 0$, where 0, $\frac{1}{2}$, 1, and 2 denote particles of spin 0, $\frac{1}{2}$, 1, and 2, respectively.

$$0 + \frac{1}{2} \rightarrow 0 + \frac{1}{2}.$$

With only rotation (or Lorentz-) invariance assumed, this reaction has four amplitudes. Some of the customary ways of writing the M-matrix are, therefore,

$$M = a_0 + a_1 \vec{\sigma} \cdot \hat{x} + a_2 \vec{\sigma} \cdot \hat{y} + a_3 \vec{\sigma} \cdot \hat{z}, \quad (2.1)$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in three non-coplanar directions, or

$$M = b_0 + b_1 \vec{\sigma} \cdot \vec{q} + b_2 \vec{\sigma} \cdot \vec{q}' + b_3 \vec{\sigma} \cdot \vec{q} \times \vec{q}', \quad (2.2)$$

where q and q' are two momenta determining the kinematics of the reaction (e.g. initial and final center-of-mass momenta), or

$$M = c_0 + c_1 \vec{\sigma} \cdot \hat{\ell} + c_2 \vec{\sigma} \cdot \hat{m} + c_3 \vec{\sigma} \cdot \hat{n}, \quad (2.3)$$

where $\hat{\ell}$, $\hat{m}$, and $\hat{n}$ are defined as

$$\hat{\ell} = \frac{\vec{q}' - \vec{q}}{|\vec{q}' - \vec{q}|}, \quad \hat{m} = \frac{\vec{q} \times \vec{q}'}{|\vec{q} \times \vec{q}'|}, \quad \hat{n} = \hat{\ell} \times \hat{m}, \quad (2.4)$$

with q and q' defined as before, or

$$M = d_0 + d_1 K + d_2 \gamma_5 + d_3 \gamma_5 K, \quad (2.5)$$

where K is some four-momentum in the kinematics of the reaction.

In contrast, we want to define the amplitudes for this reactions as

$$M = \begin{pmatrix} A_{11} & A_{1-1} \\ A_{-11} & A_{-1-1} \end{pmatrix} \quad (2.6)$$

where the labels of the rows are interpreted as meaning "up" and "down" orientation of the initial spin in terms of some fixed z direction, and the labels of the columns mean "up" and "down" orientation of the final spin in terms of some fixed z' direction. For the moment z and z' can be left completely arbitrary, and they may or may not be identical.

For a specified z and z' , it is easy to construct the linear relations that give the A's in terms of the a's, or the b's, or the c's, or the d's.

As far as the customary polarization quantities are concerned, for such a reaction (with parity conservation and with the product of the four intrinsic parities being +1) they are called P, A and R. They are defined³ with respect to $\vec{k}$, the beam momentum, $\vec{k}'$, the recoil nucleon momentum, $\hat{n} = \vec{k} \times (-\vec{k}') / |\vec{k} \times \vec{k}'|$, and $\hat{s} = \hat{n} \times \vec{k}'$. Then, for an unpolarized target, the recoil polarization is $P = (N_+^n - N_-^n) / (N_+^n + N_-^n)$, where N_+^n (N_-^n) is the number of particles with spin "up" ("down") along $\hat{n}$. For the target polarized along $\hat{k}$, with $P_i = (N_+^k - N_-^k) / (N_+^k + N_-^k)$, the recoil polarization is

$$AP_i = \frac{N_+^s - N_-^s}{N_+^s + N_-^s}, \text{ along } \hat{s}, \quad RP_i = \frac{N_+^{k'} - N_-^{k'}}{N_+^{k'} + N_-^{k'}}, \text{ along } \hat{k}'. \quad (2.7)$$

As mentioned earlier, we want to deemphasize the intermediate polarization quantities and connect the primary entities in the measurements with the amplitudes. Thus we will use directly the number of particles (either in the initial or in the final state) with spins in the "up" and "down" directions along some quantization directions. We will show in the next section that one needs no more (but no less) than two such directions to determine the amplitudes if one measures all the N_m^s 's in those directions. Although the two directions can be chosen arbitrarily (as long as they are not collinear), simplicity suggests choosing them orthogonally to each other. If then the rows and the columns of the M-matrix are labeled by the first of these two

(which we will call z in the initial and z' in the final state), and the second direction is chosen to be the y or y' directions, then the initial and final state operators (density matrices) we will consider are

$$\begin{aligned}
 Q_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \frac{d\sigma}{d\Omega} , \\
 Q_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow N_+^z, N_+^{z'} \\
 Q_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow N_-^z, N_-^{z'} \\
 Q_4 &= \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \rightarrow N_+^y, N_+^{y'} \\
 Q_5 &= \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \rightarrow N_-^y, N_-^{y'}
 \end{aligned} \tag{2.8}$$

where, just for reference, the operator corresponding to the usual differential cross section (Q_1) is also included.

We see that while our M-matrix has an "optimal" form,⁴ and so do Q_2 and Q_3 , the other two Q 's, namely Q_4 and Q_5 do not. In an optimal formalism they should have only two non-zero elements so that these satisfy the requirement of Hermiticity. Actually Q_4 and Q_5 also have diagonal elements, however. Thus they will result in a more complicated relationship than the optimal representation. In the latter, for this reaction, the observable-bicom⁵ table contains only 1×1 and 2×2 submatrices, so with Q_4 and Q_5 we can expect submatrices larger than that. This is the necessary price one has to pay for dealing with quantities which are the results of actual experiments.

These expectations are borne out by the actual traces pertaining to the "observables", which are given in Table 1. We see that the observables containing only Q_2 or Q_3 give only 1×1 submatrices, while those containing one of Q_2 or Q_3 and one of Q_4 or Q_5 contain three bicombs each. Sums and differences

of these, however, break up the relationship into groups of two bicombs and one bicom. Finally, the observables containing only Q_4 or Q_5 are related to 8 or 10 bicombs, and even sums and differences resolve this only into groups of 2, 4, or 6.

