Note on CVL Plasma Field Equations

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UCRL-ID--105180

DE91 002572

October 5, 1990

In [1], Boley describes the discretized equations for the magnetic field diffusion in the CVL plasma. In this note, these equations are derived in a different way (avoiding ghost points), and an error analysis is given.

Neglecting axial diffusion, the azimuthal field $B(r, z, t)$ is given by the following diffusion equation on $a_1 \leq r \leq a_2$, $0 \leq z \leq \ell$, $t \geq 0$:

$$\frac{\partial B}{\partial t} = \frac{\partial E}{\partial r} = \frac{\partial}{\partial r} \left[\frac{D}{r} \frac{\partial(rB)}{\partial r} \right]. \quad (1)$$

The diffusion coefficient $D = D(r, z, B)$ is related to the conductivity σ by $D = 1/\mu_0\sigma$. The boundary conditions are related to the plasma current I_p and return current I_c (which depend only on t) by

$$B(a_1, z, t) = \frac{-\mu_0}{2\pi a_1} I_c, \quad B(a_2, z, t) = \frac{\mu_0}{2\pi a_2} (I_p - I_c). \quad (2)$$

These currents are in turn related to the laser voltage V_ℓ and the voltages V_{a_1} , V_{a_2} at the radial edges by

$$\left. \begin{aligned} \dot{I}_c &= M_{10}V_\ell + M_{11}V_{a_1} - M_{12}V_{a_2} \\ \dot{I}_p - \dot{I}_c &= M_{20}V_\ell + M_{21}V_{a_1} - M_{22}V_{a_2} \end{aligned} \right\} \quad (3)$$

These voltages are related to I_p and E_z by

$$V_\ell = V_{pk} + R_{pk}(I_{ns} - I_p) \quad (4)$$

and

$$V_{a_i} = \int_0^\ell E_z(a_i, z) dz + V_{e_i} \quad (i = 1, 2). \quad (5)$$

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The discretization of (1) would be straightforward if it were not for the boundary conditions. Implicit in (2) is the requirement that at each radial edge $r = a_i$ the field is independent of z . Using a mesh of N_r radial points r_i ,

$$a_1 = r_1 < r_2 < \dots < r_{N_r} = a_2,$$

and N_z axial points z_j , let B_{ij} be the discrete approximation to $B(r_i, z_j, t)$. For $i = 1$ or N_r , $B_{ij} = B_i$ is independent of j . Denoting

$$h_i = r_{i+1} - r_i, \quad r_{i+1/2} = (r_i + r_{i+1})/2 \quad (6)$$

and

$$d(r, z) = D(r, z, B(r, z))/r,$$

we discretize (1) at the radially interior points by setting

$$E_{i+1/2,j} = d_{i+1/2,j} \left(\frac{r_{i+1} B_{i+1,j} - r_i B_{i,j}}{h_i} \right) \quad (7)$$

to approximate $E_z(r_{i+1/2}, z_j)$, and

$$\dot{B}_{ij} = \frac{E_{i+1/2,j} - E_{i-1/2,j}}{(h_i + h_{i-1})/2} \quad (8)$$

for $i = 2, \dots, N_r - 1$. Here we define

$$\left. \begin{aligned} d_{i+1/2,j} &= D(r_{i+1/2}, z_j, B_{i+1/2,j})/r_{i+1/2} \\ B_{i+1/2,j} &\equiv \frac{1}{2}(B_{i,j} + B_{i+1,j}) \end{aligned} \right\} \quad (9)$$

to get $d = D/r$ at the radial midpoints. The equations (7)–(9) correspond to the standard second-order central difference form for (1). (There are alternatives with the same order of accuracy, but no clear reason to prefer any one choice.)

The difficulty at the radial edges is in getting edge values for E_z ,

$$E_z(a_i, z) = d(a_i, z) \left. \frac{\partial(rB)}{\partial r} \right|_{a_i} \quad (i = 1, 2). \quad (10)$$

A two-point one-sided difference for (10) is only first-order accurate, while a three-point one-sided difference would increase the bandwidth of the coupling in (8), which is tridiagonal. Instead, we look at a modification of a two-point difference. Consider the expression in (7) with $i = 1$:

$$E_{3/2,j} = d_{3/2,j} \left(\frac{r_2 B_{2,j} - r_1 B_1}{h_1} \right).$$

Assuming B_1 and $B_{2,j}$ to be exact, this is a second-order accurate value for E_z at $r_{3/2}$:

$$E_{3/2,j} = E_z(r_{3/2}, z_j) + O(h_1^2). \quad (11)$$

For the exact values of E_z , we also have

$$E_z(r_{3/2}, z_j) = E_z(a_1, z_j) + \frac{1}{2} h_1 \frac{\partial E_z}{\partial r} \Big|_{(a_1, z_j)} + O(h_1^2). \quad (12)$$

