BNL--51927

DE86 005303

BNL 51927

UC-28

(Particle Accelerators and
High-Voltage Machines — TFC-4500)

PARTICLE DYNAMICS FOR MOTION IN A NON-PLANAR ACCELERATOR

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August 20, 1985

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Printed in the United States of America
Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

NTIS price codes:

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Abstract

The basic dynamics of a planar accelerator is extended to the non-planar case. This is done using the geometrical concept of torsion and extending the Hamiltonian formalism. A generalized non-planar reference orbit is adopted which introduces torsion in appropriately chosen drift spaces. The parameters of the reference orbit are associated with uncoupled and coupled betatron parameters currently in use.

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1. Introduction

In the development of circular accelerators, the choice of a reference system has played an important role in the analysis of the particle dynamics. In particular, the choice has been such that a particle displaced from the reference orbit will describe, around this, a pseudoharmonic oscillation¹, in a coordinate system moving with the particle. Thus, implicit in our choice is the fact that the reference orbit is closed (periodic) and that it represents a real trajectory of a particle.

Using Floquet's theorem, the periodic parameters relative to the closed reference orbit for linear magnetic systems have been obtained. However, the size of planar circular accelerators is growing tremendously--the circumference of future accelerators will be of the order of a hundred kms. or more--and perhaps it will be desirable to follow the ground surface. This implies that the accelerator will no longer be in a plane and planar dynamics in particular the periodic orbit parameters, will have to be modified. A nonplanar situation will actually exist in the operating so-called Main Ring at Fermilab in which it is planned to have an orbit which passes over the experimental facilities for colliding beams in the Tevatron. To better understand such problems, we consider here a formalism for analysis of the particle dynamics of non-planar motion and in particular show that in general such motion can be described in terms of a superimposed torsion of the reference orbit in appropriately chosen straight sections.

To accomplish this, we use a Hamiltonian formalism. We recognize that the geometrical parameter, torsion, allows us in general to extend the Courant-Snyder formalism to non-planar curves in R^3 , but as we will see, the reference orbit for this type of accelerator can be taken to be a curve in which the torsion is particularly simple. In fact, we define the reference curve to be such that it is perpendicular to the field of each dipole and the dipole entrance so that, inside the dipoles the torsion is zero. That is, we restrict ourselves to cases where the torsion is non-zero only between dipoles. Thus, the motion within each dipole is planar. Also, we can arbitrarily select a reference orbit with no torsion in the quadrupoles. Thus, we can choose, without loss of generality, the case in which the torsion inside any element is zero and only has a value different from zero in drift spaces.

In Chapter 2 we obtain the general Hamiltonian for the non-planar case. In Chapter 3 we study the motion of a particle moving in a transverse field. In Section 3.1 we determine the vector potential in the introduced coordinate system, and in Section 3.2 we obtain the equations of motion around non-planar orbits. In Chapter 4, the transfer matrix for a generalized drift space (i.e. one with torsion) is analyzed. We will see that the transfer matrices with respect to the coordinate sets (x, x', y, y') and (x, p_x, y, p_y) only depend on the total angle of rotation of the coordinate frame and not on the torsion (i.e. the rate of

coordinate spin rotation). However, the connection between these coordinate sets is dependent explicitly on the torsion. This has the implication that the physics as observed in regions of non-zero torsion is dependent on the torsion, τ . To avoid confusion in this situation, we have taken $\tau \neq 0$ only in the straight sections and using the fact that the transfer matrix depends only on the rotation angle and not on τ explicitly. It is interesting that a solenoid magnet introduces similar effects, specifically an associated rotation and drift. However, due to the varying longitudinal field at the solenoid ends, there must be an accompanying transverse magnetic field. This transverse field in fact results in a radial focusing term at the solenoid ends which is of course not present in the coordinate spinning arising from the torsion. Finally, in Chapter 5 we describe the resulting Hamiltonian which has the form of a linearly coupled system. We review the parametrization of such a system and discuss its stability.

2. Hamiltonian Formulation

In this chapter we will review the basic tools of Hamiltonian dynamics used in connection with planar accelerators.¹⁻⁴

Specification of Coordinates

A relativistic charged particle moving in a magnetic field is described by the Hamiltonian

$$\bar{H}(\vec{p}, \vec{q}, t) = c \left[\left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 + m_0^2 c^2 \right]^{1/2} + e\phi \quad (2.1)$$

where ϕ and $\vec{A}$ are the scalar and vector potentials, and

$$\vec{q} = (X, Y, Z), \quad \vec{p} = (p_X, p_Y, p_Z) \quad (2.2)$$

We introduce a local coordinate system $\vec{Q} = (x, y, s)$ with canonical momenta

$$\vec{P} = (P_x, P_y, P_s) \quad (2.3)$$

attached to a reference curve $\vec{r}_0(s)$ as shown in Fig. 1. The curve $\vec{r}_0(s)$ is defined so that it is a possible particle orbit, where

Fig. 1.

$\vec{r}_0(s)$ is a reference curve

$\vec{r}(x, y, z)$ is any point in space

$\hat{t}(s)$ is the unit vector tangent to the reference curve

$\hat{n}(s)$ is the principal normal vector

$\hat{b}(s)$ is the binormal unitary vector.

