MASTER

TOWARD PRECISE SOLUTION OF
ONE-DIMENSIONAL VELOCITY INVERSE PROBLEMS

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ABSTRACT

A family of one-dimensional inverse problems are considered with the goal of reconstructing (say) velocity profiles to reasonably high accuracy. The travel-time variable change is used together with an iteration scheme to produce an effective algorithm for computation. Under modest assumptions the scheme is shown to be convergent.

I. INTRODUCTION

Our purpose is to find approximate solutions to one-dimensional inverse problems. A typical mathematical description is

$$\begin{aligned}y''(x) + \frac{\omega^2}{\tilde{c}^2} y(x) &= 0, \\y'(0) + \frac{i\omega}{\tilde{c}(0)} y(0) &= 2i\omega, \\y'(X) - \frac{i\omega}{\tilde{c}(X)} y(X) &= 0\end{aligned}\tag{1'}$$

where X could be finite or $+\infty$. Parameter ω came from an earlier Fourier transform and we assume our "data", described below, is available at any desired ω value. Our actual goal is to find the unknown $\tilde{c}$ (e.g. velocity) to at least two significant figures as economically as possible; however, we show in Section 3 that the iteration scheme employed is convergent under realistic conditions.

Hagin [1] attacked this problem for $X < \infty$ using a mesh-adapting scheme which yielded Fourier series-like solutions. Gray [2] treated the $X = \infty$ case by exploiting the so-called "travel time" independent variable and the Fourier transform. This approach produces surprisingly good results at very low cost when $\tilde{c}'(x)$ is "small". We also pursue the approach of [2] here and extend it to cases in which $\tilde{c}'$ is not small or when more accuracy is required.

The travel-time variable is given by

$$\tau = \int_0^X \frac{ds}{\tilde{c}(s)}.$$

We now let $u(\tau) = y(x)$ and $c(\tau) = \tilde{c}(x)$; the above model becomes

$$\begin{aligned} u''(\tau) - \frac{c'(\tau)}{c(\tau)} u'(\tau) + \omega^2 u(\tau) &= 0, \\ u'(0) + i\omega u(0) &= c(0) 2i\omega, \\ u'(T) - i\omega u(T) &= 0 \end{aligned} \tag{1}$$

where $T = \tau(X) \leq \infty$. Hereafter we assume $c(0)$ is known and, without loss of generality, assume it has been normalized to one.

For a fixed ω let $u(x; \omega)$ denote the solution to (1); and let $u_I(x; \omega)$ denote the "incident wave", some known approximate to $u(x; \omega)$. In every case herein u_I will be chosen to satisfy the boundary conditions in (1), but it will generally not satisfy the differential equation. For example, in [1] u_I was computed by using one's best estimate for $\tilde{c}(x)$ and solving (1') numerically; whereas in [2] $u_I = e^{i\omega\tau}$ was chosen which, incidentally, does solve the problem in case $c(\tau) \equiv c(0) = 1$ (i.e. there is no "reflected wave").

Define operator L_0 by $L_0 v = v'' + \omega^2 v$ and choose $u_I = e^{i\omega\tau}$; note $L_0 u_I = 0$. We apply the Lagrange identity (or integrate-by-parts) to obtain

$$\begin{aligned} \int_0^T (u_I L_0 u - u L_0 u_I) d\tau &= \int_0^T u_I L_0 u d\tau = (u_I u' - u_I' u) \Big|_0^T \\ &= (u_I' u - u_I u') (0) = 2i\omega(u - u_I)(0) \\ &= 2i\omega u_S(0; \omega) \end{aligned}$$

where we define the "reflected wave" by $u_S = u - u_I$. The data for the

problem is $u_S(0 ; \omega)$ which is assumed known. Using $u = u_I + u_S$ and the fact that $L_0 u = \gamma u'$, where $\gamma = c'/c$, we have

$$\int_0^T \gamma u_I u_I' d\tau = 2i\omega u_S(0 ; \omega) - \int_0^T \gamma u_I u_S' d\tau . \quad (2)$$