It is also evident, however, that in order to determine the amplitudes one does not need these fancy and complicated observables. Using the notation of $L(Q_i, Q_j) = L_{ij}$, we see that for example the simple set of L_{22} , L_{23} , L_{32} , L_{33} , L_{24} , L_{34} , and L_{42} determine seven bicombs individually, which form a "good" set.⁵ Thus in this case not only do observables in two polarization directions suffice in the determination of the amplitudes, but in fact a quite simple subset of such observables will do the job, and the determination is maximally direct in that the direct results of experiments or simple differences of two such experiments yield the amplitudes individually.

$$1 + 0 + 0 + 0$$

With only rotation (or Lorentz-) invariance assumed, this reaction has three amplitudes, so that the M-matrix can be written, for example, as

$$M = a_0 S_x + a_1 S_y + a_2 S_z \tag{2.9}$$

or

$$M = b_0 \epsilon_\mu k_\mu + b_1 \epsilon_\mu q_\mu + b_2 \epsilon_{\lambda\mu\nu\rho} \epsilon_\lambda k_\mu q_\nu p_\rho \tag{2.10}$$

where the S 's are 1×3 matrices and ϵ_μ is the polarization fourvector of the spin-1 particle, $\epsilon_{\lambda\mu\nu\rho}$ is the four-dimensional Levi-Civita tensor, and k_μ , q_μ , and p_μ are three four-momenta characterizing the kinematics.

In contrast, we want to define our amplitudes as

$$M = \begin{pmatrix} A_1 & A_0 & A_{-1} \end{pmatrix} \tag{2.11}$$

where the labels of the columns are interpreted as denoting the 1, 0, and

-1 projections on a fixed but arbitrary z axis.

The customary polarization quantities for a spin-1 particle are somewhat lengthy but readily available in the literature.¹ Instead, however, we want to use again the direct particle counts with projections 1, 0, and -1 in the z direction, and, since these are not sufficient for the complete determination of the three amplitudes, also the direct particle counts with projections 1, 0, and -1 in the y direction. The latter can be calculated as follows: Since the operator corresponding to N_{+1}^z can be obtained for the spin-1 case from $\frac{1}{2} [S_z + S_z S_z]$, where S_z and S_{zz} are the customary vector and tensor spin operators, and similarly the operator corresponding to N_0^z is

$I - S_z S_z$, hence, by analogy, we get

$$\begin{aligned} N_{\pm 1}^y &\rightarrow \frac{1}{2} [\pm S_y + S_y S_y] = \frac{1}{4} \begin{pmatrix} 1 & \mp \sqrt{2}i & -1 \\ \pm \sqrt{2}i & 2 & \mp \sqrt{2}i \\ -1 & \pm \sqrt{2}i & 1 \end{pmatrix} \\ N_0^y &\rightarrow \frac{1}{2} [I - S_y S_y] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (2.12)$$

This is admittedly an ad hoc method of obtaining the operators in the y direction, and indeed, for our last example of spin 2, we will resort to a more general procedure to perform the analogous calculation.

The results for the spin-1 case for the operators in question are (again including, just for reference, also the operator corresponding to differential cross section)

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \frac{d\sigma}{d\Omega}$$

$$Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow N_{+1}^z, \quad Z_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow N_0^z,$$

$$Z_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow N_{-1}^z,$$

$$Z_5 = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{2}i & -1 \\ +\sqrt{2}i & 2 & -\sqrt{2}i \\ -1 & \sqrt{2}i & 1 \end{pmatrix} \rightarrow N_{+1}^y,$$

$$Z_6 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow N_0^y, \quad (2.13)$$

$$Z_7 = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{2}i & -1 \\ -\sqrt{2}i & 2 & \sqrt{2}i \\ -1 & -\sqrt{2}i & 1 \end{pmatrix} \rightarrow N_{-1}^y.$$

Again we see that while the N^z 's have the "optimal form", the N^y do not, and hence we can expect observable-bicom relationships which are considerably more cumbersome than what we would expect in an optimal formalism (which in this case would yield all 1×1 submatrices).

The observable table in this case is given in Table 2. The complete determination of the amplitudes from these observables is again easy: L_2, L_3 , and L_4 give the magnitudes, then L_6 gives the angle between A_1 and A_{-1} , and L_5 the angle between A_0 and $(A_1 - A_{-1})$, which is all that is needed.

$$\underline{2 + 0 + 0 + 0}$$

With only rotation (or Lorentz) invariance assumed, this reaction has five amplitudes. They can be written, for example, as

$$M = a_0 S_x + a_1 S_y + a_2 S_z + a_3 S_{xx} + a_4 S_{yy} \quad (2.14)$$

or

$$M = b_0 \epsilon_{\mu\nu} k_\mu k_\nu + b_1 \epsilon_{\mu\nu} q_\mu q_\nu + b_2 \epsilon_{\mu\nu} k_\mu q_\nu + b_3 \epsilon_{\mu\alpha\beta\gamma} k_\alpha q_\beta p_\gamma + b_4 \epsilon_{\mu\alpha\beta\gamma} k_\alpha q_\beta p_\gamma \epsilon_{\mu\nu} q_\nu \quad (2.15)$$

where the notation is similar to the spin-1 case, except that we also have second rank tensorial S_{ij} 's and $\epsilon_{\mu\nu}$'s.

In contrast, we want to write the M-matrix as

$$M = (A_2 \ A_1 \ A_0 \ A_{-1} \ A_{-2}) \quad (2.16)$$

where again the labels on the columns refer to the 2, 1, 0, -1, and -2 projections along some fixed but arbitrary z direction.

