Turbulent "polarization" terms and the Balescu-Lenard operator

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Certain unfamiliar terms in renormalized plasma turbulence theory are interpreted in terms of the familiar physics of the Balescu-Lenard collision operator. Specifically, it is argued that the so-called polarization parts of the operator which renormalizes the particle propagator are related to fluctuations of the Fokker-Planck coefficient which describes polarization drag, and to fluctuations of the effective dielectric function of the medium.

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1. Introduction

Recently, several authors (DuBois & Espedal 1978; Krommes & Kleva 1979; Krommes 1979a,b; Krommes & Similon 1980; Krommes 1980; DuBois 1981) concerned with systematic renormalizations of the Klimontovich or Vlasov equations have argued for the likely importance (or, at least, against the cavalier neglect) of certain nonlinear terms formally present but generally omitted in heuristic approaches. These terms were called "polarization terms" by Krommes & Kleva (1979). For example, the renormalized particle propagator g has, for Vlasov plasma, the general form (DuBois & Espedal 1978; Krommes & Kleva 1979; Krommes 1980)

$$g_{\tilde{k},\omega}(\tilde{v};\tilde{v}') = [-i(\omega - \tilde{k} \cdot \tilde{v} + i\tilde{\Sigma}_{\tilde{k},\omega}^{(d)} + i\tilde{\Sigma}_{\tilde{k},\omega}^{(p)})]^{-1} \delta(\tilde{v} - \tilde{v}') , \quad (1)$$

where $\tilde{\Sigma}^{(d)}$ is an operator in velocity space well-understood in the Markovian limit $(\tilde{k},\omega) \rightarrow 0$, where it reduces to a diffusion operator. The polarization term $\tilde{\Sigma}^{(p)}$ is not well-understood and is often ignored. It is, however, well-known in weak turbulence theory (Sagdeev & Galeev 1969), where it contributes to nonlinear scattering from screening clouds. In part, it was this observation which motivated Krommes & Kleva to attach the name "polarization" to $\tilde{\Sigma}^{(p)}$, since the establishment of a screening cloud is a polarization process (Tsytovich 1970). However, the fundamental intuition underlying the choice by Krommes & Kleva of the attribute "polarization" was the intimate connection of $\tilde{\Sigma}^{(p)}$ to the well-known theory of collisional effects on small amplitude fluctuations excited in a stable plasma for which the plasma parameter $\epsilon_p \equiv 1/n\lambda_D^3$ is very small, but not zero. For fluctuations in the kinetic or hydrodynamic regimes (wavelengths long compared to the Debye length, frequencies small compared to the plasma frequency), such effects are described

by the linearized Balescu-Lenard operator. The formal kinetic (in the sense of two-time correlations) derivation of that operator is due to Oberman (1972). It has been discussed by Williams (1973), Krommes (1975), and Krommes & Oberman (1976); there are earlier, related kinetic calculations in the theory of neutral fluids (Hinton 1970). Krommes & Kleva did not develop the intuition based on discreteness effects because they were concerned with collisionless turbulence. However, the fact that debate continues on the meaning and importance of the polarization terms suggests that it may be useful to describe their connection to the Balescu-Lenard theory in detail; that is the goal of this paper. Specifically, we will show by direct calculation how to reduce the appropriate renormalized equation for the infinitesimal plasma response function to the classical linearized Balescu-Lenard operator. In so doing, one can follow the reduction of the polarization term $\Sigma^{(p)}$ in detail and understand explicitly how $\Sigma^{(p)}$ is related to various aspects of the dielectric response of the medium — i.e., to polarization phenomena. These relations are more general than one might infer from the specific nature of the calculation.

Since the linearized Balescu-Lenard operator is central to the discussion, it is appropriate to make a few introductory remarks on its derivation and interpretation. [The more approximate Landau form of the operator (Landau 1937), often used in applications, is not adequate for our purposes.] The form of the linearized operator can be obtained by linearizing (Oberman 1972; Williams 1973; Krommes 1975; Krommes & Oberman 1976a) the nonlinear operator (Balescu 1960; Lenard 1960; Ichimaru 1973)

$$\left(\frac{\partial f}{\partial t}\right)_c = \left(\frac{e^2}{m}\right) \frac{1}{s} \frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{s} (ne^2) \frac{1}{s} \int d\mathbf{v}' \frac{\mathbf{\tilde{q}} \cdot \mathbf{\tilde{q}}'}{|\mathbf{D}(\mathbf{q}, \mathbf{q}', \mathbf{v})|^2}$$

$$\times \delta[\underline{q} \cdot (\underline{v} - \underline{\bar{v}})] \left(\frac{1}{m_s} \frac{\partial}{\partial \underline{v}} - \frac{1}{m_s} \frac{\partial}{\partial \underline{\bar{v}}} \right) f_s(\underline{v}) f_s(\underline{\bar{v}}) \quad (2)$$

around a background distribution F to give

$$\left(\frac{\partial}{\partial t} \delta f_s \right)_c = -(\nu \delta f)_s, \quad (3a)$$

