PIECEWISE CUBIC INTERPOLATION METHODS

F.N. Fritsch, R.E. Carlson

November 1978

ABSTRACT

This paper deals with interpolation of one-dimensional data using piecewise cubic interpolants. Methods are presented for modifying the derivative values in the Hermite representation in order to eliminate the "bumps" and "wiggles" that frequently plague the more common cubic spline or Akima interpolants. The resulting interpolant is $C^1$, but generally not $C^2$. 

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Present address: Grove City College, Grove City, PA 16127
*Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under contract number W-7405-ENG-48.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED
DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.
DISCLAIMER

 Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
This report consists of a reproduction of a poster prepared for the SIAM 1978 Fall Meeting. A more complete description of our new algorithm is being prepared for publication.

The poster contained a two-dimensional display of methods vs data sets to facilitate comparison of the six methods on four sets of data. This is simulated here by numbering the figures i,j. as follows:

**Data Sets:**

- **i = 1.** LLL data set RPN 12*
- **i = 2.** LLL data set RPN 14*
- **i = 3.** Example 3 from Akima's paper (see page 5).
- **i = 4.** A nonmonotone example. These data are from S. Pruess, "An Algorithm for Computing Smoothing Splines in Tension", *Computing 19* (1978), 365-373.

**Methods:**

- **j = 1.** Cubic Splines.
- **j = 2.** 3-Point Difference Formula.
- **j = 3.** Ellis-McLain Method.
- **j = 4.** Akima Method.
- **j = 5.** Zero Derivatives.
- **j = 6.** A new method by the authors that guarantees a monotone interpolant when the data are monotone.

*Actual data from a radiochemical calculation.*
PIECEWISE CUBIC INTERPOLANT

\[ f(x) = \phi_i(x), \quad x_i \leq x \leq x_{i+1} \quad \text{(cubic polynomial)} \]

\[ \phi_i(x_i) = y_i, \quad \phi_i'(x_{i+1}) = y_{i+1} \quad (i = 1, \ldots, N-1) \]

\[ \phi_i'(x_{i+1}) = d_{i+1} = \phi_{i+1}'(x_{i+1}) \quad (i = 1, \ldots, N-2) \]
Cubic Spline

- $f(x) \in C^2[x_i,x_N]; \ f(x_i) = y_i, \ i=1,...,N$.

- Two degrees of freedom, generally used to specify endpoint first or second derivative values.

- This version uses 3-point (non-centered) difference formulas to approximate end derivatives.

- Interpolants can have unphysical wiggles.

(See Figures i.1)
3-Point Difference

- $d_i$ is the derivative at $x_i$ of the quadratic that passes through $(x_{i-1}, y_{i-1})$, $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$.

- $d_i$ is a convex combination of the slopes of the adjacent data:
  
  $$d_i = \frac{h_i}{h_{i-1} + h_i} \Delta i_{i-1} + \frac{h_{i-1}}{h_{i-1} + h_i} \Delta i$$

  where
  
  $$h_j = x_{j+1} - x_j,$$
  
  $$\Delta j = (y_{j+1} - y_j) / h_j.$$
• $d_i$ is the derivative at $x_i$ of the cubic that passes through $(x_{i-1}, y_{i-1})$, $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$ and provides best (weighted) least squares fit to $(x_{i-2}, y_{i-2})$ and $(x_{i+2}, y_{i+2})$: 

$$\min \left( \frac{\delta_{i-2}^2}{h_{i-2}} + \frac{\delta_{i+2}^2}{h_{i+1}} \right).$$

• Results are generally quite similar to 3PD.