These vectors fulfill the following properties⁵

$$\begin{aligned} \frac{d\vec{r}_0}{ds} &= \hat{t} \quad ; \quad \frac{d\hat{t}}{ds} = \kappa \hat{n} \\ \frac{d\hat{n}}{ds} &= \tau \hat{b} - \kappa \hat{t} \quad ; \quad \frac{d\hat{b}}{ds} = -\tau \hat{n} \end{aligned} \quad (2.4)$$

where

$$\kappa = |\hat{t}'|, \quad \tau = |\hat{b}'| \quad (2.5)$$

are the curvature and the torsion of the curve at s .

In this coordinate system, the Hamiltonian (2.1) takes the form

$$\begin{aligned} \bar{H}(\vec{P}, \vec{Q}, t) = & c[(P_x - \frac{e}{c} A_x)^2 + (P_y - \frac{e}{c} A_y)^2 \\ & + \left(\frac{(P_s - \frac{e}{c} A_s) - \tau x(P_y - \frac{e}{c} A_y) + \tau y(P_x - \frac{e}{c} A_x)}{1 - \kappa x} \right)^2 + m_0^2 c^2]^{1/2} + e\phi \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} P_x &= \vec{p} \cdot \hat{n} = p_n \\ P_y &= \vec{p} \cdot \hat{b} = p_b \\ P_s &= (1 - \kappa x) p_t + \tau(x p_b - y p_n) \end{aligned} \quad (2.7)$$

and the vector potential components A_x, A_y, A_s fulfill similar relations.

In order to use the path length s as independent parameter, we make the canonical transformation

$$\begin{aligned} s &= t \\ -H &= P_s = -\bar{H} \end{aligned} \quad (2.8)$$

where $H(\vec{P}, \vec{Q}, s)$ is the new Hamiltonian, given by

$$\begin{aligned} H(\vec{P}, \vec{Q}, s) = & -\frac{e}{c} A_s - \tau x(P_y - \frac{e}{c} A_y) + \tau y(P_x - \frac{e}{c} A_x) \\ & - (1 - \kappa x) \left[P_s^2 - (P_x - \frac{e}{c} A_x)^2 - (P_y - \frac{e}{c} A_y)^2 \right]^{1/2} \end{aligned} \quad (2.9)$$

Here we have taken the case $\Phi=0$ so that

$$(H/c)^2 - m_0^2 c^2 = |\vec{p}|^2 \approx p_s^2 \quad (2.10)$$

The equations of motion are obtained from (2.9) through the Hamiltonian equations

$$x' = \frac{\partial H}{\partial p_x} \quad ; \quad y' = \frac{\partial H}{\partial p_y} \quad (2.11)$$

$$p'_x = - \frac{\partial H}{\partial x} \quad ; \quad p'_y = - \frac{\partial H}{\partial y}$$

where the derivative is now with respect to the parameter s .

3. Motion with a Transverse Magnetic Field.

3.1 Determination of the Vector Potential

In this section we are going to find the vector potential that gives the transverse dipole and quadrupole magnetic field components for the case in which the torsion $\tau \neq 0$. For $\tau = 0$ we obtain the well known equations of motion.

We have,

$$\begin{aligned} \vec{r}_0 &= x_0 \hat{i} + y_0 \hat{j} + z_0 \hat{k} \\ \hat{n} &= \alpha_1 \hat{i} + \alpha_2 \hat{j} + \alpha_3 \hat{k} \\ \hat{b} &= \beta_1 \hat{i} + \beta_2 \hat{j} + \beta_3 \hat{k} \\ \hat{t} &= \gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k} \end{aligned} \quad (3.1)$$

and further,

$$\begin{aligned} X &= x_0(s) + \alpha_1(s)x + \beta_1(s)y \\ Y &= y_0(s) + \alpha_2(s)x + \beta_3(s)y \\ Z &= z_0(s) + \alpha_3(s)x + \beta_3(s)y \end{aligned} \quad (3.2)$$

Taking into account that

$$\alpha'_i = \tau\beta_i - \kappa\gamma_i$$

$$\text{and} \quad \beta'_i = -\tau\alpha_i, \quad \gamma'_i = \kappa\alpha_i \quad (3.3)$$

we have that

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial s \end{bmatrix} = S \begin{bmatrix} \partial/\partial X \\ \partial/\partial Y \\ \partial/\partial Z \end{bmatrix} \quad (3.4a)$$

where

$$S = \begin{bmatrix} \alpha_1 & \cdot & \alpha_2 & \cdot & \alpha_3 \\ \beta_1 & \cdot & \beta_2 & \cdot & \beta_3 \\ \gamma_1(1-\kappa x) + \tau(\beta_1 x - \alpha_1 y) & \cdot & \gamma_2(1-\kappa x) + \tau(\beta_2 x - \alpha_2 y) & \cdot & \gamma_3(1-\kappa x) + \tau(\beta_3 x - \alpha_3 y) \end{bmatrix} \quad (3.4b)$$