where

$$\begin{aligned} (\nu \delta f)_s &\equiv -(a^2)_s \frac{\partial}{\partial \underline{p}} \int \frac{d\underline{q}}{(2\pi)^3} \int \frac{(ne^2)_s}{s} \int d\underline{\bar{v}} \frac{\underline{\varepsilon}_{\underline{q}} \underline{\varepsilon}_{-\underline{q}}}{|D(\underline{q}, \underline{q}, \underline{v})|^2} \delta[\underline{q} \cdot (\underline{v} - \underline{\bar{v}})] \\ &\times \left[\bar{F} \frac{\partial}{\partial \underline{p}} \delta f + \frac{\partial \bar{F}}{\partial \underline{p}} \delta \bar{f} - \frac{\partial \bar{F}}{\partial \underline{p}} \delta f - F \frac{\partial}{\partial \underline{p}} \delta \bar{f} - 2 \operatorname{Re} \left[\frac{\delta D(\underline{q}, \underline{q}, \underline{v})}{D(\underline{q}, \underline{q}, \underline{v})} \right] \left(\frac{\partial}{\partial \underline{p}} - \frac{\partial}{\partial \underline{\bar{p}}} \right) F \bar{F} \right]. \quad (3b) \end{aligned}$$

In the above, "c" stands for collision, s is a species label, $\underline{p} \equiv m_s \underline{v}$, $\bar{F} \equiv F_s$, $\underline{\varepsilon}_{\underline{q}} \equiv -4\pi i q / q^2$ is the Fourier transform of the Coulomb field from a unit charge, the Vlasov dielectric is

$$D(\underline{q}, \omega; \{F\}) \equiv 1 + \int \frac{\omega_p^2}{s q^2} \int d\underline{\bar{v}} \frac{\underline{q} \cdot \partial \bar{F} / \partial \underline{\bar{v}}}{\omega - \underline{q} \cdot \underline{\bar{v}} + i0}, \quad (4)$$

$\{\dots\}$ indicates functional dependence, $\omega_p^2 \equiv 4\pi(ne^2/m)_s$, and $\delta D \equiv D(\delta f) - 1$.

The form of (3b) can be understood in terms of a linearized Fokker-Planck equation. It is well-known (Ichimaru 1973) that (2) has the form

$$\left(\frac{\partial f_s}{\partial t} \right)_c = - \frac{\partial}{\partial \underline{v}} \cdot \left(\frac{\langle \Delta \underline{v} \rangle}{\Delta t} f_s \right)_p + \frac{\partial}{\partial \underline{v}} \cdot \left(\frac{\langle \Delta \underline{v} \Delta \underline{v} \rangle}{2 \Delta t} \right) \frac{\partial f_s}{\partial \underline{v}},$$

where $\Delta \underline{v}$ is the random velocity increment in time Δt (long compared to the autocorrelation time but short compared to the collision time) and where "p" denotes the part due to polarization drag. The coefficients $\langle \Delta \underline{v} \rangle$ and $\langle \Delta \underline{v} \Delta \underline{v} \rangle$ are themselves functionals of f and participate in any linearization. Thus, the term in $\partial(\delta f)/\partial \underline{p}$ represents velocity-space diffusion of the fluctuations, while the term in δf represents polarization drag on the

fluctuations. The remaining terms arise from fluctuations in the Fokker-Planck coefficients around the unperturbed state: the two terms in $\delta F/\delta p$ stem from fluctuations in the spectral intensity of the electric field (in terms of which the velocity space diffusion coefficient is defined), the two terms in F stem from fluctuations in the shielding cloud of a test particle experiencing polarization drag. Alternatively, one can say that the terms in $\bar{F}\delta f/\delta p$ and $\delta f\delta \bar{F}/\delta p$ represent respectively the diffusion of, and drag on the fluctuations from their collisions with the background, whereas the terms in $\delta \bar{F}\delta F/\delta p$ and $F\delta \delta \bar{F}/\delta p$ represent the diffusion of, and drag on the background from its collisions with the fluctuations. The presence of such inverse processes is necessary, of course, in order that the momentum and energy conservation laws are properly maintained. The terms in $\delta D/D$ describe the first-order part of the diffusion and drag in an effective background with dielectric $D+\delta D$.

The terms in $\delta D/D$ may be unfamiliar (Oberman 1972; Williams 1973; Krommes 1975; Krommes & Oberman 1976a). They vanish by virtue of detailed balance when the plasma is perturbed around absolute thermal equilibrium, as is the case in certain (projection operator) approaches (Krommes 1975; Krommes & Oberman 1976b) to collisional transport. Furthermore, they are not obviously present in the Landau approximation, where the dielectric D is replaced by unity. (Presumably, they would emerge from complicated perturbations in the effective wavenumber cutoffs defining the Landau operator; however, these seem never to be considered.)

In conventional approaches, the perturbation δf is interpreted as the (smooth) non-equilibrium correction to a local equilibrium state F . In these terms the linearization procedure previously described is easily justifiable. However, by a technically trivial but philosophically profound

re-interpretation (Williams 1973; Mori 1965), δf can also be thought of as describing (random, spiky) fluctuations of infinitesimal amplitude in (or near) absolute equilibrium. In this case a more formal and systematic justification of the form (3b) is required. It was apparently Williams and Oberman (Williams 1972) who first noticed that the above linearization procedure could be formally justified in terms of an appropriately-truncated hierarchy of correlation functions, the two-time generalization of the BBGKY hierarchy. Krommes and Oberman (Krommes 1975; Krommes & Oberman 1976a) performed the solution of the two-time hierarchy in detail and explicitly verified Williams' observation — namely, that the operator v defined by (3b) describes the evolution of the two-time phase space density correlation function [defined by (11)] in the kinetic and hydrodynamic regimes.