Combining (11), (12), and (1) at $r = a_1$, we have

$$E_{3/2,j} = E_z(a_1, z_j) + \frac{1}{2} h_1 \dot{B}_1 + O(h_1^2). \quad (13)$$

This says that a value for $E_z(a_1, z_j)$, accurate to second order, is given by

$$\tilde{E}_{1,j} \equiv E_{3/2,j} - \frac{1}{2} h_1 \dot{B}_1. \quad (14)$$

In a similar way, a Taylor series for (7) with $i = N_r - 1$ gives

$$E_{N_r-1/2,j} = E_z(a_2, z_j) - \frac{1}{2} h_{N_r-1} \dot{B}_{N_r} + O(h_{N_r-1}^2),$$

and thus leads to

$$\tilde{E}_{N_r,j} \equiv E_{N_r-1/2,j} + \frac{1}{2} h_{N_r-1} \dot{B}_{N_r} \quad (15)$$

as an accurate value for $E_z(a_2, z_j)$.

The appearance of the edge values of $\dot{B}$ in (14) and (15) means that the discrete representation of the boundary equations (2)–(5) will not be explicit at first, and will have to be solved for those

values of $\dot{B}$. First, we form the discrete approximation to the edge voltages, with discrete sums in (5):

$$V_{a_1} = \sum_j \ell_j \tilde{E}_{1,j} + V_{e_1}, \quad V_{a_2} = \sum_j \ell_j \tilde{E}_{N_r,j} + V_{e_2}.$$

If we define explicit edge voltages

$$\tilde{V}_1 = \sum_j \ell_j E_{3/2,j}, \quad \tilde{V}_2 = \sum_j \ell_j E_{N_r-1/2,j} \quad (16)$$

then we have

$$V_{a_1} = \tilde{V}_1 - \frac{\ell}{2} h_1 \dot{B}_1 + V_{e_1}, \quad V_{a_2} = \tilde{V}_2 + \frac{\ell}{2} h_{N_r-1} \dot{B}_{N_r} + V_{e_2}. \quad (17)$$

Next, differentiate (2) and combine with (3) to get

$$\begin{bmatrix} \frac{-2\pi a_1}{\mu_0} \dot{B}_1 \\ \frac{2\pi a_2}{\mu_0} \dot{B}_{N_r} \end{bmatrix} = V_\ell \begin{bmatrix} M_{10} \\ M_{20} \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} V_{a_1} \\ -V_{a_2} \end{bmatrix}. \quad (18)$$

Inserting (17) into (18) and combining the $\dot{B}$ terms, we have

$$\begin{aligned} & \begin{bmatrix} \frac{-2\pi a_1}{\mu_0} + \frac{\ell}{2} h_1 M_{11} & \frac{\ell}{2} h_{N_r-1} M_{12} \\ \frac{\ell}{2} h_1 M_{21} & \frac{2\pi a_2}{\mu_0} + \frac{\ell}{2} h_{N_r-1} M_{22} \end{bmatrix} \begin{bmatrix} \dot{B}_1 \\ \dot{B}_{N_r} \end{bmatrix} \\ &= V_\ell \begin{bmatrix} M_{10} \\ M_{20} \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \tilde{V}_1 + V_{e_1} \\ -\tilde{V}_2 - V_{e_2} \end{bmatrix}. \end{aligned} \quad (19)$$

Finally, eliminating I_p from (2) and inserting into (4) gives

$$I_p = \frac{2\pi}{\mu_0} (a_2 B_{N_r} - a_1 B_1)$$

and

$$V_\ell = V_{pk} + R_{pk} I_{ns} - R_{pk} \frac{2\pi}{\mu_0} (a_2 B_{N_r} - a_1 B_1). \quad (20)$$

The 2×2 system (19), together with (16) and (7) for $\tilde{V}_1$ and $\tilde{V}_2$, and (20) for V_ℓ , gives the time derivatives of the radial edge field values in terms of all the discrete field values. The equations (19) and the interior equations (8) comprise an ODE system in the $(N_r - 2) \cdot N_z + 2$ unknowns $\{B_{ij}, B_i, B_{N_r}\}$.