(See Figures i.3)
Akima

- $d_i$ is a convex combination of the slopes of the adjacent data, derived by a geometric argument:

$$d_i = \frac{a_i}{a_i + b_i} \Delta_{i-1} + \frac{b_i}{a_i + b_i} \Delta_i,$$

where

$$a_i = |\Delta_{i+1} - \Delta_i|, \quad b_i = |\Delta_{i-1} - \Delta_{i-2}|.$$


(See Figures 1.4)
Zero Derivatives

- The simplest possible algorithm is to just set $d_i = 0$, $i = 1, \ldots, N$. As noted in the references, this turns out to be piecewise monotone.

- As the examples illustrate, this produces interpolants that are extremely "unphysical". Thus, while being monotone where the data are may be necessary, it is not sufficient to produce interpolants that "look good".


(See Figures 1.5)
Toward a piecewise monotone interpolant that "looks good".

- \( \Delta_i := (y_{i+1} - y_i)/h_i, \ h_i := x_{i+1} - x_i \). If \( \Delta_i = 0 \), \( f(x) \) cannot be monotone on \([x_i, x_{i+1}]\) unless \( d_i = d_{i+1} = 0 \).
- Suppose \( \Delta_i \neq 0 \) and let \( d = d_i/\Delta_i \), \( \beta = d_{i+1}/\Delta_i \).

Then

\[
\begin{align*}
\quad \xi_i (x) &= \frac{\Delta_i}{h_i^2} \left[ (d+\beta-2) (x-x_i)^3 - (2d+\beta-3) h_i (x-x_i)^2 \\
&\quad + \alpha h_i^2 (x-x_i) \right] + y_i, \\
\quad \xi_i' (x) &= \frac{\Delta_i}{h_i^2} \left[ 3(d+\beta-2) (x-x_i)^2 - 2(2d+\beta-3) h_i (x-x_i) \right]
\end{align*}
\]

- Fritsch-Carlson
\[ \varphi''(x) = \frac{2\Delta i}{h_i^2} \left[ 3 (\alpha+\beta-2)(x-x_i) - (2\alpha+\beta-3)h_i \right]. \]

- It is easy to show that a necessary condition for monotonicity of \( \varphi_i(x) \) on \([x_i, x_{i+1}]\) is \( \alpha > 0, \beta > 0 \).

- Because \( \varphi_i'(x) \) is quadratic, monotonicity of \( \varphi_i(x) \) is directly related to the location of the extremum of \( \varphi_i' \):

  \[ x^* = x_i + \frac{h_i}{3} \left( \frac{2\alpha+\beta-3}{\alpha+\beta-2} \right) \quad (\alpha+\beta \neq 2) \]

  and its value there:

  \[ \varphi_i'(x^*) = \phi(\alpha, \beta) \cdot \Delta_i, \]

  \[ \phi(\alpha, \beta) = \alpha - \frac{1}{3} \frac{(2\alpha+\beta-3)^2}{\alpha+\beta-2}. \]

- Thus, we can characterize the monotonicity properties of \( f(x) \) on \([x_i, x_{i+1}]\) by the following diagram.
Monotone because
\( \phi(d, \beta) \geq 0 \)

\[ \beta = \frac{d_i}{\Delta_i} \]

\[ 2d + \beta = 3 \]

\[ \phi(d, \beta) = 0 \]

\[ \phi(d, \beta) = \frac{1}{3} \]

\[ \phi(d, \beta) = \frac{2}{3} \]

\[ d + 2\beta = 3 \]

Monotone because
\( f'(d) \geq \max_{d 
\in \Delta_i} d \cdot \Delta_i \), \( \forall d \in (x_i, x_{i+1}) \).

Monotone because
\( x \neq \phi(x_i, x_{i+1}) \).
• Let \(M\) be the region bounded by \(\phi(\alpha,\beta) = 0\) and the coordinate axes. This is the set of monotone interpolants on \([x_i, x_{i+1}].\)

• Let \(S\) be a subset of \(M\) with the property:
  
  Let \((\alpha, \beta) \in S.\) If \(0 \leq x^* \leq \alpha, 0 \leq \beta^* \leq \beta,\) then \((x^*, \beta^*) \in S.\) (ii) \((\beta, \alpha) \in S\) (symmetry).