In this paper we shall adopt the latter interpretation of δf . Formally, relaxation of small-amplitude fluctuations is described by the infinitesimal response function (Martin, Siggia & Rose 1973)

$$R(1;1') \equiv \delta f(1)/\delta \eta(1'), \quad (5)$$

where η is a source of particles added to the right-hand side of the Klimontovich equation, $\delta/\delta \eta$ denotes a functional derivative, we use the notation $"1" \equiv \{\tilde{x}_1, \tilde{v}_1, s_1, t_1\}$, and $f(1) \equiv \langle \tilde{N}(1) \rangle$, $\tilde{N}$ being the Klimontovich density

$$\tilde{N}(1) \equiv \frac{1}{n_{s_1}} \prod_{i=1}^{N_{s_1}} \delta[\tilde{x}_i - \tilde{x}_1(t_1)] \delta[\tilde{v}_i - \tilde{v}_1(t_1)]. \quad (6)$$

Various authors (DuBois & Espedal 1978; Krommes & Kleva 1979; Krommes 1979a,b; Krommes & Similon 1980; Krommes 1980; DuBois 1981; Krommes 1975; Krommes & Oberman 1976a; Martin, Siggia & Rose 1973; Krommes 1978; Jensen 1981; Rose 1979) have discussed systematically renormalized equations for R . In particular, Rose (1979) has provided an elegant theory of classical

renormalization taking explicit account of particle discreteness (which, since it corresponds to non-Gaussian initial conditions, was difficult to treat by earlier approaches). However, Rose was forced to introduce a cubic nonlinearity, which tends to obscure the relation to the quadratic nonlinearity treated in Vlasov theory. Thus, we will not discuss Rose's general results here but will, rather, adopt a much more modest goal: we will show how the linearized Balescu-Lenard operator ν arises from a quasilinear description of R . Though this is a substantial approximation, it is sufficiently nontrivial that it will lead us to further understanding of the complicated structure of the statistical theory of plasma.

The paper is organized as follows. In Sec. 2 we review the procedure which renormalizes the equation for the response function consistently to lowest order in ϵ_p . We then show how this renormalization reduces to the linearized Balescu-Lenard equation in the long wavelength, low frequency limit. We discuss the physics of this reduction in Sec. 3. Intermediate algebraic steps are collected in two appendices.

2. Polarization terms and the Balescu-Lenard operator

The operator ν is closely related to the Markovian limit $(\underline{k}, \omega) \rightarrow 0$ of the polarization term $\Sigma_{\underline{k}, \omega}^{(p)}$ as written, for example, by DuBois & Espedal (1978), Krommes & Kleva (1979), or Krommes (1980) in the direct-interaction approximation; however, they are not identical. Of course, it should not be expected that the Vlasov results reviewed by Krommes (1980) would be directly applicable to a problem with particle discreteness, because initial conditions were ignored in that work and the equal-time Klimontovich two-point correlation contains the singular, non-Gaussian self-energy contribution $\{\epsilon_n^{-1} = 0(\epsilon_p)\}$:

$$\langle \delta N(\underline{1}, t) \delta N(\underline{2}, t) \rangle = n_s^{-1} \delta_{s_1, s_2} \delta(\underline{x}_1 - \underline{x}_2) \delta(\underline{v}_1 - \underline{v}_2) F(\underline{2}, t) + G(\underline{1}, \underline{2}, t). \quad (7)$$

(Here G is the smooth pair correlation function, and we distinguish all arguments except the time by the notation " $\underline{1}$ ".) However, we can easily proceed from first principles with finite ε_p and recover the collisionless result as a special case. The mean Klimontovich equation is, in the presence of a source η ,

$$(\partial_t + \underline{v} \cdot \underline{\nabla}) f + \langle \underline{E} \rangle \cdot \underline{\nabla} f + \underline{\nabla} \cdot \langle \delta \underline{E} \delta \underline{N} \rangle = \eta, \quad (8)$$

where the usual self-energy interaction is assumed to be omitted in the fluctuation term. We can find a formal equation for the response function by differentiating Eq. (8) functionally (Martin, Siggia & Rose 1973) with respect to η :

$$(\partial_t + \underline{v} \cdot \underline{\nabla}) R(\underline{1}; \underline{1}') + \langle \underline{E} \rangle \cdot \underline{\nabla} R + \underline{\nabla} f \cdot \hat{\underline{E}} R + \underline{\nabla} \cdot \hat{\underline{E}}(\underline{1}, \underline{1}') \frac{\delta}{\delta \eta(\underline{1}')} C(\underline{1}, \underline{1}') = \delta(\underline{1} - \underline{1}'), \quad (9)$$

where

$$\hat{\underline{E}}(\underline{1}, \underline{1}') \equiv -\underline{\nabla}(|\underline{x}_1 - \underline{x}_{1'}|^{-1})(n\mathbf{q}) \delta(t_1 - t_{1'}) \quad (10)$$

is the operator which solves Poisson's equation for $\underline{E}$ in terms of f ($\underline{E} = \hat{\underline{E}} f$), we use an integration-summation convention over repeated arguments, and where

$$C(\underline{1}, \underline{2}) \equiv \langle \delta N(\underline{1}) \delta N(\underline{2}) \rangle. \quad (11)$$

When expressed in terms of R and C , the term in $\delta C / \delta \eta$ in Eq. (9) effects the renormalization of the linear response and will reduce, in the appropriate limit, to the linearized Balescu-Lenard operator.