The operators we want to consider here are the ones corresponding to the various projection substates in the z direction and also in one additional direction, for example y. Again we will also add, for reference, the customary differential cross section. We then have eleven operators

$$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11} \quad (2.17)$$

corresponding to the experimental quantities of

$$\frac{d\sigma}{d\Omega}, N_{+2}^z, N_{-1}^z, N_0^z, N_{-1}^z, N_{-2}^z, N_{+2}^y, N_{-1}^y, N_{-2}^y, N_{-1}^y, \text{ and } N_{-2}^y \quad (2.18)$$

respectively.

The explicit forms of the $Z_2 - Z_6$ are obvious again

$$Z_{4-j} = \begin{pmatrix} \dots & j & \dots & \dots \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \quad (2.19)$$

but the $Z_7 - Z_{11}$ are, of course, more cumbersome. As mentioned earlier, in this case we will calculate them by a general method applicable to all values of spins which also has the advantage that in calculating the observables we will never even have to write out explicitly the forms of these operators. We use the fact that, for example, Z_7 can be obtained from Z_2 by the rotation operator corresponding to a rotation by $\pi/2$ in the three-dimensional space, i.e. we have

$$Z_{n+5} = R^{(2)} \left(\frac{\pi}{2} \right) Z_n R^{(2)} \left(\frac{\pi}{2} \right)^{-1} \quad (2.20)$$

Using the usual representations of the rotation group, we have⁷

$$D^{(J)}(\alpha, \beta, \gamma)_{m', m} = e^{-im'\alpha} d^{(J)}(\beta)_{m', m} e^{-im\gamma} \quad (2.21)$$

$$d^{(J)}(\beta)_{m', m} = \sum_{\kappa} (-1)^{\kappa} \frac{((J+m)!(J-m)!(J+m')!(J-m')!)^{1/2}}{\kappa!(J+m-\kappa)!(J-m'-\kappa)!(\kappa+m'-m)!} \times$$

$$\times \left(\cos \frac{\beta}{2} \right)^{2J-2\kappa-m'+m} \left(-\sin \frac{\beta}{2} \right)^{2\kappa-m+m'}$$

from which we obtain

$$R^{(J)}\left(\frac{\pi}{2}\right)_{m', m} = D^{(J)}\left(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right)_{m', m} \quad (2.22)$$

$$= \frac{i^{m'-m}}{2^J} \sum_{\kappa=0,1,\dots} (-1)^{\kappa} \frac{((J+m)!(J-m)!(J+m')!(J-m')!)^{1/2}}{\kappa!(J+m-\kappa)!(J-m'-\kappa)!(\kappa+m'-m)!}$$

which gives us the following rotation matrix for the spin-2 case

$$R^{(2)}\left(\frac{\pi}{2}\right) = \frac{1}{4} \begin{pmatrix} 1 & +2i & -\sqrt{6} & -2i & 1 \\ +2i & -2 & 0 & -2 & -2i \\ -\sqrt{6} & 0 & -2 & 0 & -\sqrt{6} \\ -2i & -2 & 0 & -2 & +2i \\ 1 & -2i & -\sqrt{6} & +2i & 1 \end{pmatrix} \quad (2.23)$$

The observables containing Z_7 , or Z_8 , ..., or Z_{11} are therefore of the following form

$$\begin{aligned} \text{Tr}[MZ; M^\dagger] &= \text{Tr}[M R^{(a)}(\frac{\pi}{2}) Z_{i-s} R^{(a)}(\frac{\pi}{2})^\dagger M^\dagger] \\ &= \text{Tr}[R^{(a)}(\frac{\pi}{2})^\dagger M^\dagger M R^{(a)}(\frac{\pi}{2}) Z_{i-s}] \end{aligned} \quad (2.24)$$

which suggests that it is more convenient to calculate

$$R^{(a)}(\frac{\pi}{2})^\dagger M^\dagger M \quad (2.25)$$

which appears in all these traces, and the traces themselves will consist of the individual diagonal elements of the product of (2.25) with $R^{(2)}(\frac{\pi}{2})$. We get the diagonal elements singly because of the special simple form of Z_{i-s} given in Eq. (2.19). The results of all these calculations for the observable-bicom table are shown in Table 3. In it those ten bicom which never appear in these particular observables have been omitted.

We see again that the magnitudes are directly and individually measured by some of the experimental results themselves. The other observables, those referring to the second polarization direction, have a much more cumbersome relationship to the bicom, and even appropriately chosen sums and differences of two of them yield only linear combinations of at least two bicom. One can, however, show by inspecting the Jacobians that there are a number of sets of nine observables (excluding the differential cross section) which determine the amplitudes completely. This result is just another illustration of the general theorem which will be proven in the next section.

of these amplitudes are the measured quantities $N_{s_z}^z$,

$$|a_{s_z}| = N_{s_z}^z, \quad s_z = s, \dots, -s. \quad (3.1)$$

Consider a second direction ζ , obtained from the z direction by a rotation of the space axes by Euler angles (α, β, γ) . This rotation induces a transformation $D^{(s)}(\alpha, \beta, \gamma)$ on the $(2s+1)$ dimensional complex vector a_{ζ} , which transformation defines a new amplitude vector b_{ζ} ,

$$b_{\zeta} = D^{(s)}(\alpha, \beta, \gamma) a_{\zeta},$$

whose components are the amplitudes

$$b_{s_{\zeta}} = \sum_{s_z=-s}^{+s} D_{s_{\zeta} s_z}^{(s)}(\alpha, \beta, \gamma) a_{s_z}, \quad (3.2)$$

defined with respect to eigenstates of $\vec{S} \cdot \vec{\zeta}$. The magnitudes of these "rotated" amplitudes are the measured quantities

$$|b_{s_{\zeta}}| = N_{s_{\zeta}}^{\zeta}, \quad s_{\zeta} = s, \dots, -s. \quad (3.3)$$

Let the phases of the original amplitudes be ϕ_{s_z} ,

$$a_{s_z} = |a_{s_z}| e^{i\phi_{s_z}}, \quad (3.4)$$

and assume $\phi_{-s} = 0$ which fixes the overall arbitrary phase. To prove the theorem, we must show that, given the sets of fixed real numbers $\{N_{s_z}^z\}$ and $\{N_{s_{\zeta}}^{\zeta}\}$, the phases $\{\phi_{s_z}\}$ are determined (up to discrete ambiguities).