Thus,

$$\begin{bmatrix} \partial/\partial X \\ \partial/\partial Y \\ \partial/\partial Z \end{bmatrix} = S^{-1} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial s \end{bmatrix} \quad (3.5a)$$

where

$$S^{-1} = \begin{bmatrix} \alpha_1 + \frac{\tau\gamma_1 y}{1-\kappa x} & \beta_1 - \frac{\tau\gamma_1 x}{1-\kappa x} & \frac{\gamma_1}{1-\kappa x} \\ \alpha_2 + \frac{\tau\gamma_2 y}{1-\kappa x} & \beta_2 - \frac{\tau\gamma_2 x}{1-\kappa x} & \frac{\gamma_2}{1-\kappa x} \\ \alpha_3 + \frac{\tau\gamma_3 y}{1-\kappa x} & \beta_3 - \frac{\tau\gamma_3 x}{1-\kappa x} & \frac{\gamma_3}{1-\kappa x} \end{bmatrix} \quad (3.5b)$$

The momentum $\vec{p}$ and the vector potential $\vec{A}$ also will have the same transformation as (3.4a) and (3.5a), i.e.

$$\begin{bmatrix} p_x \\ p_y \\ p_s \end{bmatrix} = S \begin{bmatrix} p_X \\ p_Y \\ p_Z \end{bmatrix} ; \quad \begin{bmatrix} A_x \\ A_y \\ A_s \end{bmatrix} = S \begin{bmatrix} A_X \\ A_Y \\ A_Z \end{bmatrix} \quad (3.6)$$

Calculating $(\vec{p} - \frac{e}{c} \vec{A})^2$ from eq. (3.6) we obtain the same expression that leads to the Hamiltonian (2.6).

Now we are interested in evaluating $\nabla X \vec{A}$ in the new coordinate system (x,y,s). Using (3.5a), (3.6b) and the inverse transformation of eq. (3.1),

$$\begin{aligned} \nabla X \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \\ &= \begin{vmatrix} \alpha_1 \hat{n} + \beta_1 \hat{b} + \gamma_1 \hat{t} & \alpha_2 \hat{n} + \beta_2 \hat{b} + \gamma_2 \hat{t} & \alpha_3 \hat{n} + \beta_3 \hat{b} + \gamma_3 \hat{t} \\ (\alpha_1 + \frac{\tau\gamma_1 y}{1-\kappa x}) \frac{\partial}{\partial x} + (\beta_1 - \frac{\tau\gamma_1 x}{1-\kappa x}) \frac{\partial}{\partial y} + \frac{\gamma_1}{1-\kappa x} \frac{\partial}{\partial s} & (\alpha_2 - \frac{\tau\gamma_2 y}{1-\kappa x}) \frac{\partial}{\partial x} + (\beta_2 - \frac{\tau\gamma_2 x}{1-\kappa x}) \frac{\partial}{\partial y} + \frac{\gamma_2}{1-\kappa x} \frac{\partial}{\partial s} & \dots \\ (\alpha_1 + \frac{\tau\gamma_1 y}{1-\kappa x}) A_x + (\beta_1 - \frac{\tau\gamma_1 x}{1-\kappa x}) A_y + \frac{\gamma_1}{1-\kappa x} A_s & (\alpha_2 + \frac{\tau\gamma_2 y}{1-\kappa x}) A_x + (\beta_2 - \frac{\tau\gamma_2 x}{1-\kappa x}) A_y + \frac{\gamma_2}{1-\kappa x} A_s & \dots \\ \alpha_3 \hat{n} + \beta_3 \hat{b} + \gamma_3 \hat{t} & & \\ (\alpha_3 + \frac{\tau\gamma_3 y}{1-\kappa x}) \frac{\partial}{\partial x} + (\beta_3 - \frac{\tau\gamma_3 x}{1-\kappa x}) \frac{\partial}{\partial y} + \frac{\gamma_3}{1-\kappa x} \frac{\partial}{\partial s} & & \\ (\alpha_3 + \frac{\tau\gamma_3 y}{1-\kappa x}) A_x + (\beta_3 - \frac{\tau\gamma_3 x}{1-\kappa x}) A_y + \frac{\gamma_3}{1-\kappa x} A_s & & \end{vmatrix} \\ &= \{ \frac{\tau\gamma}{1-\kappa x} (\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}) + \frac{1}{1-\kappa x} (\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}) \} \hat{n} \\ &+ \{ \frac{\tau x}{1-\kappa x} (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) + \frac{1}{1-\kappa x} (\frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x}) \} \hat{b} \\ &+ (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \hat{t} \end{aligned} \quad (3.7)$$

In particular, if we only have a transverse field, that is,

$$B_t = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0 \quad (3.8)$$

then in the case of nonzero torsion in magnet elements, we have

$$B_n = \frac{1}{1-\kappa x} \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s} \right) \quad (3.9a)$$

$$B_b = \frac{1}{1-\kappa x} \left(\frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x} \right) \quad (3.9b)$$

Now, although we restrict ourselves to zero torsion [$\tau=0$] inside any magnetic elements, orbit torsion is produced by having τ different from zero in field free regions. Thus, in this coordinate system, longitudinal fields are introduced in the "field free" regions. We find that the general expression (3.7) can be simplified and can be expressed as,

$$\nabla X \hat{A} = \frac{1}{1-\kappa x} \left[\left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s} \right) \hat{n} + \left(\frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x} \right) \hat{b} \right] + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{t} \quad (3.9c)$$

We can choose, in eq. (3.8)

$$A_x = A_y = 0 \quad (3.10)$$

and we further assume the power series for A_t ,

$$A_t = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + \dots \quad (3.11)$$

