LA-UR--85-3766

DE86 002436

CONF-8509203--1

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

NOV 08 1985

TITLE: DIMENSION AND ENTROPY FOR QUASIPERIODIC AND CHAOTIC CONVECTION

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SUBMITTED TO: Proceedings of the International Conference on Dimensions and Entropies in Chaotic Systems, September 11-16, 1985
Pecos River, NM

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Dimension and Entropy for Quasiperiodic and Chaotic Convection

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Abstract

High quality experimental data have been taken on a convection cell containing a dilute ^3He - ^4He solution. We discuss some problems with the determination of dimension and entropy for experimental data, and compare the results to detailed Poincaré sections. At the chaotic transition, we show the behavior of dimension and entropy as a function of Rayleigh number.

Rayleigh-Bénard convection has been perhaps the most intensively studied non-linear experimental system. Dimension and entropy calculations on convection data have been carried out by MALRAISON, ATTEN, BERGE and DUBOIS [1] as well as GIGLIO et. al. [2]. Work done on dimensionality of attractors for other hydrodynamic systems includes that of BRANDSTÄTER, et. al. [3] for Couette-Taylor flow and GUCKENHEIMER and BUZYNA [4] for geostrophic flow. Theoretical and practical aspects of dimension and entropy measurement are discussed in a recent review article by ECKMANN and RUELLE [5]. We report here on systematic observations of convection in a ^3He -superfluid ^4He solution, in which we have achieved extremely low levels of noise and drift. Measurements were made over the transition region for the breakdown of a two-torus into a chaotic attractor. Calculations of dimension and entropy as a function of an external stress parameter are presented.

The fluid used in this work is a dilute solution of 1.46% ^3He in superfluid ^4He . One can treat this solution as a one-component fluid and derive a Rayleigh number R , analogous to the Rayleigh number used as the stress parameter for convection in ordinary fluids, and proportional to the top-bottom temperature difference. A second dimensionless parameter, the Prandtl number, is required to fully characterize the system in the one-component Boussinesq approximation. Theoretical [6] and experimental [7,8] justification for the one-component treatment of these solutions, and definitions of the appropriate Prandtl and Rayleigh numbers may be found in previous publications [7].

In this work we are using a rectangular geometry with height $d = 0.80$ cm, length $2.0 d$, and width $1.4 d$. The sidewalls are thermally insulating relative to the fluid, whereas the top and bottom boundaries are thermally conducting. In operation, the bottom boundary is maintained at a fixed temperature by feedback control. A fixed heat current is applied to the top boundary. Details of cell construction and operation can be found elsewhere [7].

The time series discussed below are all derived from a single differential thermocouple sensor. One end of the thermocouple is attached to the copper plate forming the upper boundary of the cell; the other to a small, cylindrical, thermally isolated copper plug inserted into the center of the upper plate. The sensor is sensitive to local temperature gradients but not to fluctuations in the mean cell temperature. This is one reason for our very low noise level.

With a small heat flow through the cell, the fluid is motionless and conducts heat diffusively. Increasing the heat flow increases the temperature difference between the top and bottom boundaries and thus the Rayleigh number. When the critical Rayleigh number R_c is reached the fluid begins to convect. Further increasing the Rayleigh number excites first one oscillatory instability and then a second, as shown in Fig. 1. In the two-frequency region we observe mode-locking. As the Rayleigh number is increased through this region, a large discontinuous change in both oscillatory

frequencies is observed and the shape of the attractor changes drastically. We will concentrate henceforth on the region above this discontinuity and below the chaotic onset.

Fig. 1 Phase diagram of transitions on the Rayleigh-Prandtl number plane

We define a winding number $W = f_2/f_1$, where $f_1 \sim 0.7$ Hz and $f_2 \sim 0.1$ Hz are the fundamental frequencies of the first and second instabilities seen with increasing R . Figure 2 shows a plot of W versus normalized Rayleigh number. Many flat steps are visible. At the upper right the system has become chaotic. While we find that

Fig. 2. A plot of winding number as a function of normalized Rayleigh number showing a number of mode-locking steps. $R_c = 2017$

the chaotic transition is "continuous" or gradual and thus hard to identify precisely, on the scale of Figs. 1 or 2 this is not a problem. For each point in Fig. 2 the frequencies f_1 and f_2 are determined by acquiring a time series, generating the power spectrum, and fitting peak frequencies. A more detailed discussion of the mode-locking will appear elsewhere [9] but a brief overview is given here. As an example of simple quasiperiodic data, we show in Fig. 3 the attractor for a state well below the chaotic onset. This projection is produced using the usual "delay coordinate" method [10]. The delay time used in constructing the phase space we denote by τ ; $\tau = 5.81$ seconds. for Figs. 3-5. Despite its rather contorted appearance, this is a nice two-torus. By making a suitable cut (illustrated by the shaded plane in Fig. 3) we produce the simple Poincaré section of Fig. 4. Using the parametrization shown yields the return map, Fig. 5.