Our method will be to construct $2s$ equations that are bilinear in the amplitudes (i.e. linear combinations of $(a_k^* a_l)$'s). These equations will determine the $2s$ phases, providing the Jacobian is non-zero. By considering

a particular set of terms in the Jacobian, we then show that the non-vanishing of the determinant of the rotation matrix insures the non-vanishing of the Jacobian.

Consider the square moduli of the b_m 's which can be written in terms of the a 's via equation 3.2

$$|b_m|^2 = \sum_{k,l=-s}^s D_{ml}^* D_{mk} a_l^* a_k, \quad m = +s, \dots, -s, \quad (3.5)$$

where we use the notation $D_{mk} \equiv D_{mk}^{(s)}(\alpha, \beta, \gamma)$. Define the $2s+1$ real functions of the phases $\{\phi_{sz}\}$

$$K_m(\{\phi_{sz}\}) = |b_m|^2 = \sum_{l=-s}^s |D_{ml}|^2 |a_l|^2,$$

which can be written as

$$K_m = \sum_{\substack{k,l \\ k \neq l}} D_{ml}^* D_{mk} a_l^* a_k. \quad (3.6)$$

The K_m 's are equal to fixed numbers k_m given by equations (3.1) and (3.3)

$$K_m(\{\phi_{sz}\}) = k_m, \quad (3.7)$$

$$\text{where } k_m = (N_m^z)^2 = \sum_l |D_{ml}|^2 (N_l^z)^2.$$

At most $2s$ of these equations (3.7), can be independent, since $\sum_m |b_m|^2 = \sum_l |a_l|^2$

(which follows from eqn. (3.5) or the unitarity of the rotation matrices), and

so

$$\sum_m K_m = 0.$$

$$J = \det \begin{pmatrix} \frac{\partial K_s}{\partial \phi_s} & \frac{\partial K_s}{\partial \phi_{s-1}} & \dots & \frac{\partial K_s}{\partial \phi_{-s+1}} \\ \frac{\partial K_{s-1}}{\partial \phi_s} & \frac{\partial K_{s-1}}{\partial \phi_{s-1}} & \dots & \frac{\partial K_{s-1}}{\partial \phi_{-s+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial K_{-s}}{\partial \phi_s} & \frac{\partial K_{-s}}{\partial \phi_{s-1}} & \dots & \frac{\partial K_{-s}}{\partial \phi_{-s+1}} \end{pmatrix}, \quad (3.9)$$

where one unspecified row has been deleted (i.e. one value of the index m on K_m has been removed). Now, because each element is homogeneous in terms of A_{jk} 's, the expansion of the determinant can be partitioned into a sum of terms, each characterized by a particular distinct product of $2s$ factors of A_{jk} 's.

That is, in general

$$J = \sum_{\{j_1, k_1, \dots, j_{2s}, k_{2s}\}} C_{j_1, k_1, \dots, j_{2s}, k_{2s}} A_{j_1, k_1} A_{j_2, k_2} \dots A_{j_{2s}, k_{2s}}.$$

However, since $A_{jk} a_j |a_k|$ the terms in the sum can be combined so that each distinct set of powers (e.g. $|a_s|^m |a_{s-1}|^{m_{s-1}} \dots |a_{-s}|^{m_{-s}}$, where

$m_s + m_{s-1} + \dots + m_{-s} = 2(2s)$ labels one "partition" term. If the Jacobian is to be zero, (for any arbitrary a) each one of these partition terms must have a

Suppose, then, any one of the equations (3.6) is removed from the set. We thus have $2s$ equations in the $2s$ unknowns $\{\phi_{s_z}\}$. These equations have discrete solutions if and only if the Jacobian relating $\{K_m\}$ to $\{\phi_{s_z}\}$ is non-zero.

The elements of the Jacobian are obtained using eqn. (3.4),

$$\frac{\partial K_m}{\partial \phi_j} = \sum_{\substack{k, l \\ k \neq l}} D_{ml}^* D_{mk} (-i\delta_{jl} a_j^* a_k + i\delta_{jk} a_l^* a_j), \quad j = s, \dots, -s+1;$$

which can be rewritten in the form

$$\frac{\partial K_m}{\partial \phi_j} = D_{mj}^* \sum_{k \neq j} D_{mk} A_{jk}(\gamma), \quad (3.8)$$

where

$$A_{jk}(\gamma) \equiv [1 + e^{2i(k-j)\gamma}] \operatorname{Im} a_j^* a_k - i[1 - e^{2i(k-j)\gamma}] \operatorname{Re} a_j^* a_k,$$

and we have used the property

$$D_{mk}^{(s)}(\alpha, \beta, \gamma) = e^{-i(m\alpha + k\gamma)} d_{mk}^{(s)}(\beta),$$

to combine terms. The Jacobian is the determinant of the $(2s) \times (2s)$ array formed by the elements of eqn. (3.8),

III. THEOREM ON THE MINIMUM NUMBER OF POLARIZATION DIRECTIONS

In order to determine the $2s+1$ complex amplitudes for a spin S particle in the initial or final state of a reaction at least $2(2s+1)-1$ independent measurements must be made. The observables, suitably chosen, will yield the $2s+1$ magnitudes and $2s$ phases (relative to one overall phase), up to certain discrete ambiguities for the phases. In a single experiment, the measurements generally consist of the determination of some polarization parameters with respect to a fixed direction, that is, the expectation values of the spin along a fixed axis are determined