In general, the correlation function C obeys a complicated equation and is coupled in a complex way to the response function R . However, to lowest

order in the plasma parameter one can easily show (DuBois & Espedal 1978) that C is simply propagated by R from its initial condition at some time t_0 :

$$C(\bar{1}, 1) = R(\bar{1}; \bar{2}) R(1; 2) \Delta(\bar{2}, 2 | t_0), \quad (12)$$

where

$$\Delta(\bar{2}, 2 | t_0) \equiv \delta(t_{\bar{2}} - t_0) \delta(t_2 - t_0) C(\bar{2}, 2, t_0) \quad (13)$$

and we assume that t_0 is less than both t_1 and $t_{\bar{1}}$. Schematically, we find from Eq. (12)

$$\frac{\delta C}{\delta \eta} = \left(\frac{\delta R}{\delta \eta} \right) R \Delta + R \left(\frac{\delta R}{\delta \eta} \right) \Delta + R R \left(\frac{\delta \Delta}{\delta \eta} \right), \quad (14a)$$

Employing the result $\delta R = -R \delta(R^{-1}) R$, which follows from varying the identity $R R^{-1} = 1$, and using (13) and (7) in (14a), we find

$$\begin{aligned} \frac{\delta C(\bar{1}, 1)}{\delta \eta(1')} &= \left[-R(\bar{1}; \bar{3}) \frac{\delta R^{-1}(\bar{3}; \bar{2})}{\delta \eta(1')} C(\bar{2}, 1) + (1 - \bar{1}) \right] \\ &+ R(\bar{1}; \bar{2}, t_0) R(1; 2, t_0) \left[n_2^{-1} R(2, t_0; 1') + O\left(\frac{\delta G}{\delta \eta}\right) \right]. \end{aligned} \quad (14b)$$

We will ignore the term in $\delta G / \delta \eta$ by appealing to standard phase mixing arguments. We may compute $\delta R^{-1} / \delta \eta$ by first multiplying Eq. (9) on the right by R^{-1} to get

$$R^{-1}(1; 1') \approx [\partial_t + \mathbf{v} \cdot \nabla + (\hat{\mathbf{E}} \mathbf{f}) \cdot \hat{\partial}]_1 \delta(1 - 1') + \hat{\partial} \mathbf{f} \cdot \hat{\mathbf{E}}(1, 1')$$

[to lowest order in ϵ_p , we may ignore the last term on the left-hand side of Eq. (9)], then by differentiating with respect to η :

$$\frac{\delta R^{-1}(\bar{3}; \bar{2})}{\delta \eta(1')} \approx \hat{E}R(\bar{3}, 1') \cdot \hat{\partial}_3 \delta(\bar{3}-\bar{2}) + \hat{\partial}R(\bar{3}; 1') \cdot \hat{E}(\bar{3}, \bar{2}). \quad (15)$$

Combining Eqs. (14) and (15), one then readily finds that

$$\begin{aligned} \frac{5}{\delta \eta(1')} \hat{\partial} \cdot < \delta E(1) \delta \tilde{N}(1) > = \Sigma'(1; \bar{1}) R(\bar{1}; 1') + \hat{\partial} \delta \bar{E}(1, \bar{1}) \cdot \hat{E}R(\bar{1}; 1') \\ + \hat{\partial}_1 \cdot \hat{E}(1, \bar{1}) R(\bar{1}; \bar{2}, t_0) R(1; \bar{2}, t_0) n_2^{-1} R(\bar{2}, t_0; 1'), \end{aligned} \quad (16)$$

where

$$\Sigma'(1; \bar{1}) \equiv -\hat{\partial}_1 \cdot \hat{E}(1, 2) [R(2; \bar{2}) < \delta E(1) \delta E(\bar{2}) > \cdot \hat{\partial}_2 \delta(\bar{2}-\bar{1}) + (1-2)], \quad (17a)$$

$$\hat{\partial} \delta \bar{E}(1, \bar{1}) \equiv \hat{\partial}_1 \cdot \hat{E}(1, 2) [R(2; \bar{1}) \hat{\partial}_1 C(\bar{1}, 1) + (1-2)]. \quad (17b)$$

The objects Σ' and $\delta \bar{E}$ as given by (17) are the well-known direct-interaction approximations for the renormalizing terms in the collisionless approximation (DuBois & Espedal 1978; Krommes & Kleva 1979; Krommes 1980). (Corrections to the direct-interaction approximation follow by retaining the term $\delta C / \delta \eta$ in the equation for $\delta R^{-1} / \delta \eta$.) The remaining term of (16) (i.e. RFP) is explicitly proportional to ϵ_p and, thus, is ignored in Vlasov turbulence theories.