The analogous approach taken in [1], [2] is to assume that u_S' is small compared to u_I' in which case dropping the right integral in (2) leads to the approximation

$$\int_0^T \gamma u_I u_I' d\tau \approx 2i\omega u_S(0 ; \omega) . \quad (3)$$

For example, if $T = \infty$ and $u_I = e^{i\omega\tau}$; (3) leads to the especially amenable integral equation for γ

$$\int_0^\infty \gamma(\tau) e^{2i\omega\tau} d\tau = 2u_S(0 ; \omega) . \quad (4)$$

If $u_S(0 ; \omega)$ is known for all real ω , then (4) can be solved very economically for c'/c via the FFT, then integrated, etc. to obtain an estimate for c . In [2] it was shown that if $\gamma = c'/c$ is of order $\epsilon \ll 1$, then the error in going from (2) to (3) is of order ϵ^3 . Hence this approach gives very adequate results if γ is indeed "small". Moreover, it was demonstrated in [2] that "small" can be interpreted rather generously.

In spite of the effectiveness of solving the simple problem (4), there are occasions when it is necessary to refine the procedure in order to produce more accurate results. One possibility is to use a more descriptive estimate for u_I ; for example, the WKB approximation $u_I = \sqrt{c} e^{i\omega\tau}$. It turns out that this is effective in certain select cases, but we do not pursue this here.

A far more generally applicable and robust approach is to iterate on (2) in order to involve the term neglected in (3). In the next section we demonstrate how this can be done and in Section 3 we prove convergence.

II. ITERATION

We turn to (2) with $u_I = e^{i\omega\tau}$ and obtain

$$i\omega \int_0^T \gamma e^{2i\omega\tau} d\tau = 2i\omega u_S(0; \omega) - \int_0^T \gamma u'_S(\tau; \omega) e^{i\omega\tau} d\tau \quad (5)$$

with $\gamma = c'/c$. When $T = \infty$ Fourier transform methods (and analysis) are effective; for example, the convergence proof in Section 3 is carried out for this case. In computational practice, however, one must truncate the infinite intervals and use some discrete approximation (like the FFT) which turns out to be equivalent to the $T < \infty$ case. Hence in discussing the implementation of the iteration scheme, we consider the $T < \infty$ case and, without loss of generality, assume $T = 1$.

For simplicity of presentation let us assume that $\gamma = c'/c$ is well represented by its Fourier sine series (e.g. suppose $c'(0) = c'(1) = 0$ and c is relatively smooth). If we put $T = 1$ in (5), divide out the $i\omega$, set $\omega = \omega_k = k\pi/2$ and take the imaginary part we obtain

$$\begin{aligned} \int_0^1 \gamma \sin k\pi\tau d\tau &= 2 \operatorname{Im} u_S(0; \omega_k) + \frac{1}{\omega} \operatorname{Re} \int_0^1 \gamma u'_S(\tau; \omega_k) e^{ik\pi\tau/2} d\tau \\ &= \frac{1}{2} b_k + \frac{1}{\omega} \operatorname{Re} \int_0^1 \gamma u'_S(\tau; \omega_k) e^{ik\pi\tau/2} d\tau \\ &= \frac{1}{2} \tilde{b}_k. \end{aligned} \quad (6)$$

In (6) the b_k provide a first estimate of the desired Fourier coefficients for γ and are available from the data. The $\tilde{b}_k$ represent the exact coefficients in

$$\gamma(\tau) \sim \sum_{k=1}^{\infty} \tilde{b}_k \sin k\pi\tau \quad (7)$$

Of course the $\tilde{b}_k$ are not known since the integral in (6) involves the unknown γ explicitly (and implicitly through $u_S' = u_S'(\tau; \omega_k; \gamma)$); hence iteration is necessary. To start the procedure, an appropriately large n (typically $10 \leq n \leq 40$) is chosen and γ_0 and c_0 computed by

$$\begin{aligned} \gamma_0(\tau) &= \sum_{k=1}^n b_k \sin k\pi\tau = 4 \sum_{k=1}^n \operatorname{Im} u_S(0; \omega_k) \sin k\pi\tau \\ c_0(\tau) &= \exp \left[\frac{1}{\pi} \sum_{k=1}^n b_k (1 - \cos k\pi\tau)/k \right]. \end{aligned} \quad (8)$$