The number of spin S particles measured in a particular experiment, polarized along the $\vec{\zeta}$ direction, with spin projection $\vec{S} \cdot \vec{\zeta} = s_{\zeta}$, will be denoted by $N_{s_{\zeta}}^{\zeta}$. If all $N_{s_{\zeta}}^{\zeta}$'s are measured for a fixed direction, ζ , this gives $2s+1$ observables, which are obviously not sufficient for a complete specification of the amplitudes. Measuring the spin projections along a second axis, η , determines $2s+1$ more quantities, $N_{s_{\eta}}^{\eta}$ ($s_{\eta} = s, \dots, -s$). Only $2s$ of these can provide new information, since $\sum_{\eta} N_{s_{\eta}}^{\eta} = \sum_{\zeta} N_{s_{\zeta}}^{\zeta}$ is the total number of particles. Does this larger set of $(2s+1)+2s=2(2s+1)-1$ measurements constitute a complete set? Does the completeness depend on the choice of direction for the second axis? The answer to these questions is provided by the following theorem.

Theorem: The measurements of all projections of a particle's spin along two non-parallel axes allows a complete determination of the amplitudes for that particle up to discrete ambiguities and an overall phase.

To prove this theorem, we first define various sets of amplitudes and measurements. Let $\underline{a}$ be the $(2s+1)$ -tuple of complex amplitudes for a particle of spin S

$$\underline{a} = \{a_s, a_{s-1}, \dots, a_{-s}\},$$

where the a_{s_z} 's are defined with respect to z -axis eigenstates. The magnitudes

vanishing coefficient. Hence, to show that $J \neq 0$, we need only demonstrate that one such term is non-zero. Which one of the many possible choices will yield a simple result?

Consider the partition term proportioned to

$$A_{s,s-1} A_{s,s-2} \cdots A_{s,-s}$$

or

$$|a_s|^{2s} |a_{s-1}| |a_{s-2}| \cdots |a_{-s}| \quad (3.10)$$

This term comes from the product of entries in the array (3.9) each of which contains an $A_{s,j}$ or $A_{j,s}$. Note that elements in the first column are of the form

$$\frac{\partial K_m}{\partial \phi_s} = D_{m,s}^* (D_{m,s-1} A_{s,s-1} + D_{m,s-2} A_{s,s-2} + \cdots + D_{m,-s} A_{s,-s}) \quad (3.11)$$

Every term in the sum contains an s index, whereas, in all other columns

$$j = s-1, \cdots, -s+1$$

$$\frac{\partial K_m}{\partial \phi_j} = D_{mj}^* (D_{m,s} A_{js} + D_{m,s-1} A_{j,s-1} + \cdots + D_{m,-s} A_{j,-s}) \quad (3.12)$$

the s index occurs only in A_{js} in the first term. Then the terms in J that combine to form the desired partition (3.10) arise from the $A_{s,-s}$ terms in the first column and the $A_{j,s}$ in all other columns. Hence, using (3.11) and (3.12), we can write

$$J = \det \begin{pmatrix} D_{s,s}^* D_{s,s} A_{s,-s} + \cdots & D_{s,s-1}^* D_{s,s} A_{s-1,s} + \cdots & \cdots & D_{s,-s}^* D_{s,s} A_{-s,s} + \cdots \\ D_{s-1,s}^* D_{s-1,s} A_{s,s} + \cdots & D_{s-1,s-1}^* D_{s-1,s} A_{s-1,s} + \cdots & \cdots & D_{s-1,-s}^* D_{s-1,s} A_{-s-1,s} + \cdots \\ \vdots & \vdots & \ddots & \vdots \\ D_{-s,s}^* D_{-s,s} A_{s,s} + \cdots & D_{-s,s-1}^* D_{-s,s} A_{s-1,s} + \cdots & \cdots & D_{-s,-s}^* D_{-s,s} A_{-s,s} + \cdots \end{pmatrix}$$

Note that the D 's in the first column can be rewritten

$$D_{m,s}^* D_{m,-s} = e^{4isy} D_{m,-s}^* D_{m,s}$$

so that the m th row now has a factor of D_{ms} , which becomes a factor of the determinant. The first column has the common factor $e^{4isy} A_{s,-s}$, and the remaining columns each have their $A_{j,s}$ factor.

Thus

$$J = (D_{ss} D_{s-1,s} D_{s-2,s} \cdots D_{-s,s}) e^{4isy} A_{s,-s} A_{s-1,s} A_{s-2,s} \cdots A_{-s+1,s} \times \det \begin{pmatrix} D_{s,-s}^* & D_{s,s-1}^* & \cdots & D_{s,-s+1}^* \\ D_{s-1,-s}^* & D_{s-1,s-1}^* & \cdots & D_{s-1,-s+1}^* \\ \vdots & \vdots & \ddots & \vdots \\ D_{-s,-s}^* & D_{-s,s-1}^* & \cdots & D_{-s,-s+1}^* \end{pmatrix} + (\text{other partition terms}) \quad (3.13)$$

Note that, except for the order of the columns, the determinant that remains is the determinant of the matrix D^* with the first column removed and one row deleted. Therefore, the required determinant in eqn. (3.13) is a minor of an element in the first column of $\det(D^*)$. Since $\det(D^*) = +1$, at least one such minor must be non-zero. Recall that the choice of which row to delete from the original set of equations (3.7) was arbitrary. Choose the deleted row so that the determinant in eqn. (3.13) is one of the non-zero minors of $\det(D^*)$. That is all we need to demonstrate that the particular partition term is non-zero (for arbitrary values of amplitudes and rotation angles) and, therefore, that the Jacobian is non-zero. This completes the proof.