In conventional Balescu-Lenard theory for $\epsilon_p \ll 1$ (Williams 1973; Krommes 1975; Krommes & Oberman 1976a), the term in $\delta \bar{E}$ is ignored. We comment on this approximation in Sec. 3. To simplify Σ' , recall (DuBois & Espedal 1978; Krommes & Kleva 1979; Krommes 1980) that the response function can be written in the form

$$R = g(1 - \hat{\partial}^{-1} \hat{\partial} \bar{F} \cdot \hat{E}g), \quad (18)$$

where

$$g \equiv (\hat{\partial}_t + \hat{v} \cdot \hat{\nabla} + \Sigma')^{-1}, \quad \bar{F} \equiv f + \delta \bar{F}, \quad (19a, b)$$

and

$$D \equiv 1 + \hat{E}g \cdot \hat{\partial} \bar{f} \quad (20)$$

is the dielectric operator. Notice that in the operator notation used in (18)-(20) a superbar on f does not indicate species dependence, but rather signifies the definition (19b). An important identity is

$$\hat{\Sigma}R = D^{-1} \hat{E}g, \quad (21)$$

which states that the field of a dressed test particle (R) is the shielded field of a bare test particle (g). To recover the linearized Balescu-Lenard operator, we must invoke the quasilinear approximation, in which we replace $\bar{f}$ by f and $g_k(v; v')$ by $g_k^{(0)}(v) \delta(v-v')$, where

$$g_k^{(0)}(v) \equiv [-i(\omega_k - k \cdot v + i0)]^{-1}. \quad (22)$$

(We write $k \equiv (k, \omega_k)$.) Although it is not necessary, we will assume that approximate stationarity in space and time prevails. Then, inserting (18) into (17a) and Fourier analyzing, we find

$$\begin{aligned} (\Sigma'R)_k = & -\hat{\partial} \cdot \int \frac{d\alpha}{(2\pi)} g_{k-q}^{(0)}(1) \hat{\Sigma}_q \cdot \hat{\partial}_1 R_k(1; 1') + \hat{\partial} \cdot \int \frac{d\alpha}{(2\pi)} \frac{\hat{\Sigma}_{k-q}^{(ne)} g_{k-q}^{(u)}(\bar{1})}{D(k-q, \omega_k - \omega_q)} \\ & \times [\langle \delta f(1) \delta E \rangle_q - g_{k-q}^{(0)}(1) (\hat{\partial} f)_1 \hat{\Sigma}_q] \cdot \hat{\partial}_1 R_k(\bar{1}; 1'), \end{aligned} \quad (23)$$

where the first term is the quasilinear version of $\Sigma^{(d)}R$, the second term is the quasilinear version of $\Sigma^{(p)}R$, and

$$\hat{\Sigma}_q \equiv \langle \delta E \delta E \rangle_q \quad (24)$$

is the electric field spectrum. The latter follows straightforwardly by evaluating $B_{\vec{q}}$ (12) in the asymptotic limit $t-t_0 \rightarrow \infty$ with $t-\bar{t}$ held finite (App. A) and is a standard result (Rostoker 1961) often obtained from the test particle superposition principle (Rostoker 1964a,b):

$$\vec{E}_{\vec{q}} = 2\pi \frac{1}{s} \overline{(ne^2)} \int d\vec{v} f(\vec{v}) \frac{\vec{q} \cdot \vec{v}}{|D(\vec{q}, \omega_{\vec{q}})|^2} \delta(\omega_{\vec{q}} - \vec{q} \cdot \vec{v}). \quad (25)$$

In the limit $k \rightarrow 0$ only the real part of $g^{(0)}$ contributes to $\Sigma^{(d)}$, so, using (25) in (23), we can reduce $\Sigma^{(d)}_R$ to

$$(\Sigma^{(d)}_R)_k = \pi(e^2) \frac{\partial}{\partial p} \int \frac{d\vec{q}}{(2\pi)^3} \overline{(ne^2)} \int d\vec{v} \frac{\vec{q} \cdot \vec{v}}{|D(\vec{q}, \vec{q} \cdot \vec{v})|^2} \delta(\vec{q} \cdot (\vec{v} - \vec{v}')) \vec{v} \frac{\partial}{\partial p} R_k(\vec{v}; \vec{v}'). \quad (26)$$

One can compute the term $\langle \delta f(i) \delta E(\vec{r}) \rangle$, needed for $\Sigma^{(p)}$, by operating on (12) with $\hat{E}(\vec{r})$. Using (18) for $R(1)$, noting (21), and retaining only the singular part of (7), one finds

$$\langle \delta f \delta E \rangle = g(\vec{R}) n^{-1} f - g \hat{\partial} f \cdot \vec{E}. \quad (27)$$