Then, appropriate magnet placement with respect to the defined reference orbit, together with the assumption of linear fields, lead to the magnetic field form,

$$\begin{aligned} B_n &= B' y \\ B_b &= -B_0 + B' x \end{aligned} \quad (3.12)$$

which then leads to

$$A_s = a_o + (a_1 - \kappa a_o)x + (a_3 - \kappa a_1)x^2 + a_5 y^2 \quad (3.13)$$

with

$$a_1 - \kappa a_s = B_o \quad (3.14)$$

$$-2a_3 + \kappa a_1 + \kappa^2 a_o = B'$$

To obtain a complete description of the vector potential we need a third condition, which is obtained by comparison with the well known equations of motion in torsion free regions. This is done in the next section.

3.2 Equations of Motion

Since $A_x = A_y = 0$, eq. (2.9) is reduced to

$$H = -\frac{e}{c} A_s + \tau(yP_x - xP_y) - (1-\kappa x) [p_s^2 - p_x^2 - p_y^2]^{1/2} \quad (3.15)$$

If we consider additionally a spread in the momentum, then for an off-momentum particle we can express

$$P_s = P_o \left(1 + \frac{\Delta P}{P}\right) \quad (3.16)$$

and

$$(1-\kappa x) [p_s^2 - p_x^2 - p_y^2] \approx P_o \left[1 + (1-\kappa x) \frac{\Delta P}{P} - \kappa x - \frac{1}{2} \left(\frac{p_x}{P_o}\right)^2 - \frac{1}{2} \left(\frac{p_y}{P_o}\right)^2\right] \quad (3.17)$$

Using (3.13) and (3.17) in (3.15)

$$\begin{aligned} H = & -\frac{e}{c} (a_o + (a_1 - \kappa a_o)x + (a_3 - \kappa a_1)x^2 + a_5 y^2) + \tau(yP_x - xP_y) \\ & - P_o \left[1 + (1-\kappa x) \frac{\Delta P}{P} - \kappa x - \frac{1}{2} \left(\frac{p_x}{P_o}\right)^2 - \frac{1}{2} \left(\frac{p_y}{P_o}\right)^2\right] \end{aligned} \quad (3.18)$$

The Hamiltonian equations are

$$\begin{aligned}
 P'_x &= \frac{e}{c} \left[(a_1 - \kappa a_0) + 2(a_3 - \kappa a_1)x \right] + \tau P_y - P_0 \kappa \left(\frac{\Delta P}{P} + 1 \right) \\
 P'_y &= \frac{e}{c} B'y - \tau P_x \\
 x' &= \frac{P_x}{P_0} + \tau y \\
 y' &= \frac{P_y}{P_0} - \tau x
 \end{aligned} \tag{3.19}$$

From (3.19) we get the equations

$$x'' = \left(\frac{2e}{P_0} (a_3 - \kappa a_1) + \tau^2 \right) x = \frac{e}{P_0} (a_1 - \kappa a_0) - \kappa \left(\frac{\Delta P}{P} + 1 \right) + 2\tau y' + \tau' y$$

and

$$y'' = \left(\frac{e}{P_0} B' + \tau^2 \right) y = -2\tau x' - \tau' x \tag{3.20}$$

For $\tau=0$ they must agree with

$$\begin{aligned}
 x'' + (K + \kappa^2)x &= -\kappa \frac{\Delta P}{P} \\
 y'' - Ky &= 0
 \end{aligned} \tag{3.21}$$

where $K = \frac{B'}{B_0 \rho}$. Taking into account that $P_0 = eB_0 \rho$, the coefficients must satisfy

$$a_3 - \kappa a_1 = -\frac{1}{2} (B' + \kappa B_0) \tag{3.22}$$

Solving (3.14) and (3.22) we get

$$\begin{aligned}
 a_0 &= 0 \\
 a_1 &= B_0 \\
 a_3 &= \frac{1}{2} (B_0 \kappa - B')
 \end{aligned} \tag{3.23}$$

and then eq. (3.13) is

$$A_s = B_0 x - \frac{1}{2} (B_0 \kappa + B') x^2 + \frac{B'}{2} y^2 \quad (3.24)$$

and the equations of motion (3.20) take the form

$$\begin{aligned} x'' + (K + \kappa^2 - \tau^2)x &= -\kappa \frac{\Delta P}{P} + 2\tau y' + \tau' y \\ y'' - (K + \tau^2)y &= -2\tau x' - \tau' x \end{aligned} \quad (3.25)$$

4. Transfer Matrix for a Drift Space with Torsion

We are now going to derive the equations of motion in our coordinate system for a straight segment with constant torsion τ and length s .

For this drift space we have

$$K = 0 = \kappa \quad (4.1)$$

and then equations (3.25) are simply

$$\begin{aligned} x'' - \tau^2 x &= 2\tau y' + \tau' y \\ y'' - \tau^2 y &= -2\tau x' - \tau' x \end{aligned} \quad (4.2)$$

We can easily see that the solution of equations (4.2) are

$$\begin{aligned} x(s) &= (a + bs) \cos \theta + (c + ds) \sin \theta \\ y(s) &= (c + ds) \cos \theta - (a + bs) \sin \theta \end{aligned} \quad (4.3a)$$

where a, b, c, d are constants, and

$$\frac{d\theta}{ds} = \tau(s) \quad (4.3b)$$

We can see from the appendix that having torsion in the straight sections is a direct consequence, having bending dipoles in intersecting planes. We will show that the transfer matrix is independent of τ explicitly. Thus, we can choose the length over which $\tau \neq 0$ arbitrarily. In particular we can choose $\tau=0$ in all elements (quadrupoles) between dipoles. Also, since the transfer matrix is independent of τ explicitly, we can replace the transfer matrix by the product of a drift followed by a zero drift rotation.