Fig. 3. Delay coordinates are used to generate a phase space. This shows a two-dimensional projection of a quasiperiodic attractor and the cutting surface used to produce Fig. 4

As R/R_c is increased the system becomes increasingly complex and a 1-D mapping no longer is adequate to describe the system. Hysteresis and Hopf bifurcations from locked states are observed [9], and indicate a 2-D mapping is necessary to characterize the state. Continuing upwards in R/R_c , we observe chaotic states having broadband spectral noise and Poincaré sections with no apparent structure. We have used several dimension algorithms, described below, to characterize the states near the chaotic onset.

In all our dimension work we rely on the standard "delay coordinate" method [10] for generating a phase space. In all of what follows we have used 33,000 point time series 38.5 minutes in length. We have applied three different dimension algorithms to our data. In all of these, we first randomly select a subset of points on the attractor as our reference points (enough for good statistics but not excessive computation time). Our most extensively used method is due to GRASSBERGER et. al. [11]. In this case, for each reference point the distance to all other points is calculated. These separations are then accumulated for all the reference points and the number of separations greater than or equal to ϵ , called $C(\epsilon)$, is calculated. A second algorithm follows TERMONIA [12]. Given some number of points n , $R(n)$ is the radius of the smallest ball around a reference point that contains n points. $R(n)$ is

Fig. 4. This Poincaré section illustrates the simple behavior present well below the chaotic onset

Fig 5. A return map corresponding to the data in Fig. 4

determined for each reference point and then averaged. It is the averaging process that distinguishes this algorithm from Grassberger's. Finally, one can calculate the number of points N inside a ball of radius ϵ circumscribed about a reference point and average $\ln(N(\epsilon))$ over the chosen set of reference points. This is known as the "pointwise" dimension. Additionally, three definitions of distance may be used: the max norm, defined as the largest of all the coordinate differences; the mod norm, defined as the sum of the absolute values of each coordinate difference; and, finally,

the standard Euclidian distance, defined as the square-root of the sum of the squares of the coordinate differences.

We explored the variation of dimension using different methods of calculating point separations, and using the different dimension algorithms as described above. We found no systematic variation of dimension with the different norm definitions. However, a systematic variation was found in a comparison of the three different algorithms. The Grassberger-Proccacia algorithm yielded dimension values which were usually 10 to 15% below the values obtained using the other two algorithms. The parameters used in the algorithms were a delay of $\tau = 1.05$ seconds, five hundred reference points and embedding dimensions ranging from 3 to 10. The convergence with increasing embedding dimension was approximately $\pm 5\%$ for all three algorithms. None of the algorithms proved superior in the sense of more apparent scaling regions or less scatter in the extracted dimension. Therefore, we recommend the Grassberger-Proccacia algorithm with the max norm for ease and speed of computation.

As noted above we used a randomly selected subset of reference points to calculate the dimension. We found that there was appreciable variation in the dimension when this subset was too small and that the variance in the average dimension decreased as the square root of N_{ref} , the number of reference points. In Fig. 6, the variation of dimension with the choice of a particular set of reference points and the number in that set is shown. In both cases $\tau = 7$ seconds and max norm distances were used in computation of the Grassberger-Proccacia algorithm for an embedding dimension of five. For each histogram 25 different sets of reference points were used. The left pair of histograms are for a simple quassipericodic state which yields an average dimension of 2.25. The variance in the average dimension is 0.15 and 0.05 for $N_{ref} = 10$ and 100, respectively. For a chaotic state (the right pair of histograms), the variance is 0.23 and 0.07 for $N_{ref} = 10$ and 100. Although the absolute variance is greater for the chaotic data, the fractional effect is about the same, 7% and 2% for $N_{ref} = 10$ and 100 respectively.

Fig. 6. Variation of distribution of dimensions with size of reference subset for simple quasiperiodic data (left) and chaotic data (right). $N_{ref} = 10$ (shaded). $N_{ref} = 100$ (outlined). Twenty-five different subsets were used