After inserting (27) into (23), one sees that $\Sigma^{(p)}$ reduces as $k \rightarrow 0$ to

$$\begin{aligned} (\Sigma^{(p)}_R)_k &= 2 \int \frac{d\vec{q}}{(2\pi)^4} \left[\frac{\delta D(-\vec{q}, -\omega_{\vec{q}})}{D(-\vec{q}, -\omega_{\vec{q}})} \right] 2\pi \delta(\omega_{\vec{q}} - \vec{q} \cdot \vec{v}) \left[\frac{\vec{q} \cdot \vec{v}}{D(-\vec{q}, -\omega_{\vec{q}})} - \vec{E}_{\vec{q}}(\vec{q} \cdot \vec{v}) \right] \\ &= 2 \int \frac{d\vec{q}}{(2\pi)^3} \delta(\omega_{\vec{q}} - \vec{q} \cdot \vec{v}) \left[\frac{\delta D(-\vec{q}, -\omega_{\vec{q}})}{D(-\vec{q}, -\omega_{\vec{q}})} \right] \\ &\quad \times \left[2 \text{Re} \left[\frac{\vec{q} \cdot \vec{v}}{D(-\vec{q}, -\omega_{\vec{q}})} \right] - \vec{E}_{\vec{q}}(\vec{q} \cdot \vec{v}) - \frac{\vec{q} \cdot \vec{v}}{D(\vec{q}, \omega_{\vec{q}})} \right] \\ &= \pi(e^2) \frac{\partial}{\partial p} \int \frac{d\vec{q}}{(2\pi)^3} \overline{(ne^2)} \int d\vec{v} \frac{\vec{q} \cdot \vec{v}}{|D(\vec{q}, \vec{q} \cdot \vec{v})|^2} \delta(\vec{q} \cdot (\vec{v} - \vec{v}')) \end{aligned}$$

$$\times \{2\text{Re}[\frac{\delta D(\underline{q}, \underline{q}, \underline{v})}{D(\underline{q}, \underline{q}, \underline{v})}](\frac{\partial}{\partial \underline{p}} - \frac{\partial}{\partial \underline{p}}) \underline{f} \underline{f} + \underline{f} \frac{\partial \underline{R}}{\partial \underline{p}}(\underline{v}; \underline{v}')\}, \quad (28)$$

where in arriving at the last step we used (25) and have written here

$$\delta D(\underline{q}, \omega_{\underline{q}}) \equiv \delta D(\underline{q}, \omega_{\underline{q}}; \{R_k(\underline{v}; \underline{v}')\}). \quad (29)$$

Finally, the last term of Eq. (16) can be evaluated. The details are presented in App. B; one finds

$$\begin{aligned} (\partial \cdot \hat{E} R R^{-1})_k &= -\pi(e^2) \frac{\partial}{\partial \underline{p}} \int \frac{d\underline{q}}{(2\pi)^3} \int \frac{d\underline{v}}{\underline{v}} \frac{\underline{v} \cdot \underline{q} \underline{q} \cdot \underline{v}}{|D(\underline{q}, \underline{q}, \underline{v})|^2} \\ &\times \delta[\underline{q} \cdot (\underline{v} - \underline{v}')] \left[\underline{R}_k \frac{\partial \underline{f}}{\partial \underline{p}} - R_k \frac{\partial \underline{f}}{\partial \underline{p}} \right]. \end{aligned} \quad (30)$$

The results (26), (28), and (30) sum to the linearized Balescu-Lenard operator (3b) with δf replaced by R . The long wavelength, low frequency (kinetic) limit of Eq. (11) is thus

$$(\partial_t + \underline{v} \cdot \underline{\nabla} + \langle \underline{E} \rangle \cdot \underline{\nabla} + \partial \underline{f} \cdot \hat{\underline{E}} + \underline{v}) R(1; 1') = \delta(1 - 1'). \quad (31)$$

3. Discussion

The physical content of these manipulations is summarized in Table 1. On the first line of Table 1 we have recorded the terms in braces in Eq. (3b) (with δf replaced by R). On the second line we indicate the physical interpretation of each term, as relating either to a diffusion process in velocity space or to polarization drag. All terms can be found from a Fokker-Planck equation for the total one-body distribution

$$\langle \tilde{N}(\eta) \rangle = \langle \tilde{N} \rangle + \eta \frac{\partial \langle \tilde{N} \rangle}{\partial \eta} + \dots = f + \eta R + \dots \quad (32)$$

in the presence of fluctuations by extracting the first-order response. The terms qualified by "F.-P." arise from the linearization of the Fokker-Planck coefficients around the background f ; the other terms are the conventional ones one might naively write down using the Fokker-Planck coefficients of the background.

On the third line of Table 1, we record which part of the renormalized theory gives rise to the indicated term. Here we follow earlier notation (Krommes 1980; Martin, Siggia & Rose 1973) and write the general renormalized theory coupling f , C , and R in operator form as

$$C = R \tilde{F} R^T, \quad (33a)$$

$$R^{-1} = \left(g^{(0)} \right)^{-1} + \partial \tilde{F} \cdot \tilde{E} + \Sigma^{(d)} + \Sigma^{(p)}. \quad (33b)$$

The function $\tilde{F}$ has been called (DuBois & Espedal 1978; Krommes 1980) the source of incoherent noise. In Sec. 2 we used only the lowest order discreteness contribution to $\tilde{F}$, namely Δ . In Table 1 the phrase "with $\tilde{F}$ " indicates that the explicit form of that term emerges only after inserting the near-equilibrium form of the spectral function $\tilde{E}$, which is computed from $\tilde{F}$ (specifically Δ).