IV. GENERAL FORMALISM

The examples in section II and the general theorem in section III allow us now to describe the situation also for a reaction containing particles with arbitrary spins.

To start on a negative note, it is clear that a general four-particle reaction

$$s_1 + s_2 \rightarrow s_3 + s_4, \quad (4.1)$$

where all s_i 's are non-zero, will have a substantial polarization structure even if everything is done to simplify it. After all, even if one could attain the "optimal" type of formalism and still retain directly measurable quantities, the observable-bicom matrix would contain⁴ submatrices in size up to and including 8×8 . On the other hand, the structure of such "composite" reactions can be derived from simpler reactions through the factorization rule,⁸ so that one needs to be concerned, in a primary way, only with reactions containing two non-zero and two zero-spin particles.

Considering, therefore, the reaction

$$s_1 + 0 \rightarrow s_2 + 0 \quad (4.2)$$

we define its M matrix and its reaction amplitudes as

$$M = \begin{pmatrix} A_{s_1 s_2} & A_{s_1(s_2-1)} & \dots & A_{s_1-s_2} \\ A_{s_1-1 s_2} & A_{s_1-1 s_2-1} & \dots & A_{s_1-1 -s_2} \\ \dots & \dots & \dots & \dots \\ A_{-s_1 s_2} & A_{-s_1 s_2-1} & \dots & A_{-s_1 -s_2} \end{pmatrix} \quad (4.3)$$

where the rows of the M-matrix are labeled by the polarization projections of s_2 in some fixed but arbitrary z' direction, and the columns by spin

projections of s_1 in some fixed but arbitrary z direction which may or may not be the same as z' .

The "observable operators" for the initial particle with non-zero spin will be $(2s_1+1) \times (2s_1+1)$ matrices. Their rows and columns will be labeled by the $2s_1+1$ spin substates with respect to the polarization direction z . Similarly, the observable operators for the final particle with non-zero spin will be $(2s_2+1) \times (2s_2+1)$ matrices with rows and columns labeled by the spin projections on the z' axis.

The set of observable operators for the initial state will include those corresponding to N_m^z 's, that is, the number of particles in each of the spin-projection states. These operators will have a form in our representation in which their matrices will have all 0's except for a 1 in the m th position along the principal diagonal.

The $(2s_1+1)$ operators described in the previous paragraphs are, however, by themselves insufficient to determine all the amplitudes, even if combined with an ample number of operators pertaining to the final state particles. We need $2s_1$ more operators. These will be the ones corresponding to N_m^w , where w is some quantization direction non-collinear with z . Choosing w to be y (so that z and y are orthogonal) may bring about some simplification but most likely not a substantial amount, and such orthogonality is certainly not necessary for the choice of the second axis.

Similarly we define $2s_2+1$ observable operators for the final state particle also. There the quantization axes are z' and w' . These might or might not be the same as z and w , and might or might not be orthogonal to each other.

According to the theorem in section III, the set of observables thus described, pertaining to two directions for the initial state and two

directions for the final state, are always sufficient for a complete determination of the amplitudes.

The actual matrix connecting these observables and the bicomps of the amplitudes defined above has a distinct structure. An observable corresponding to a definite N_m^z and a definite $N_m^{z'}$ gives directly one $|A_{ij}|^2$ by itself. An observable corresponding to a definite N_m^z but to an unpolarized final state gives one of the $\frac{1}{2}|A_{ij}|^2$. To determine all magnitudes individually, therefore, we must perform at least one measurement in which both the initial and the final particles are in definite polarization projection states. This is, of course, just a special case of the theorem by Simonius.⁹

In any case, the determination of the magnitudes of the amplitudes is thus rather easy and direct.

The relationship between the observables corresponding to N_m^y and $N_m^{y'}$ are more involved since these operators are not at all diagonal or even optimal in this representation. This is unavoidable if one wishes to use observables that are direct products of measurement along the second polarization axis.

The taking of various sums and differences of these observables of the second type, however, can reduce drastically the number of bicomps to which these sums or differences connect. In many cases this simplification is sufficient to allow an easy, explicit, and analytic solution of the bicomps and of the amplitudes in terms of the observables. In more complicated cases the solution, while known to exist, is more cumbersome.

Observables corresponding to N_m^v and/or $N_m^{v'}$, where v is a third axis of polarization not collinear with z or v' (z' or w' in the case of v') can be constructed similarly. They will also be significantly non-diagonal and

non-optimal. Our theorem tells us, however, that such a third direction is not necessary for the determination of the amplitudes.

In all this discussion we did not mention polarization vectors, correlation tensors, depolarizations, polarization transfer tensors and the like.¹ From our point of view, such quantities are not needed and if interposed between the amplitudes and the experimental measurements, may serve to complicate the connections and misdirect both the designs of experiments and the choice of amplitudes.

If a complete set of measurements cannot be made or has not been performed yet, the most useful type of measurement is a set of observables corresponding to a N_m^z and a $N_m^{z'}$, because each of them determines a single parameter by itself, namely the magnitude of an amplitude. Such magnitudes can be calculated easily from theoretical models and hence can serve as very convenient contact points between experiment and theory, offering a far superior test of theoretical predictions than the traditional single contact point through the differential cross section alone.

To conclude this general discussion, we want to prove a theorem that pertains specifically to how a particular theoretical model can be easily tested by polarization experiments. Let us consider a model which makes a specific prediction for the reaction amplitudes as given in some specified but arbitrary formalism. We will now show that by an appropriate choice of the quantization direction one can always convert this prediction to a prediction that a certain single polarization experiment yield a zero result. Thus we always have an easy test of any theoretical model in the form of a null experiment.