Our assumptions about the reference orbit and magnet placement imply that $\kappa(s)$ and $\tau(s)$ are step functions. So, we must keep in mind that there are δ -function contributions to their derivatives.

In what follows we will study in detail the case $\tau'(s) = \text{constant} \rightarrow \infty$, in two situations:

- a) when the transfer matrix relates to the canonical phase space,
- b) when the transfer matrix relates to the (x, x', y, y') space.

Let us assume that $\tau(s)$ is of the form

and $x_1, x'_1, p_{x1}, y_1, y'_1, p_{y1}$ be initial values at s_1 (4.4)

Case a.

First we will find the transfer matrix for the region where $\tau = \tan \alpha(s-s_1)$. Putting initial conditions (4.4) in eqs. (4.3), we have

$$\begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & -G(s-s_1) \cos \theta_2 & \sin \theta_2 & -G(s-s_1) \sin \theta_2 \\ 0 & \cos \theta_2 & 0 & \sin \theta_2 \\ -\sin \theta_2 & G(s-s_1) \sin \theta_2 & \cos \theta_2 & -G(s-s_1) \cos \theta_2 \\ 0 & -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ P_{x1} \\ y_1 \\ P_{y1} \end{bmatrix}$$

(4.5a)

where

$$\theta_2 = \frac{\tau_0}{2} \frac{(s-s_1)^2}{s_2-s_1} \quad (4.5b)$$

If we now take $s=s_2$ and take the limit $s_2 \rightarrow s_1$

$$\begin{bmatrix} x_2 \\ P_{x2} \\ y_2 \\ P_{y2} \end{bmatrix} = I \begin{bmatrix} x_1 \\ P_{x1} \\ y_1 \\ P_{y1} \end{bmatrix} \quad (4.6)$$

Since $I^{-1} = I$, we expect that in the region where $\tau = \tan \alpha(s_4-s)$ the transfer matrix will be again I , and therefore, in going from s_1 to s_4 , the only significant matrix in the play is in the region where $\tau=\tau_0$, which now calculates.

Using (4.3) and (4.6), we have

$$\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} = \begin{bmatrix} \cos \theta_3 & G(s-s_2) \cos \theta_3 & \sin \theta_3 & G(s-s_2) \sin \theta_3 \\ 0 & \sin \theta_3 & 0 & \sin \theta_3 \\ -\sin \theta_3 & -G(s-s_2) \sin \theta_3 & \cos \theta_3 & G(s-s_2) \cos \theta_3 \\ 0 & -\sin \theta_3 & \cos \theta_3 & \cos \theta_3 \end{bmatrix} \begin{bmatrix} x_2 \\ p_{x2} \\ y \\ p_{y2} \end{bmatrix} \quad (4.7a)$$

where

$$\theta_3 = \tau_o(s-s_2). \quad G = \frac{1}{P_o} \quad (4.7b)$$

If in (4.7a) we make $s=s_3$ and $\ell=s_3-s_2$ and $\bar{\theta}_3 = \tau_o(s_3-s_2)$, we can write

$$\begin{bmatrix} \cos \bar{\theta}_3 & G\ell \cos \bar{\theta}_3 & \sin \bar{\theta}_3 & G\ell \sin \bar{\theta}_3 \\ 0 & \cos \bar{\theta}_3 & 0 & \cos \bar{\theta}_3 \\ -\sin \bar{\theta}_3 & -G\ell \sin \bar{\theta}_3 & \cos \bar{\theta}_3 & G\ell \cos \bar{\theta}_3 \\ 0 & -\sin \bar{\theta}_3 & 0 & \cos \bar{\theta}_3 \end{bmatrix} = R(\bar{\theta}_3)D(\ell) = D(\ell)R(\bar{\theta}_3) \quad (4.8a)$$

where

$$R(\theta_3) = \begin{bmatrix} \cos \bar{\theta}_3 & 0 & \sin \bar{\theta}_3 & 0 \\ 0 & \cos \bar{\theta}_3 & 0 & \sin \bar{\theta}_3 \\ -\sin \bar{\theta}_3 & 0 & \cos \bar{\theta}_3 & 0 \\ 0 & -\sin \bar{\theta}_3 & 0 & \cos \bar{\theta}_3 \end{bmatrix} \quad (4.8b)$$

is a rotation in each phase space, and

$$D(\ell) = \begin{bmatrix} 1 & G\ell & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & G\ell \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.8c)$$

is the matrix corresponding to a normal drift space.

Case b.