In a stochastic acceleration problem in which the field is not self-consistent with the distribution ($\delta E / \delta n = 0$), only the term in $\Sigma^{(d)}$ would survive (and would emerge from a standard Fokker-Planck analysis applied to the fluctuations in the presence of a statistically specified background).

The other five terms are consequences of self-consistency. We see that the incoherent noise (particle discreteness here) gives rise to polarization drag on the fluctuations, in accord with intuition. [One needs a source (particle) to disturb the medium.] It also gives rise to a less obvious effect, arising from a fluctuation of the velocity space diffusion coefficient and understandable as an inverse process of the fluctuations affecting the background. The term $\Sigma^{(p)}$ gives rise to several terms, each related to a polarization effect: a fluctuation in the Fokker-Planck coefficient describing polarization drag, also understandable as an inverse process, and fluctuations in the effective dielectric response of the medium, which through the field spectrum affects both the effective diffusion coefficient and the effective drag. This interpretation of $\Sigma^{(p)}$ strongly suggests that the attribute "polarization" is reasonable.

It seems clear that the neglect of $\Sigma^{(p)}$ (and $\delta\bar{F}^{(p)}$) in most heuristic approaches arises because of its interpretation as describing fluctuations in the Fokker-Planck coefficients, an effect which is perhaps difficult to grasp intuitively. It is not clear that the neglect is justified. As is quite clear from the discussions by Williams (1973), Krommes (1975), and Krommes & Oberman (1976a), as well as from the analysis presented here and in formal renormalizations like the direct-interaction approximation, the polarization terms enter on a par with the diffusion terms in a nominal ordering. They may, in fact, be small in certain applications; however, it would appear that these must be studied on a case-by-case basis. Krommes & Simion (1980) have studied the instructive example of the guiding center plasma, where the terms are important.

Both in the Balescu-Lenard theory and in collisionless turbulence, symmetries dictate that if the polarization terms are retained, the incoherent

noise $\tilde{F}$ must be (a priori) also retained. For example, in the calculation presented here the effects of polarization drag emerge both from $\Sigma^{(p)}$ and from $\tilde{F}$. Both terms are necessary in order to ensure, a priori, the appropriate conservation laws. Similon and Krommes (1980, 1981) have discussed this point explicitly for the important case of the direct-interaction approximation. This means that when the incoherent noise is explicitly important, the "coherent approximation" of Krommes & Kleva is inconsistent, as those authors noted.

We have ignored $\delta\tilde{F}$ completely. In the near-equilibrium limit with small ϵ_p , this does not seem to be serious, since the resulting operator ν has reasonable conservation laws (number, momentum, and kinetic energy). Had $\delta\tilde{F}$ been properly retained, the operator would have conserved total (kinetic plus potential) energy. This is of importance in the theory of dense plasmas (Baus 1973). The correction $\delta\tilde{F}$ is also related to the direct correlation function, well-known in the theory of dense systems (Boley 1974) where it replaces the bare Coulomb interaction. It would be very instructive to make a detailed comparison between the direct-interaction form of ν and extant approximations for strongly coupled plasmas.

The thrust of this work was to show the role of the polarization term $\Sigma^{(p)}$ in a limit which is well-known and noncontroversial. Practically, this is not very interesting in itself. However, similar considerations hold for collisionless Vlasov turbulence, although the general form of the noise $\tilde{F}$ is, of course, different from its discreteness approximation Δ . Indeed, Dupree (1972) has derived a Balescu-Lenard type of operator for the one-body distribution. As far as the equations for the fluctuations (DuBois & Espedal 1978; Krommes & Kleva 1979; Krommes 1980) are concerned, the interpretation of $\Sigma^{(p)}$ as related to the fluctuations in Fokker-Planck coefficients persists and should be of help in understanding the physical content of those equations.

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Appendix A: The near-equilibrium electric field spectrum

To find the conventional result (25), one assumes that approximate spatial and temporal stationarity hold, and that the plasma is stable so that the pair correlation can be neglected in (7). Equation (12) then becomes

$$C_{\vec{q}}(\vec{1}, 1) = R_{\vec{q}}(\vec{1}, \vec{2}; 2, t_0) R_{-\vec{q}}(\vec{1}, \vec{2}; 2, t_0) n_2^{-1} f(\vec{2}, t_0), \quad (A1)$$

$$C_{-\vec{q}}(\vec{1}, 1) = \int \frac{d\omega}{2\pi} \exp[-i\omega(t-\bar{t})] \int \frac{d\Omega}{2\pi} \exp[-i\Omega(\bar{t}-t_0)] \\ \times R_{-\vec{q}, \Omega-\omega}(\vec{1}, 2) R_{\vec{q}, \omega}(\vec{1}, 2) n_2^{-1} f(\vec{2}). \quad (A2)$$