The first step is to transform the predicted amplitudes from the formalism in which they were given to a formalism of our type; that is,

to a notation in which the amplitudes are gathered into a matrix the elements of which are labeled by the spin projections in one (or two) specified but arbitrary direction(s) (z and z'). This is a straightforward procedure involving, perhaps, conversions of gamma matrices to sigma matrices, or similar well known algebraic manipulations. We can therefore assume that the model predictions can easily be stated in terms of predictions for the elements of an M-matrix in our form.

Since we know from the previous results of this paper that in any reference frame of our type the N_m^z type observables are given in terms of just one single absolute value squared of one of the amplitudes, what remains to be shown is only that one can always perform a rotation of the reference axis z (or axes z and z') to some new direction ξ (or ξ and ξ') such that in that new frame one of the elements of the matrix is zero; (i.e., one of the amplitudes vanishes). If so, the observable (of the N_m^z type) which is given by the absolute value squared of that amplitude alone will vanish.

This remaining step is established by the following theorem:

Theorem: Let us be given the set of amplitudes $\{a_s, \dots, a_{-s}\}$ with fixed complex values.

It is always possible to choose a rotation, $R(\alpha, \beta, 0)$, that induces a unitary transformation on $\mathcal{A}$,

$$aD_{ms}^s(\alpha, \beta, 0) = b = \{b_s, \dots, b_{-s}\}$$

such that $b_s = 0$.

Proof: $b_s = \sum_{m=-s}^s a_m D_{ms}^s(\alpha, \beta, 0)$

We must show that there exist values of α and β for which

$b_s = 0$. From Eq. (2.21) we have

$$D_{ms}^s(\alpha, \beta, 0) = e^{-im\alpha} \sum_x \frac{(-1)^x}{x!} \frac{\sqrt{(2s)!(s+m)!(s-m)!}}{(2s-x)!(s-m-x)!(x+m-s)!} \times \\ \times \left(\cos \frac{\beta}{2}\right)^{3s-2x-m} \left(-\sin \frac{\beta}{2}\right)^{2x-s+m}$$

The sum over x vanishes except for $x = s-m$, and so

$$D_{ms}^s(\alpha, \beta, 0) = e^{-im\alpha} (-1)^{s-m} \frac{\sqrt{(2s)!}}{\sqrt{(s-m)!(s+m)!}} \left(\cos \frac{\beta}{2}\right)^{s+m} \left(-\sin \frac{\beta}{2}\right)^{s-m} \\ = \left[\left(\frac{1}{2} \sin \beta\right)^s \frac{\sqrt{(2s)!}}{\sqrt{(s-m)!(s+m)!}}\right] \left[e^{-i\alpha} \cot \frac{\beta}{2}\right]^m$$

Hence the sum b_s is

$$b_s = \left(\frac{1}{2} \sin \beta\right)^s \sqrt{(2s)!} \sum_{m=-s}^s \frac{a_m}{\sqrt{(s-m)!(s+m)!}} \left[e^{-i\alpha} \cot \frac{\beta}{2}\right]^m$$

and

$b_s = 0$ requires

$$\sum_{m=-s}^s \frac{a_m}{\sqrt{(s-m)!(s+m)!}} \left(e^{-i\alpha} \cot \frac{\beta}{2}\right)^m = 0,$$

or, factoring out $\left(e^{-i\alpha} \cot \frac{\beta}{2}\right)^s$,

$$0 = \sum_{m=-s}^s \frac{a_m}{\sqrt{(s-m)!(s+m)!}} \left(e^{i\alpha} \tan \frac{\beta}{2}\right)^{s-m}$$

This is a polynomial of order $2s$ in the complex variable $(e^{i\alpha} \tan \frac{\beta}{2})$ with complex coefficients. By the Euler-Gauss Theorem,¹⁰ such a polynomial has complex roots. Hence there exist solutions

$$e^{i\alpha} \tan \frac{\beta}{2} = z_0, z_1, \dots$$

or $\tan \frac{\beta}{2} = |z_0|, \alpha = \arg z_0. \quad \text{Q.E.D.}$

Thus, given a theoretical model prediction for the reaction amplitudes of some particle process at a given set of kinematic variables, the experimentalist can always find a direction such that measuring the number of particles in one particular polarization state with respect to that quantization direction provides a null experiment as to whether the model prediction is correct or not. Naturally, the one experiment, if it yields a zero result, does not uniquely establish the model in question as the only correct explanation, but the experiment provides a relatively easy and unambiguous first step in the complete testing of the model in question.

V. CONCLUDING REMARKS

The main results of this paper can be summarized in a few paragraphs.

1) It seems advisable to relate the direct results of experiments to reaction amplitudes in a way which contains as few intermediate steps as possible and hence simplifies the connections. This is shown to be possible by a judicious choice of amplitudes and "observables".

2) The resulting formalism is unavoidably more cumbersome than the "ideal", "optimal" formalisms in which the observable-bicom matrix is diagonalized to a maximal extent. The reason for the added complexity is that one needs more than one polarization direction for the complete determination of the amplitudes, and the quantities in the second

quantization direction which are direct results of experiments are necessarily more complicated in structure than "optimal" observables would be.

3) Yet one can show in general that observables using only two quantization directions are always sufficient for a complete determination of the amplitudes. This fact might have an impact on the design of polarization experiments, even though in some circumstances it might be easier to go to a third polarization direction than to measure all measurable quantities in the first two.

4) The differential cross section is always a cumbersome observable in its relationship to the amplitudes and contributes little toward the determination of amplitudes.¹¹

5) The simplest formalism directly connecting amplitudes and the results of actual experiments uses the same basis (i.e., the same labels) for the M-matrix as for the matrices describing the operators the expectation values of which we want to measure. To have a sufficient number of the latter, however, one must include both diagonal operators and operators referring to a second quantization axis which therefore are not diagonal in that representation.