For this case, the transfer matrix for going from s_1 to s , for $s_1 \leq s \leq s_2$ is,

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & -(s-s_1) \cos \theta_2 & \cdot & \sin \theta_2 & -(s-s_1) \sin \theta_2 \\ -\tau \sin \theta_2 & -(s-s_1) \tau \sin \theta_2 + \cos \theta_2 & \vdots & \tau \cos \theta_2 & \sin \theta_2 + (s-s_1) \tau \cos \theta_2 \\ -\sin \theta_2 & -(s-s_1) \sin \theta_2 & \vdots & \cos \theta_2 & -(s-s_1) \cos \theta_2 \\ -\tau \cos \theta_2 & -\sin \theta_2 - (s-s_1) \tau \cos \theta_2 & \vdots & -\tau \sin \theta_2 & \cos \theta_2 - (s-s_1) \tau \sin \theta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \end{bmatrix} \quad (4.9)$$

Where θ_2 is given by eq. (4.5b).

Taking $s=s_2$ and $s_2 \rightarrow s_1$,

$$\lim_{\substack{s \rightarrow s_2 \\ 2 \quad 1}} [\text{matrix (4.9)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \tau & 0 \\ 0 & 0 & 1 & 0 \\ \tau & 0 & 0 & 1 \end{bmatrix} \quad (4.10)$$

Direct calculations show that the matrix for going from s_2 to s_3 is

$$\begin{bmatrix} \cos\bar{\theta}_3 + \bar{\theta}_3 \sin\bar{\theta}_3 & (s_3-s_2) \cos\bar{\theta}_3 & \sin\bar{\theta}_3 - \bar{\theta}_3 \cos\bar{\theta}_3 & (s_3-s_2) \sin\bar{\theta}_3 \\ \tau_0 \bar{\theta}_3 \cos\bar{\theta}_3 & \cos\bar{\theta}_3 - \bar{\theta}_3 \sin\bar{\theta}_3 & \tau_0 \bar{\theta}_3 \sin\bar{\theta}_3 & \sin\bar{\theta}_3 + \bar{\theta}_3 \cos\bar{\theta}_3 \\ -\sin\bar{\theta}_3 + \bar{\theta}_3 \cos\bar{\theta}_3 & -(s_3-s_2) \sin\bar{\theta}_3 & \cos\bar{\theta}_3 + \bar{\theta}_3 \sin\bar{\theta}_3 & (s_3-s_2) \cos\bar{\theta}_3 \\ -\tau_0 \bar{\theta}_3 \sin\bar{\theta}_3 & -\sin\bar{\theta}_3 - \bar{\theta}_3 \cos\bar{\theta}_3 & \tau_0 \bar{\theta}_3 \cos\bar{\theta}_3 & \cos\bar{\theta}_3 - \bar{\theta}_3 \sin\bar{\theta}_3 \end{bmatrix} \quad (4.11)$$

For the section between s_3 and s_4 , the matrix must be the inverse of (4.10), i.e.,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\tau_0 & 0 \\ 0 & 0 & 1 & 0 \\ \tau_0 & 0 & 0 & 1 \end{bmatrix} \quad (4.12)$$

Thus, the transfer matrix from s to s_4 is the product of (4.10), (4.11) and (4.12). Performing the product, we obtain

$$\begin{bmatrix} \cos\bar{\theta}_3 & (s_3-s_2) \cos\bar{\theta}_3 & \sin\bar{\theta}_3 & (s_3-s_2) \sin\bar{\theta}_3 \\ 0 & \cos\bar{\theta}_3 & 0 & \sin\bar{\theta}_3 \\ -\sin\bar{\theta}_3 & -(s_3-s_2) \sin\bar{\theta}_3 & \cos\bar{\theta}_3 & (s_3-s_2) \cos\bar{\theta}_3 \\ 0 & -\sin\bar{\theta}_3 & 0 & \cos\bar{\theta}_3 \end{bmatrix} \quad (4.13)$$

which is similar to (4.7a), with $G=1$, and therefore can be factorized as (4.8a), i.e., it is the product of a rotation matrix times a drift space.

Thus, we have proved that the transfer matrix depends only on the coordinate rotation angle and not on τ explicitly and also that the observed physics in regions of non-zero τ depend on the rate of the spinning coordinate system, τ .

5. Analytic Treatment of Equations of Motion

Using eq. (3.13) we can express the Hamiltonian (3.15) as

$$H = \frac{eB'}{2c} (x^2 - y^2) + \tau(yP_x - xP_y) + \frac{1}{2} \frac{p_x^2}{P_0} + \frac{1}{2} \frac{p_y^2}{P_0} \quad (5.1a)$$

$$\text{where we make } B_0=0, \kappa=0, \Delta P/P=0 \quad (5.1b)$$

and we ignore the constant 1.

Hamiltonians of the type (5.1a) have already been studied⁶, and hence, we will only give the solution for our case.