One then iterates as follows:

1. Using current estimates for γ and c and with $\omega_k = k\pi/2$ for $k = 1, \dots, n$: numerically solve the initial value problem

$$\begin{aligned} (u'/c)' + \frac{\omega_k^2}{c} u &= 0 \\ u(0; \omega_k) &= u_S(0; \omega_k) + 1 \\ u'(0; \omega_k) &= u_S'(0; \omega_k) + i\omega_k = i\omega_k[1 - u_S(0; \omega_k)]. \end{aligned} \quad (9)$$

Denote this solution by $\tilde{u}$ and use $\tilde{u}_S = \tilde{u} - e^{i\omega\tau}$ in the right side of (6) to compute $\tilde{b}_k$, $k = 1, \dots, n$.

2. Compute a new approximation to γ and c as in (8), but using instead the $\tilde{b}_k$ from Step 1.

Steps 1 and 2 are repeated until c has converged to the desired accuracy. For example, in seeking two or three figure results this typically takes three or four iterations. Note in Step 1, we recommend solving the initial value problem rather than the equivalent boundary value problem (1). This is, of course, more efficient and is possible by taking advantage of the data $u_S(0; \omega_k)$ and the left boundary condition in (1). An important computational detail involves avoiding the use of γ explicitly in the iteration when c is not smooth. For example, if c were discontinuous then $\gamma = c'/c$ is a distribution and the series for γ in (8) would probably be divergent as $n \rightarrow \infty$. Notice we have avoided γ in (9) above by putting the differential equation in self-adjoint form. Less obvious is the fact that γ can be avoided in the calculation of (6) by integration-by-parts. We do not provide these details.

Following are two examples illustrating the effectiveness of the iteration scheme in seeking c to two figures. In these examples the data $u_S(0; \omega_k)$ were computed artificially by solving (1) numerically for $\omega_1, \dots, \omega_n$ using the exact c . Then the "inverse" procedure just described was used to reconstruct c . The value of n depends on the smoothness of c (unknown in real problems) and the desired accuracy. In these examples $n = 10$ was adequate for our purposes.

Example 1: Consider

$$c(\tau) = \begin{cases} 1 + 2 \sin^2 \pi x/2 & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 1 & \text{elsewhere} \end{cases} \quad (10)$$

First (8) was used to compute c_0 ; then four iterations (Steps 1 - 2 above) were taken. As an economy measure we only altered $b_1 - b_5$ during the iteration (clearly for a smooth function $b_6 - b_{10}$ are less important and typically one figure accuracy in these is adequate). The results are summarized in Figure 1. Note that two figure accuracy was achieved. The c_0 shown comes from (8) and is analogous to the final results in [2] where $c'/c = \gamma$ was assumed to be "small".

Our final example is a piecewise linear function (a "ramp") whose smoothness is perhaps more representative of practical problems.

Example 2: Consider

$$c(\tau) = \begin{cases} 1 & \text{for } 0 \leq \tau \leq .2 \\ 1 + \frac{10}{3} (\tau - .2) & \text{for } .2 \leq \tau \leq .8 \\ 3 & \text{for } .8 \leq \tau \leq 1.0 \end{cases} \quad (11)$$

In spite of the "corners" in this function the Fourier expansion truncated at $n = 10$ is adequate for our demands. Again we only corrected $b_1 - b_5$ and make three iterations. The computation took about 14 seconds on a Burroughs B6800. The results are shown in Figure 2.

In the next section the proof of the iteration scheme is established. This, of course, suggests that c could be computed to any required accuracy. However, if c were needed to, say, six or seven figures some formidable computational hurdles would be encountered. For example, in Step 1 n would have to be large (e.g. $n > 100$) and this would necessitate solving highly oscillatory differential equations numerically to high accuracy or, possibly, at some stage using instead some large parameter approximation (e.g. WKB).

We do not pursue these computational matters here.