6) Such simplest formalisms can use any z direction as the first quantization axis. A special case is the helicity frame in which the z axis is chosen to coincide with the particle momentum, but this is not necessary and in fact the helicity frame has no particular advantage over other frames, at least from the point of view of aiding the determination of amplitudes.

7) The second polarization axis can be anything not collinear with

the first. It need not be perpendicular to the first, though if it is the structure of relations between those observables and the bicomps might be somewhat simpler or at least more transparent and regular. If, however, experimental limitations such as bulky magnets used in auxiliary equipment suggest a second quantization axis that is not perpendicular to the first, such a possibility should be explored without apprehension.

8) It is always much easier to measure the magnitudes of amplitudes than their relative phases. This is, at least, true for the type of amplitudes we chose to define. In terms of other amplitudes, these magnitudes will involve both magnitudes and phases. It is therefore advisable to calculate from theoretical models the magnitudes of our amplitudes and use them as the initial tests of the model since such measurements can be performed more easily and yet constitute a much more stringent test than the comparison with only differential cross sections.

9) Relative phases of amplitudes, while more difficult to measure than magnitudes, can be obtained from suitably chosen sums and differences of the directly measured N 's. It should therefore be the task of an experimentalist to try to design experimental procedures to measure directly the required sums or differences of N 's. These sums and differences may or may not coincide with the traditional polarization tensors and coefficients, but which of these sums and differences are of interest should not be determined by the combinations that happen to appear in the customary polarization quantities but rather by the requirement that the sums or differences be optimally interfaced with the bicomps we want to determine. For example, Table 3 shows that $L(Z_8) - L(Z_{10})$ yields directly a sum of only two bicomps, even though each of the L 's by itself is related to 10 bicomps.

10) If a theoretical model provides a prediction for the values of the reaction amplitudes of an arbitrary particle process in any formalism, it is always possible to find a spacial direction so that the measurement of the number of particles in the reaction which have a spin projection $m_s = s$ in that quantization direction constitutes a null experiment to test the validity of the predictions of the model.

11) Even when utilizing the most appropriate set of amplitudes tailored to be compatible with the most thoughtfully designed experiments, a number of general theorems, given in previous papers,^{4,10} limit the degree of simplicity and ease one can attain. Thus the utilization of partial information from a partial set of measurements is very important.¹² The results of the present paper, as outlined above, provide numerous general rules about the choice of amplitudes and the selection of experiments so that maximal use can be made of such incomplete situations.

Acknowledgement

The authors appreciate a relevant communication from Prof. David Roper.

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Table 1.

Relationship between bicomps (amplitude products) and some of the observables for the reaction $0+2 \rightarrow 0+2$. For details, see the text.

[illegible]

Table 3. Relationship between bicombs (amplitude products) and some of the experimental observables in the reaction $2 + 0 \rightarrow 0 + 0$. For details, see the text.

	2	2	2	2	2	$\text{Re} A_1 A_j$			$\text{Im} A_1 A_j$						
	$ A_2 $	$ A_1 $	$ A_0 $	$ A_{-1} $	$ A_{-2} $	2 0	-1 1	2 -2	-2 0	2 1	1 0	0 -1	-1 -2	2 -1	1 -2
$L(Z_1) \rightarrow \frac{d\sigma}{d\Omega}$	1	1	1	1	1										
$L(Z_2) \rightarrow N_2^z$	1														
$L(Z_3) \rightarrow N_1^z$		1													
$L(Z_4) \rightarrow N_0^z$			1												
$L(Z_5) \rightarrow N_{-1}^z$				1											
$L(Z_6) \rightarrow N_{-2}^z$					1										
$L(Z_7) \rightarrow N_2^y$	1/16	1/4	3/8	1/4	1/16	$-\frac{\sqrt{6}}{8}$	$-\frac{1}{2}$	$\frac{1}{8}$	$-\frac{\sqrt{6}}{8}$	$-\frac{1}{4}$	$-\frac{\sqrt{6}}{4}$	$-\frac{\sqrt{6}}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$L(Z_8) \rightarrow N_1^y$	1/4	1/4	0	1/4	1/4	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$L(Z_9) \rightarrow N_0^y$	3/8	0	1/4	0	3/8	$\frac{\sqrt{6}}{4}$	0	$\frac{3}{4}$	$\frac{\sqrt{6}}{4}$	0	0	0	0	0	0
$L(Z_{10}) \rightarrow N_{-1}^y$	1/4	1/4	0	1/4	1/4	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$L(Z_{11}) \rightarrow N_{-2}^y$	1/16	1/4	3/8	1/4	1/16	$-\frac{\sqrt{6}}{8}$	$-\frac{1}{2}$	$\frac{1}{8}$	$-\frac{\sqrt{6}}{8}$	$\frac{1}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$

Table 2. Relationship between bicombs (amplitude products) and some of the observables for the reaction $1 + 0 \rightarrow 0 + 0$. For details, see the text.

	$\begin{matrix} 2 & 2 & 2 \\ A_1 & A_0 & A_{-1} \end{matrix}$			$\text{Re} A_1 A_j^*$			$\text{Im} A_1 A_j^*$		
				1 0	1 -1	0 -1	1 0	1 -1	0 -1
$L(Z_1)$	1	1	1						
$L(Z_2)$	1								
$L(Z_3)$		1							
$L(Z_4)$				1					
$L(Z_5)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		$-\frac{1}{2}$		$-\frac{\sqrt{2}}{2}$		$-\frac{\sqrt{2}}{2}$
$L(Z_6)$	$\frac{1}{2}$		$\frac{1}{2}$		1				
$L(Z_7)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		$-\frac{1}{2}$		$+\frac{\sqrt{2}}{2}$		$+\frac{\sqrt{2}}{2}$