We define X as

$$X \equiv \begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix} \quad (5.2)$$

A similar analysis to ref. 6 shows that X can be expressed as

$$X = RV$$

where

$$R = \begin{bmatrix} I \cos\phi & D^{-1} \sin\phi \\ -D \sin\phi & I \cos\phi \end{bmatrix} \quad (5.3)$$

and

$$V \equiv \begin{bmatrix} u \\ P_u \\ v \\ P_v \end{bmatrix} = \begin{bmatrix} (W_1 \beta_1)^{1/2} \cos \psi_1 \\ -(W_1 / \beta_1)^{1/2} (\sin \psi_1 + \alpha_1 \cos \psi_1) \\ (W_2 \beta_2)^{1/2} \cos \psi_2 \\ -(W_2 / \beta_2)^{1/2} (\sin \psi_2 + \alpha_2 \cos \psi_2) \end{bmatrix} \quad (5.4)$$

In eq. (5.3) ϕ and the 3 independent parameters α^+

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ; \quad (ad-bc = 1) \quad (5.5)$$

are parameters of the group $Sp(4)$ and fulfill the conditions:

$$\begin{aligned} \phi' &= \frac{1}{2} \tau(a+d) \\ a' &= bF + cZ + \tau(2-a(a+d)) \cot 2\phi \\ b' &= -(a-d)Z - \tau b(a+d) \cot 2\phi \\ c' &= (a+d)F - \tau c(a+d) \cot 2\phi \\ d' &= bF - cZ + \tau(2-d(a+d)) \cot 2\phi \end{aligned} \quad (5.6)$$

where

$$F = \frac{eB'}{c} \quad \text{and} \quad Z = \frac{1}{P_0} \quad .$$

In eq. (5.4) there appear the extended Courant-Snyder¹ parameters which here satisfy the conditions

$$\begin{aligned} \alpha_1' &= -Z\gamma_1 + F\beta_1 + \tau(b\gamma_1 + c\beta_1) \tan \phi \\ \beta_1' &= -2Z\alpha_1 + \tau(2b\alpha_1 - (a-d)\beta_1) \tan \phi \\ \gamma_1' &= 2F\alpha_1 + \tau(2c\alpha_1 + (a-d)\gamma_1) \tan \phi \end{aligned}$$

$$\alpha'_2 = -Z\gamma_2 - F\beta_2 - \tau(b\gamma_2 + c\beta_2) \tan\phi \quad (5.7)$$

$$\beta'_2 = -2Z\alpha_2 - \tau(2b\alpha_2 - (a-d)\beta_2) \tan\phi$$

$$\gamma'_2 = -2F\alpha_2 - \tau(2c\alpha_2 + (a-d)\gamma_2) \tan\phi$$

$$\psi_1 = \int \frac{(Z - \tau b \tan\phi)}{\beta_1} ds - \delta_1$$

$$\psi_2 = \int \frac{(Z + \tau b \tan\phi)}{\beta_2} ds - \delta_2$$

W_1, W_2 in (5.4) and δ_1, δ_2 in (5.7) are the constants of integration. W_1 and W_2 are given by

$$W_1 = \tilde{V} \, \tilde{\mathcal{S}} \begin{bmatrix} c_1 & \beta_1 & \cdot & 0 \\ -\gamma_1 & -\alpha_1 & \cdot & 0 \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{bmatrix} V, \quad W_2 = \tilde{V} \, \tilde{\mathcal{S}} \begin{bmatrix} 0 & \cdot & 0 & \cdot \\ \cdot & \cdot & \alpha_2 & \beta_2 \\ 0 & \cdot & -\gamma_2 & -\alpha_2 \end{bmatrix} V \quad (5.8)$$

Stability of Motion

The transfer matrix for one turn, can be expressed as⁶

$$T = RUR^{-1} \quad (5.9)$$

where R is given by (5.3) and U is

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad (5.10)$$

The stability of the motion can be studied for each normal coordinate separately, since the motion is decoupled in this system. Note that although the motion is simply harmonic in the normal coordinate system, when viewed in the real frame of reference (x, x', y, y') , this simplicity becomes hidden in coupled 4D phase space motion. The matrices, A, B , are the 2x2 symplectic matrices parametrized by

$$\begin{aligned}
 A &= I \cos \mu_1 + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin \mu_1 \\
 B &= I \cos \mu_2 + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \sin \mu_2
 \end{aligned} \tag{5.11}$$

For n revolutions the transfer matrix is

$$T^n = R U^n R^{-1} = R \begin{pmatrix} A^n & 0 \\ 0 & B^n \end{pmatrix} R^{-1} \tag{5.12}$$

But

$$A^n = I \cos n\mu_1 + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin n\mu_1 \tag{5.13}$$

and there is similar relation for B^n . So the motion will be stable if μ_1, μ_2 are real, i.e., if

$$| \operatorname{Tr} A | \leq 2 \quad \text{and} \quad | \operatorname{Tr} B | \leq 2. \tag{5.14}$$

Appendix A

In this appendix we analyze the torsion for a drift space between dipoles, as is shown in the next figure.

where

$$\vec{r}_0 = x_0 \hat{i} + y_0 \hat{j} + z_0 \hat{k}$$

$$\hat{i}' = \cos \alpha \hat{i} - \sin \alpha \hat{k}$$

$$\hat{j}' = \hat{j}$$

(A.1)

For the left circle

$$\vec{r}(s) = \rho_1 \cos \frac{s}{\rho_1} \hat{i} + \rho_1 \sin \frac{s}{\rho_1} \hat{j}$$

and

$$\vec{r}(0) = \vec{p}_1 = \rho_1 \hat{i}$$

and

$$\begin{aligned}\hat{t} &= \frac{d\vec{r}}{ds} = -\sin \frac{s}{\rho_1} \hat{i} + \cos \frac{s}{\rho_1} \hat{j} \\ \hat{t}_1 &\equiv \hat{t}(0) = \hat{j}\end{aligned}\quad (\text{A.2})$$