Recall that we have chosen to attack the original inverse problem in the travel-time form (1) and have illustrated how (say) velocity $c(\tau)$ can be reconstructed. It is very possible that one would prefer to have c as a function of (say) "depth"; $\tilde{c}(x)$ in the original problem statement. Recall that

$$\tilde{c}(x) = c(\tau) \quad \text{and} \quad \tau = \int_0^x 1/\tilde{c}.$$

It follows that $x(\tau) = \int_0^\tau c$ could be approximated once $c(\tau)$ is available, and finally, $\tilde{c}(x) = c(\tau(x))$ could be estimated for a finite set of x values.

III. CONVERGENCE OF ITERATION

We now address the question of convergence of the iteration scheme just discussed. We treat the $T = \infty$ case in (2) and assume $\gamma \in L^1(0, \infty)$ and that $\gamma = 0$ for $\tau < 0$. Using $T = \infty$ in (2) we get

$$i\omega \int_0^\infty e^{2i\omega\tau} \gamma(\tau) d\tau = 2i\omega u_S(0; \omega) - \int_0^\infty \gamma u_S'(\tau; \omega) e^{i\omega\tau} d\tau. \quad (12)$$

Consider these two forms of the Fourier integral theorem:

$$\underset{\sim}{F}(G) = \int_{-\infty}^\infty e^{i\omega t} G(t) dt = g; \quad \underset{\sim}{F}^{-1}(g) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\omega t} g(\omega) d\omega = G$$

$$\underset{\sim}{\tilde{F}}(G) = \int_{-\infty}^\infty e^{2i\omega t} G(t) dt = \tilde{g}; \quad \underset{\sim}{\tilde{F}}^{-1}(\tilde{g}) = \frac{1}{\pi} \int_{-\infty}^\infty e^{-2i\omega t} \tilde{g}(\omega) d\omega = G$$

Now $\underline{F}^{-1}(u_S(0; \omega)) = U_S(0; t)$ represents the data in the time domain.

Hence applying $\underline{F}^{-1}$ to (12) leads to:

$$\begin{aligned} \gamma(\tau) &= \underline{F}^{-1}(F(2U_S(0; \cdot))) + \underline{F}^{-1}\left(\frac{i}{\omega} \int_0^\infty \gamma u'_S e^{i\omega\tau_1} d\tau_1\right) \\ &= \frac{1}{\pi} \int_{-\infty}^\infty d\omega e^{-i\omega(2\tau)} \underline{F}(2U_S(0; \cdot)) + \underline{F}^{-1}\left(\frac{i}{\omega} \int_0^\infty \gamma u'_S e^{i\omega\tau_1} d\tau_1\right) \quad (13) \\ &= 4 U_S(0; 2\tau) + \frac{i}{\pi} \int_{-\infty}^\infty d\omega e^{-2i\omega\tau} \omega^{-1} \int_0^\infty \gamma u'_S(\tau_1; \omega) e^{i\omega\tau_1} d\tau_1 \end{aligned}$$

Analogous to the iteration in Section 2, we will define $\gamma_0(\tau) = 4 U_S(0; 2\tau)$ and use (13) for iteration:

$$\gamma_{m+1}(\tau) = 4 U_S(0; 2\tau) + \frac{i}{\pi} \int_{-\infty}^\infty d\omega \omega^{-1} e^{-2i\omega\tau} \int_0^\infty \gamma_m u'_S(\tau_1; \omega; \gamma_m) e^{i\omega\tau_1} d\tau_1 \quad (14)$$

where $u'_S(\tau; \omega; \gamma_m)$ denotes the solution of the boundary (or initial) value problem for u'_S using the latest estimate of γ . In operator notation we denote this $\gamma_{m+1} = \underline{T} \gamma_m$ and wish to show that $\underline{T}$ has a fixed point within some ball in $L^1(0, \infty)$. To that end, we now work to show that $\underline{T}$ is a contraction.