The development of (19) suggests that the edge $\dot{B}$ values have second-order spatial accuracy, i.e., their errors go as the square of the mesh spacings. It is possible to prove this claim rigorously. Given a continuous field $B(r, z, t)$, and assuming that the discrete values $B_{i,j}$ etc. are exact, we can define exact edge values of E_z by (10), form exact values $V_{a_1}^e, V_{a_2}^e$ of the edge voltages defined by (5), and ask how the resulting exact values $\dot{B}_1^e$ and $\dot{B}_{N_r}^e$, defined by (18), compare with the approximate values from (19). Let ϵ_1 and ϵ_2 be the errors in $\dot{B}_1$ and $\dot{B}_{N_r}$, respectively. Then on subtracting the exact version of (18) from (19), we have

$$\begin{bmatrix} \frac{-2\pi a_1}{\mu_0} \epsilon_1 \\ \frac{2\pi a_2}{\mu_0} \epsilon_2 \end{bmatrix} + \frac{\ell}{2} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} h_1 \dot{B}_1 \\ h_{N_r-1} \dot{B}_{N_r} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \tilde{V}_1 + V_{e_1} - V_{a_1}^e \\ -\tilde{V}_2 - V_{e_2} + V_{a_2}^e \end{bmatrix}. \quad (21)$$

We have, from (5) and (16),

$$\tilde{V}_1 + V_{e_1} - V_{a_1}^e = \sum_j \ell_j E_{3/2,j} - \int_0^\ell E_z(a_1, z) dz.$$

The Taylor series (13) for $E_{3/2,j}$ (with exact values for B and $\dot{B}$) gives

$$\sum_j \ell_j E_{3/2,j} = \sum_j \ell_j E_z(a_1, z_j) + \frac{1}{2} \ell h_1 \dot{B}_1^e + O(h_1^2).$$

Next, we assume that the quadrature rule

$$\int_0^\ell f(z) dz \approx \sum_j \ell_j f(z_j)$$

has at least second order, i.e., for smooth $f(z)$ it has error $= O((\Delta z)^2)$, where $\Delta z = \max(z_{j+1} - z_j)$.

(For example, the Trapezoid Rule gives this accuracy.) Then we have

$$\begin{aligned} \tilde{V}_1 + V_{e_1} - V_{a_1}^e &= \sum_j \ell_j E_z(a_1, z_j) - \int_0^\ell E_z(a_1, z) dz + \frac{1}{2} \ell h_1 \dot{B}_1^e + O(h_1^2) \\ &= \frac{1}{2} \ell h_1 \dot{B}_1^e + C(h_1^2) + O((\Delta z)^2). \end{aligned}$$

Similarly, the analogous Taylor series for $E_{N_r-1/2,j}$ gives

$$-\tilde{V}_2 - V_{e_2} + V_{a_2}^e = \frac{1}{2} \ell h_{N_r-1} \dot{B}_{N_r}^e + O(h_{N_r-1}^2) + O((\Delta z)^2).$$

Substituting into (21), and defining $\Delta r = \max(h_i)$, we obtain

$$\begin{aligned} \begin{bmatrix} \frac{-2\pi a_1}{\mu_0} \epsilon_1 \\ \frac{2\pi a_2}{\mu_0} \epsilon_2 \end{bmatrix} &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\ell h_1 \dot{B}_1^e - \frac{1}{2}\ell h_1 \dot{B}_1 \\ \frac{1}{2}\ell h_{N_r-1} \dot{B}_{N_r}^e - \frac{1}{2}\ell h_{N_r-1} \dot{B}_{N_r} \end{bmatrix} + O((\Delta r)^2) + O((\Delta z)^2) \\ &= -\frac{1}{2}\ell \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} h_1 \epsilon_1 \\ h_{N_r-1} \epsilon_2 \end{bmatrix} + O((\Delta r)^2 + (\Delta z)^2), \end{aligned}$$

or

$$\begin{bmatrix} \frac{-2\pi a_1}{\mu_0} + \frac{\ell}{2} h_1 M_{11} & \frac{\ell}{2} h_{N_r-1} M_{12} \\ \frac{\ell}{2} h_1 M_{21} & \frac{2\pi a_2}{\mu_0} + \frac{\ell}{2} h_{N_r-1} M_{22} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = O((\Delta r)^2 + (\Delta z)^2).$$

On applying the inverse of this 2×2 matrix (which is clearly nonsingular for all sufficiently small values of h_1 and h_{N_r-1}), we conclude that ϵ_1 and ϵ_2 are $O((\Delta r)^2 + (\Delta z)^2)$. The central difference expressions in (8) are also accurate to $O((\Delta r)^2)$. Thus we can conclude that *all of the discrete $\dot{B}$ values from (8) and (19) are second-order accurate*, i.e., have errors $= O((\Delta r)^2 + (\Delta z)^2)$.

Reference

- [1] C. D. Boley, "New Capabilities in the Circe Model of the Copper Laser Modulator and Plasma Load," Memorandum CLT 90-12, February 15, 1990.

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