For the right circle

$$\begin{aligned}\vec{p}_2(s) &= x'\hat{i} + y'\hat{k}' = \rho_2 \cos \frac{s-s_2}{\rho_2} \cos \alpha \hat{j} + \rho_2 \sin \frac{s-s_2}{\rho_2} \hat{j} \\ &\quad - \rho_2 \cos \frac{s-s_2}{\rho_2} \sin \alpha \hat{k}\end{aligned}\quad (\text{A.3})$$

We can express the position as

$$\vec{r}(s) = \begin{cases} \rho_1 \cos \frac{s}{\rho_1} \hat{i} + \rho_1 \sin \frac{s}{\rho_1} \hat{j} & -s_1 \leq 0 \\ \vec{p}_1 + s\hat{t}_1 & 0 < s < s_2 \\ \vec{r}_0 + \vec{p}_2 & s_2 \leq s \leq s_3 \end{cases} \quad (\text{A.4})$$

Using (A.1), (A.2) and (A.3) into (A.4) and asking for $\vec{r}(s)$ to be a continuous curve, we get

$$x_0 = \rho_1 - \rho_2 \cos \alpha$$

$$y_0 = s_2$$

$$z_0 = \rho_2 \sin \alpha \quad (\text{A.5})$$

The tangent vector is

$$\hat{t}(s) = \begin{cases} -\sin \frac{s}{\rho_1} \hat{i} + \cos \frac{s}{\rho_1} \hat{j} & -s_1 \leq 0 \\ \hat{j} & 0 < s < s_2 \\ -\sin \frac{s-s_2}{\rho_2} \cos \alpha \hat{i} + \cos \frac{s-s_2}{\rho_2} \hat{j} + \sin \frac{s-s_2}{\rho_2} \sin \alpha \hat{k} & s_2 \leq s \leq s_3 \end{cases} \quad (\text{A.6})$$

and its derivative is

$$\frac{d\hat{t}}{ds} = \begin{cases} \frac{1}{\rho_1} \left(-\cos \frac{s}{\rho_1} \hat{i} - \sin \frac{s}{\rho_1} \hat{j} \right) = \kappa_1 \hat{n}_1 & -s \leq 0 \\ 0 \left(\cos \beta \hat{i} - \sin \beta \hat{k} \right) = \kappa_0 \hat{n}_0 & 0 < s < s_0 \\ \frac{1}{\rho_2} \left(-\cos \frac{s-s_2}{\rho_2} \cos \alpha \hat{i} - \sin \frac{s-s_2}{\rho_2} \hat{j} + \cos \frac{s-s_2}{\rho_2} \sin \alpha \hat{k} \right) = \kappa_2 \hat{n}_2 & s_2 \leq s \leq s_3 \end{cases} \quad (A.7)$$

The binormal vector $\hat{b} = \hat{t} \times \hat{n}$ is

$$\hat{b}(s) = \begin{cases} \hat{k} & -s_1 \leq 0 \\ -\sin \beta \hat{i} - \cos \beta \hat{k} & 0 < s < s_2 \\ \sin \alpha \hat{i} + \cos \alpha \hat{k} & s_2 \leq s \leq s_3 \end{cases} \quad (A.8)$$

and its derivative is

$$\frac{d\hat{b}}{ds} = \begin{cases} 0 \\ -\beta' \left(\cos \beta \hat{i} - \sin \beta \hat{k} \right) = -\tau_0 \hat{n}_0 \\ 0 \end{cases}$$

and then

$$\beta' = \tau_0 \quad \text{or} \quad \beta(s) = \tau_0 s + c \quad (A.9)$$

Consequently ,

$$\cos \beta = A \cos \tau_0 s - B \sin \tau_0 s \quad (A.10a)$$

and

$$\sin \beta = A \sin \tau_0 s + B \cos \tau_0 s \quad (A.10b)$$

where

$$A = \cos c \quad , \quad B = \sin c \quad (A.10c)$$

From eqs. (A.7) we see that continuity of $\hat{n}$ implies that at $s=0$

$$\cos \beta(0) = -1$$

$$\sin \beta(0) = 0 \quad (\text{A.11})$$

and at $s=s_2$, $\cos \beta(s_2) = -\cos \alpha$

$$\sin \beta(s_2) = \sin \alpha \quad (\text{A.12})$$

Equations (A.10) and (A.11) give

$$\Lambda = -1 \quad , \quad B = 0 \quad (\text{A.13})$$

and (A.10) with condition (A.13) and (A.12) give

$$\cos \tau_0 s_2 = \cos \alpha \quad (\text{A.14})$$

$$\sin \tau_0 s_2 = \sin \alpha$$

from which we find the torsion

$$\boxed{\tau_0 = \frac{\alpha}{s_2}} \quad (\text{A.15})$$

and thus

$$\hat{b}(s) = \begin{cases} \hat{k} \\ \sin \tau_0 s \hat{i} + \cos \tau_0 s \hat{k} \\ \hat{k}' \end{cases} \quad \begin{matrix} -s_1 \leq 0 \\ 0 < s < s_2 \\ s_2 \leq s \leq s_3 \end{matrix} \quad (\text{A.16})$$

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