First we find a more tractable expression for $\underline{T}$ by examining a series for u'_S (usually attributed to Born, e.g. see [3]). If we write the differential equation in (1) in the form $u'' + \omega^2 u = \gamma u'$ and apply the appropriate Green's function we get the solution to (1), and its derivative, as

$$u(\tau; \omega) = e^{i\omega\tau} + \frac{1}{2i\omega} \int d\tau_1 \gamma u' e^{i\omega|\tau - \tau_1|}, \quad (15)$$

$$u'(\tau, \omega) = i\omega e^{i\omega\tau} + \frac{1}{2} \int d\tau_1 \gamma u' \operatorname{sgn}(\tau - \tau_1) e^{i\omega|\tau - \tau_1|},$$

where $\int$ will denote integration over $(0, \infty)$ hereafter. If one iterates on

the last integral equation by starting with $u'_0 = i\omega e^{i\omega\tau}$ in the right side, a series for u' is produced (the Born series):

$$u'(\tau_1; \omega) = i\omega e^{i\omega\tau_1} + 2i\omega \sum_2^{\infty} 2^{-n} I_n(\omega),$$

$$\tilde{I}_n(\omega) = \int d\tau_2 \gamma \dots \int d\tau_{n-1} \gamma \int d\tau_n \gamma \operatorname{sgn}[(\tau_1 - \tau_2) \dots (\tau_{n-1} - \tau_n)] e^{i\omega\tau^*}$$

where $\tau^* = |\tau_1 - \tau_2| + \dots + |\tau_{n-1} - \tau_n| + \tau_n$. Using this expression for u' and the fact that $u'_S = u' - i\omega e^{i\omega\tau}$ in (13) we get

$$\begin{aligned} \gamma(\tau) &= \underline{T} \gamma \\ &= 4 U_S(0; 2\tau) - \frac{2}{\pi} \int_{-\infty}^{\infty} d\omega e^{-2i\omega\tau} \int d\tau_1 \gamma e^{i\omega\tau_1} \sum_2^{\infty} 2^{-n} \tilde{I}_n(\omega) \\ &= 4 U_S(0; 2\tau) - \frac{2}{\pi} \sum_2^{\infty} 2^{-n} I_n(\tau) \end{aligned} \quad (16)$$

where

$$\begin{aligned} I_n(\tau) &= \int d\tau_1 \gamma \dots \int d\tau_{n-1} \gamma \int d\tau_n \gamma \operatorname{sgn}[(\tau_1 - \tau_2) \dots (\tau_{n-1} - \tau_n)] \int_{-\infty}^{\infty} d\omega e^{i\omega(\tau' - 2\tau)} \\ &= \int d\tau_1 \gamma \dots \int d\tau_{n-1} \gamma \int d\tau_n \gamma \operatorname{sgn}[(\tau_1 - \tau_2) \dots (\tau_{n-1} - \tau_n)] \delta(\tau' - 2\tau) \end{aligned} \quad (17)$$

with $\tau' = \tau_1 + |\tau_1 - \tau_2| + \dots + |\tau_{n-1} - \tau_n| + \tau_n$. In the I_n derivation we took the ω integration first in order to produce the δ -function as this form will be convenient to work with below. Note in (17) that $I_n(\tau) = I_n(\tau; \gamma)$, but we frequently surpress the γ dependence.

We now examine the I_n series in (17) rather carefully. Physically $2^{-n} I_n(\tau)$ can be viewed as the contribution to $U_S(0; 2\tau)$ resulting from n reflections of a left-to-right moving wave; e.g. see Foster [4]. From that point of view the following result is not surprising.

1. For n even, $I_n(\tau) \equiv 0$ for $\tau \geq 0$. This can be argued very simply as follows. Think of dividing the domain of integration in (17) into $n!$ subregions determined by the $n!$ orders of $\tau_1, \dots, \tau_n$. (For example, for $n = 4$ ordering $\tau_1 \geq \tau_2 \geq \tau_4 \geq \tau_3$ defines $\frac{1}{24}$ of the domain of integration in the definition of I_4). Define the "complement" of an ordering to be that ordering constructed by switching τ_1 and τ_n , τ_2 and τ_{n-1} , etc. (For example, complement $(\tau_1 \geq \tau_2 \geq \tau_4 \geq \tau_3) = (\tau_4 \geq \tau_3 \geq \tau_1 \geq \tau_2)$). Compare the two contributions to I_n associated with these two orderings. Except for the sgn (which will be constant throughout a particular subregion), the numerical contribution will be precisely the same since one has simply renamed the variables $\tau_1, \dots, \tau_n$. However, note that $\text{sgn}[(\tau_n - \tau_{n-1}) \dots (\tau_2 - \tau_1)] = -\text{sgn}[(\tau_1 - \tau_2) \dots (\tau_{n-1} - \tau_n)]$ since there are an odd number of terms. Hence the contributions to I_n due to a particular ordering and its complement ordering will cancel. It follows that $I_n(\tau) \equiv 0$ when n is even.