  If we adjust \((\alpha, \beta)\) to lie inside \(S\) by decreasing \(x\) and/or \(\beta,\) (i) insures that we do not destroy monotonicity in an adjacent interval.

• Regions \(S\) we have considered:

  \(S_1\) bounded by \(\alpha = 3, \beta = 3,\) and \(\phi(\alpha, \beta) = 1/3.\)
  
  \(S_2\) bounded by \(\alpha = 3, \beta = 3,\) and \(\alpha + \beta = 4.\)
  
  \(S_3\) bounded by \(\alpha + \beta = 3.\)
  
  \(S_4\) bounded by \(2\alpha + \beta = 3\) and \(\alpha + 2\beta = 3.\)

  (All bounded below by coordinate axes.)
Suggested Monotonicity Subregions $\delta_i$
• A family of piecewise monotone, piecewise cubic interpolation schemes:

Step 1. Initialize \( d_i \) to some convex combination of \( \Delta_{i-1} \) and \( \Delta_i \). [For these examples we use 3PD.]

Step 2a. If \( \Delta_i = 0 \), set \( d_i = d_{i+1} = 0 \) and go to next interval.
Step 2b. If \( \Delta_i \neq 0 \), compute \( \alpha \) and \( \beta \).

Step 3a. If \( \alpha \geq 0 \), \( \beta \geq 0 \) and \( (\alpha, \beta) \in S \), go to next interval.
Step 3b. If \( \alpha > 0 \), \( \beta > 0 \) and \( (\alpha, \beta) \notin S \), compute the largest \( \tau \), \( 0 \leq \tau \leq 1 \), such that \( (\tau \alpha, \tau \beta) \in S \). Set \( d_i := \tau d_i \), \( d_{i+1} := \tau d_{i+1} \).
[Here we use \( S = S_3 \).]

Step 3c. If \( \alpha < 0 \) or \( \beta < 0 \) the data are nonmonotone. Possible procedures:
(a) Leave \( d_i \), \( d_{i+1} \) unchanged and go to next interval.
(b) Set \( d_i = d_{i+1} = 0 \). [This insures exact piecewise monotonicity.]
[Procedure (a) was used for these examples.]

(See Figures i.6)
Figure 1.1. Cubic Spline on Data Set 1.
Figure 1.2. 3-Point Difference Formula on Data Set 1.
Figure 1.3. Ellis-McLain Method on Data Set 1.
Figure 1.4. Akima Method on Data Set 1.
Figure 1.5. Zero Derivatives on Data Set 1.
Figure 1.6. Fritsch-Carlson Method on Data Set 1.
Figure 2.1. Cubic Spline on Data Set 2.
Figure 2.2. 3-Point Difference Formula on Data Set 2.
Figure 2.3. Ellis-McLain Method on Data Set 2.
Figure 2.4. Akima Method on Data Set 2.
Figure 2.5. Zero Derivatives on Data Set 2.
Figure 2.6. Fritsch-Carlson Method on Data Set 2.
Figure 3.1. Cubic Spline on Data Set 3.
Figure 3.2. 3-Point Difference Formula on Data Set 3.
Figure 3.3. Ellis-McLain Method on Data Set 3.
Figure 3.4. Akima Method on Data Set 3.
Figure 3.5. Zero Derivatives on Data Set 3.
Figure 3.6. Fritsch-Carlson Method on Data Set 3.
Figure 4.1. Cubic Spline on Data Set 4.
Figure 4.2. 3-Point Difference Formula on Data Set 4.
Figure 4.3. Ellis-McLain Method on Data Set 4.
Figure 4.4. Akima Method on Data Set 4.
Figure 4.5. Zero Derivatives on Data Set 4.
Figure 4.6. Fritsch-Carlson Method on Data Set 4.
NOTICE

"This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately-owned rights."

NOTICE

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.