The variation of the dimension with the value of the delay used in the phase space reconstruction was also investigated. Again we used the Grassberger-Proccacia algorithm with max norm distance, an embedding dimension of five, and 200 reference points, and fit over a fixed region of $C(\epsilon)$. As seen in Fig. 7, there was only slight variation for the simple quasiperiodic state but a substantial increase in the dimension of the chaotic state for $\tau > 7$ seconds. In addition there was a dip at $\tau > 1$ second in both data sets corresponding to a delay time commensurate with the shorter period of ~ 1.2 seconds. Peaks in the dimension for the chaotic data at $\tau = 7.5$ seconds and 15 seconds may be due to commensuration with the long period of ~ 8 sec. We would in general expect that picking a value of the delay close to a oscillation period would reduce rather than increase the dimension. However, the general trend of increasing dimension after some finite delay is reached is typical for states having positive entropy. Positive entropy requires that the attractor has some finite width to its probability density along the d^{th} coordinate even when d is much greater than the fractal dimension. For consistency, we use GRASSBERGER'S [13] measure of entropy K_2 . $K_{2,d}(\epsilon) = (1/\tau) \ln(C_d(\epsilon)/C_{d+1}(\epsilon))$ and K_2 is the limiting value for large d and small ϵ . For sufficiently small ϵ this gives $\ln(C_d) - \ln(C_{d+1}) = \tau K_2$. As Grassberger has described, this produces an offset at small ϵ on the usual $\log(C_d(\epsilon))$ vs $\log(\epsilon)$ plot (Fig. 8). Since all the curves intersect at $\epsilon = 1$ this distorts the slopes close to the origin, and for large enough τ the scaling region is driven below the limits of resolution.

Fig. 7. Variation of fractal dimension with delay τ for an embedding dimension of five and $K_{\text{ref}} = 200$. Lines are guides to the eye for quasiperiodic data (Δ) and chaotic data ($\circ$)

Extraction of the dimension from three different data sets is illustrated in Fig. 8 along with their associated Poincaré sections. The scaling regions chosen for the $\log_{10}(C(\epsilon))$ vs. $\log_{10}(\epsilon)$ plots are indicated. The Grassberger algorithm with $K_{\text{ref}} = 500$ and max norm were used. Cases a) and b) correspond to the previous quasi-periodic and chaotic data sets; c) is a particularly difficult instance. The ability of the dimension algorithms to distinguish locked, quasiperiodic and chaotic states was comparable to one's own ability based on original inspection of power spectral density plots and Poincaré sections for the three different kinds of states.

Fig. 8. The left-hand column shows plots of $\log_{10} C(\epsilon)$ vs. $\log_{10} (\epsilon)$ for three data sets. We evaluated the Grassberger dimension by fitting a straight line to the indicated portions of the plots. On the right are the corresponding Poincaré sections. These sections were produced in the same manner as Fig. 4. a) Simple quasiperiodic data, $d = 3-7$; b) Chaotic data, $d = 3-10$; c) An example of a data set where the dimension is difficult to determine, $d = 5-7$.

In Fig. 9 we show the behavior of dimension with increasing R/R_c calculated just as for Fig. 8. One can see an average trend of increasing dimension, as well as rapid

variations produced by periodic windows. This is similar to behavior found in the supercritical circle map [14]. We intend to explore the scaling of dimension with R/R_c and the periodic window structure in the supercritical regime in some future study.

Fig. 9. The Grassberger dimension is shown as a function of normalized Rayleigh number

In our entropy calculations we benefited from a paper by CAPUTO and ATTEN [15]. This prompted us to try the algorithm put forth by GRASSBERGER [13]; we prefer it over TERMONIA'S method [16]. Grassberger's algorithm calculates a quantity K_2 which is a lower bound on the Kolomogorov entropy. We used the maximum norm in our calculations as suggested in refs. [9] and [13]. In Fig. 10 we show plots of $K_{2,d}(e)$ for the same

Fig. 10. $K_{2,d}(e)$ is plotted as function of the embedding dimension d for quasi-periodic (Δ) and chaotic (O) data sets. Note the plateau in the chaotic case

quasiperiodic and chaotic data as Figs. 6-8, with $N_{\text{ref}} = 3000$. The chaotic data gives a conspicuous plateau region and then falls off as the amount of data becomes insufficient, whereas the quasiperiodic data decreases smoothly. A value of $\epsilon = 0.15$ was used (with the total excursion of the time series normalized to unity). Figure 11 shows the behavior of K_2 with varying R/R_c . Three types of points are used. The open and closed circles denote values obtained at $d \approx 100$, $\tau = 7$ sec., $N_{\text{ref}} = 300$, and $\epsilon = 0.15$ (averaged over $90 \leq d \leq 100$). In some cases the data seemed to indicate a small positive entropy but no plateau was found, and these are the open circles. The x points are values obtained by fitting to a plateau region. An attempt to check the scaling with τ showed that the entropy tended to increase somewhat as τ was reduced. This is probably due to the fact that as τ is decreased one examines more phase space points per unit time, which has an effect similar to decreasing ϵ . We are probably underestimating the entropy noticeably for the x cases due to the relatively large value of ϵ .

Fig. 11. K_2 versus normalized Rayleigh number

We are pleased to acknowledge valuable conversations with D. Farmer and E. Jen.

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