When n is odd, the above argument does not apply and one will get contributions corresponding to an odd number of reflections. It is necessary that we bound these terms as follows:

2. For $n \geq 3$ and odd:

$$\int d\tau |I_n(\tau)| \leq ||\gamma||^n = (\int |\gamma|)^n. \quad (18)$$

We show this for $n = 3$.

$$\begin{aligned} \int d\tau |I_3| &= \int d\tau \left| \int d\tau_1 \gamma \int d\tau_2 \gamma \int d\tau_3 \gamma \operatorname{sgn}[(\tau_1 - \tau_2)(\tau_2 - \tau_3)] \delta(-2\tau + \tau') \right| \\ &\leq \int d\tau_1 |\gamma| \int d\tau_2 |\gamma| \int d\tau_3 |\gamma| \int d\tau \delta(-2\tau + \tau') \\ &= ||\gamma||^3 \end{aligned}$$

where $\tau' = \tau_1 + |\tau_1 - \tau_2| + |\tau_2 - \tau_3| + \tau_3 \geq 0$. The same argument clearly applies for arbitrary odd n .

We can now show that

$$\underline{T}\gamma = 4 U_S(0; 2\tau) - \frac{1}{\pi} \sum_{n=1}^{\infty} 2^{-2n} I_{2n+1}(\tau; \gamma) \quad (19)$$

is a contraction on $S(\epsilon) = \{f : ||f|| \leq \epsilon\}$ for some ϵ if the data $4U_S(0; 2\tau) = \gamma_0(\tau)$ is sufficiently small.

Let $\gamma_1 = \underline{T}\gamma_0$, $\gamma_2 = \underline{T}\gamma_1$, etc. and denote $\delta_0 = ||\gamma_0||$. It follows from (19) and (18) that

$$\begin{aligned} \delta_1 &= ||\gamma_1|| = ||\underline{T}\gamma_0|| \leq \delta_0 + \frac{1}{\pi} (2^{-2} \delta_0^3 + 2^{-4} \delta_0^5 + \dots) \\ &= \delta_0 + \frac{1}{\pi} \frac{\delta_0^3}{4 - \delta_0^2}. \end{aligned}$$

Similarly $\delta_2 = ||\gamma_2|| \leq \delta_0 + \frac{1}{\pi} \delta_1^3 (4 - \delta_1^2)^{-1}$, etc. Since we need $\delta_m \leq \epsilon$

for all $m \geq 1$ we are led to this restriction on δ_0 :

$$\delta_0 \leq \varepsilon \left[1 - \frac{\varepsilon^2/\pi}{4 - \varepsilon^2} \right] . \quad (20)$$

Since this requires $\delta_0 < \varepsilon$ we have the somewhat unusual requirement that the γ_0 must be strictly inside the boundary of $S(\varepsilon)$. This becomes more precise shortly.

We now show that, for the proper choice of ε , $\underline{T}$ is a contraction on $S(\varepsilon)$. Let f and g be in $S(\varepsilon)$. We need

$$||\underline{T}f - \underline{T}g|| \leq \rho ||f - g|| \quad \text{for } \rho < 1$$

where

$$\pi(\underline{T}f - \underline{T}g) = -2^{-2}[I_3(\tau; f) - I_3(\tau; g)] - 2^{-4}[I_5(\tau; f) - I_5(\tau; g)]$$

- ...