This expression is to be evaluated for finite $t-\bar{t}$ in the limit $\bar{t}-t_0 \rightarrow \infty$. It is convenient to take this limit by invoking the theorem

$$\lim_{T \rightarrow \infty} F(T) = \lim_{\Omega+i0 \rightarrow 0} [-i(\Omega+i0) F_{+\Omega}] \quad (A3)$$

for any function F which approaches a finite limit as $T \rightarrow \infty$. Here $F_{+}(T) \equiv H(T)F(t)$ (H being the Heaviside function) and F_{Ω} denotes the Fourier transform. Applying the electric field operator to (A2) and recalling (19), we find

$$\epsilon_{\vec{q}} = \frac{\epsilon_{\vec{q}} \epsilon_{-\vec{q}} (n e^2)_2 f(\vec{2})}{|\mathcal{D}(\vec{q}, \omega)|^2} \left\{ \lim_{\Omega+i0 \rightarrow 0} [-i(\Omega+i0)] \right. \\ \left. \times [-i(\omega - \vec{q} \cdot \vec{v}_2 + i0)]^{-1} [-i(\Omega - \omega + \vec{q} \cdot \vec{v}_2 + i0)]^{-1} \right\}.$$

By expanding the term in braces in partial fractions and using the Plemelj formula, one can readily show that the term in braces is just $2\pi\delta(\omega - \underline{q} \cdot \underline{v}_2)$. The result (23) follows immediately.

Appendix B: Reduction of the noise term

We have

$$\begin{aligned} I &\equiv [\hat{\partial} \cdot \hat{\underline{E}}(1, \bar{1}) R(\bar{1}; 2) R(1; 2) n_2^{-1} \delta(t_2 - t_0) R(2, t_0; 1')]_{\underline{k}} \\ &= \hat{\underline{z}} \cdot \int_{\underline{q}} \frac{d\omega}{2\pi} \frac{d\Omega}{2\pi} \exp[-i(\Omega(t - t_0))] \\ &\quad \times R_{\underline{q}, \omega}(\underline{1}; 2) (\hat{\underline{E}}\underline{R})_{\underline{k} - \underline{q}, \Omega - \omega}(\underline{2}) n_2^{-1} R_{\underline{k}}(2, t_0; 1'). \end{aligned} \quad (B1)$$

Inserting (16) and (19) into (B1), and recalling (A3), we find

$$\begin{aligned} I &= \hat{\underline{z}} \cdot \int_{\underline{q}} \frac{d\omega}{2\pi} \left[\delta(1 - 2) - g_{\underline{q}, \omega}(\underline{1}) (\hat{\partial} \underline{f})_1 \frac{\underline{\varepsilon}_{\underline{q}}^{(ne)}{}_2}{D(\underline{q}, \omega)} \right] \\ &\quad \times \left(\lim_{\Omega \rightarrow i0+0} [-i(\Omega + i0)] g_{\underline{q}, \omega}(\underline{2}) g_{-\underline{q}, \Omega - \omega}(\underline{2}) \frac{\underline{\varepsilon}_{-\underline{q}}^{(eR)}{}_2}{D(-\underline{q}, -\omega)} \right) \\ &= \hat{\underline{z}} \cdot \int_{\underline{q}} \frac{d\omega}{2\pi} \left[2\pi\delta(\omega - \underline{q} \cdot \underline{v}_2) \right] \frac{\underline{\varepsilon}_{\underline{q}}^{(eR)}}{D(-\underline{q}, -\omega)} - g_{\underline{q}, \omega}(\underline{1}) (\hat{\partial} \underline{f})_1 \frac{\underline{\varepsilon}_{\underline{q}} \underline{\varepsilon}_{-\underline{q}}}{|D(\underline{q}, \omega)|^2} (ne^2)_2 R_2. \end{aligned} \quad (B2)$$

Symmetries dictate that only the real parts of the terms involving $\underline{q}$ and ω contribute. The first term of (B2) can be simplified by multiplying both numerator and denominator by $D(\underline{q}, \omega)$; the second term reduces because $\text{Re}[g_{\underline{q}, \omega}(\underline{1})] = \pi\delta(\omega - \underline{q} \cdot \underline{v}_1)$. The result (28) then follows immediately.

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$\bar{f} \frac{\partial}{\partial p} R$	$\frac{\partial f}{\partial p} \bar{R}$	$-\frac{\partial \bar{f}}{\partial p} R$	$-f \frac{\partial}{\partial p} \bar{R}$	$-2\text{Re}(\frac{\delta D}{D}) \frac{\partial f}{\partial p} \bar{f}$	$2\text{Re}(\frac{\delta D}{D}) f \frac{\partial \bar{f}}{\partial p}$
diffusion (of fluctuations)	diffusion (of background) [$\delta(F.-P.)$]	drag (on fluctuations)	drag (on background) [$\delta(F.-P.)$]	(dielectric fluctuations) diffusion (on background) [$\delta(F.-P.)$]	drag (on background) [$\delta(F.-P.)$]
$\Sigma^{(d)}$ (with $\tilde{f}$)	$\tilde{f}$	$\tilde{f}$	$\Sigma^{(p)}$	$\Sigma^{(p)}$ (with $\tilde{f}$)	$\Sigma^{(p)}$

Table 1: Origin and Interpretation of the terms of the linearized Balescu-Lenard operator. See Sec. III for discussion.