We concentrate on the first term. Let $F = f(\tau_1) f(\tau_2) f(\tau_3)$ and $G = g(\tau_1) g(\tau_2) g(\tau_3)$; then

$$\begin{aligned} & \int d\tau |I_3(\tau; f) - I_3(\tau; g)| \\ &= \int d\tau \left| \int d\tau_1 \int d\tau_2 \int d\tau_3 \operatorname{sgn}(\cdot) (F - G) \delta(-2\tau + \tau') \right| \\ &\leq \int d\tau_1 \int d\tau_2 \int d\tau_3 |F - G| \\ &\leq \int d\tau_1 \int d\tau_2 \int d\tau_3 \left\{ \left| (f(\tau_1) - g(\tau_1)) f(\tau_2) f(\tau_3) \right| \right. \\ &\quad \left. + \left| (f(\tau_2) - g(\tau_2)) f(\tau_3) g(\tau_1) \right| \right. \\ &\quad \left. + \left| (f(\tau_3) - g(\tau_3)) g(\tau_1) g(\tau_2) \right| \right\} \\ &= ||f - g|| (||f||^2 + ||f|| \cdot ||g|| + ||g||^2) . \end{aligned}$$

Hence

$$\int d\tau \left| I_3(\tau; f) - I_3(\tau; g) \right| \leq 3 \epsilon^2 \|f - g\| .$$

In the same way it is shown that

$$\int d\tau \left| I_n(\tau; f) - I_n(\tau; g) \right| \leq n \epsilon^{n-1} \|f - g\| .$$

It therefore follows that

$$\begin{aligned} \int d\tau \left| \underline{T}f - \underline{T}g \right| &\leq \frac{1}{\pi} \left[3\left(\frac{\epsilon}{2}\right)^2 + 5\left(\frac{\epsilon}{2}\right)^4 + 7\left(\frac{\epsilon}{2}\right)^6 + \dots \right] \|f - g\| \\ &= \frac{1}{\pi} \frac{\epsilon^2(12 - \epsilon^2)}{(4 - \epsilon^2)^2} \|f - g\| = \rho \|f - g\| . \end{aligned}$$

Since we need $\rho < 1$, this provides our bound on ϵ . It is easy to show that this requires $\epsilon^2 < \epsilon_0^2 \approx 1.662$ or $\epsilon < \epsilon_0 \approx 1.289$. Using this ϵ bound in (20) leads to the (approximate) data requirement $4\|U_S(0; 2\tau)\| = \delta_0 < .997$. We summarize the results to this point in the following:

Theorem. Assume data $U_S(0; \cdot)$ is in $L^1(0, \infty)$ and satisfies $4\|U_S(0, 2\tau)\| \leq .997$. Using $\gamma_0 = 4U_S(0; 2\tau)$ and iterating $\gamma_{m+1} = \underline{T} \gamma_m$ where $\underline{T}$ is defined by (14) leads to a sequence $\{\gamma_m\}$ which converges to the (unique in $L^1(0, \infty)$) solution to (13). This solution satisfies $\|\gamma\| < 1.290$.

An interesting by-product of the proof is that γ_0 itself is a good approximation to γ when $\|\gamma\| \ll 1$. This is true since the error $\gamma - \gamma_0 = O(\|\gamma\|^3)$. Hence, as illustrated in [2], if $\gamma = c'/c$ is indeed

small, the immediately available $\gamma_0 = 4U_S(0; 2\tau)$ is often very adequate. Computing experience also shows that when $||\gamma|| \approx 1$, γ_0 is typically good to at least one figure accuracy; and this suggests the restriction $||\gamma_0|| < .997 < 1.289$ is probably an artificial restraint due to the method of proof.

The work of this section is similar in spirit, but not in detail, to the work of Prosser [5] - [7]. In particular, we have made no decay or smoothness requirements on c except, implicitly, that $\gamma = c'/c$ must be defined (a.e.) and lie in $L^1(0, \infty)$ and $S(\epsilon)$ for $\epsilon < 1.289$. On the other hand, the claims of the theorem are rather modest. It does not, for example, guarantee for a given c (restricted as above) that the boundary value problem (1) is solvable for all real ω . (This question can be studied via the Born series.) We have shown that if this is the case, then γ (hence c) can be reconstructed by iteration.

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Figure 1. $c = 1 + 2 \sin^2 \pi x/2$, c_0 and c_4 (the result of four iterations);
 $\max|c - c_0| = 0.194$, $\max|c - c_4| = 0.018$

Figure 2. c = "ramp function", c_0 and c_3 (the result of three iterations);
 $\max |c - c_0| = 0.258$, $\max |c - c_3| = 0.023$